Chapter 1

Electrostatic Field in Vacuum

1.1. Electric charges

We now know that matter is made up of negatively charged electrons and positively charged protons, that is, the Universe contains two types of matter that we can call *negative* and *positive*. Particles of the same species repel each other and those of different species attract each other by a force called *electrostatic* or *Coulomb force*, similar to the gravitational force, which varies inversely with the square of the distance. However, the electrostatic force is much more intense than the gravitational force. The existence of these two types of charges was revealed in Franklin's experiments. For example, when a glass rod is rubbed with a dry cloth, some electrons from the glass shift onto the cloth. The cloth is then negatively charged, as it has more electrons than protons. The glass rod becomes positively charged because it has lost electrons (excess protons). The glass rod is then said to undergo an *electrification effect*. If this operation is repeated with a wax rod, the same electrification phenomenon is observed, except that this time, the electrons shift from the cloth to the wax. The wax rod is thus negatively charged. In addition, if two sufficiently electrified glass rods are brought together, they repel each other, that is, there is *repulsion*. The same repulsion is observed with two wax rods. On the other hand, if a glass rod is brought near a wax rod, they attract each other. It can thus be concluded that charges with the same sign repel each other and those with opposite signs attract each other. If the positively charged glass rod is brought close to a metallic ball attached to a wire (free to move), the electrons in the ball are attracted to the area on its surface that is close to the glass rod, leaving the other side of the ball positively charged: this is the phenomenon of *electrostatic induction*. An attractive force then appears between the ball and the glass rod. When it comes into contact with the ball, the positive charges on the rod are neutralized by part of the negative charges on the ball. This allows the ball-rod assembly to conserve a net positive charge. The ball, which is now positively charged, is then repelled by another positively charged rod. This transfer of charges is called the *conduction effect*. The metallic ball can also be called a *conductor*, as the charges on the ball are easily induced and conducted.

An electron bears an electric charge $-e$, whereas a proton has an electric charge $+e$. For a given quantity of ordinary matter, we can proceed with an algebraic sum of the elementary electric charges within the material. We therefore conclude that every body contains an *integer* quantity of an electric charge, denoted by e. This elementary charge represents the smallest possible value of an electric charge. This is the *charge quantification principle*. This quantization of electric charges means that any charged body is presented as a discontinuous distribution of its constituent charges. On a large scale, this distribution of constituent charges can be considered to be continuous (the elementary charges are very small). For an insulated system (no material can cross the system's boundaries), the total electric charge, that is, the algebraic sum of positive and negative charges present in the body at any given moment, remains constant. Charges are neither created nor destroyed.

1.2. Electrostatic force

Between two electric point-like charges q and q', located, respectively, at points $M\,(x, y, z)$ and $M'\,(x', y', z')$, a force is exerted which is inversely proportional to the square of the distance separating them and is carried by the line joining the two points (central force):

$$\overrightarrow{\mathbf{F}}_q = -\overrightarrow{\mathbf{F}}_{q'} = \frac{1}{4\pi\varepsilon_0} \frac{qq'\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} \qquad [1.1]$$

where $\overrightarrow{\mathbf{r}}$ (respectively, $\overrightarrow{\mathbf{r}}'$) is the vector that joins the origin O and the point $M\,(x, y, z)$ (respectively, $M'\,(x', y', z')$), as depicted in Figure 1.1. The forces applied to q and q' are of the same magnitude, but opposite in direction. The quantity ε_0 is called the *electric permittivity of vacuum* or *vacuum permittivity* and is related to the *speed of light* in vacuum c and the *magnetic permeability of vacuum* (or *vacuum permeability*) μ_0 through the relation $\varepsilon_0 = 1/\mu_0 c^2$. $\overrightarrow{\mathbf{F}}_q$ (respectively, $\overrightarrow{\mathbf{F}}_{q'}$) is the force applied on the charge q (respectively, q'). In particular, if $qq' > 0$, that is, of the same sign (respectively, $qq' < 0$, i.e. of the opposite sign), the force is attractive (respectively, repulsive). This is *Coulomb's law*.

According to the *superposition principle*, if there are several point-like charges $q'_1, \ldots, q'_n$ located, respectively, at $\overrightarrow{\mathbf{r}_1}', \ldots, \overrightarrow{\mathbf{r}_n}'$, the total force applied on the charge q is the resultant of all individual forces:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q'\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}_i}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}_i}'\right|^3} \qquad [1.2]$$

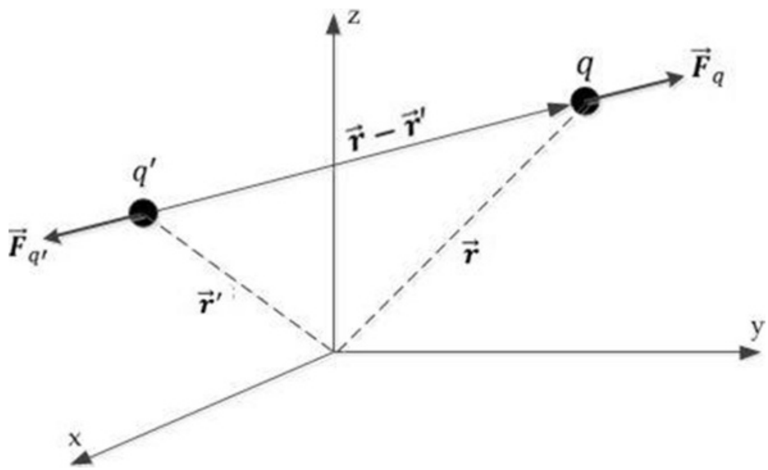

Figure 1.1. *Electrostatic force between two point-like charges q and q'*

EXAMPLE 1.1. (Multi-charge System).– Two identical electric charges, denoted here by q, located at A and B on the Oz axis, are separated by a distance $2d$. The origin O is chosen in the middle of the segment AB. Let us calculate the total force exerted on a test charge q' located at a point with coordinates $(x_0, 0, 0)$ on the Ox axis, as depicted in Figure 1.2. Using the superposition principle, we obtain:

$$\vec{\mathbf{F}} = \frac{q'q\left(x_0\,\vec{\mathbf{i}} - d\,\vec{\mathbf{k}}\right)}{4\pi\varepsilon_0\left(x_0^2+d^2\right)^{3/2}} + \frac{q'q\left(x_0\,\vec{\mathbf{i}} + d\,\vec{\mathbf{k}}\right)}{4\pi\varepsilon_0\left(x_0^2+d^2\right)^{3/2}} = \frac{2q'qx_0\,\vec{\mathbf{i}}}{4\pi\varepsilon_0\left(x_0^2+d^2\right)^{3/2}} \qquad [1.3]$$

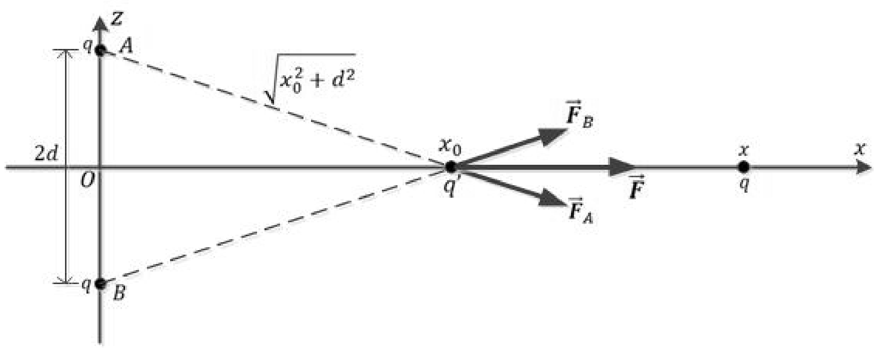

Figure 1.2. *Superposition principle for a system with multiple charges*

It is seen that the resultant force at $(x_0, 0, 0)$ only has a non-zero component along the Ox axis. This property can be explained by the fact that the resulting distribution is rotational symmetrical around Ox. In simpler terms, Ox is a symmetry axis. Note that if the charges at points A and B were of opposite polarities, then axis O would be an antisymmetry axis for the system and the total force exerted on charge q' would be normal to Ox. We will return to these properties in subsequent sections. At the origin, O $(x_0 = 0)$, the force is zero (the forces applied by the charges are mutually annihilated); this is an unstable equilibrium position for the system: when moving away from $x_0 = 0$, q' tends to move further away. At a point that is very far from the origin, that is, when $x_0 \gg d$, the force $\overrightarrow{\mathbf{F}}$ takes the approximate value: $\overrightarrow{\mathbf{F}} \simeq \frac{q'(2q)\,\overrightarrow{\mathbf{i}}}{4\pi\varepsilon_0 x_0^2}$; this corresponds to a force applied by a charge $2q$ concentrated at O. The force [1.3] reaches its maximum at point $x_0 = \frac{d}{\sqrt{2}}$, that is, $\overrightarrow{\mathbf{F}}_{\text{max}} = \frac{q'q\,\overrightarrow{\mathbf{i}}}{3\sqrt{3}\pi\varepsilon_0 d^2}$. Let us now add another charge q at the point $(x, 0, 0)$, such that $x > x_0$. The total force becomes:

$$\overrightarrow{\mathbf{F}} = \frac{q'q\,\overrightarrow{\mathbf{i}}}{4\pi\varepsilon_0}\left(\frac{2x_0}{\left(x_0^2 + d^2\right)^{3/2}} - \frac{1}{\left(x - x_0\right)^2}\right) \qquad [1.4]$$

This force cancels out for $x = x_0 + \frac{1}{\sqrt{2}}\frac{\left(x_0^2+d^2\right)^{3/4}}{\sqrt{x_0}}$. The position x_0 is here a stable equilibrium position for the system: when moving away from x_0, q' tends to return to x_0. ■

Consider a differential element of free charges dq' (located at $\overrightarrow{\mathbf{r}}'$) within a macroscopic distribution (of charges). The total force $\overrightarrow{\mathbf{F}}_q$ applied by the distribution on a test charge q (located at $\overrightarrow{\mathbf{r}}$) has the value:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0}\int_{\text{sys.}} \frac{\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right) dq'}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} \qquad [1.5]$$

The total charge in the distribution is obtained by summing over all differential elements, that is, by integrating over the whole system $Q' = \int_{\text{sys.}} dq'$. In general, there are three types of distribution, which are discussed below.

1.2.1. *Distribution along a line (linear) or a curve (curvilinear)*

When the electric charge is distributed along a line or a curve $\mathcal{C}'$, equation [1.5] is reduced to:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0}\int_{\mathcal{C}'} \frac{\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)\lambda\left(\overrightarrow{\mathbf{r}}'\right) dl'}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} \qquad [1.6]$$

where $\lambda\left(\overrightarrow{\mathbf{r}}'\right)$ is the *linear, line charge* or *curvilinear* (charge per unit length). Here, dl' represents an infinitesimal element of the line or curve. The total charge located along the line or curve is obtained from the integral $Q = \int_{C'} \lambda\left(\overrightarrow{\mathbf{r}}'\right) dl'$.

EXAMPLE 1.2. (Total Charge).– Let us determine the total charge in each of the following linear distributions: (a) a linear charge λ uniformly distributed along an infinite line; (b) a linear charge $\lambda(z) = \frac{qa}{\pi(a^2+z^2)}$ distributed along an infinite line; (c) a curvilinear charge (constant) λ distributed over a circle with radius R. For (a), we find $Q = \int_{-\infty}^{\infty} \lambda dz = \infty$; for (b), $Q = \int_{-\infty}^{\infty} \frac{qadz}{\pi(a^2+z^2)} = \frac{q}{\pi} \tan^{-1} \frac{z}{a}\Big|_{-\infty}^{\infty} = q$; for (c), $Q = \int_0^{2\pi} \lambda R d\varphi = 2\pi R\lambda$.

EXAMPLE 1.3. (Uniformly Charged Infinitely Long Wire).–

1) Consider an infinitely long wire that is uniformly charged (constant λ). For reasons of symmetry, we will calculate the force applied to a test charge q located in the Oxy plane, as shown in Figure 1.3. The charge element $\lambda dz'$ exerts the elementary force $d\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0} \frac{\left(x\overrightarrow{\mathbf{i}}+y\overrightarrow{\mathbf{j}}-z'\overrightarrow{\mathbf{k}}\right)\lambda dz'}{(x^2+y^2+z'^2)^{3/2}}$. After integration, this gives:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{\left(x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} - z'\overrightarrow{\mathbf{k}}\right)\lambda dz'}{\left(x^2+y^2+z'^2\right)^{3/2}} \qquad [1.7]$$

The third integral is zero, so $\int_{-\infty}^{\infty} \frac{\left(-z'\overrightarrow{\mathbf{k}}\right)\lambda dz'}{(x^2+y^2+z'^2)^{3/2}} = 0$, because the function under the sign of the integral is odd. Consequently, we obtain:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{\left(x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}}\right)\lambda dz'}{\left(x^2+y^2+z'^2\right)^{3/2}} = \frac{q\lambda\left(x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}}\right)}{2\pi\varepsilon_0\left(x^2+y^2\right)} \equiv \frac{q\lambda\overrightarrow{\mathbf{r}}}{2\pi\varepsilon_0 r^2} \qquad [1.8]$$

2) If, instead of λ, we take the linear charge (b) from Example 1.2, then:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{\left(\overrightarrow{\mathbf{r}} - z'\overrightarrow{\mathbf{k}}\right)}{\left(r^2+z'^2\right)^{3/2}} \frac{q'adz'}{\pi\left(a^2+z'^2\right)} = \frac{qq'a\overrightarrow{\mathbf{r}}}{4\pi^2\varepsilon_0} \int_{-\infty}^{\infty} \frac{dz'}{\left(r^2+z'^2\right)^{3/2}\left(a^2+z'^2\right)} \qquad [1.9]$$

To calculate this integral, let us first carry out the change variable $z' = r\tan\theta'$:

$$\overrightarrow{\mathbf{F}}_q = \frac{qq'\overrightarrow{\mathbf{r}}}{4\pi^2\varepsilon_0 ar^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta' d\theta'}{1+\frac{r^2}{a^2}\tan^2\theta'} = \frac{qq'\overrightarrow{\mathbf{r}}}{4\pi^2\varepsilon_0 ar^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3\theta' d\theta'}{\cos^2\theta' + \frac{r^2}{a^2}\sin^2\theta'} \qquad [1.10]$$

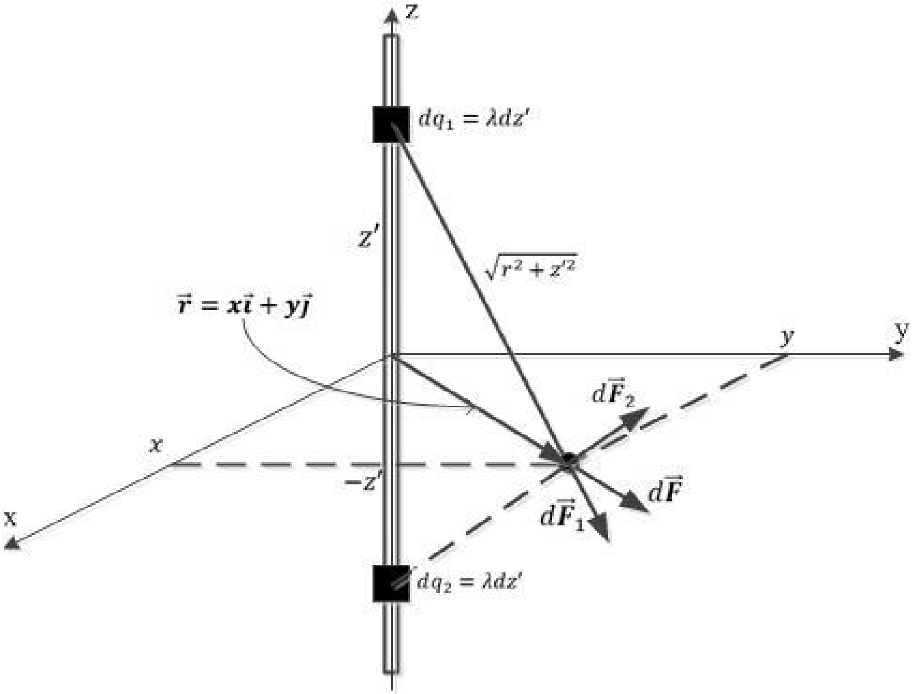

Figure 1.3. *Uniformly charged infinitely long wire dimensionless: by symmetry, the total force exerted on the test charge q is radial. Differential charges dq_1 and dq_2 located symmetrically at z' and $-z'$ produce forces whose projections add up in the Oxy plane and cancel each other out in the Oz direction*

