1

The Concept of Logic

CONCEPTS COVERED IN THIS CHAPTER.–

Logic is essential, not only for computer operations, but also for artificial intelligence. The concepts of logic outlined in this chapter are therefore of fundamental importance.

Beginning with syllogistic reasoning, the exploration examines propositional calculus and its fundamental operations: negation, conjunction, disjunction, conditional, biconditional, as well as their combinations.

The last section focuses on the tools of deductive reasoning.

References: [EXE 59, MEN 64, QUI 84, WOO 80, LIP 83].

1.1. Syllogisms

Syllogisms are a type of logical reasoning that dates back to the time of Aristotle in Ancient Greece. They are constructed from logical propositions, known as premises, which are linked by a conclusion. The conclusions are derived from the logical of these propositions, allowing the deduction of a conclusion from the given premises. In what follows, the example of a classic syllogism will be considered:

(i) all men are mortal,

(ii) all Greeks are men,

(iii) therefore, all Greeks are mortal.

The representation of such a logic is termed "Barbara's syllogism", named after a figure in medieval logic who popularized it. In the above example, there are two

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premises and a conclusion. The first premise is represented by (i), the second by (ii) and the conclusion by (iii).

The described syllogism is considered valid because it respects the rules of formal logic. The conclusion follows necessarily from the two premises and cannot be false if the premises are true. Indeed, each premise of a syllogism must be evaluated according to its truth value to determine whether the conclusion that follows from it is also true or false. In the example, TRUE or FALSE (a truth value) can be assigned to each premise; then it is said that a proposition is a statement which takes one and only one truth value.

EXAMPLE 1.1.– Some illustrations:

"The sky is blue,"
"Money cannot buy happiness,"
"The house is burning and the firefighters are not there,"
"Computer scientists are all crazy,"

are "propositions".

"Eat your soup!,"
"Good heavens!,"
"Where are they all running?"

do not correspond to propositions.

A syllogism, such as the one considered above, can be represented by the following notation:

all Xs are Ys (major premise),
all Zs are Xs (minor premise),
so, all Zs are Ys (conclusion).

And especially,

$$Premises\ \{Every\ X\ is\ Y, Every\ Z\ is\ X)$$

$$Conclusion: every\ Z\ is\ Y$$

The structure is also known as the "canonical form" of a syllogism. The terms X, Y and Z represent categories or classes of things (identified objects), and the syllogism explores the logical relationship between these categories. The canonical form of a syllogism facilitates highlighting the logical structure of reasoning and verifying its validity using the rules of formal logic. Note that a proposition can be placed in one of the following four categories, as shown in Table 1.1.

A	Universal and affirmative	e.g. “all men are mortals”
E	Universal and negative	e.g. “no man is mortal”
I	Particular and affirmative	e.g. “some men are mortal”
O	Particular and negative	e.g. “some men are not mortal”

Table 1.1. *Categories of propositions*

Note that a syllogism is composed of three propositions, each belonging to one of the four classes of propositions (A, E, I and O). The middle term connects the two premises of the syllogism, and there are four possible figures depending on the position of this term. The number of possible categories for a syllogism is $4^3 = 64$ possible combinations (four categories for each of the three propositions). Furthermore, depending on the position of X, Y, Z in the premises, there are four classes of syllogisms (see Figure 1.1). The number of categories of syllogisms then becomes 4^3x4 = 256. For each possible combination of premises, there are two possible formulations (A implies B or B implies A), effectively doubling the number of categories of syllogisms: 512.

category	I	II	III	IV
premise	XY	YX	XY	YX
premise	ZX	ZX	XZ	XZ
conclusion	ZY	ZY	ZY	ZY

Figure 1.1. *Syllogism classes*

In summary, there are 512 possible categories of syllogisms, but the number of admissible syllogisms is smaller due to the presence of “rules of reasoning”.

EXAMPLE 1.2.– Some illustrations:

– A syllogism with two negative premises is not valid. In a valid syllogism, if one of the two premises is negative, the conclusion must be negative.

– In a valid syllogism, if both premises are affirmative, then the conclusion must be affirmative.

– A syllogism is not valid if it contains two particular premises.

These rules, some of which are presented above, actually reduce the number of valid forms of syllogisms to 19.

As previously discussed, syllogisms follow a general form:

(premise 1, premise 2) implies (conclusion).

We can generalize and acknowledge that syllogisms can involve more than two premises, as demonstrated in Lewis Caroll's "kangaroo argument".

EXAMPLE 1.3.– The premises:

1. The only animals in this house are cats.
2. Any animal can become a favorite animal that loves to watch the moon.
3. When I hate an animal, I avoid it.
4. No animal is carnivorous unless it prowls at night.
5. No cat fails to kill mice.
6. No animal pleases me, except those in this house.
7. Kangaroos cannot become pets.
8. Only carnivores kill mice.
9. I hate animals that I do not like.
10. Animals that roam at night always like to look at the moon.

Conclusion: I always avoid kangaroos.

The objective of the chapter is to provide an overview of the mechanisms for reasoning based on the manipulation of mathematical concepts.