Now taking $t' = \sin\theta'$, we obtain:

$$\overrightarrow{\mathbf{F}}_q = \frac{qq'\overrightarrow{\mathbf{r}}}{4\pi^2\varepsilon_0 ar^2}\int_{-1}^{1}\frac{\left(1-t'^2\right)dt'}{1+\left(\frac{r^2}{a^2}-1\right)t'^2}$$
$$\begin{cases} \dfrac{qq'\overrightarrow{\mathbf{r}}}{4\pi^2\varepsilon_0 ar^2}\displaystyle\int_{-1}^{1}\left(1-t'^2\right)dt' = \dfrac{qq'\overrightarrow{\mathbf{r}}}{3\pi^2\varepsilon_0 ar^2} & r = a \\ \dfrac{qq'a\,\overrightarrow{\mathbf{r}}}{2\pi^2\varepsilon_0 r^2\left(r^2-a^2\right)}\left[\dfrac{r^2}{a^2\sqrt{\frac{r^2}{a^2}-1}}\tan^{-1}\sqrt{\dfrac{r^2}{a^2}-1}-1\right] & r > a \\ \dfrac{qq'a\,\overrightarrow{\mathbf{r}}}{2\pi^2\varepsilon_0 r^2\left(a^2-r^2\right)}\left[1-\dfrac{r^2}{2a^2\sqrt{1-\frac{r^2}{a^2}}}\ln\left(\dfrac{1+\sqrt{1-\frac{r^2}{a^2}}}{1-\sqrt{1-\frac{r^2}{a^2}}}\right)\right] & r < a \end{cases} \quad [1.11]$$

EXAMPLE 1.4. (Uniformly Charged Wire of Finite Length L).– For a uniformly charged wire of finite length L, as in Figure 1.4, the total force experienced by a charge q located at the point $\left(\overrightarrow{\mathbf{r}}, z\right)$ has the value:

$$\begin{aligned}\overrightarrow{\mathbf{F}}_q(\overrightarrow{\mathbf{r}}, z) &= \frac{q}{4\pi\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\left[\overrightarrow{\mathbf{r}} + (z-z')\,\overrightarrow{\mathbf{k}}\right]\lambda dz'}{\left[r^2+(z-z')^2\right]^{3/2}} \\ &= \frac{q\lambda}{4\pi\varepsilon_0}\left[\frac{\overrightarrow{\mathbf{r}}\left(\frac{L}{2}-z\right)+r^2\overrightarrow{\mathbf{k}}}{r^2\sqrt{r^2+\left(\frac{L}{2}-z\right)^2}} + \frac{\overrightarrow{\mathbf{r}}\left(\frac{L}{2}+z\right)-r^2\overrightarrow{\mathbf{k}}}{r^2\sqrt{r^2+\left(\frac{L}{2}+z\right)^2}}\right]\end{aligned} \quad [1.12]$$

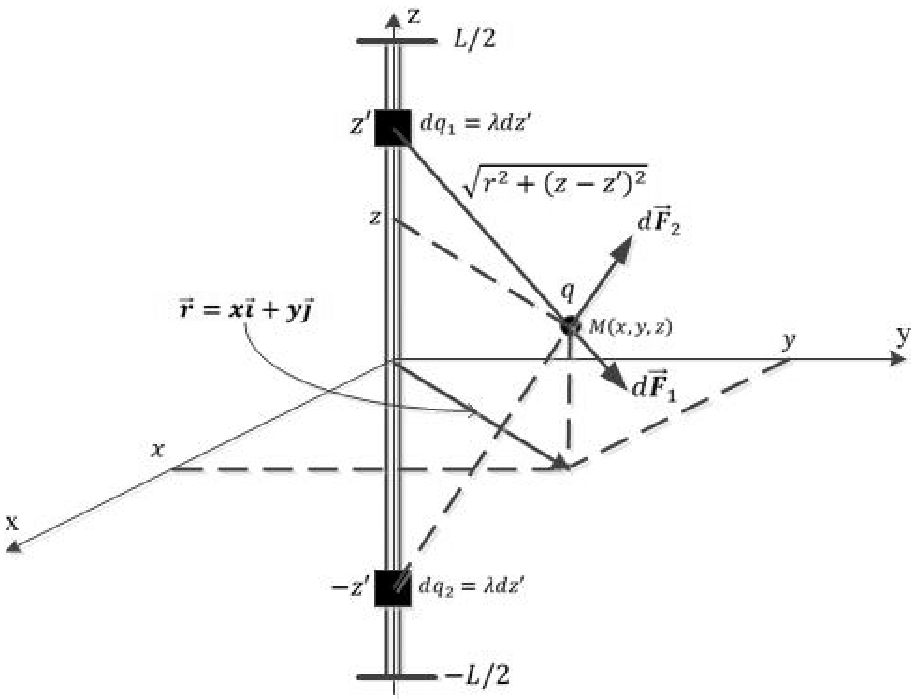

Figure 1.4. *For a uniformly loaded wire of finite length L, the force exerted by a test load q is not radial, except at the middle of the wire*

This is reduced in the middle of the wire, for $z = 0$, at $\overrightarrow{\mathbf{F}}_q\left(\overrightarrow{\mathbf{r}}, 0\right) = \frac{q\lambda}{4\pi\varepsilon_0}\frac{L\overrightarrow{\mathbf{r}}}{r^2\sqrt{r^2+\frac{L^2}{4}}}$. When $L \longrightarrow \infty$, equation [1.8] is found.

1.2.2. *Distribution over a surface (surface charge density)*

When the charge is distributed over a surface S', the force experienced on the test charge q is written as:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0} \int_{S'} \frac{\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right) \sigma\left(\overrightarrow{\mathbf{r}}'\right) dS'}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} \quad [1.13]$$

where $\sigma\left(\overrightarrow{\mathbf{r}}'\right)$ is the *surface charge density* (charge per unit area). dS' is an infinitesimal element of the surface. We have $dS' = dx'dy'$ in Cartesian coordinates, $dS' = r'dr'd\varphi'$ in polar coordinates, etc. The total charge on the surface S' is $Q = \int_{S'} \sigma\left(\overrightarrow{\mathbf{r}}'\right) dS'$.

EXAMPLE 1.5. (Infinite Plane Around a Disc).–

1) Consider an infinite plane of negligible thickness, which is charged as follows:

$$\sigma\left(\overrightarrow{\mathbf{r}}'\right) = \begin{cases} 0 & \text{for}\, x'^2 + y'^2 < R^2 \\ \sigma & \text{for}\, x'^2 + y'^2 \geq R^2 \end{cases} \quad [1.14]$$

As shown in Figure 1.5, the force applied by $dq' = \sigma\left(r', \varphi'\right) dS' = \sigma r'dr'd\varphi'$ on a charge q located at $(0, 0, z)$ has the value $d\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0} \frac{\sigma\left(z\,\overrightarrow{\mathbf{k}} - r'\cos\varphi'\,\overrightarrow{\mathbf{i}} - r'\sin\varphi'\,\overrightarrow{\mathbf{j}}\right)}{\left(z^2+r'^2\right)^{3/2}} r'dr'd\varphi'$. By summing, we obtain:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0} \int_R^{\infty} \int_0^{2\pi} \frac{\sigma\left(z\,\overrightarrow{\mathbf{k}} - r'\cos\varphi'\,\overrightarrow{\mathbf{i}} - r'\sin\varphi'\,\overrightarrow{\mathbf{j}}\right)}{\left(z^2 + r'^2\right)^{3/2}} r'dr'd\varphi' \quad [1.15]$$

The integrals on $\sin\varphi$ and $\cos\varphi$ give 0. Therefore, it comes that:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{2\varepsilon_0} \int_R^{\infty} \frac{\sigma z\,\overrightarrow{\mathbf{k}}}{\left(z^2 + r'^2\right)^{3/2}} r'dr' = \frac{q\sigma z\,\overrightarrow{\mathbf{k}}}{2\varepsilon_0\sqrt{z^2 + R^2}} \quad [1.16]$$

2) Instead of [1.14], we choose:

$$\sigma\left(\overrightarrow{\mathbf{r}}'\right) = \begin{cases} 0 & \text{for}\, x'^2 + y'^2 < R^2 \\ \frac{Rq'}{2\pi r'^3} & \text{for}\, x'^2 + y'^2 \geq R^2 \end{cases} \quad [1.17]$$

We find:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0}\int_R^\infty\int_0^{2\pi}\frac{\left(z\overrightarrow{\mathbf{k}} - r'\cos\varphi'\overrightarrow{\mathbf{i}} - r'\sin\varphi'\overrightarrow{\mathbf{j}}\right)}{\left(z^2+r'^2\right)^{3/2}}\frac{Rq'}{2\pi r'^3}r'dr'd\varphi'$$
$$= \frac{qq'Rz\overrightarrow{\mathbf{k}}}{4\pi\varepsilon_0}\int_R^\infty\frac{dr'}{r'^2\left(z^2+r'^2\right)^{3/2}} = \frac{qq'R\overrightarrow{\mathbf{k}}}{4\pi\varepsilon_0 z^3}\left[\frac{z^2+2R^2}{R\sqrt{z^2+R^2}} - 2\right] \quad [1.18]$$

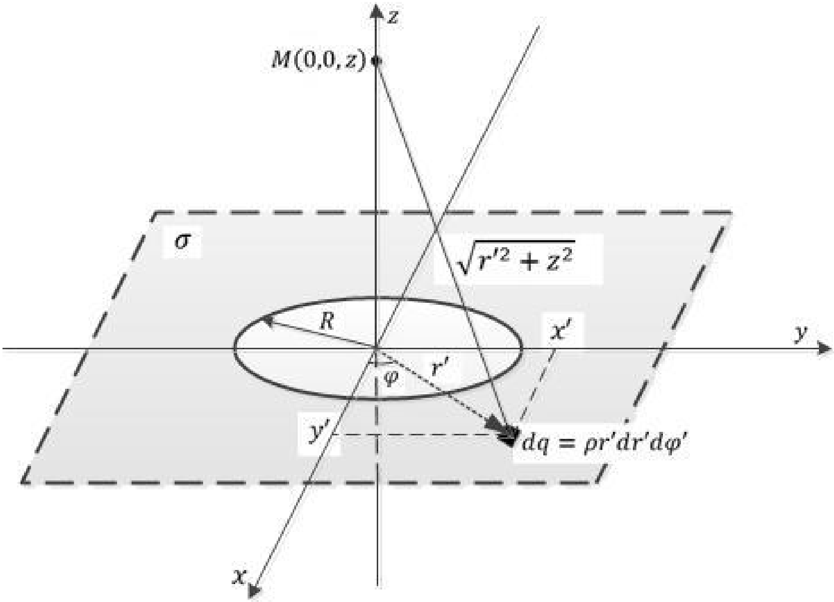

Figure 1.5. *Infinite plane distribution around a disc of radius R: By symmetry, the force experienced along the Oz axis is perpendicular to the plane*

1.2.3. *Distribution over a volume (volume charge density)*

When the charge is distributed within a volume τ', we get:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0}\int_{\tau'}\frac{\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)\rho\left(\overrightarrow{\mathbf{r}}'\right)d\tau'}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} \quad [1.19]$$

where $\rho\left(\overrightarrow{\mathbf{r}}'\right)$ is the volume charge density (charge per unit volume) in τ'. $d\tau'$ is an infinitesimal element of the volume. In Cartesian coordinates, $d\tau'$ is equal to

$dx'dy'dz'$; in cylindrical coordinates, $d\tau'$ is $r'dr'd\varphi'dz'$; in spherical coordinates, it is $r'^2 \sin\theta' dr' d\theta' d\phi'$; etc. The total charge measured inside τ' is $Q = \int_{\tau'} \rho\left(\overrightarrow{\mathbf{r}}'\right) d\tau'$.

EXAMPLE 1.6. (Electron Cloud).– The electronic cloud of charge $+e$ around the hydrogen atom's nucleus is modeled by the spherically symmetric distribution:

$$\rho\left(\overrightarrow{\mathbf{r}}\right) = -\frac{e}{\pi r_0^3} e^{-\frac{2r}{r_0}} \qquad [1.20]$$

where r_0 is called the *Bohr radius*. The cloud's total charge is:

$$\begin{aligned} Q &= \int_\tau \rho d\tau = -\int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{e}{\pi r_0^3} e^{-\frac{2r}{r_0}} r^2 \sin\theta dr d\theta d\varphi \\ &= -\int_0^\infty \frac{4e}{\pi r_0^3} e^{-\frac{2r}{r_0}} r^2 dr = \frac{2e}{\pi r_0^2} e^{-\frac{2r}{r_0}} \left[r^2 - \frac{r_0^2}{2}\left(-\frac{2r}{r_0} - 1\right)\right]\Bigg|_0^\infty = -e \end{aligned} \qquad [1.21]$$

EXAMPLE 1.7. (Uniformly Charged Infinite Sheet of Thickness $2a$).–

(1) Consider an infinite sheet of thickness $2a$, which is uniformly charged, as shown in Figure 1.6. The total force experienced by a charge q, located at the point $(0, 0, z)$, is written as:

$$\overrightarrow{\mathbf{F}}_q = \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-a}^{a} \frac{\rho\left(z\overrightarrow{\mathbf{k}} - x'\overrightarrow{\mathbf{i}} - y'\overrightarrow{\mathbf{j}} - z'\overrightarrow{\mathbf{k}}\right)}{\left[(z-z')^2 + x'^2 + y'^2\right]^{3/2}} dx'dy'dz' \qquad [1.22]$$

In cylindrical coordinates, it yields (we integrate with respect to r' and then with respect to z'):

$$\begin{aligned} \overrightarrow{\mathbf{F}}_q &= \frac{q\rho}{4\pi\varepsilon_0} \int_0^\infty \int_0^{2\pi} \int_{-a}^{a} \frac{(z-z')\overrightarrow{\mathbf{k}} - r'\cos\varphi\overrightarrow{\mathbf{i}} - r'\sin\varphi'\overrightarrow{\mathbf{j}}}{\left[(z-z')^2 + r'^2\right]^{3/2}} r'dr'd\varphi'dz' \\ &= \frac{q\rho}{2\varepsilon_0} \int_{-a}^{a} \frac{(z-z')\overrightarrow{\mathbf{k}}}{|z-z'|} dz' = \begin{cases} \dfrac{q\rho a}{\varepsilon_0}\overrightarrow{\mathbf{k}} & z \geq a \\ -\dfrac{q\rho a}{\varepsilon_0}\overrightarrow{\mathbf{k}} & z \leq -a \\ \dfrac{q\rho\overrightarrow{\mathbf{k}}}{2\varepsilon_0}\left[\displaystyle\int_{-a}^{z} dz' + \int_z^a (-dz')\right] = \dfrac{q\rho z}{\varepsilon_0}\overrightarrow{\mathbf{k}} & |z| < a \end{cases} \end{aligned}$$

[1.23]

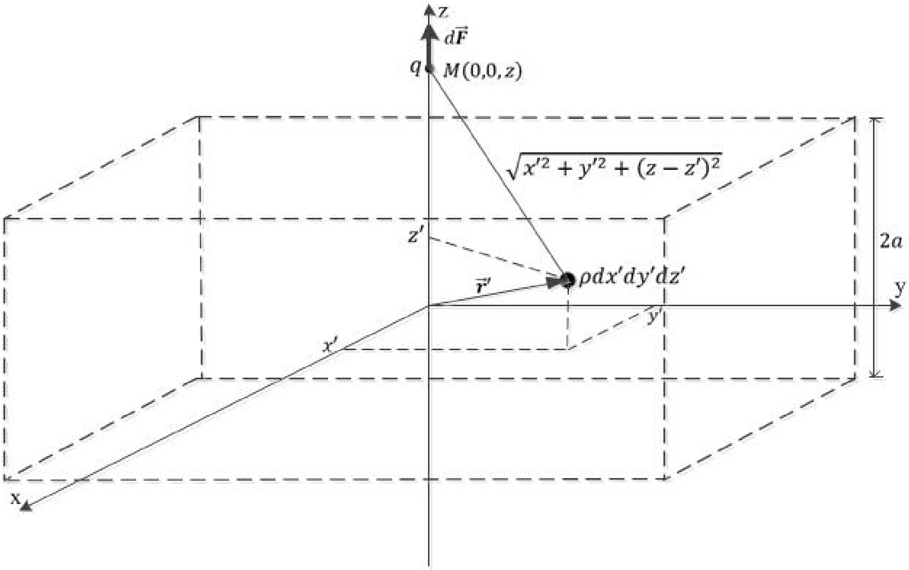

Figure 1.6. *Infinite sheet of thickness $2a$: the total force felt on the charge q is perpendicular to the sheet. Outside the sheet, the force does not depend on the distance between the charge and the sheet, while it depends on z inside*

We can see that $\overrightarrow{\mathbf{F}}_q$ is continuous on the boundaries $z = \pm a$.