According to N. Bourbaki (Elements of the History of Mathematics):

> We consider that the intervention of metamathematics in the presentation of logic and mathematics can and must be reduced to the very elementary part which deals with the handling of operator symbols and deductive criteria.

1.2. Elementary operations of propositional calculus

1.2.1. *Negation*

Consider the following propositions P and Q:

P: Amiens is in France.

Q: it is false that Amiens is in France.

The proposition Q is derived from the proposition P by prefixing the phrase “it is false that” to it. Q is regarded as the negation of P, denoted as ¬P. The symbol “¬” represents negation and can thus be interpreted as “it is false that”. The truth table illustrating negation is simple (see Figure 1.2).

P	¬P
T	F
F	T

Figure 1.2. *Negation truth table*

Expressing negation in a literary manner often poses difficulties. While the previous example, where ¬P translates to “Amiens is not in France” is straightforward, more complex cases exist. To illustrate such complexity, consideration will be given to a few examples.

Beginning with the first:

EXAMPLE 1.4.–

P: all integers are real numbers.

Q_1: some integers are not real numbers.

Q_2: no integer is a real number.

Which of Q_1 and Q_2 represents the negation of P?

After reflection, it becomes evident that P and Q_2 can be false simultaneously. Thus, Q_2 is not the negation of P. Conversely, Q_1 represents ¬P.

Here is a second example:

EXAMPLE 1.5.–

P: all angles can be trisected (i.e. divided into three equal parts) using only the ruler and the compass (notoriously wrong!).

Q_1: no angle can be trisected using only the ruler and compass.

Q_2: there is at least one angle that cannot be trisected using only the ruler and compass.

What is the negation of P, Q_1 or Q_2? P is false because it is impossible to trisect an angle of 37° with only a ruler and compass.

Q_1 is false because we can trisect an angle of 180° with a ruler and compass (see Figure 1.3).

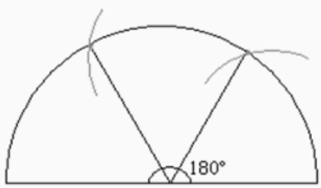

Figure 1.3. *Trisection of an angle of 180°*

It is therefore Q_2 which is the negation of P.

It should be noted that difficulties often arise when propositions contain words such as “All,” “Some” or similar expressions. However, as will be discussed later in this chapter, these difficulties can be resolved by using quantifiers and the rules associated with them.

1.2.2. *Conjunction*

A conjunction is a logical connector that combines two propositions to form a third one, using the word “and” to link clauses. The conjunction of two propositions P and Q is denoted by $P \wedge Q$.

EXAMPLE 1.6.–

P: the number 12 is rational.

Q: the number 12 is positive.

P∧Q: the number 12 is rational and positive.

Conjunction is a logical operation that can be represented by a truth table, as shown in Figure 1.4.

P	Q	P∧Q
T	T	T
T	F	F
F	T	F
F	F	F

Figure 1.4. *Conjunction truth table*

It is important to emphasize that the conjunction P ∧ Q is true only if both propositions P and Q are true. In other words, the conjunction is true only when both initial propositions are simultaneously true. If either proposition is false, then the conjunction will be false.

1.2.3. *Disjunction*

Disjunction is a logical connector that combines two propositions to form a third, using the word "or" to link clauses. The disjunction of two propositions P and Q is denoted by P ∨ Q.

EXAMPLE 1.7.–

P: a price reduction is granted to women.

Q: a price reduction is granted to soldiers.

P∨Q: a price reduction is granted to women or soldiers.

By definition, the disjunction truth table is given in Figure 1.5.

P	Q	P∨Q
T	T	T
T	F	T
F	T	T
F	F	F

Figure 1.5. *Disjunction truth table*

It is important to emphasize that the disjunction P ∨ Q is false only if both propositions P and Q are simultaneously false. The disjunction, which is defined as the *inclusive or* and is represented by the symbol ∨, comes from the Latin "vel" and means "either or both". In other words, if either of the initial propositions is true, or if both propositions are true, then the disjunction will be true.

There is another type of "or", called *exclusive or*, which comes from the Latin "aut" and which means "either one or the other". In the case of the logical operation of "exclusive or", the first row of the truth table would be T, T, F. However, there is no need to create a special symbol for "exclusive or", because it can be expressed using the already defined symbols of conjunction (∧), disjunction (∨) and negation (¬), as shown in Figure 1.6.

P	Q	P∨Q	P∧Q	¬(P∧Q)	(P∨Q)∧(¬(P∧Q))
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Figure 1.6. *"Exclusive or" truth table*

Therefore, the expression (P ∨ Q) ∧ (¬(P ∧ Q)) represents the "exclusive or", where P and Q are two logical propositions. There are other ways of expressing "exclusive or" using different combinations of conjunction, disjunction and negation, which will be discussed later.

1.2.4. *Conditional*

Common language often uses the conditional form: "if P then Q".

This proposition is an implication constructed from the propositions P and Q, denoted as P → Q or "if P then Q". It can be associated with a truth table, which is, by definition, that of Figure 1.7

P	Q	P→Q
T	T	T
T	F	F
F	T	T
F	F	T

Figure 1.7. *Conditional truth table*

Note that the implication P → Q is always true, except when P is true and Q is false. In all other cases, the implication is considered true.