(2) If the charge is now distributed over a maximum radius $r'_{\max}$ around Oz with the density $\rho(r', z') = \frac{\mu}{r'}$ (μ is a constant), the total force experienced by the charge q is:

$$\begin{aligned}
\overrightarrow{\mathbf{F}}_q &= \frac{q\mu\overrightarrow{\mathbf{k}}}{2\varepsilon_0}\int_{r'=0}^{r'_{\max}}\int_{z'=-a}^{a}\frac{(z-z')dr'dz'}{\left[(z-z')^2+r'^2\right]^{3/2}} \\
&= \frac{q\mu\overrightarrow{\mathbf{k}}}{2\varepsilon_0}\int_0^{r'_{\max}}\left[\frac{1}{\sqrt{(z-a)^2+r'^2}} - \frac{1}{\sqrt{(z+a)^2+r'^2}}\right]dr' \\
&= \frac{q\mu\overrightarrow{\mathbf{k}}}{2\varepsilon_0}\ln\left[\frac{r'+\sqrt{(z-a)^2+r'^2}}{r'+\sqrt{(z+a)^2+r'^2}}\right]\Bigg|_{r'=0}^{r'_{\max}} \\
&= \frac{q\mu\overrightarrow{\mathbf{k}}}{2\varepsilon_0}\ln\left[\frac{r'_{\max}+\sqrt{(z-a)^2+r'^2_{\max}}}{r'_{\max}+\sqrt{(z+a)^2+r'^2_{\max}}}\left|\frac{z+a}{z-a}\right|\right]
\end{aligned} \qquad [1.24]$$

For $r'_{\text{max}} \to \infty$, equation [1.24] reduces $\overrightarrow{\mathbf{F}}_q = \frac{q\mu \overrightarrow{\mathbf{k}}}{2\varepsilon_0} \ln \left| \frac{z+a}{z-a} \right|$. The total charge held in the sheet is equal to $q' = \int_{-a}^{a} \int_0^{r'_{\text{max}}} \frac{\mu}{r'} 2\pi r' dr' dz' = 4\pi\mu a r'_{\text{max}}$; this gives $\mu = \frac{q'}{4\pi a r'_{\text{max}}}$ and therefore $\overrightarrow{\mathbf{F}}_q = \frac{qq'\overrightarrow{\mathbf{k}}}{8\pi\varepsilon_0 a r'_{\text{max}}} \ln \left[\frac{r'_{\text{max}}+\sqrt{r'^2_{\text{max}}+(z-a)^2}}{r'_{\text{max}}+\sqrt{r'^2_{\text{max}}+(z+a)^2}} \left| \frac{z+a}{z-a} \right| \right]$. ■

The Coulomb force [1.1] can be recovered from equation [1.19] using the Dirac distribution:

$$\begin{aligned} \rho\left(\overrightarrow{\mathbf{r}}'\right) &= q'\delta\left(\overrightarrow{\mathbf{r}}' - \overrightarrow{\mathbf{r}}'_{q'}\right) = q'\delta\left(x' - x'_{q'}\right)\delta\left(y' - y'_{q'}\right)\delta\left(z' - z'_{q'}\right) \\ &= q'r'^{-2}\sin^{-1}\theta'\delta\left(r' - r'_{q'}\right)\delta\left(\theta' - \theta'_{q'}\right)\delta\left(\varphi' - \varphi'_{q'}\right) \\ &= q'r'^{-2}\delta\left(r' - r'_{q'}\right)\delta\left(\cos\theta' - \cos\theta'_{q'}\right)\delta\left(\varphi' - \varphi'_{q'}\right) \end{aligned} \quad [1.25]$$

1.3. Electric field

The *electric field* $\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right)$ is defined as being the force exerted on a test charge equal to unity. Thus, the electric field created by a charge q' is equal to:

$$\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) = \frac{\overrightarrow{\mathbf{F}}_q}{q} = \frac{q'}{4\pi\varepsilon_0} \frac{\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} \quad [1.26]$$

In the same way, the electric fields created by (a) several point-like charges $\{q'_i\}$, (b) a linear charge λ distributed over a curve $\mathcal{C}'$, (c) a surface charge density σ distributed over an area S' and (d) a volume charge density ρ distributed within a volume τ' are, respectively:

$$\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \displaystyle\sum_{i=1}^{n} \dfrac{q'_i\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}_i'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}_i'\right|^3} & \text{(dist. of charges)} \\ \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_{\mathcal{C}'} \dfrac{\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)\lambda\left(\overrightarrow{\mathbf{r}}'\right) dl'}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} & \text{(line dist.)} \\ \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_{S'} \dfrac{\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)\sigma\left(\overrightarrow{\mathbf{r}}'\right) dS'}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} & \text{(area density dist.)} \\ \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_{\tau'} \dfrac{\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)\rho\left(\overrightarrow{\mathbf{r}}'\right) d\tau'}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} & \text{(volume density dist.)} \end{cases} \quad [1.27]$$

If the charge distribution is known, the electric field is obtained by integrating one of the formulas [1.27]. A few simple rules must be observed:

– A distinction must be made between the coordinates of the field points and the source points (differential charges). The integration must always be carried out over the source coordinates.

– Equations [1.27] are vector equations: they generally have three components, that is, three integrals to be carried out. Certain symmetries can be used to show that certain field components are zero.

– The distance $\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|$ is always positive. When square roots must be taken, it must be ensured that the positive root is taken.

– The solution to a particular problem can often be obtained by integrating the contributions of simpler differential systems (see Example 1.9).

EXAMPLE 1.8. (Uniformly Charged Infinitely Long Wire (see Figure 1.3)).– The electric field measured at the point (x, y, z) is obtained by dividing the force [1.8] by q:

$$\overrightarrow{\mathbf{E}} = \frac{\overrightarrow{\mathbf{F}}_q}{q} = \frac{\lambda \overrightarrow{\mathbf{r}}}{2\pi\varepsilon_0 r^2} \qquad [1.28]$$

The field $\overrightarrow{\mathbf{E}}$ has no component along the Oz axis, since the two charge elements dq_1' and dq_2', symmetrically located at a distance z on either side of the O point, produce fields of equal magnitude, oppositely directed in this direction and therefore cancel each other out, leaving only the radial components, which add to each other. If the wire has a length L (see Figure 1.4), formula [1.28] must be replaced, according to [1.12], by:

$$\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}, z\right) = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{\overrightarrow{\mathbf{r}}\left(\frac{L}{2} - z\right) + r^2\overrightarrow{\mathbf{k}}}{r^2\sqrt{r^2 + \left(\frac{L}{2} - z\right)^2}} + \frac{\overrightarrow{\mathbf{r}}\left(\frac{L}{2} + z\right) - r^2\overrightarrow{\mathbf{k}}}{r^2\sqrt{r^2 + \left(\frac{L}{2} + z\right)^2}}\right] \qquad [1.29]$$

EXAMPLE 1.9. (Uniformly Charged Infinite Plane).– Consider a uniformly charged infinite plane located at $z = 0$, as shown in Figure 1.7. The plane is divided into an infinite number of linear charges along the Oy axis of thickness dx with $d\lambda = \sigma dx$. It is also possible to divide the plane into linear charges along the Ox axis of thickness dy; this does not change the result. Each linear charge produces an infinitesimal electric field whose projection along the Oz axis has the value:

$$dE_z = \frac{\sigma dx}{2\pi\varepsilon_0\left(x^2 + z^2\right)^{1/2}}\cos\theta = \frac{\sigma z dx}{2\pi\varepsilon_0\left(x^2 + z^2\right)} \qquad [1.30]$$

We do not take into account the field component along the Ox axis, because the linear charge at $-z$ induces a field in this direction of the same amplitude but in the opposite direction, which thus cancels out the first field.

The total field is then obtained by integration over all linear charges:

$$\overrightarrow{\mathbf{E}} = \frac{\sigma z \overrightarrow{\mathbf{k}}}{2\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{dx}{(x^2+z^2)} = \begin{cases} \dfrac{\sigma \overrightarrow{\mathbf{k}}}{2\varepsilon_0} & z \geq 0 \\ -\dfrac{\sigma \overrightarrow{\mathbf{k}}}{2\varepsilon_0} & z < 0 \end{cases} \qquad [1.31]$$

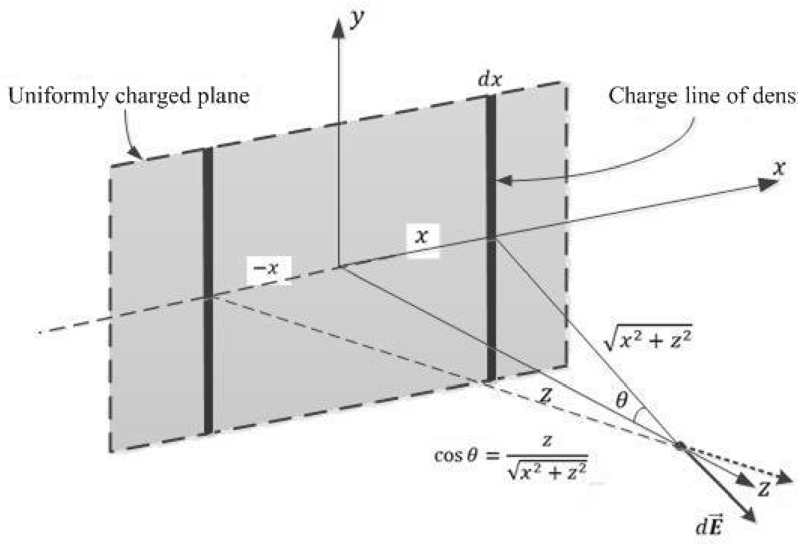

Figure 1.7. *Uniformly charged infinite plane composed of an infinite number of parallel charged lines (λdx) along Oy: the field created by lines symmetrically placed at x and $-x$ has components that cancel out in the Ox direction and components that add up in the Oz direction*

Note that the field does not decrease with distance from the plane.

EXAMPLE 1.10. (An Infinite Plane Around a Disc (see Figure 1.5)).– Along the Oz axis of the disc, a test charge q is subject to the force $\overrightarrow{\mathbf{F}}_q = \frac{q\sigma z \overrightarrow{\mathbf{k}}}{2\varepsilon_0\sqrt{z^2+R^2}}$. It therefore follows that:

$$\overrightarrow{\mathbf{E}} = \frac{\sigma z \overrightarrow{\mathbf{k}}}{2\varepsilon_0\sqrt{z^2+R^2}} \qquad [1.32]$$

When $R \longrightarrow 0$ (uniformly loaded infinite plane), we return to [1.31].

EXAMPLE 1.11. (Uniformly Charged Infinite Sheet of Thickness $2a$ (see Figure 1.6)).–

$$\overrightarrow{\mathbf{E}}(z) = \begin{cases} \frac{\rho a}{\varepsilon_0}\overrightarrow{\mathbf{k}} & z \geq a \\ \frac{\rho z}{\varepsilon_0}\overrightarrow{\mathbf{k}} & -a < z < a \\ -\frac{\rho a}{\varepsilon_0}\overrightarrow{\mathbf{k}} & z \leq -a \end{cases} \qquad [1.33]$$

EXAMPLE 1.12. (Uniformly Charged Disc with Negligible Thickness).– Let us calculate the electric field, created by a flat disk of negligible thickness, radius R and charge Q (uniformly distributed over the disk), at point M located along axis Oz of the disk at distance z from its center, as shown in Figure 1.8. The charge density is given by $\rho\left(\overrightarrow{\mathbf{r}}'\right) = \frac{Q}{\pi R^2}\delta\left(z'\right)$ for $x'^2 + y'^2 \leq R^2$ and 0 elsewhere. It therefore follows that:

$$\overrightarrow{\mathbf{E}}(0,0,z) = \frac{Q}{4\pi^2 R^2 \varepsilon_0}\int_{-R}^{R}\int_{-\sqrt{R^2-x'^2}}^{\sqrt{R^2-x'^2}} \frac{z\overrightarrow{\mathbf{k}} - x'\overrightarrow{\mathbf{i}} - y'\overrightarrow{\mathbf{j}}}{\left(x'^2 + y'^2 + z^2\right)^{3/2}} dx'dy' \qquad [1.34]$$

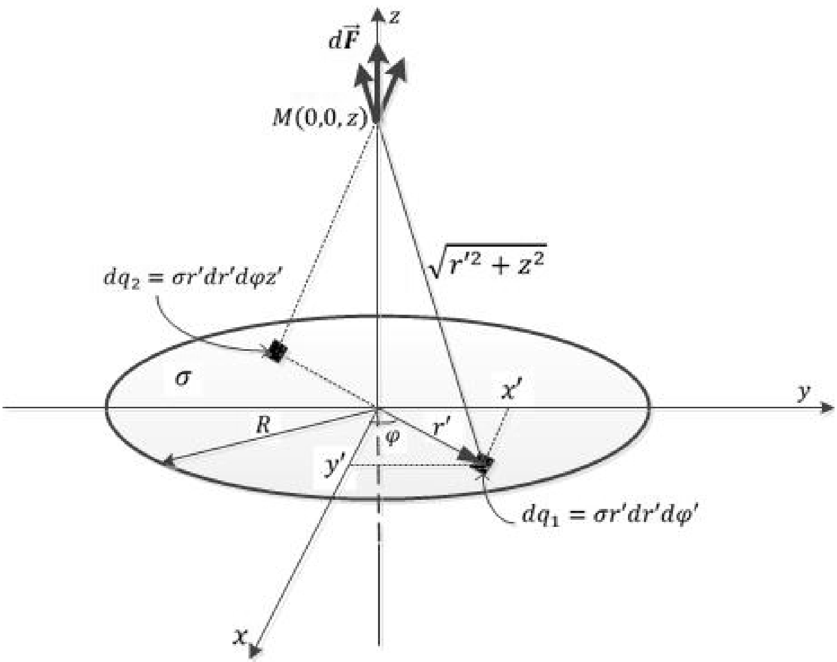

Figure 1.8. *By symmetry, the electric field created by a uniformly charged disk with radius R and no thickness is normal to the disk along points on the Oz axis*

By shifting to cylindrical coordinates, we obtain:

$$\overrightarrow{\mathbf{E}}(0,0,z) = \frac{Q}{4\pi^2 R^2 \varepsilon_0} \int_0^R \int_0^{2\pi} \frac{z\overrightarrow{\mathbf{k}} - r' cos\varphi' \overrightarrow{\mathbf{i}} - r' \sin\varphi' \overrightarrow{\mathbf{j}}}{\left(r'^2 + z^2\right)^{3/2}} r' dr' d\varphi' \qquad [1.35]$$

The integral on the variable φ' eliminates the terms in $\cos\varphi'$ and $\sin\varphi'$. Therefore, we have:

$$\overrightarrow{\mathbf{E}}(0,0,z) = \int_0^R \frac{Qz\overrightarrow{\mathbf{k}} r' dr'}{2\pi\varepsilon_0 R^2 \left(r'^2 + z^2\right)^{3/2}} = \frac{Q}{2\pi\varepsilon_0 R^2} \left[\frac{z\overrightarrow{\mathbf{k}}}{|z|} - \frac{z\overrightarrow{\mathbf{k}}}{\sqrt{R^2 + z^2}}\right] \qquad [1.36]$$

When $z \ll R$, the electric field is reduced to $\frac{\sigma\overrightarrow{\mathbf{k}}}{2\varepsilon_0} \frac{z}{|z|}$ (i.e. to [1.31]).

1.3.1. *Distribution with spherical symmetry*

1.3.1.1. *Sphere charged in volume*

For simplicity's sake, let us start by considering a sphere of radius R uniformly charged in volume. Let us use a spherical coordinate system whose point of origin is the center of the sphere. We propose to calculate the electric field $\overrightarrow{\mathbf{E}}$ at a point M (inside or outside the sphere). By symmetry, it is clear that $\overrightarrow{\mathbf{E}}$ does not depend on the coordinates (θ, φ), as shown in Figure 1.9. We will start by writing the contribution to $\overrightarrow{\mathbf{E}}$ of the elementary charge $\rho d\tau'$ contained in a volume element $d\tau'$ ($\rho = \frac{Q}{\text{Volume of the sphere}} = \frac{3Q}{4\pi R^3}$, where Q is the total charge of the sphere) and then integrate over the sphere. The volume element is $r'^2 dr' \sin\theta' d\theta' d\varphi'$ (for simplicity, the point M is chosen along the Oz axis). The charge contained in this elementary volume creates the differential field $d\overrightarrow{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0} \rho r'^2 dr' \sin\theta' d\theta' d\varphi' \frac{\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3}$ at point M. For reasons of symmetry, it is obvious that the electric field $\overrightarrow{\mathbf{E}}$ is radial. Consequently, we will consider only the radial component:

$$\begin{aligned} \overrightarrow{\mathbf{E}} &= \frac{1}{4\pi\varepsilon_0} \int_0^R \int_0^\pi \int_0^{2\pi} \frac{\rho r'^2 dr' \sin\theta' d\theta' d\varphi'}{\sqrt[3]{r^2 + r'^2 - 2rr'\cos\theta'}} \left(1 - \frac{r'}{r}\cos\theta'\right) \overrightarrow{\mathbf{r}} \\ &= \frac{1}{2\varepsilon_0} \int_0^R \int_0^\pi \frac{\rho r'^2 dr' \sin\theta' d\theta'}{\sqrt[3]{r^2 + r'^2 - 2rr'\cos\theta'}} \left(1 - \frac{r'}{r}\cos\theta'\right) \overrightarrow{\mathbf{r}} \end{aligned} \qquad [1.37]$$

In order to compute this integral, let us introduce the variable $s^2 = r^2 + r'^2 - 2rr'\cos\theta'$:

$$\overrightarrow{\mathbf{E}} = \frac{\rho}{4\varepsilon_0} \left[\int_{r'=0}^R \int_{s=|r-r'|}^{r+r'} r' dr' ds \left(1 + \frac{r^2 - r'^2}{s^2}\right)\right] \frac{\overrightarrow{\mathbf{r}}}{r^3} \qquad [1.38]$$

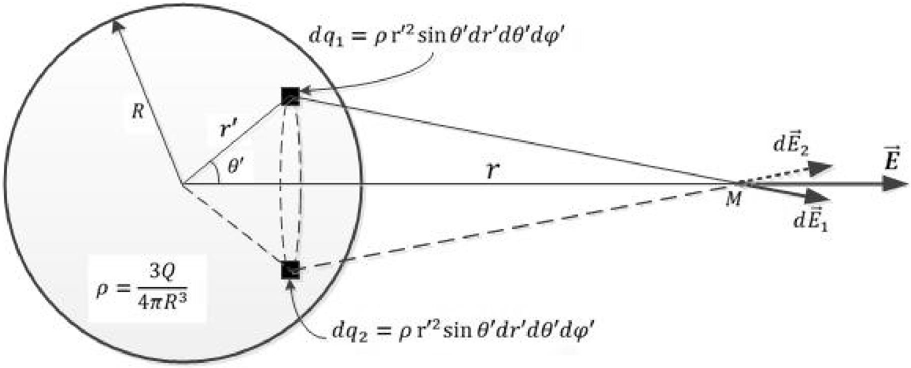

Figure 1.9. *Electric field created by a uniformly charged sphere in volume*

There are two possible cases:

– *Electric field outside the sphere*: when the point M lies outside the sphere, we always have $r > r'$:

$$\begin{aligned}\overrightarrow{\mathbf{E}}_{\text{ext}} &= \frac{\rho}{4\varepsilon_0}\left[\int_0^R \int_{r-r'}^{r+r'} r' dr' ds \left(1 + \frac{r^2 - r'^2}{s^2}\right)\right] \frac{\overrightarrow{\mathbf{r}}}{r^3} \\ &= \frac{\rho}{\varepsilon_0}\left[\int_0^R r'^2 dr'\right] \frac{\overrightarrow{\mathbf{r}}}{r^3} = \frac{\rho R^3}{3\varepsilon_0} \frac{\overrightarrow{\mathbf{r}}}{r^3} \equiv \frac{Q}{4\pi\varepsilon_0} \frac{\overrightarrow{\mathbf{r}}}{r^3}\end{aligned} \quad [1.39]$$

The electric field of a sphere of radius R and total charge Q coincides outside the sphere with the field of a point-like charge Q placed at the center of this sphere.

– *Electric field within the sphere*: when the point M lies within the sphere, the integration interval $\int_0^R dr'$ must be divided into two intervals, namely, $\int_0^r dr'$ and $\int_r^R dr'$:

$$\begin{aligned}\overrightarrow{\mathbf{E}}_{\text{int}} &= \frac{\rho}{4\varepsilon_0}\left[\int_0^r \int_{r-r'}^{r+r'} \left[1 + \frac{r^2 - r'^2}{s^2}\right] r' dr' ds \right. \\ &\quad \left. + \int_r^R \int_{r'-r}^{r+r'} \left[1 + \frac{r^2 - r'^2}{s^2}\right] r' dr' ds\right] \frac{\overrightarrow{\mathbf{r}}}{r^3} \\ &= \frac{\rho}{\varepsilon_0}\left[\int_0^r r'^2 dr' + 0\right] \frac{\overrightarrow{\mathbf{r}}}{r^3} = \frac{\rho}{3\varepsilon_0}\overrightarrow{\mathbf{r}} + 0 \equiv \frac{Q}{4\pi\varepsilon_0 R^3}\overrightarrow{\mathbf{r}}\end{aligned} \quad [1.40]$$

EXAMPLE 1.13.– Let us now choose the spherically symmetric charge density described as follows: $\rho(r') = \rho_0\left(1 - \frac{r'^2}{R^2}\right)$ if $r \leq R$ and 0 if $r > R$. Repeating the same calculation steps, we find the following:

– *Electric field outside the sphere*:

$$\vec{\mathbf{E}}_{\text{ext}} = \frac{\rho_0}{\varepsilon_0}\left[\int_0^R \left(1 - \frac{r'^2}{R^2}\right) r'^2 dr'\right]\frac{\vec{\mathbf{r}}}{r^3} = \frac{2\rho_0 R^3}{15\varepsilon_0}\frac{\vec{\mathbf{r}}}{r^3} \qquad [1.41]$$

– *Electric field inside the sphere*:

$$\vec{\mathbf{E}}_{\text{int}} = \frac{\rho_0}{\varepsilon_0}\left[\int_0^r \left(1 - \frac{r'^2}{R^2}\right) r'^2 dr'\right]\frac{\vec{\mathbf{r}}}{r^3} = \frac{\rho_0}{\varepsilon_0}\left(\frac{1}{3} - \frac{r^2}{5R^2}\right)\vec{\mathbf{r}} \qquad [1.42]$$

1.3.1.2. *Sphere charged on its surface*

Here, we will consider a charge Q distributed over the surface of the sphere, as shown in Figure 1.10. The radial component of the differential electric field due to the differential charge element $\sigma R^2 \sin\theta' d\theta' d\varphi'$ is $d\vec{\mathbf{E}} = \frac{\sigma R^2 \left(r - R\cos\theta'\right)\sin\theta' d\theta' d\varphi'}{4\pi\varepsilon_0 [r^2 + R^2 - 2rR\cos\theta]^{3/2}} \frac{\vec{\mathbf{r}}}{r}$. Thus, the electric field created by the entire sphere is given as:

$$\begin{aligned}\vec{\mathbf{E}} &= \int_0^{\pi}\int_0^{2\pi} \frac{\sigma R^2 \sin\theta' \left(r - R\cos\theta'\right) d\theta' d\varphi'}{4\pi\varepsilon_0 \left[r^2 + R^2 - 2rR\cos\theta\right]^{3/2}}\frac{\vec{\mathbf{r}}}{r} \\ &= \int_0^{\pi} \frac{\sigma R^2 \sin\theta' \left(r - R\cos\theta'\right) d\theta'}{2\varepsilon_0 \left[r^2 + R^2 - 2rR\cos\theta\right]^{3/2}}\frac{\vec{\mathbf{r}}}{r}\end{aligned} \qquad [1.43]$$

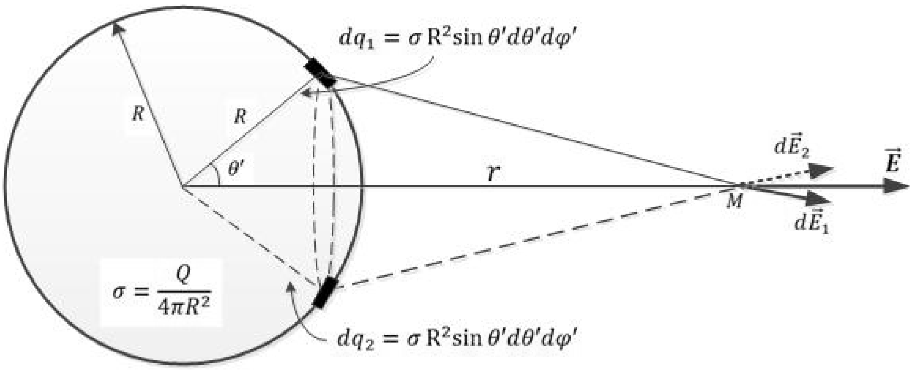

Figure 1.10. *A sphere of radius R with charge Q uniformly distributed on its surface. Symmetrically located differential charge elements indicate that the electric field is radial. The electric field inside the sphere is zero, while outside the sphere, the field is the same as if all the charge Q were concentrated in a point-like charge at the origin*

By introducing the variable change $s^2 = r^2 + R^2 - 2r\cos\theta$ in the same way, it is possible to rewrite the electric field integral as follows:

$$\begin{aligned}\overrightarrow{\mathbf{E}} &= \frac{\sigma R\overrightarrow{\mathbf{r}}}{4\varepsilon_0 r^3}\int_{|r-R|}^{r+R}\left[\frac{r^2-R^2}{s^2}+1\right]ds \\ &= \frac{\sigma R\overrightarrow{\mathbf{r}}}{4\varepsilon_0 r^3}\left[r+R-|r-R|-\left(r^2-R^2\right)\left(\frac{1}{r+R}-\frac{1}{|r-R|}\right)\right] \\ &= \begin{cases}\dfrac{\sigma R^2\overrightarrow{\mathbf{r}}}{\varepsilon_0 r^3} \equiv \dfrac{Q\overrightarrow{\mathbf{r}}}{4\pi\varepsilon_0 r^3} & \text{(outside the sphere } (r>R))\\ 0 & \text{(inside the sphere } (r<R))\end{cases}\end{aligned} \qquad [1.44]$$

1.3.2. *Thomson's model*

According to Thomson's model, a hydrogen atom is seen as a solid sphere with radius r_0 and a uniformly distributed positive charge inside. The force $\overrightarrow{\mathbf{F}}_e\left(\overrightarrow{\mathbf{r}}\right)$ acting on the electron placed at $\overrightarrow{\mathbf{r}}$ (inside the sphere) is:

$$\overrightarrow{\mathbf{F}}_e\left(\overrightarrow{\mathbf{r}}\right) = -e\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) = -\frac{e^2}{4\pi\varepsilon_0 r_0^3}\overrightarrow{\mathbf{r}} = -k\overrightarrow{\mathbf{r}} \qquad [1.45]$$

where $k = \frac{e^2}{4\pi\varepsilon_0 r_0^3}$. The electrical force acting on the electron is therefore analogous to the restoring force of a spring (of zero length in a vacuum): a central, attractive force proportional to the distance to the center (analogous to the elongation of the spring). The electron is thus elastically linked to the center of the atom. What is more, the moment of this force relative to O is zero. Using the angular momentum theorem, we can deduce that the electron's angular momentum $\overrightarrow{\mathbf{L}}$ with respect to O is constant. Since $\overrightarrow{\mathbf{L}} = m_e\overrightarrow{\mathbf{r}}\times\overrightarrow{\mathbf{v}}$, we can conclude that the vectors $\overrightarrow{\mathbf{r}}$ and $\overrightarrow{\mathbf{v}}$ are both perpendicular to a constant vector (i.e. they lie in the same plane): the electron's motion therefore takes place in a plane. Applying Newton's second law, that is, $m_e\frac{d^2\overrightarrow{\mathbf{r}}}{dt^2} = -k\overrightarrow{\mathbf{r}}$, we get:

$$\frac{d^2\overrightarrow{\mathbf{r}}}{dt^2} + \omega_0^2\overrightarrow{\mathbf{r}} = 0 \qquad [1.46]$$

where $\omega_0^2 = \frac{k}{m_e}$. If ω_0 (or ν_0) corresponds to a frequency of one of the lines in the Lyman spectrum of the hydrogen atom λ_0, we write that $\omega_0 = 2\pi\nu_0 = \frac{2\pi c}{\lambda_0}$; which implies that $r_0 = \left(\frac{e^2\lambda_0^2}{16\pi^3\varepsilon_0 c^2}\right)^{1/3}$. Differential equation [1.46] has the solution:

$$\overrightarrow{\mathbf{r}}(t) = r_0 \cos\omega_0 t\,\overrightarrow{\mathbf{i}} + \frac{v_0}{\omega_0}\sin\omega_0 t\,\overrightarrow{\mathbf{j}} \qquad [1.47]$$

which corresponds to the ellipse $\frac{x^2}{r_0^2} + \frac{y^2}{\frac{v_0^2}{\omega_0^2}} = 1$ with center O and axis $\left(r_0, \frac{v_0}{\omega_0}\right)$.

In addition to the restoring force, if the electron is also subjected to a *frictional force* $\overrightarrow{\mathbf{f}} = -h\overrightarrow{\mathbf{v}}$, where $h > 0$ (an opposing force proportional to speed), the electron's equation of motion becomes:

$$m_e\frac{d^2\overrightarrow{\mathbf{r}}}{dt^2} = -k\overrightarrow{\mathbf{r}} - h\frac{d\overrightarrow{\mathbf{r}}}{dt} \qquad [1.48]$$

Let us put $\frac{h}{m_e} = 2\lambda$ and $\frac{k}{m_e} = \omega_0^2$. We obtain the *characteristic equation* $\alpha^2 + 2\lambda\alpha + \omega_0^2 = 0$ and its discriminant reduced to $\Delta = \lambda^2 - \omega_0^2$. There are three possible cases, depending on the sign of Δ:

a) $\Delta > 0$ *or* $\lambda > \omega_0$ *(high friction)*: the two solutions of the characteristic equation are real and negative: $\alpha_1 = -\lambda + \sqrt{\lambda^2 - \omega_0^2}$ and $\alpha_2 = -\lambda - \sqrt{\lambda^2 - \omega_0^2}$. The general solution of the differential equation is:

$$\overrightarrow{\mathbf{r}} = \frac{\overrightarrow{\mathbf{v}}_0 - \alpha_2\overrightarrow{\mathbf{r}}_0}{\alpha_1 - \alpha_2}e^{\alpha_1 t} + \frac{\alpha_1\overrightarrow{\mathbf{r}}_0 - \overrightarrow{\mathbf{v}}_0}{\alpha_1 - \alpha_2}e^{\alpha_2 t} \qquad [1.49]$$

b) $\Delta = 0$ *or* $\lambda = \omega_0$ *(critical regime)*: we have a double solution for the characteristic equation, $\alpha = -\lambda$. The general solution is:

$$\overrightarrow{\mathbf{r}} = \left[\overrightarrow{\mathbf{r}}_0 + \left(\overrightarrow{\mathbf{v}}_0 + \lambda\overrightarrow{\mathbf{r}}_0\right)t\right]e^{-\lambda t} \qquad [1.50]$$

c) $\Delta < 0$ *or* $\lambda < \omega_0$ *(low friction)*: we have two distinct complex solutions $\alpha_1 = -\lambda + \mathrm{i}\sqrt{\omega_0^2 - \lambda_0}$ and $\alpha_2 = -\lambda - \mathrm{i}\sqrt{\omega_0^2 - \lambda_0}$. The solution is therefore:

$$\overrightarrow{\mathbf{r}} = \left(\overrightarrow{\mathbf{r}}_0\cos\sqrt{\omega_0^2 - \lambda_0}t + \frac{\left(\overrightarrow{\mathbf{v}}_0 + \lambda\overrightarrow{\mathbf{r}}_0\right)}{\sqrt{\omega_0^2 - \lambda_0}}\sin\sqrt{\omega_0^2 - \lambda_0 t}\right)e^{-\lambda t} \qquad [1.51]$$

Even if the friction is small, the electron will inevitably whirl toward the center O along an elliptical trajectory with a smaller and smaller surface area. The atom described by Thomson's model is not stable.