EXAMPLE 1.8.– Consider x and y as angles, along with the following propositions:

P: x and y have their parallel sides.

Q: x = y.

The four rows of the truth table are shown in Figure 1.8.

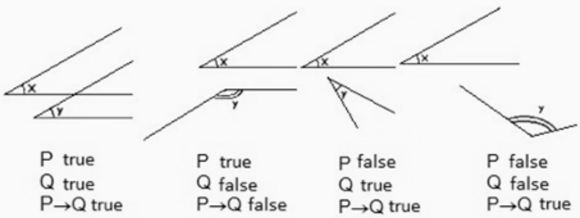

Figure 1.8. *Truth table*

EXAMPLE 1.9.– Consider the following propositions:

P: it is raining.

Q: I will drive you home.

P→Q: if it rains, then I will drive you back.

Note that if P is true (it is raining), this implies that if the proposition Q (I will drive you home) is false, then the conditional P→Q is false. Indeed, if P is true and

Q is false, this means that although it is raining, I am not taking you back by car, which contradicts the implication of the conditional. This situation is "abnormal" and justifies that P→Q is false in this specific case.

1.2.5. *Biconditional*

P	Q	P→Q	Q→P	(P→Q)∧(Q→P)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Figure 1.9. *Truth table*

Let us examine the proposition (P→Q) ∧ (Q→P) whose truth table (see Figure 1.9) is constructed from the previous truth tables (see Figure 1.7). This new operation is denoted as P↔Q and is called *biconditional*. It can be expressed by "P if and only if Q".

The biconditional is true if P and Q have the same truth value, and false otherwise. Therefore, the truth table in Figure 1.10 illustrates this relationship.

P	Q	P↔Q
T	T	T
T	F	F
F	T	F
F	F	T

Figure 1.10. *Biconditional truth table*

By comparing the truth table of the "biconditional" to that of the "exclusive or", we can observe that the "exclusive or" can be represented by ¬(P↔Q).

1.2.6. *Tautologies*

Given two propositions P and Q, let us examine all the binary operations that can be considered and draw up the corresponding truth tables (see Figure 1.11):

– The right half of Figure 1.11 is obtained by taking the negation of the left part.

– Table 14 in Figure 1.11 corresponds to a result which is always true; this is a *tautology*.

– Table 16 corresponds to a result which is always false; this is *a contradiction*. The negation of a tautology is a contradiction.

1

P	Q	P
T	T	T
T	F	T
F	T	F
F	F	F

2

P	Q	Q
T	T	T
T	F	F
F	T	T
F	F	F

3

P	Q	$\neg P$
T	T	F
T	F	F
F	T	T
F	F	T

4

P	Q	$\neg Q$
T	T	F
T	F	T
F	T	F
F	F	T

5

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

6

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

7

P	Q	$\neg(P \wedge Q)$
T	T	F
T	F	T
F	T	T
F	F	T

8

P	Q	$\neg(P \vee Q)$
T	T	F
T	F	F
F	T	F
F	F	T

9

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

10

P	Q	$Q \rightarrow P$
T	T	T
T	F	T
F	T	F
F	F	T

11

P	Q	$\neg(P \rightarrow Q)$
T	T	F
T	F	T
F	T	F
F	F	F

12

P	Q	$\neg(Q \rightarrow P)$
T	T	F
T	F	F
F	T	T
F	F	F

13

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

14

P	Q	T
T	T	T
T	F	T
F	T	T
F	F	T

15

P	Q	$\neg(P \leftrightarrow Q)$
T	T	F
T	F	T
F	T	T
F	F	F

16

P	Q	$\neg T$
T	T	F
T	F	F
F	T	F
F	F	F

Figure 1.11. *Possible binary operations*

From the propositions P, Q, …, and the elementary operations $\wedge$, $\vee$, $\neg$, it is possible to construct propositional forms f(P, Q, …). Depending on the values of P, Q, …, the form can be true or false. If f(P, Q, …) is always true independently of the truth values of P, Q, …, the form f is a tautology. Conversely, if f(P, Q, …) is always false independently of the truth values of P, Q, …, the form f is a contradiction.

The notion of tautology is important in propositional logic because it simplifies complex logical expressions and verifies the validity of reasoning by establishing that a proposition is a tautology.

EXAMPLE 1.10.– $P\vee(\neg P)$, $\neg(P\wedge(\neg P))$, $(\neg(P\wedge Q))\leftrightarrow((\neg P)\vee(\neg Q))$ are tautologies. This can be simply by drawing up the truth tables (see the different truth tables grouped in Figure 1.12).

The following propositions are contradictions: $\neg(P\vee(\neg P))$, $P\wedge(\neg P)$, $(P\wedge Q)\leftrightarrow((\neg P)\vee(\neg Q))$.

The first two are contradictions because they are the negations of the first two previously presented tautologies. The third proposition can be verified by setting up a truth table.