1.3.3. *Symmetry in an electric field*

Since the field $\overrightarrow{\mathbf{E}}$ is created by a charge distribution τ', then if we know the symmetry properties of this distribution, we will know those of $\overrightarrow{\mathbf{E}}$. These properties simplify calculations considerably. If, in a homogeneous and isotropic medium, we apply a geometric transformation to a charge distribution, then its $\overrightarrow{\mathbf{E}}$ field will undergo the same transformation. If a charge distribution has a certain degree of symmetry, it is possible to deduce the field at one point from its value at another. The following can be observed:

– *Translation invariance*: a scalar function Φ or vector function $\overrightarrow{\mathbf{V}}$ is *translation invariant* along the Oz-axis, if, for any point, M and any translation parallel to Oz (regardless of displacement), we have $\Phi\left(M\right) = \Phi\left(M'\right)$ and $\overrightarrow{\mathbf{V}}\left(M\right) = \overrightarrow{\mathbf{V}}\left(M'\right)$, where M' is the point obtained from M through this translation. If the distribution of charges is translation invariant along the Oz-axis, the associated field $\overrightarrow{\mathbf{E}}$ does not depend on z. On the other hand, $\overrightarrow{\mathbf{E}}$ can have a component along Oz.

– *Axial symmetry*: a scalar function Φ or a vector function $\overrightarrow{\mathbf{V}}$ is said to be rotation invariant around an axis, denoted here by Oz; if for any point M and any rotation $\mathcal{R}_\varphi$ around Oz (regardless of the angle), we have $\Phi\left(M\right) = \Phi\left(M'\right)$ and $\overrightarrow{\mathbf{V}}\left(M\right) = \overrightarrow{\mathbf{V}}\left(M'\right)$, where $M' = \mathcal{R}_\varphi\left(M\right)$. If the distribution of charges is invariant through the rotation around the Oz-axis, then the associated field, expressed here in cylindrical coordinates, does not depend on φ. We have:

$$\overrightarrow{\mathbf{E}}\left(M\right) = \int_{\tau(A)} \frac{\rho\left(A\right)\overrightarrow{\mathbf{AM}}d\tau\left(A\right)}{4\pi\varepsilon_0 AM^3} \tag{1.52}$$

$$\overrightarrow{\mathbf{E}}\left(M'\right) = \int_{\tau(A')} \frac{\rho\left(A'\right)\overrightarrow{\mathbf{A'M'}}d\tau\left(A'\right)}{4\pi\varepsilon_0 A'M'^3} \tag{1.53}$$

For any point A', there exists a single point A such that $A' = \mathcal{R}_\varphi\left(A\right) \Longrightarrow \overrightarrow{\mathbf{A'M'}} = \mathcal{R}_\varphi\left(\overrightarrow{\mathbf{AM}}\right)$, $A'M' = AM$ and $d\tau\left(A\right) = d\tau\left(A'\right)$. It then follows that:

$$\overrightarrow{\mathbf{E}}\left(M'\right) = \int_{\tau(A)} \frac{\rho\left(\mathcal{R}_\varphi\left(A\right)\right)\mathcal{R}_\varphi\left(\overrightarrow{\mathbf{AM}}\right)d\tau\left(A\right)}{4\pi\varepsilon_0 AM^3} \tag{1.54}$$

Since the distribution ρ is rotation invariant (i.e. $\rho\left(\mathcal{R}_\varphi\left(A\right)\right) = \rho\left(A\right)$), it comes that:

$$\begin{aligned}\overrightarrow{\mathbf{E}}\left(M'\right) &= \int_{\tau(A)} \frac{\rho\left(A\right)\mathcal{R}_\varphi\left(\overrightarrow{\mathbf{AM}}\right)\tau d\left(A\right)}{4\pi\varepsilon_0 AM^3} \\ &= \mathcal{R}_\varphi\left(\int_{\tau(A)} \frac{\rho\left(A\right)\overrightarrow{\mathbf{AM}}d\tau\left(A\right)}{4\pi\varepsilon_0 AM^3}\right) = \mathcal{R}_\varphi\left(\overrightarrow{\mathbf{E}}\left(M\right)\right)\end{aligned} \tag{1.55}$$

– *Cylindrical symmetry*: if the charge distribution is *invariant by translation along Oz and by rotation about the same axis*, then the field depends only on the distance r from the axis Oz.

– *Spherical symmetry*: if the distribution of charges is *invariant through any rotation around a fixed point*, then $\overrightarrow{\mathbf{E}}$ only depends on the distance r measured between the center O and the field point.

– *Symmetry plane and antisymmetry plane*: we will look at symmetries and anti-symmetries with respect to a plane P. To do this, let us introduce a few useful definitions: (1) if M is any point in space, the point M' that is symmetric to M with respect to the plane P is denoted by $M' = \text{sym}_P M$ (P will appear here as the median plane of the segment MM'); (2) any vector $\overrightarrow{\mathbf{V}}$ can be written as $\overrightarrow{\mathbf{V}} = \overrightarrow{\mathbf{V}}_{\|} + \overrightarrow{\mathbf{V}}_{\perp}$, where $\overrightarrow{\mathbf{V}}_{\|}$ is the component parallel to P and $\overrightarrow{\mathbf{V}}_{\perp}$ is the normal component. The symmetric of $\overrightarrow{\mathbf{V}}$ with respect of the plane P is the vector:

$$\overrightarrow{\mathbf{V}}' = \text{sym}_P \overrightarrow{\mathbf{V}} = \overrightarrow{\mathbf{V}}_{\|} - \overrightarrow{\mathbf{V}}_{\perp} \quad [1.56]$$

a) *Symmetry plane*: P is said to be a *symmetry plane* of the scalar function Φ if, for any point M, we have $\Phi(M') = \Phi(M)$, where $M' = \text{sym}_P M$. Similarly, P is a plane symmetry of the vector function $\overrightarrow{\mathbf{V}}$ if, for any M, we have: $\overrightarrow{\mathbf{V}}' = \overrightarrow{\mathbf{V}}(M') = \text{sym}_P \overrightarrow{\mathbf{V}}(M) = \overrightarrow{\mathbf{V}}_{\|} - \overrightarrow{\mathbf{V}}_{\perp}$, that is, $\overrightarrow{\mathbf{V}}'_{\|} = \overrightarrow{\mathbf{V}}_{\|}$ and $\overrightarrow{\mathbf{V}}'_{\perp} = -\overrightarrow{\mathbf{V}}_{\perp}$, as shown in Figure 1.11.

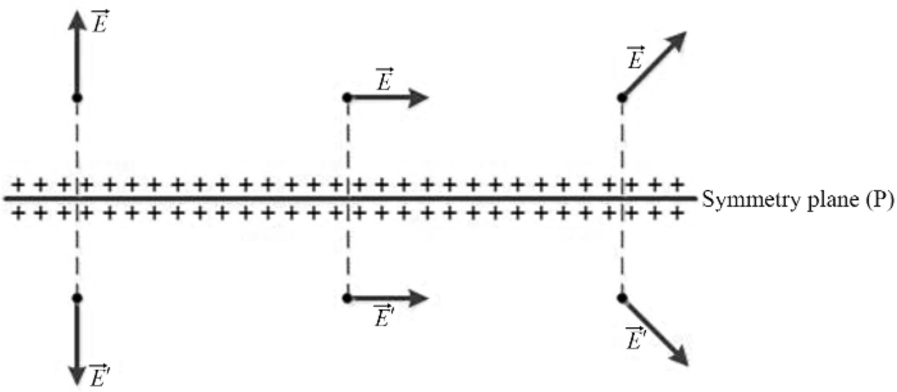

Figure 1.11. *Transformation of $\overrightarrow{\mathbf{E}}$ with respect to the symmetry plane (P): symmetry preserves the tangential component $\overrightarrow{\mathbf{E}}_{\|}$ and inverts the normal component $\overrightarrow{\mathbf{E}}_{\perp}$*

b) *Antisymmetry plane*: P is said to be a *antisymmetry plane* of Φ if, for any point M, we have $\Phi(M') = -\Phi(M)$. P is also a antisymmetry plane of $\overrightarrow{\mathbf{V}}$ if, for any

M, we have: $\vec{\mathbf{V}}' = \vec{\mathbf{V}}(M') = -\mathrm{sym}_P \vec{\mathbf{V}}(M) = -\vec{\mathbf{V}}_{\|} + \vec{\mathbf{V}}_{\perp}$, that is, $\vec{\mathbf{V}}'_{\|} = -\vec{\mathbf{V}}_{\|}$ and $\vec{\mathbf{V}}'_{\perp} = \vec{\mathbf{V}}_{\perp}$, as given in Figure 1.12.

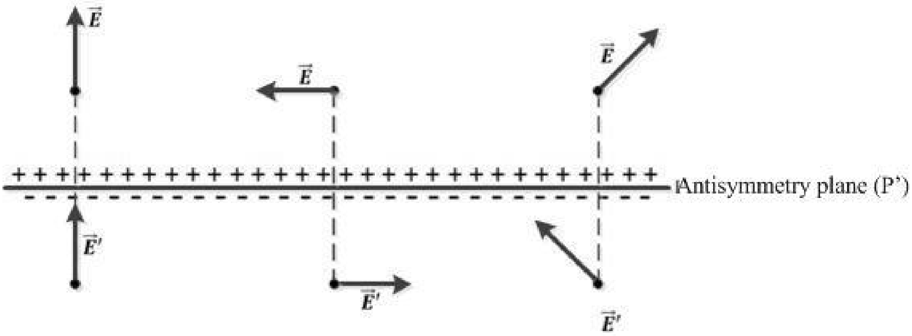

Figure 1.12. *Transformation of $\vec{\mathbf{E}}$ with respect to the antisymmetry plane (P): antisymmetry preserves the component normal $\vec{\mathbf{E}}_{\perp}$ and inverts the tangential component $\vec{\mathbf{E}}_{\|}$*

Consider two charges q and q' located at two points A and A' symmetrical with respect to a plane P, then calculate the fields created at two points M and M also symmetrical with respect to the same plane, as shown in Figure 1.13. The value of the electric field measured at point M is:

$$\vec{\mathbf{E}}(M) = \frac{1}{4\pi\varepsilon_0}\left(q\frac{\overrightarrow{\mathbf{AM}}}{AM^3} + q'\frac{\overrightarrow{\mathbf{A'M}}}{A'M^3}\right) = \vec{\mathbf{E}}_{\|}(M) + \vec{\mathbf{E}}_{\perp}(M) \qquad [1.57]$$

with:

$$\begin{aligned} \vec{\mathbf{E}}_{\|}(M) &= \frac{1}{4\pi\varepsilon_0}\left(q\frac{\overrightarrow{\mathbf{AM}}_{\|}}{AM^3} + q'\frac{\overrightarrow{\mathbf{A'M}}_{\|}}{A'M^3}\right) \\ \vec{\mathbf{E}}_{\perp}(M) &= \frac{1}{4\pi\varepsilon_0}\left(q\frac{\overrightarrow{\mathbf{AM}}_{\perp}}{AM^3} + q'\frac{\overrightarrow{\mathbf{A'M}}_{\perp}}{A'M^3}\right) \end{aligned} \qquad [1.58]$$

We also have

$$\vec{\mathbf{E}}(M') = \frac{1}{4\pi\varepsilon_0}\left(q\frac{\overrightarrow{\mathbf{AM'}}}{AM'^3} + q'\frac{\overrightarrow{\mathbf{A'M'}}}{A'M'^3}\right) = \vec{\mathbf{E}}_{\|}(M') + \vec{\mathbf{E}}_{\perp}(M') \qquad [1.59]$$

with:

$$\overrightarrow{\mathbf{E}}_{\parallel}(M') = \frac{1}{4\pi\varepsilon_0}\left(q\frac{\overrightarrow{\mathbf{AM'}}_{\parallel}}{AM'^3} + q'\frac{\overrightarrow{\mathbf{A'M'}}_{\parallel}}{A'M'^3}\right)$$
$$\overrightarrow{\mathbf{E}}_{\perp}(M') = \frac{1}{4\pi\varepsilon_0}\left(q\frac{\overrightarrow{\mathbf{AM'}}_{\perp}}{AM'^3} + q'\frac{\overrightarrow{\mathbf{A'M'}}_{\perp}}{A'M'^3}\right) \qquad [1.60]$$

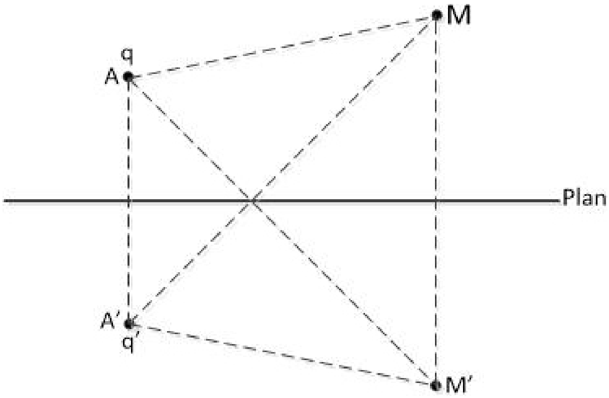

Figure 1.13. *Fields created by symmetric (respectively, antisymmetric) charges, $q' = q$ (respectively, $q' = -q$), at two symmetric points M and M' (with respect to a plane P). The symmetry (respectively, antisymmetry) of the charges implies the symmetry (respectively, antisymmetry) of the electric field with respect to the same plane. On the symmetry (respectively, antisymmetry) plane, the field $\overrightarrow{\mathbf{E}}$ is contained in this plane (respectively, is perpendicular to this plane)*

Further, since $AM' = A'M$, $\overrightarrow{\mathbf{AM'}}_{\parallel} = \overrightarrow{\mathbf{A'M}}_{\parallel}$, $\overrightarrow{\mathbf{AM'}}_{\perp} = -\overrightarrow{\mathbf{A'M}}_{\perp}$, $A'M' = AM$, $\overrightarrow{\mathbf{A'M'}}_{\parallel} = \overrightarrow{\mathbf{AM}}_{\parallel}$ and $\overrightarrow{\mathbf{A'M'}}_{\perp} = -\overrightarrow{\mathbf{AM}}_{\perp}$, it can be derived from this that:

$$\overrightarrow{\mathbf{E}}_{\parallel}(M') = \frac{1}{4\pi\varepsilon_0}\left(q\frac{\overrightarrow{\mathbf{A'M}}_{\parallel}}{A'M^3} + q'\frac{\overrightarrow{\mathbf{AM}}_{\parallel}}{AM^3}\right)$$
$$\overrightarrow{\mathbf{E}}_{\perp}(M') = \frac{1}{4\pi\varepsilon_0}\left(-q\frac{\overrightarrow{\mathbf{A'M}}_{\perp}}{A'M^3} - q'\frac{\overrightarrow{\mathbf{AM}}_{\perp}}{AM^3}\right) \qquad [1.61]$$

It can therefore be concluded that:

1) If $q = q'$ (P is a symmetry plane), we therefore have $\overrightarrow{\mathbf{E}}_{\parallel}(M') = \overrightarrow{\mathbf{E}}_{\parallel}(M)$ and $\overrightarrow{\mathbf{E}}_{\perp}(M') = -\overrightarrow{\mathbf{E}}_{\perp}(M)$: the symmetry of the distribution of charges with respect to

P also implies that the field is symmetric with respect to the same plane. In particular, if M is in the plane, that is, $M' = M$, then we would have $\overrightarrow{\mathbf{E}}_{\perp}(M) = -\overrightarrow{\mathbf{E}}_{\perp}(M)$, that is, $\overrightarrow{\mathbf{E}}_{\perp}(M) = 0$ and $\overrightarrow{\mathbf{E}}(M) = \overrightarrow{\mathbf{E}}_{||}(M)$. If the distribution of charges admits a plane of symmetry (P), then at all points in this plane, $\overrightarrow{\mathbf{E}}$ is contained in this field.

2) If $q' = -q$ (P is a antisymmetry plane), we thus have $\overrightarrow{\mathbf{E}}_{||}(M') = -\overrightarrow{\mathbf{E}}_{||}(M)$ and $\overrightarrow{\mathbf{E}}_{\perp}(M') = \overrightarrow{\mathbf{E}}_{\perp}(M)$: the antisymmetry of the distribution of charges implies the antisymmetry of the field. In particular, if $M \in P$, we will then have $\overrightarrow{\mathbf{E}}_{||}(M) = -\overrightarrow{\mathbf{E}}_{||}(M)$, that is, $\overrightarrow{\mathbf{E}}_{||}(M) = 0$ and $\overrightarrow{\mathbf{E}}(M) = \overrightarrow{\mathbf{E}}_{\perp}(M)$. If the distribution of charges admits an antisymmetry plane (P), then at all points in this plane, $\overrightarrow{\mathbf{E}}$ is perpendicular to this plane.

These results can be easily generalized to more complex distributions.

1.3.4. *Field lines and tubes*

The concept of a *field line* or force line is very useful to form an idea of the general shape of the electric field or even a magnetic field in a region of space. It represents the orientation of the resultant electric field at a point in space. The field line of $\overrightarrow{\mathbf{E}}$ is a curve Γ defined in space so that at each of its points, the vector $\overrightarrow{\mathbf{E}}$ is tangent to this curve. It is conventionally oriented in the direction of $\overrightarrow{\mathbf{E}}$. Let $d\overrightarrow{\mathbf{l}}$ be an elementary displacement along a field line Γ. The fact that at every point on Γ, the field $\overrightarrow{\mathbf{E}}$ is parallel to $d\overrightarrow{\mathbf{l}}$ and translated by:

$$\overrightarrow{\mathbf{E}} \times d\overrightarrow{\mathbf{l}} = 0 \qquad [1.62]$$

This vector equation gives three coupled differential equations (these differential equations define the field line) that must be solved. For example, in Cartesian coordinates, we get:

$$\begin{aligned} &E_y(x,y,z)\,dz - E_z(x,y,z)\,dy = 0 \\ &E_z(x,y,z)\,dx - E_x(x,y,z)\,dz = 0 \\ &E_x(x,y,z)\,dy - E_y(x,y,z)\,dx = 0 \end{aligned} \qquad [1.63]$$

Solving these differential equations brings up three indeterminate constants. The values of these constants are fixed by choosing any point on the field line being considered. Another form of writing can also be adopted: if Γ is a curve that is parameterized by $\overrightarrow{\mathbf{r}}(t) = x(t)\overrightarrow{\mathbf{i}} + y(t)\overrightarrow{\mathbf{j}} + z(t)\overrightarrow{\mathbf{k}}$, with $t \in \mathbb{R}$, the vector that is tangent to Γ at the point $\overrightarrow{\mathbf{r}}(t)$ is the vector $\frac{d\overrightarrow{\mathbf{r}}(t)}{dt} = \frac{dx(t)}{dt}\overrightarrow{\mathbf{i}} + \frac{dy(t)}{dt}\overrightarrow{\mathbf{j}} + \frac{dz(t)}{dt}\overrightarrow{\mathbf{k}}$.

Thus, Γ is a field line for $\overrightarrow{\mathbf{E}} = E_x\left[x\left(t\right), y\left(t\right), z\left(t\right)\right]\overrightarrow{\mathbf{i}} + E_y\left[x\left(t\right), y\left(t\right), z\left(t\right)\right]\overrightarrow{\mathbf{j}} + E_z\left[x\left(t\right), y\left(t\right), z\left(t\right)\right]\overrightarrow{\mathbf{k}}$ if, for any t, we have:

$$\begin{aligned} \frac{dx\left(t\right)}{dt} &= E_x\left[x\left(t\right), y\left(t\right), z\left(t\right)\right] \\ \frac{dy\left(t\right)}{dt} &= E_y\left[x\left(t\right), y\left(t\right), z\left(t\right)\right] \\ \frac{dz\left(t\right)}{dt} &= E_z\left[x\left(t\right), y\left(t\right), z\left(t\right)\right] \end{aligned} \qquad [1.64]$$

For any fixed point $\overrightarrow{\mathbf{r}}_0 = \overrightarrow{\mathbf{r}}\left(t_0\right)$, there is only one field line that passes through it.

EXAMPLE 1.14.– Let us determine the field lines of the vector field $\overrightarrow{\mathbf{V}} = -y\overrightarrow{\mathbf{i}} + x\overrightarrow{\mathbf{j}}$: $\overrightarrow{\mathbf{r}}\left(t\right) = x\left(t\right)\overrightarrow{\mathbf{i}} + y\left(t\right)\overrightarrow{\mathbf{j}} + z\left(t\right)\overrightarrow{\mathbf{k}}$ describes a field line if $\frac{d\overrightarrow{\mathbf{r}}(t)}{dt} = \frac{dx(t)}{dt}\overrightarrow{\mathbf{i}} + \frac{dy(t)}{dt}\overrightarrow{\mathbf{j}} + \frac{dz(t)}{dt}\overrightarrow{\mathbf{k}} = \overrightarrow{\mathbf{V}}\left[x\left(t\right), y\left(t\right), z\left(t\right)\right] = -y\left(t\right)\overrightarrow{\mathbf{i}} + x\left(t\right)\overrightarrow{\mathbf{j}}$, that is:

$$\begin{aligned} \frac{dx\left(t\right)}{dt} &= -y\left(t\right) \\ \frac{dy\left(t\right)}{dt} &= x\left(t\right) \\ \frac{dz\left(t\right)}{dt} &= 0 \end{aligned} \qquad [1.65]$$

Thus, $\frac{dx(t)}{dt}x\left(t\right) + \frac{dy(t)}{dt}y\left(t\right) = \frac{1}{2}\frac{d}{dt}\left[x^2\left(t\right) + y^2\left(t\right)\right] = 0 \Longrightarrow x^2\left(t\right) + y^2\left(t\right) = C_1^2$ and $z\left(t\right) = C_2$, where C_1 and C_2 are constants. Γ describes a circle in the horizontal plane centered on the Oz axis.

EXAMPLE 1.15.– Let us find the field lines for the central field $f\left(r\right)\overrightarrow{\mathbf{e}}_r$: in spherical coordinates, a parameterized curve Γ is given by $r\left(t\right) \in \left]0, \infty\right[$, $\theta\left(t\right) \in \left[0, \pi\right]$ and $\varphi\left(t\right) \in \left[0, 2\pi\right[$. The points along the curve are marked by their position vectors $\overrightarrow{\mathbf{r}}\left(t\right) = r\left(t\right)\overrightarrow{\mathbf{e}}_r\left(t\right)$, where $\overrightarrow{\mathbf{e}}_r\left(t\right)$ also depends on t, since it changes direction with the point $\overrightarrow{\mathbf{r}}\left(t\right)$. The tangent vector to Γ at the point $\overrightarrow{\mathbf{r}}\left(t\right)$ is therefore $\frac{d\overrightarrow{\mathbf{r}}}{dt} = \frac{dr}{dt}\overrightarrow{\mathbf{e}}_r + r\frac{d\overrightarrow{\mathbf{e}}_r}{dt}$. Consequently, Γ is a field line if $\frac{dr}{dt}\overrightarrow{\mathbf{e}}_r + r\frac{d\overrightarrow{\mathbf{e}}_r}{dt} = f\left(r\right)\overrightarrow{\mathbf{e}}_r$ ($\frac{d\overrightarrow{\mathbf{e}}_r}{dt}$ is written as a function of $\overrightarrow{\mathbf{e}}_\theta\left(t\right)$ and $\overrightarrow{\mathbf{e}}_\varphi\left(t\right)$), that is:

$$\begin{aligned} \frac{dr\left(t\right)}{dt} &= f\left(r\right) \\ \frac{d\overrightarrow{\mathbf{e}}_r}{dt} &= 0 \end{aligned} \qquad [1.66]$$

The second equation indicates that $\overrightarrow{\mathbf{e}}_r\left(t\right)$ is constant. The field lines are therefore radial lines centered on the source of the field. The first equation gives the distance $r\left(t\right)$ to the source:

$$\frac{dr\left(t\right)}{dt} = f\left(r\right) \Longrightarrow t = \int_{r_0}^{r} \frac{dr}{f\left(r\right)} \tag{1.67}$$

where r_0 is the initial distance of the source point. In particular, for $f\left(r\right) = \frac{q}{4\pi\varepsilon_0 r^2}$, we will have $r^3 = r_0^3 + \frac{4\pi\varepsilon_0 t}{3q}$. For $q > 0$, the field lines are oriented toward the exterior: the electric field is repulsive. In contrast, if $q < 0$, the field lines converge toward the source: the electric field is attractive, as depicted in Figure 1.14. ■

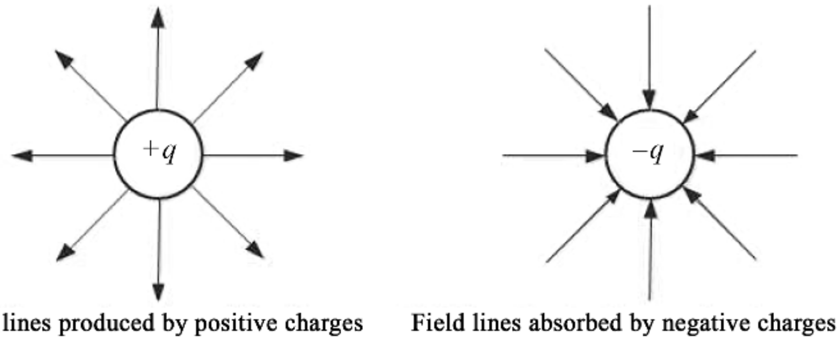

Figure 1.14. *The field lines are oriented curves that translate the characteristics (direction, orientation and also intensity) of a vector field in a region in space: the electric field lines are produced by the positive charges (repulsion) and absorbed by the negative charges (attraction)*

A *field tube* is a bundle of field lines that go across a surface. Here are a few observations:

– The field lines are continuous between the positive and negative charges. The field lines are produced by the positive charges and absorbed by the negative charges.

– The number of field lines produced or absorbed by a charge is proportional to the value of the charge (the greater the charge, the more it produces or absorbs lines).

– The field lines must respect the symmetry of the distribution of charges.

– The field lines must not intersect.

– Moving away from the charge distribution, the field lines appear to come from a point-like charge whose value is equal to the net charge of the distribution.

1.4. Electric potential

In the presence of two charges with opposite signs, work must be done against the force of attraction to separate them. This work can be recovered if the charges are allowed to approach each other. Similarly, if the charges are of the same sign, work must be done against the force of repulsion to bring them together. This work can be recovered if the charges are allowed to move away from each other. Therefore, a charge gains energy when it is moved in the opposite direction to the force that keeps it in a system (attraction) or pushes it away from a system (repulsion). In this case, we use the term *potential energy*, as the quantity of energy depends on the position of the charge in a force field.

Consider an electric field $\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right)$ created by a single point-like charge q' located at the coordinate origin ($\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right)$ and the field measured at position (x, y, z)). Let us take a test charge q small enough that its presence does not disturb the first charge q'. The work required to move this test charge at constant speed from M to another point N along a given trajectory will be:

$$W_{M\to N} = -\int_M^N q\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right)\cdot d\overrightarrow{\mathbf{r}} \qquad [1.68]$$

The negative sign indicates that the work carried out is against the field. The scalar product in [1.68] indicates that no work is required to displace the test charge perpendicular to the electric field (along spheres with a constant radius in our case). These surfaces are called *equipotential surfaces*. Non-zero work is required to move q to a different radius. Consequently, it can be seen that the work [1.68] depends only on the start and end positions (r_M and r_N) of the trajectory, and not on the form of the trajectory itself, as shown in Figure 1.15:

$$W_{M\to N} = -\frac{qq'}{4\pi\varepsilon_0}\int_{\overrightarrow{\mathbf{r}}_M}^{\overrightarrow{\mathbf{r}}_N}\frac{\overrightarrow{\mathbf{r}}\cdot d\overrightarrow{\mathbf{r}}}{r^3} = \frac{qq'}{4\pi\varepsilon_0}\left(\frac{1}{r_N}-\frac{1}{r_M}\right) \qquad [1.69]$$

It is easy to check the adequacy of the sign in [1.69] with the previous discussion: (1) if $qq' < 0$ (charge attraction) and $r_N > r_M$ (charge separation), then $W_{M\to N} > 0$ (positive work done on q to separate the two charges); (2) if $qq' > 0$ (chare repulsion) and $r_N < r_M$ (charges coming together), then $W_{M\to N} > 0$ (positive work done on q to bring the two charges together).

For a closed path $\mathcal{C}$ (i.e. starting and ending at the same point $r_N = r_M$), the work required to displace the charge is zero:

$$W|_{\mathcal{C}} = -\oint_{\mathcal{C}} q\overrightarrow{\mathbf{E}}\cdot d\overrightarrow{\mathbf{r}} = -\frac{qq'}{4\pi\varepsilon_0}\oint_{\mathcal{C}}\frac{\overrightarrow{\mathbf{r}}\cdot d\overrightarrow{\mathbf{r}}}{r^3} = \frac{qq'}{4\pi\varepsilon_0}\oint_{\mathcal{C}} d\left(\frac{1}{r}\right) = 0 \qquad [1.70]$$

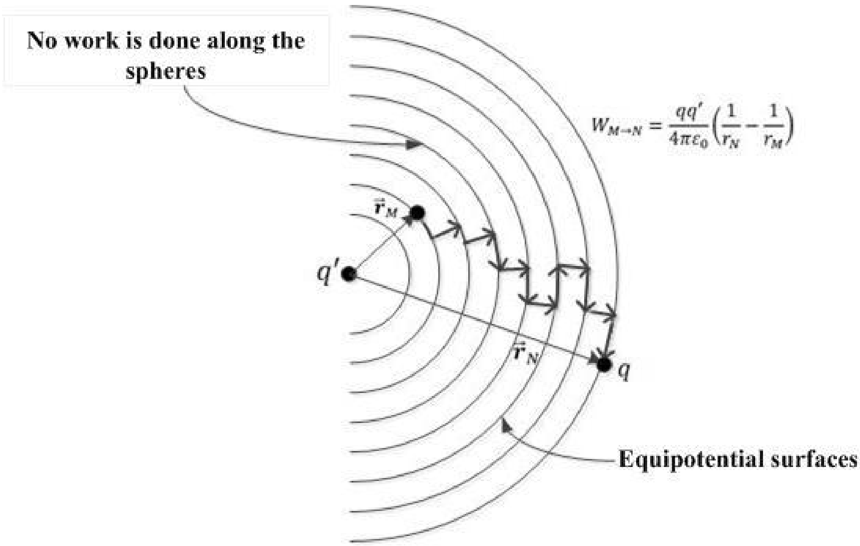

Figure 1.15. *The work required to displace a test charge q in the field of another point-like charge q' depends only on the starting and ending position, because it is zero on equipotential surfaces*

If the electric field is no longer created by a point-like charge q', but by a distribution of charges contained in τ', each curvilinear integral associated with a differential charge $\rho\left(\overrightarrow{\mathbf{r}}'\right) d\tau'$ is zero. We will therefore have for the entire distribution:

$$\oint \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) \cdot d\overrightarrow{\mathbf{r}} = 0 \qquad [1.71]$$

The electric field is therefore *conservative*. This important property comes from the fact that the electrical force is a central force. Using Stokes's theorem, it is possible to convert the integral [1.71] into a surface integral of the curl of the electric field $\oint_C \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{r}} = \int_S \left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}}\right) \cdot d\overrightarrow{\mathbf{S}}$. It follows that at any point in space, we will have:

$$\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) = 0 \qquad [1.72]$$

On the other hand, we also know that the curl of the gradient of a scalar function along a closed path is zero. This means that the electric field can be determined (or fully described) from the gradient of a scalar function of the position, called *electric potential*:

$$\vec{\mathbf{E}}\left(\vec{\mathbf{r}}\right) = -\vec{\nabla} V\left(\vec{\mathbf{r}}\right) \qquad [1.73]$$

It is easy to check that equations [1.72] and [1.73] are compatible: the field [1.73] satisfies equation [1.72]. The advantage of introducing potential is technical. If we can calculate the function V (a simple, double or triple integral to be realized), it is possible to directly obtain the three components of the electric field by applying the operator $-\vec{\nabla}$. This is often easier than calculating each component of the field separately (three integrals to be performed). It should be noted that:

1) The negative sign in equation [1.73] takes into account that the electric field is directed toward decreasing potentials.