P	$\neg P$	$P\vee(\neg P)$
T	F	T
F	T	T

P	$\neg P$	$\neg(P\wedge(\neg P))$
T	F	T
F	T	T

P	Q	$P\wedge Q$	$\neg(P\wedge Q)$	$\neg P$	$\neg Q$	$(\neg P)\vee(\neg Q)$	$(\neg(P\wedge Q))\leftrightarrow((\neg P)\vee(\neg Q))$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Figure 1.12. *Representation of the different truth tables*

A tautology of the form $P\leftrightarrow Q$ is a logical equivalence. According to the definition of $\leftrightarrow$, this means that P and Q have the same truth tables. Therefore, it is noted that

$$P \Leftrightarrow Q \text{ or more simply } P \equiv Q$$

Thus, in the previous examples, the logical proposition $\neg(P\wedge Q)$ is equivalent to the logical proposition $(\neg P)\vee(\neg Q)$, denoted as $\neg(P\wedge Q) \equiv (\neg P)\vee(\neg Q)$. It means that these two propositions have the same truth tables and are therefore logically equivalent.

The following table lists the main basic logical equivalences:

Idempotence	$P\vee P \equiv P$		$P\wedge P \equiv P$	
Associativity	$(P\vee Q)\vee R \equiv P\vee(Q\vee R)$		$(P\wedge Q)\wedge R \equiv P\wedge(Q\wedge R)$	
Commutativity	$P\vee Q \equiv Q\vee P$		$P\wedge Q \equiv Q\wedge P$	
Distributivity	$P\vee(Q\wedge R) \equiv (P\vee Q)\wedge(P\vee R)$		$P\wedge(Q\vee R) \equiv (P\wedge Q)\vee(P\wedge R)$	
Identify	$P\vee f \equiv P$	$P\wedge v \equiv P$	$P\vee v \equiv v$	$P\wedge f \equiv f$
Complementarity	$P\vee(\neg P)$	$P\wedge(\neg P) \equiv f$	$\neg v \equiv f$	$\neg f \equiv v$
Involution	$\neg(\neg P) \equiv P$			
Morgan's laws	$\neg(P\vee Q) \equiv (\neg P)\wedge(\neg Q)$		$\neg(P\wedge Q) \equiv (\neg P)\vee(\neg Q)$	

In this table, t denotes a tautology and f denotes a contradiction. The table only shows the symbols $\wedge$, $\vee$, $\neg$ which is sufficient since the other symbols can be expressed in terms of these. Logical equivalences can contain all symbols.

Note that these equivalences are fundamental in propositional logic and can be used to simplify complex logical expressions.

EXAMPLE 1.11.– The equivalence $P\rightarrow Q \equiv (\neg P)\vee Q$ is evident. This means that $P \rightarrow Q$ can be expressed as $(\neg P)\vee Q$ using negation and disjunction. This equivalence is commonly known as the law of implication or law of detachment (see the truth table shown in Figure 1.13).

P	Q	P→Q	¬P	(¬P)∨Q
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Figure 1.13. *Truth table*

Consequences:

– If it rains, I open my umbrella ⇔ It does not rain or I open my umbrella.

– If the triangle is equilateral, it is isosceles ⇔ The triangle is not equilateral or it is isosceles.

– If the man is single, he is unhappy ⇔ The man is not single or he is unhappy.

Another interesting tautological expression is represented by the form P→Q. When P→Q holds as a tautology, it signifies that P logically implies Q, symbolized as P ⇒ Q, indicating that Q holds true whenever P does. This translates to “If P is true, then Q is also true.” Consequently, if P is false, no information can be drawn regarding Q. Fundamentally, P can be viewed as a sufficient condition for Q.

P	Q	R	P→Q	Q→R	(P→Q)∧(Q→R)	P→R	(P→Q)∧(Q→R)→(P→R)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Figure 1.14. *Truth table*

A syllogism is a specific instance of logical implication. Consider, for example, the following syllogism (transitivity): P → Q, Q → R, therefore P → R which can be represented as follows: ((P→Q)∧(Q→R)) → (P→R).

Examining the truth table in Figure 1.14, it is evident that (P→Q)∧(Q→R) ⇒ (P→R).

1.3. Tools of deductive reasoning

In the domain of mathematics, the primary objective is to move from one true proposition to another through *deductive reasoning,* leading to a *demonstration.* This process is facilitated by employing various tools, which can be categorized into three main categories:

– *The replacement rule*: this rule permits the substitution of a proposition with another that is logically equivalent, maintaining the validity of the demonstration.

– *The substitution principle*: this principle enables the replacement of a variable with an equivalent expression, preserving the integrity of the demonstration.

– *Inference rules*: these rules facilitate the derivation from one or more propositions to a new proposition, employing logical forms such as implication or contrapositive.

These fundamental tools in mathematics are instrumental in constructing rigorous and precise demonstrations.

The *replacement rule* can be formulated as:

> REPLACEMENT RULE. *In a tautology, the replacement rule permits the substitution of a proposition with another having the same truth value.*

This is evident because, in a tautology, the truth values of the component propositions do not determine the truth value of the form, which is consistently TRUE. Thus, one proposition can be substituted with another of the same truth value without altering the truth value of the tautology. Hence, the tautology $[(P\rightarrow(Q\rightarrow R)]\leftrightarrow[(P\wedge Q)\rightarrow R)]$ transforms, by replacing R with P, into a new tautology: $[(P\rightarrow(Q\rightarrow P)]\leftrightarrow[(P\wedge Q)\rightarrow P]$.

The principle of substitution operates differently:

> PRINCIPLE OF SUBSTITUTION.– *For a propositional form* f (P, Q, …), *if* P *and* P' *are logically equivalent*, then f(P, Q, …) *and* f(P', Q, …) *are also logically equivalent, i.e.* $P \equiv P' \Rightarrow f(P, Q, \ldots) \equiv f(P', Q, \ldots)$.

> EXAMPLE 1.12.– Illustrating the principle of substitution with the following example, consider the propositional form $\neg(P\wedge Q) \rightarrow P$ (which is false). Using Morgan's law, knowing that $\neg(P\wedge Q) \equiv (\neg P)\vee(\neg Q)$, hence deducing $(\neg P)\vee(\neg Q)\rightarrow P$.

> EXAMPLE 1.13.– To demonstrate the distinction between the replacement rule and the substitution principle, Morgan's law $\neg(P\vee Q) \equiv (\neg P)\wedge(\neg Q)$ can be derived from Morgan's law $\neg(P\wedge Q) \equiv (\neg P)\vee(\neg Q)$. Begin with the hypothesis that $\neg(P\wedge Q)\leftrightarrow(\neg P)\vee(\neg Q)$ is a tautology. Replace P by $\neg P$: this simplifies to the tautology $\neg((\neg P)\wedge Q)\leftrightarrow(\neg(\neg P))\vee(\neg Q)$. Replacing Q by $\neg Q$: this provides the tautology $\neg((P)\wedge(\neg Q))\leftrightarrow(\neg(\neg P))\vee(\neg(\neg Q))$.
>
> With $\neg(\neg P) \equiv P$, and $\neg(\neg Q) \equiv Q$, and according to the substitution principle and the above result, thus $\neg((\neg P)\wedge(\neg Q)) \equiv P\vee Q$.
>
> Let us then consider the propositional form $\neg(P\vee Q)$. According to the principle of substitution and the previous line: $\neg(P\vee Q) \equiv (\neg((\neg P)\wedge(\neg Q)))$, or again by substitution $\neg(P\vee Q) \equiv (\neg P)\wedge(\neg Q)$.

The *rules of inference* are logical implications listed as follows:

– *Modus ponens* rule: if P is true and (P→Q) is true, then Q is true.

– *Conjunctive inference* rule: if P is true and Q is true, then P∧Q is true.

– *Conjunctive simplification* rule: if P∧Q is true, then P is true.

– *Hypothetical syllogism* rule: if (P→Q) is true and (Q→R) is true, then (P→R) is true.

– *Modus tollens* rule: if (P→Q) is true and ¬Q is true, then ¬P is true.

– Rule *of inference by case*: if (A→Q) is true and (B→Q) is true, then (A∨B)→Q is true.

All these rules are easily demonstrated using truth tables.

EXAMPLE 1.14.– Demonstrations employing truth tables can be found in Figures 1.15–1.17.

P	Q	P→Q	(P∧(P→Q))→Q
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

Figure 1.15. *Modus ponens*

P	Q	P→Q	¬Q	(P→Q)∧(¬Q)	¬P	((P→Q)∧(¬Q))→¬P
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Figure 1.16. *Modus tollens*

A	B	Q	A→Q	B→Q	A∨B	(A∨B)→Q	(A→Q)∧(B→Q)	((A→Q)∧(B→Q))→((A∨B)→Q)
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F	T
F	F	T	T	T	F	T	T	T
F	F	F	T	T	F	T	T	T

Figure 1.17. *Inference by case*

1.4. Quantification

Consider the following syllogism:

– All men are mortal.

– Socrates is a man.

– So, Socrates is mortal.

It is practical to use functional symbolism:

– H(Socrates) will mean "Socrates is a man."

– M(Socrates) will mean "Socrates is mortal."

– H(x) will mean "x is a man."

– M(x) will mean "x is mortal."

Consider that for all the values of the variable x, H(x) represents a proposition that can be either true or false:

– H(Napoleon) is true.

– H(Eiffel Tower) is false.

H(x) is termed a *predicate,* specifically *monadic* as it contains only one variable.

EXAMPLE 1.15.– P(x): 2x + 1 = 7

If x = 3, P(x) is true.

If x = 8, P(x) is false.

EXAMPLE 1.16.– Q(x) : $(x+1)^2 = x^2 + 2x + 1$

Q(x) is always true, regardless of the value assigned to x.

In the last example, Q(x) is true for all x. This fact can be expressed using the *universal quantifier* ∀x.

∀x Q(x) signifies "for all x, Q(x) holds true"

Using the universal quantifier, the syllogism concerning Socrates can be expressed as follows:

$\forall x\ H(x) \rightarrow M(x)$,
H(Socrates),
So M(Socrates).

Another quantifier is the *existential quantifier*, denoted by ∃, which means "there is at least one". Thus, in the context of example 1.15, ∃ x P(x) could be used to express "there exists at least one x such that 2x + 1 = 7".