2) According to the elementary properties of the gradient, the electric field is always perpendicular to surfaces with constant potential.

3) The potential V is determined to an additive constant. The potential V and $V + A$ (A is a constant here) give the same value for $\vec{\mathbf{E}}$. It is therefore necessary to specify a zero reference potential, which is often taken to infinity. In practice, zero potential is often attributed to the surface of the Earth (the "ground" reference point).

4) Since $\vec{\mathbf{E}} \cdot d\vec{\mathbf{r}} = -\vec{\nabla} V \cdot d\vec{\mathbf{r}} = -dV$, we deduce that electric potential is measured at a point $\vec{\mathbf{r}}$ by the integral $V\left(\vec{\mathbf{r}}\right) = -\int \vec{\mathbf{E}}\left(\vec{\mathbf{r}}\right) \cdot d\vec{\mathbf{r}} + A$ (A is an additive constant). Assuming that the potential is always zero at infinity, we finally obtain:

$$V\left(\vec{\mathbf{r}}\right) = \int_x^\infty \int_y^\infty \int_z^\infty \vec{\mathbf{E}}\left(\vec{\mathbf{r}}\right) \cdot d\vec{\mathbf{r}} \qquad [1.74]$$

The potential is a *continuous* function, except at points where the field tends to infinity. This follows directly from $dV = -\vec{\mathbf{E}} \cdot d\vec{\mathbf{r}}$: when $d\vec{\mathbf{r}}$ tends to zero, dV tends to zero if $\vec{\mathbf{E}}$ is finite in the neighborhood. We will see later that the electric field is discontinuous in the neighborhood of a charged surface.

5) The potential difference between two points $\vec{\mathbf{r}}_M$ and $\vec{\mathbf{r}}_N$ represents the work per unit charge required to move from $\vec{\mathbf{r}}_M$ to $\vec{\mathbf{r}}_N$ quasi-statically:

$$V\left(\vec{\mathbf{r}}_N\right) - V\left(\vec{\mathbf{r}}_M\right) = -\int_M^N q\vec{\mathbf{E}}\left(\vec{\mathbf{r}}\right) \cdot d\vec{\mathbf{r}} = \frac{W_{M\to N}}{q} \qquad [1.75]$$

It must be noted that equations [1.71], [1.72] and [1.75] are the *field version* of Kirchhoff's second law, which states that the algebraic sum of the potential differences around a closed loop in a circuit is zero. Using the usual convention that V is zero at infinity, it can finally be written that:

$$W_{\infty\to N} = qV\left(\vec{\mathbf{r}}_N\right) \equiv W_e\left(\vec{\mathbf{r}}_N\right) \qquad [1.76]$$

$W_e\left(\overrightarrow{\mathbf{r}}_N\right)$ is called the *electrostatic potential energy* of the charge q in the field $\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right)$ or, more accurately, it is the potential energy of the interaction between q and the sources of $\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right)$. We can adopt the following definition: *the electrostatic potential energy, W_e, of a charged particle placed in an electric field is equal to the work that must be carried out to bring this particle, quasi-statically, from infinity to its current position.*

6) The potential created at the point $\overrightarrow{\mathbf{r}}$ by a point-like charge q' placed at the origin is:

$$V\left(\overrightarrow{\mathbf{r}}\right)=\int_{\overrightarrow{\mathbf{r}}}^{\infty}\frac{q'dr}{4\pi\varepsilon_0 r^2}=\frac{q'}{4\pi\varepsilon_0 r} \qquad [1.77]$$

The potential $V\left(\overrightarrow{\mathbf{r}}\right)$ is then interpreted as the work per unit charge required to bring a charge from infinity to a certain distance r, from a point-like charge q'.

7) According to the superposition principle, the electric potential is additive. In fact, we have: $\overrightarrow{\mathbf{E}}=\overrightarrow{\mathbf{E}}_1+\cdots+\overrightarrow{\mathbf{E}}_n=-\overrightarrow{\nabla}V_1-\cdots-\overrightarrow{\nabla}V_n=-\overrightarrow{\nabla}\left(V_1+\cdots+V_n\right)$, which gives $V=V_1+\cdots+V_n$.

8) The net potential of multiple point-like charges is obtained by summing over the individual potentials of all of the charges:

$$V\left(\overrightarrow{\mathbf{r}}\right)=\sum_{i=1}^{n}\frac{q_i'}{4\pi\varepsilon_0\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}_i'\right|} \qquad [1.78]$$

If there is a continuous charge distribution, the summation becomes an integration over all of the differential charge elements dq':

$$V\left(\overrightarrow{\mathbf{r}}\right)=\int_{\text{Dist.}}\frac{dq'}{4\pi\varepsilon_0\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}'\right|} \qquad [1.79]$$

where the integral is a curvilinear integral for a linear charge, a surface integral for a surface charge density, and a volume integral for a volume charge density.

$$V\left(\overrightarrow{\mathbf{r}}\right)=\begin{cases}\displaystyle\int_{C'}\frac{\lambda\left(\overrightarrow{\mathbf{r}}'\right)dl'}{4\pi\varepsilon_0\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}'\right|} & \text{(linear charge)}\\ \displaystyle\int_{S'}\frac{\sigma\left(\overrightarrow{\mathbf{r}}'\right)dS'}{4\pi\varepsilon_0\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}'\right|} & \text{(surface charge density)}\\ \displaystyle\int_{\tau'}\frac{\rho\left(\overrightarrow{\mathbf{r}}'\right)d\tau'}{4\pi\varepsilon_0\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}'\right|} & \text{(volume charge density)}\end{cases} \qquad [1.80]$$

9) The electric field can be easily found again by applying the operator $-\vec{\nabla}$ over the potential V and by recalling that the derivatives must be taken with respect to the field positions (x, y, z), while the integration is carried out over the source positions (x', y', z'). The operator $\vec{\nabla}$ can therefore pass inside the integral and act on the quantity $\left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right|$ only:

$$\vec{\mathbf{E}} = -\vec{\nabla} V = -\int_{\text{Dist.}} \frac{dq'}{4\pi\varepsilon_0} \vec{\nabla} \left(\frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right|} \right) = \int_{\text{Dist.}} \frac{dq' \left(\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right)}{4\pi\varepsilon_0 \left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right|^3} \qquad [1.81]$$

EXAMPLE 1.16. (Uniformly Charged Infinitely Long Wire).– Let us calculate the potential created by an infinite rectilinear wire carrying a uniform linear charge λ, as shown in Figure 1.3. The potential at point M is:

$$V = \int_{-\infty}^{\infty} \frac{\lambda dz'}{4\pi\varepsilon_0 \sqrt{r^2 + z'^2}} = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(z' + \sqrt{z'^2 + r^2}\right)\Big|_{-\infty}^{\infty} \qquad [1.82]$$

This calculation is impossible because the potential cannot be zero at infinity in this case. This can be shown simply by calculating the potential difference between two points M and N from the field found in Example 1.8:

$$V(r_M) - V(r_N) = \int_{r_M}^{r_N} \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}} = \int_{r_M}^{r_N} \frac{\lambda dr}{2\pi\varepsilon_0 r} = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_N}{r_M}\right) \qquad [1.83]$$

We can clearly see that we cannot take $r_N = \infty$ and $V(r_N) = 0$. This is because the wire is infinitely long: the source is not localized and there are charges at infinity.

EXAMPLE 1.17. (Uniformly Charged Disc with Negligible Thickness).– Let us calculate the potential created by a flat disk, with no thickness, with radius R and charge Q (uniformly distributed over the disk), at a point M located on the axis of the disk denoted by Oz, a distance z from its center, as shown in Figure 1.8. The charge density is $\rho\left(\vec{\mathbf{r}}'\right) = \frac{Q}{\pi R^2}\delta(z')$ for $x'^2 + y'^2 \leq R^2$ and 0 elsewhere.

Consequently, the electric potential measured at the site M is:

$$\begin{aligned} V(0,0,z) &= \frac{1}{4\pi\varepsilon_0} \int_{\text{Disc}} \frac{\rho\left(\vec{\mathbf{r}}'\right) d\tau'}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right|} = \frac{1}{4\pi\varepsilon_0} \int_0^R \int_0^{2\pi} \frac{Q}{\pi R^2} \frac{r' dr' d\varphi'}{\sqrt{r'^2 + z^2}} \\ &= \frac{Q}{2\pi\varepsilon_0 R^2} \left(\sqrt{R^2 + z^2} - |z|\right) \end{aligned}$$

[1.84]

To once again arrive at the electric field $\overrightarrow{\mathbf{E}}(0,0,z)$, it is enough to write:

$$\overrightarrow{\mathbf{E}}(0,0,z) = -\frac{\partial V}{\partial z}\overrightarrow{\mathbf{k}} = \frac{Q}{2\pi\varepsilon_0 R^2}\frac{z}{|z|}\left(\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}}\right)\overrightarrow{\mathbf{k}} \qquad [1.85]$$

EXAMPLE 1.18. (Hollow Sphere with Uniform Surface Charge).– Consider a hollow sphere of radius R, with a charge Q that is uniformly distributed across its surface, as shown in Figure 1.10. Let us determine the potential created by the sphere at the point M. For reasons of symmetry, the field point M is chosen along the axis Oz. The differential charge $\sigma R^2 \sin\theta' d\theta' d\varphi'$ $\left(\sigma = \frac{Q}{4\pi R^2}\right)$ located on its surface produces the differential potential at the point M given by $dV = \frac{\sigma R^2 \sin\theta' d\theta' d\varphi'}{4\pi\varepsilon_0\sqrt{R^2+r'^2-2r'R\cos\theta'}}$. By integrating over the entire sphere, we obtain:

$$\begin{aligned} V(r) &= \int_{\theta'=0}^{\pi}\int_{\varphi'=0}^{2\pi}\frac{\sigma R^2\sin\theta' d\theta' d\varphi'}{4\pi\varepsilon_0\sqrt{R^2+r'^2-2r'R\cos\theta'}} \\ &= \int_0^{\pi}\frac{\sigma R^2\sin\theta' d\theta'}{2\varepsilon_0\sqrt{R^2+r'^2-2r'R\cos\theta'}} \end{aligned} \qquad [1.86]$$

Let us perform the change of variable $s^2 = R^2 + r'^2 - 2r'R\cos\theta'$

$$V(r) = \int_{|R-r|}^{R+r}\frac{\sigma R ds}{2r\varepsilon_0} = \frac{\sigma R}{2r\varepsilon_0}(R+r-|R-r|) = \begin{cases} \dfrac{\sigma R}{\varepsilon_0} = \dfrac{Q}{4\pi\varepsilon_0 R} & r \le R \\ \dfrac{\sigma R^2}{\varepsilon_0 r} = \dfrac{Q}{4\pi\varepsilon_0 r} & r > R \end{cases} \qquad [1.87]$$

and consequently:

$$\overrightarrow{\mathbf{E}}(r) = -\frac{dV}{dr}\frac{\overrightarrow{\mathbf{r}}}{r} = \begin{cases} 0 & r \le R \\ \dfrac{Q\overrightarrow{\mathbf{r}}}{4\pi\varepsilon_0 r^3} & r > R \end{cases} \qquad [1.88]$$

It can be seen that the potential is continuous along the boundary $r = R$, while the field is discontinuous.

EXAMPLE 1.19.– (Solid Sphere Uniformly Charged).– Let us return to the sphere with radius R and charge Q uniformly distributed throughout its volume, as shown in Figure 1.9. It is possible to arrive at the expressions for the electric field inside and outside of the sphere from the potential $V\left(\overrightarrow{\mathbf{r}}\right)$. The differential charge

$\rho r'^2 dr' \sin\theta' d\theta' d\varphi'$, located inside the sphere, induces the potential $dV = \frac{\rho r'^2 dr' \sin\theta' d\theta' d\varphi'}{4\pi\varepsilon_0\sqrt{r^2+r'^2-2rr'\cos\theta'}}$ at the point M. Consequently, the potential created by the whole sphere can be obtained from the following integral (for reasons of symmetry, the potential will only depend on r):

$$\begin{aligned} V(r) &= \int_0^R \int_0^\pi \int_0^{2\pi} \frac{\rho r'^2 dr' \sin\theta' d\theta' d\varphi'}{4\pi\varepsilon_0\sqrt{r^2+r'^2-2rr'\cos\theta'}} \\ &= \frac{\rho}{2\varepsilon_0 r}\int_0^R r'dr' \int_{s=|r-r'|}^{r+r'} ds = \frac{\rho}{2\varepsilon_0 r}\int_0^R r'dr' \left(r+r'-|r-r'|\right) \end{aligned} \qquad [1.89]$$

– *Electric potential outside the sphere* $(r > R)$:

$$V_{\text{ext}}(r) = \frac{\rho}{\varepsilon_0 r}\int_0^R r'^2 dr' = \frac{\rho R^3}{3\varepsilon_0 r} = \frac{Q}{4\pi\varepsilon_0 r} \qquad [1.90]$$

As with $\overrightarrow{\mathbf{E}}_{\text{ext}}$, the potential V_{ext} is the same as if the charge Q were concentrated at the center of the sphere. We arrive again at the electric field $\overrightarrow{\mathbf{E}}_{\text{ext}}$ using formula [1.77], that is, $\overrightarrow{\mathbf{E}}_{\text{ext}} = -\overrightarrow{\nabla} V_{\text{ext}} = -\frac{dV_{\text{ext}}}{dr}\frac{\overrightarrow{\mathbf{r}}}{r}$; this gives:

$$\overrightarrow{\mathbf{E}}_{\text{ext}} = \frac{Q}{4\pi\varepsilon_0}\frac{\overrightarrow{\mathbf{r}}}{r^3} \qquad [1.91]$$

– *Electric potential inside the sphere* $(r \le R)$:

$$\begin{aligned} V_{\text{int}}(r) &= \frac{\rho}{\varepsilon_0 r}\int_{r'=0}^{r} r'^2 dr' + \frac{\rho}{\varepsilon_0}\int_{r'=r}^{R} r'dr' = \frac{\rho R^2}{6\varepsilon_0}\left[3 - \left(\frac{r}{R}\right)^2\right] \\ &= \frac{Q}{8\pi\varepsilon_0 R}\left[3 - \left(\frac{r}{R}\right)^2\right] \end{aligned} \qquad [1.92]$$

and consequently:

$$\overrightarrow{\mathbf{E}}_{\text{int}} = -\overrightarrow{\nabla} V_{\text{int}} = -\frac{dV_{\text{int}}}{dr}\frac{\overrightarrow{\mathbf{r}}}{r} = \frac{Q}{4\pi\varepsilon_0 R^3}\overrightarrow{\mathbf{r}} \qquad [1.93]$$

EXAMPLE 1.20.– If the sphere's volume charge density is governed by $\rho(r') = \rho_0\left(1 - \frac{r'^2}{R^2}\right)$ for $r' \le R$ and 0 for $r' > R$, the electric potential has the value:

$$V(r) = \frac{\rho_0}{2\varepsilon_0 r}\int_0^R \left(1 - \frac{r'^2}{R^2}\right) r'\left(r + r' - |r - r'|\right) dr' \qquad [1.94]$$

– *Electric potential outside the sphere*:

$$V_{\text{ext}} = \frac{\rho_0}{\varepsilon_0 r} \int_{r'=0}^{R} \left(1 - \frac{r'^2}{R^2}\right) r'^2 dr' = \frac{2\rho_0 R^3}{15\varepsilon_0 r} \quad [1.95]$$

$$\overrightarrow{\mathbf{E}}_{\text{ext}} = -\overrightarrow{\nabla} V_{\text{ext}} = -\frac{dV_{\text{ext}}}{dr}\frac{\overrightarrow{\mathbf{r}}}{r} = \frac{2\rho_0 R^3}{15\varepsilon_0}\frac{\overrightarrow{\mathbf{r}}}{r^3} \quad [1.96]$$

– *Electric potential inside the sphere*:

$$\begin{aligned} V_{\text{int}} &= \frac{\rho_0}{\varepsilon_0 r}\left[\int_0^r \left(1 - \frac{r'^2}{R^2}\right) r'^2 dr' + r\int_r^R \left(1 - \frac{r'^2}{R^2}\right) r' dr'\right] \\ &= \frac{\rho_0 R^2}{4\varepsilon_0}\left[1 - \frac{2}{3}\left(\frac{r}{R}\right)^2 + \frac{1}{5}\left(\frac{r}{R}\right)^4\right] \end{aligned} \quad [1.97]$$

$$\overrightarrow{\mathbf{E}}_{\text{int}} = -\overrightarrow{\nabla} V_{\text{int}} = -\frac{dV_{\text{int}}}{dr}\frac{\overrightarrow{\mathbf{r}}}{r} = \frac{\rho_0}{\varepsilon_0}\left[\frac{1}{3} - \frac{1}{5}\left(\frac{r}{R}\right)^2\right]\overrightarrow{\mathbf{r}} \quad [1.98]$$

1.5. Equipotential surfaces

The set of points in space whose potential V is equal to a certain constant value V_0 constitutes an *equipotential surface* V_0.