Similarly, the assertion "Some men are intelligent" can be expressed using predicates:

H(x): x is a man.
I(x): x is intelligent.
under the form $\exists x\ [H(x) \wedge I(x)]$.

Here is a suggestion for improvement:

Note the presence of brackets (or parentheses or braces) in the application of logical quantifiers, indicating their domain of action or scope. By using the universal (∀) and existential (∃) quantifiers, we can succinctly express the four categorical forms A, E, I and O of traditional logic. Now, examine the following predicate: B(x): "x is white". Suppose the universe of x is finite, composed of three objects a, b and c for simplicity. In such a scenario, it becomes straightforward to verify the following equivalences:

$$\forall x\ B(x) \equiv B(a) \wedge B(b) \wedge B(c) \qquad [1.1]$$

$$\exists x\ B(x) \equiv B(a) \vee B(b) \vee B(c) \qquad [1.2]$$

A	Universal and affirmative	e.g. all men are intelligent	$\forall x\ [H(x) \rightarrow I(x)]$
E	Universal and negative	e.g. no man is intelligent	$\forall x\ [H(x) \rightarrow \neg I(x)]$
I	Particular and affirmative	e.g. some men are intelligent	$\exists x\ [H(x) \wedge I(x)]$
O	Particular and negative	e.g. some men are not intelligent	$\exists x\ [H(x) \wedge \neg I(x)]$

Table 1.2. *Categories of propositions*

Having established this equivalence, taking the negation of both sides of equation [1.1] yields:

$$\neg[\forall x\ B(x)] \equiv \neg[B(a) \wedge B(b) \wedge B(c)]$$

$$\text{Or } \neg[\forall x\ B(x)] \equiv (\neg B(a)) \vee (\neg B(b)) \vee (\neg B(\chi)) \qquad \text{(Morgan's law)}$$

or by replacing B by ¬B,

$$\neg[\forall x\ \neg B(x)] \equiv B(a) \vee B(b) \vee B(c) \equiv \exists x\ B(x) \text{ according to [1.2]}$$

Thus, $\exists x\ B(x) \equiv \neg[\forall x\ \neg B(x)]$ implies, on the one hand, $\exists x\ \neg B(x) \equiv \neg[\forall x\ B(x)]$ and, on the other hand, $\neg(\exists x\ B(x)) \equiv \forall x\ \neg B(x)$. This demonstrates the process of negating of a logical expression with a quantifier. Additionally, this result holds true for any universe

$$\neg[\forall x\ B(x)] \equiv \exists x\ \neg B(x)$$

$$\neg[\exists x\ B(x)] \equiv \forall x\ \neg B(x)$$

Observing the effect of negation indicates the exchange of the symbols ∀ and ∃. This observation enables us to obtain the negation of propositions containing "All", "Some", "None", etc.

To illustrate the application of these exchange rules, consider the following examples:

EXAMPLE 1.17.– Let P represent the proposition: "all that glitters is not gold", which can be expressed as $\forall x\ [B(x) \rightarrow \neg O(x)]$, where B(x) denotes that the object x shines, and O(x) denotes that object x is gold. The negation of P can be derived by applying the rules of negation and quantifier exchange:

$$\neg\{\forall x\ [B(x) \rightarrow \neg O(x)]\} \equiv \exists x\ \neg[B(x) \rightarrow \neg O(x)] \equiv \exists x\ [B(x) \wedge O(x)]$$

Thus, the negation of P becomes the proposition: "there exists at least one object which shines, and which is made of gold". In this case, ¬P can then be expressed as $\exists\ x\ [B(x) \wedge O(x)]$, signifying that some shining objects are made of gold.

EXAMPLE 1.18.– Let Q represent the proposition: "the rule applies to everyone", expressed as $\forall x\ [B(x) \rightarrow \neg O(x)]$, where R(x) denotes that the rule applies to x. The negation of Q can be derived by applying the rules of negation and quantifier exchange: $\neg\forall\ x\ R(x) \equiv \exists\ x\ \neg R(x)$. Thus, the negation of Q becomes the proposition: "there exists at least one person to whom the rule does not apply". Hence, $\neg Q$ can then be represented as $\exists x\ \neg R(x)$, indicating that the rule does not apply to certain individuals.

EXAMPLE 1.19.– Let S represent the proposition: "Every polygon has an exterior and an interior," expressed as $\forall x\ [P(x) \rightarrow I(x) \wedge E(x)]$, where P(x) denotes that object "x" is a polygon, I(x) denotes that object "x" has an interior and E(x) denotes that object "x" has an exterior. The negation of S is formulated as follows:

$$\neg[\forall x\ [P(x) \rightarrow I(x) \wedge E(x)]] \equiv \exists x\ \neg[P(x) \rightarrow I(x) \wedge E(x)] \equiv \exists x\ [\neg P(x) \vee (I(x) \wedge E(x))]$$

$$\equiv \exists x\ [P(x) \wedge (\neg(I(x) \wedge E(x)))] \equiv \exists x\ [P(x) \wedge ((\neg I(x)) \vee (\neg E(x)))]$$

$$\equiv \exists x\ [[P(x) \wedge (\neg I(x))] \vee [P(x) \wedge (\neg E(x))]]$$

Thus, $\neg S$ implies that there exist polygons with either no interior or no exterior.

Consider the expression $\exists x\ [F(x) \vee G(x)]$ where, for example, F(x) represents "*x* is a saint" and G(x) represents "x is a genius". It is evident that there are individuals who are either saints or geniuses, indicating the existence of individuals who are saints or individuals who are geniuses, such that:

$$\exists x\ [F(x) \vee G(x)] \equiv [\exists x\ F(x)] \vee [\exists x\ G(x)])$$

the generalization to be accepted. Similarly,

$$\forall x\ [F(x) \wedge G(x)] \equiv [\forall x\ F(x)] \wedge [\forall x\ G(x)]$$

1.5. Prenex forms and pure forms

A compound proposition may contain forms such as $\exists x\ [P \wedge F(x)]$, where P is a proposition unrelated to the object x and F(x) is a predicate. Clearly, the scope of the quantifier can be restricted, allowing us to write, for example:

$$\exists x\ [P \wedge F(x)] \equiv P \wedge [\exists x\ F(x)]$$

Similarly,

$$\forall x\ [P \wedge F(x)] \equiv P \wedge [\forall x\ F(x)]$$
$$\exists x\ [P \vee F(x)] \equiv P \vee [\exists x\ F(x)]$$
$$\forall x\ [P \vee F(x)] \equiv P \vee [\forall x\ F(x)]$$

Then, the following deduction can be made:

$$\exists x\, [P\rightarrow F(x)] = \exists x\, [(\neg P)\vee F(x)] = (\neg P)\vee[\exists x\, F(x)] = P \rightarrow \exists x\, F(x)$$

$$\forall x\, [P\rightarrow F(x)] = \forall x\, [(\neg P)\vee F(x)] = \neg P\vee[\forall x\, F(x)] = P \rightarrow \forall x\, F(x)$$

$$\exists x\, [P\rightarrow F(x)] \equiv P \rightarrow \exists x\, F(x)$$
$$\forall x\, [P\rightarrow F(x)] \equiv P \rightarrow \forall x\, F(x)$$

Nevertheless, the following results are less evident:

$$\exists x\, [F(x)\rightarrow P] \equiv \forall x\, F(x) \rightarrow P$$
$$\forall x\, [F(x)\rightarrow P] \equiv \exists x\, F(x) \rightarrow P$$

Indeed, $\exists x\, [F(x)\rightarrow P] \equiv \exists x\, [\neg F(x)\vee P) \equiv [\exists x\, \neg F(x)]\vee P \equiv \neg[\forall x\, F(x)]\vee P \equiv \forall x\, F(x)\rightarrow P$, with a similar demonstration for the second result.

A compound proposition can also involve multiple predicates, necessitating the distinction of several variables. For instance, $\forall x\, [F(x)\rightarrow\exists y[G(y)\wedge H(x)]$ involves two variables, x and y.

Through propositions, predicates, logical operations and quantifiers, schemas can be formed. These schemas are called *monadic* if the predicates contain only one variable:

$\forall x\, [F(x)\rightarrow \exists y\, [G(y)\wedge H(x)]$ $F(x) \rightarrow \exists y\, [G(y)\wedge H(x)]$ $\exists y\, [G(y)\wedge H(x)]$
$G(y)\wedge H(x)$ are monadic schemas.

In a schema, a variable is *free* if no quantifier applies to it; otherwise, it is *bound.* A schema is *open* if it contains at least one free variable; otherwise, it is *closed.* Thus, the schema $\forall x\, [F(x) \wedge \exists y\, [G(y)\wedge H(x)]]$ is closed, as all variables are linked to a quantifier. Conversely, the schemas $F(x) \rightarrow \exists y\, [G(y)\wedge H(x)]$, $\exists y\, [G(y)\wedge H(x)]$, $G(y)\wedge H(x)$ are open, since the variable x is free and not bound to a quantifier.

In a schema, quantifiers can be manipulated in two opposing ways:

– Broadening the scope of a quantifier leads to a *prenex form.* For example, the proposition "$\exists x\, \forall y\, (P(x,y))$" can be transformed into "$\forall y\, \exists x\, (P(x,y))$".

– Restricting the scope of a quantifier leads to a *pure form.* For example, the proposition "$\forall x\, \exists y\, (P(x,y))$" can be transformed into "$\exists y\, \forall x\, (P(x,y))$".

In both scenarios, the previously defined passing rules can be applied. However, it is important to note that there are no passing rules for the biconditional operator ($\leftrightarrow$). Thus, it becomes necessary to "eliminate" biconditionals by substituting them with equivalent expressions. Let us consider the following examples to illustrate the application of the aforementioned rules:

EXAMPLE 1.20.– Consider the schema $P \leftrightarrow \forall x\ [F(x) \rightarrow \exists y\ [F(y) \wedge G(x)]]$, which is equivalent to $\{P \rightarrow \forall x\ [F(x) \rightarrow \exists y\ [F(y) \wedge G(x)]\} \wedge \{\forall x\ [F(x) \rightarrow \exists y\ [F(y) \wedge G(x)]] \rightarrow P\}$. To prevent ambiguity, this schema is rewritten as follows:

$$\{P \rightarrow \forall x\ [F(x) \rightarrow \exists y\ [F(y) \wedge G(x)]\} \wedge \{\forall z\ [F(z) \rightarrow \exists t\ [F(t) \wedge G(zx)]] \rightarrow P\}.$$

This diagram will be transformed into the prenex form. Using the passage rules, the successive steps are as follows:

$$\{\forall x\ [P \rightarrow \exists y\ [F(x) \rightarrow F(y) \wedge G(x)]]\} \wedge \{\exists z\ [\exists t\ [F(z) \rightarrow F(t) \wedge G(z)]] \rightarrow P\}$$

$$\{\forall x\ \exists y\ [P \text{ -> } (F(x) \rightarrow F(y)) \wedge G(x)]\} \wedge \{\exists z\ \forall t\ [F(z) \rightarrow F(t) \wedge G(z)] \rightarrow P]\}$$

$$\forall x\ \exists y\ \exists z\ \forall t\ [P \rightarrow (F(x) \rightarrow F(y) \wedge G(x))] \wedge [(F(z) \rightarrow F(t) \wedge G(z)) \rightarrow P]$$

The diagram is thus converted into the prenex form.

EXAMPLE 1.21.– Consider the schema: $\{\forall x\ [(\exists y\ (F(x) \leftrightarrow G(y))) \vee (\exists y\ F(y))]\} \wedge [\forall x\ (F(x) \vee G(x))]$. After modifying the variable letters for safety, it becomes:

$$\{\forall x\ [(\exists y\ (F(x) \leftrightarrow G(y))) \vee (\exists z\ F(z))]\} \wedge [\forall t\ (F(t) \vee G(t))].$$

First, the diagram will be converted in the prenex form:

$$\exists y\ (F(x) \leftrightarrow G(y)) \vee (\exists z\ F(z)) \equiv \exists y\ \exists z\ (F(z) \leftrightarrow G(y)) \vee F(z);$$
$$\text{hence, } \forall x\ \exists y\ \exists z\ \forall t\ [(F(x) \leftrightarrow G(y)) \vee F(z)] \wedge [F(t) \vee G(t)]$$

This can be summarized in the following prenex form:

$$\forall x\ \exists y\ [(F(x) \leftrightarrow G(y)) \vee F(z)] \wedge [F(t) \vee G(x)]$$

Now, purify the induced schema by:

$$\exists y\ (F(x) \leftrightarrow G(y)) \equiv \exists y\ \{(F(x) \rightarrow G(y)) \wedge (G(y) \rightarrow F(x))]$$

$$\equiv \exists y\ \{\neg F(x) \vee G(y)) \wedge (\neg G(y) \vee F(x))\}$$

$\equiv \exists y\ \{(F(x)\wedge G(y))\vee(\neg F(x)\wedge\neg G(y))\vee(\neg F(x)\wedge F(x))\vee(G(y)\wedge\neg G(y))\}$

The last two terms evaluate to "False", leading to the following expression:

$\exists y\ (F(x) \leftrightarrow G(y)) \equiv (\exists y\ (F(x)\wedge G(y)))\vee(\exists y\ (\neg F(x)\wedge\neg G(y))) \equiv (F(x)\wedge(\exists y\ G(y))\vee(\neg F(x)\wedge(\exists y\neg G(y)))$

Therefore, the expression is: $\forall x\ \{(F(x)\wedge(\exists y\ G(y)()\vee(\neg F(x)\wedge(\exists y\ \neg G(y)()\vee(\exists z\ F(z))\}\wedge[\forall t\ F(t)\vee G(t)]$

However,

$(F(x)\wedge(\exists y\ G(y)))\vee(\neg F(x)\wedge(\exists y\ \neg G(y)))$

$\equiv (F(x)\vee(\neg F(x)))\wedge((\exists y\ G(y))\vee(\exists y\neg G(y)))\wedge(F(x)\vee(\exists y\ \neg G(y)))\wedge((\exists y\ (y))\vee\neg F(x))$ which induces

$\forall x\ [(F(x)\wedge(\exists y\ G(y)))\vee(\neg F(x)\wedge(\exists y\ \neg G(y)))\vee(\exists z\ F(z))]$

$\equiv \forall x\ [(F(x)\vee(\exists y\ \neg G(y)))\wedge((\exists y\ G(y))\vee\neg F(x))]$

$\equiv \forall x\ (F(x)\vee(\exists y\ \neg G(y)))\wedge \forall x\ (((\exists y\ G(y)))\vee\neg F(x))$

$\equiv (\forall x\ F(x)\vee(\exists y\ \neg G(y)))\wedge((\exists y\ G(y))\vee\forall x\ \neg F(x))$

Finally, the pure form is achieved:

$((\forall x\ F(x))\vee(\exists y\ \neg G(y)))\wedge((\forall x\ \neg F(x))\vee(\exists y\ G(y)))\vee(\exists z\ F(z))\wedge(\forall t\ (F(t)\vee G(t)))$