Consider an equipotential surface passed through a point $M(x, y, z)$. If we make an infinitesimal displacement on this surface toward a neighboring point $N(x + dx, y + dy + dz)$, the vector $\overrightarrow{\mathbf{MN}}$ is then in the plane tangent to the surface at M.

Moreover, $dV = -\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{MN}}$ and, by definition, the variation in potential during this displacement is zero, since $V(M) = V(N)$; this implies that $\overrightarrow{\mathbf{E}} \perp \overrightarrow{\mathbf{MN}}$. We can therefore conclude that the field $\overrightarrow{\mathbf{E}}$ is perpendicular to the equipotential at any point. Consequently, the field lines are perpendicular to equipotential surfaces.

EXAMPLE 1.21. (Equipotentials and Field Lines for Two Linear Charges of Opposite Polarity).–

1) Let us return to the example of a linear distribution with charge density λ and length L (located on Oz between $-\frac{L}{2}$ and $\frac{L}{2}$), as seen in Figure 1.16(a). The potential measured at the point $M(x, y, z)$ takes the value:

$$V(r,z) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\lambda dz'}{4\pi\varepsilon_0\sqrt{r^2+(z-z')^2}} = -\frac{\lambda \ln\left(\frac{z-\frac{L}{2}+\sqrt{r^2+\left(z-\frac{L}{2}\right)^2}}{z+\frac{L}{2}+\sqrt{r^2+\left(z+\frac{L}{2}\right)^2}}\right)}{4\pi\varepsilon_0}$$
$$= -\frac{\lambda}{4\pi\varepsilon_0}\left[\sinh^{-1}\left(\frac{z-\frac{L}{2}}{r}\right) - \sinh^{-1}\left(\frac{z+\frac{L}{2}}{r}\right)\right] \quad [1.99]$$

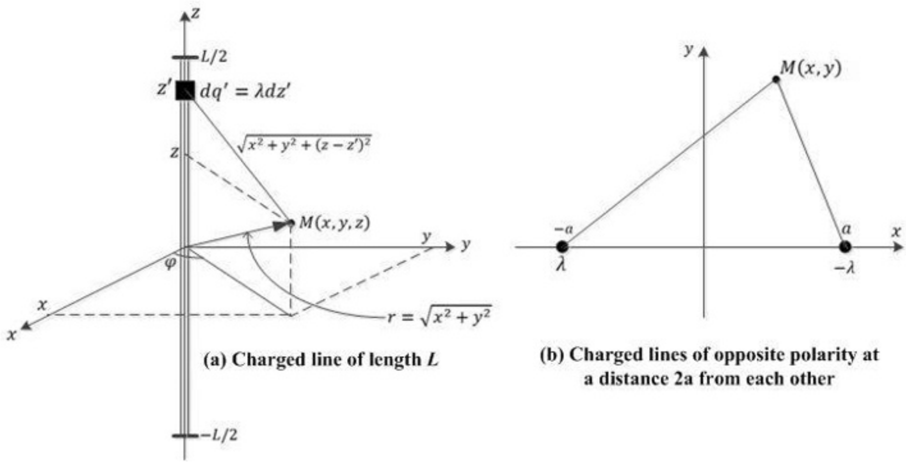

Figure 1.16. *Potential created by two linear charges with opposite polarities, which are parallel and separated by a distance of* $2a$

The components of $\overrightarrow{\mathbf{E}}$ can be obtained as follows:

$$E_z(r,z) = -\frac{\partial V}{\partial z} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+\left(z-\frac{L}{2}\right)^2}} - \frac{1}{\sqrt{r^2+\left(z+\frac{L}{2}\right)^2}}\right]$$
$$E_r(r,z) = -\frac{\partial V}{\partial r} = -\frac{\lambda}{4\pi\varepsilon_0 r}\left[\frac{z-\frac{L}{2}}{\sqrt{r^2+\left(z-\frac{L}{2}\right)^2}} - \frac{z+\frac{L}{2}}{\sqrt{r^2+\left(z+\frac{L}{2}\right)^2}}\right] \quad [1.100]$$

When L becomes large, the expressions for the field and for the potential approach the expressions of an infinitely long line of charge:

$$\lim_{L \longrightarrow \infty} \begin{cases} E_z = 0 \\ E_r = \dfrac{\lambda}{2\pi\varepsilon_0 r} \\ V = -\dfrac{\lambda}{2\pi\varepsilon_0}\left[\ln r - \ln L\right] \end{cases} \qquad [1.101]$$

We observe that the potential has a constant term that becomes infinite as L tends toward infinity. The presence of this constant is explained by the fact that the potential [1.79] is not zero here for $r \longrightarrow \infty$, since the linear charge is infinitely long and, consequently, there is a non-zero charge at infinity. This additive and infinite constant is not important, as it does not contribute to the expression of the electric field and can therefore be ignored. Very far from the charge line, the potential [1.99] approaches the potential of a point charge λL:

$$\lim_{r^2+z^2 \gg L^2} V = \frac{\lambda L}{4\pi\varepsilon_0\sqrt{r^2+z^2}}. \qquad [1.102]$$

2) If we are now in the presence of two infinite linear charges with opposite polarities, $\pm\lambda$, parallel and separated from each other by a distance $2a$ (the origin is chosen on the median), as shown in Figure 1.16(b), the total potential is just the superposition of these potentials [1.101]:

$$V(x, y) = -\frac{\lambda}{2\pi\varepsilon_0} \ln \sqrt{\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}} \qquad [1.103]$$

where the additive constants cancel each other out. Consequently, the equipotential surfaces (equipotential lines are observed in the Oxy plane) are described by the expressions:

$$\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} = e^{-4\pi\varepsilon_0 V/\lambda} \equiv K_1 \qquad [1.104]$$

where K_1 is a constant (specific to each equipotential). This last equation can also be rewritten in the form:

$$\left(x - \frac{a(1+K_1)}{K_1 - 1}\right)^2 + y^2 = \frac{4K_1 a^2}{(1-K_1)^2} \qquad [1.105]$$

This is the equation for a circle with radius $R = \frac{2\sqrt{K_1}a}{|1-K_1|}$ and center $\left(\frac{a(1+K_1)}{K_1-1}, 0\right)$, as seen in Figure 1.17.

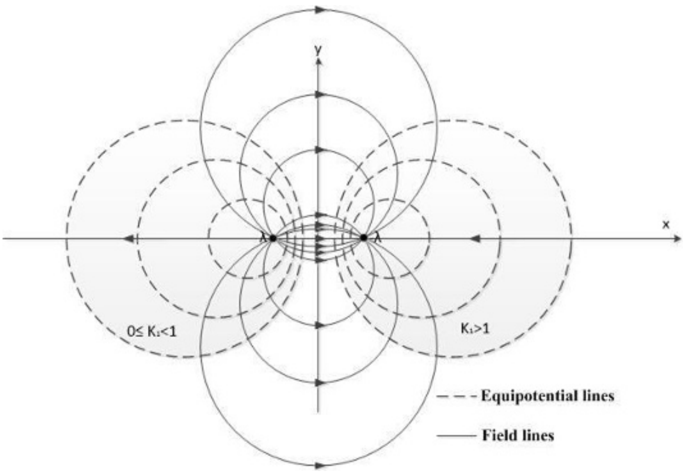

Figure 1.17. *Equipotential lines (dashed lines) and field lines (solid lines) forming a set of orthogonal circles*

For $K_1 = 1$, we obtain a circle of infinite radius whose center is at $(\pm\infty, 0)$: this therefore represents the plane $x = 0$. For $0 \leq K_1 < 1$, the equipotential circles are in the left half-plane, while for $1 < K_1 < +\infty$, the circles are in the right half-plane.

3) The electric field associated with the potential [1.103] is:

$$\overrightarrow{\mathbf{E}} = -\overrightarrow{\nabla} V = \frac{\lambda}{2\pi\varepsilon_0} \frac{2a\left(y^2 + a^2 - x^2\right)\overrightarrow{\mathbf{i}} - 4axy\overrightarrow{\mathbf{j}}}{\left[(x-a)^2 + y^2\right]\left[(x+a)^2 + y^2\right]} \quad [1.106]$$

The following differential equation must be solved to obtain the field lines:

$$\frac{dy}{dx} = \frac{E_y}{E_x} = -\frac{2xy}{y^2 + a^2 - x^2} \quad [1.107]$$

which can also be written as:

$$\frac{d\left(x^2 + y^2\right)}{a^2 - \left(x^2 + y^2\right)} + d\left(\ln y\right) = 0 \quad [1.108]$$

After integrating the above equation, we get:

$$x^2 + \left(y - \frac{a}{\tan K_2}\right)^2 = \frac{a^2}{\sin^2 K_2} \quad [1.109]$$

where K_2 is an integration constant that must be determined by specifying a point (x_0, y_0) along the chosen field line. The field lines are therefore also circles with radius $\frac{a}{\sin K_2}$ and centers at the points $\left(0, \frac{a}{\tan K_2}\right)$.

1.6. Gauss' law

Consider a closed surface S oriented toward the exterior and enclosing a volume τ. Let us take a differential surface element $d\overrightarrow{\mathbf{S}}$ (of the total surface) located around a point $M\,(x, y, z)$. The quantity:

$$d\Phi = \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{n}}\, dS = E\cos\theta dS = -\left[\frac{\partial V}{\partial x}\cos\alpha + \frac{\partial V}{\partial y}\cos\beta + \frac{\partial V}{\partial z}\cos\gamma\right] dS \quad [1.110]$$

where α, β and γ are the director angles of the outwardly directed normal $\overrightarrow{\mathbf{n}}$ to the surface dS, is called the *differential flux* of the field $\overrightarrow{\mathbf{E}}$ through the differential surface element $d\overrightarrow{\mathbf{S}}$ ($\overrightarrow{\mathbf{E}}$ is the field measured at the point $M\,(x, y, z)$). θ is the angle measured between $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{n}}$, as shown in Figure 1.18. The expression $\frac{\partial V}{\partial x}\cos\alpha + \frac{\partial V}{\partial y}\cos\beta + \frac{\partial V}{\partial z}\cos\gamma$ is called the *directional derivative* V along $\overrightarrow{\mathbf{n}}$; it is denoted by $\frac{\partial V}{\partial n}$. Consequently, the total flux of $\overrightarrow{\mathbf{E}}$ across the entire surface S is equal to the sum of the elementary fluxes across the various elements, dS, of this surface, that is, the integral:

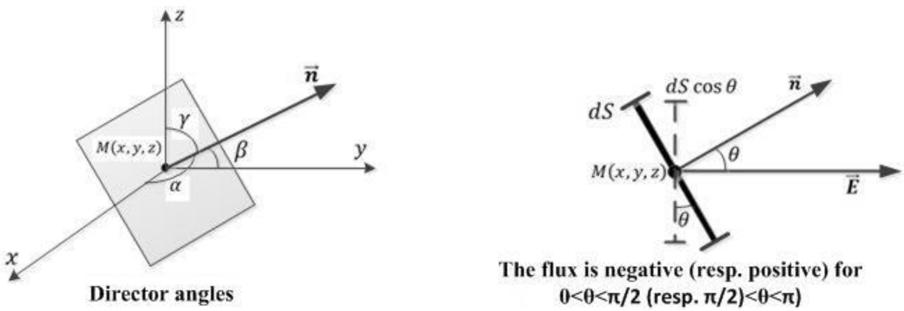

Figure 1.18. *The differential flux of the field $\overrightarrow{\mathbf{E}}$ across the surface element $d\overrightarrow{\mathbf{S}}$ is equal to the product of the magnitude of $\overrightarrow{\mathbf{E}}$ and $dS\cos\theta$ (the projection of dS onto the perpendicular plane to the field direction)*

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot \vec{\mathbf{n}} dS = \oint_S E \cos\theta dS = -\oint_S \left[\frac{\partial V}{\partial x}\cos\alpha + \frac{\partial V}{\partial y}\cos\beta + \frac{\partial V}{\partial z}\cos\gamma\right] dS \qquad [1.111]$$

1.6.1. *Integral form*

Let us now consider the two cases depicted in Figure 1.19, where the volume of an arbitrary shape τ (the surface surrounding τ is denoted S) surrounds a point-like charge q (i.e. q is located inside S) or is in the proximity of a point-like charge q (q is located outside the surface S). The point-like charge q is at position $M'(x', y', z')$. In both cases, the electric field emanates radially from the point-like charge. We wish to calculate the flux of the electric field across the surface S surrounding the volume:

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \oint_S \frac{q\left(\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right)}{4\pi\varepsilon_0 \left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right|^3} \cdot d\vec{\mathbf{S}} = -\frac{q}{4\pi\varepsilon_0}\oint_S \vec{\nabla}\left(\frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right|}\right) \cdot d\vec{\mathbf{S}} \qquad [1.112]$$

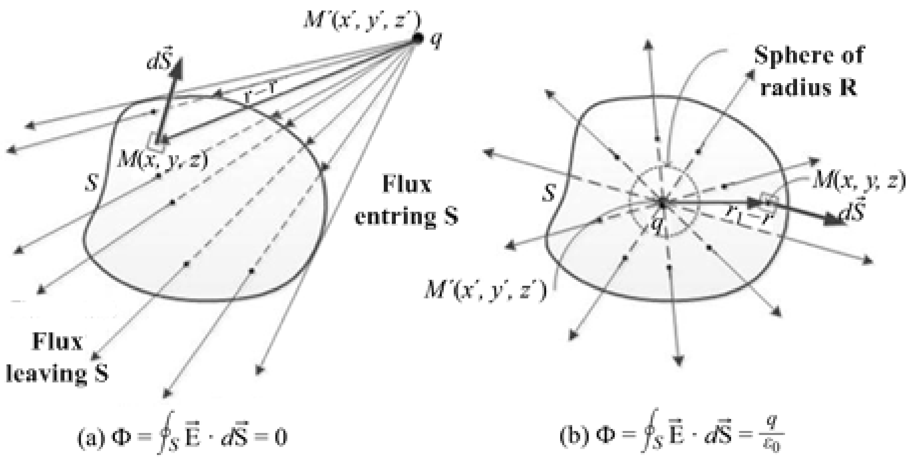

Figure 1.19. *(a) The net flux of the electric field across a closed surface due to a point-like charge outside the surface is zero. (b) The flux of the electric field from a point-like charge enclosed within a surface travels entirely across this surface*

Let us now use the divergence theorem to convert the surface integral [1.112] into a volume integral:

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = -\frac{q}{4\pi\varepsilon_0} \int_\tau \vec{\nabla} \cdot \left[\vec{\nabla} \left(\frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \right) \right] d\tau$$
$$= -\frac{q}{4\pi\varepsilon_0} \int_\tau \Delta \left(\frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \right) d\tau \qquad [1.113]$$

1) If the point-like charge q is located outside the surface (i.e. outside the volume), the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}'$ never coincide (i.e. we always have $\vec{\mathbf{r}} - \vec{\mathbf{r}}' \neq 0$); this implies that $\Delta \left(\frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \right) = 0$ (see Appendix A.1, which will be available online at: www.iste.co.uk/abdesselam/electrodynamics1.zip). It is thus seen that the net flux $\vec{\mathbf{E}}$ across the surface is equal to zero:

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = 0 \qquad [1.114]$$

This result can be explained qualitatively in Figure 1.19(a). The electric field coming from q on the part of surface S closest to q (i.e. the part of the surface facing q) has its normal component directed in the opposite direction to $d\vec{\mathbf{S}}$; this makes a negative contribution to the flux. On the other hand, on the other part of the surface, the electric field has its normal component in the same direction as $d\vec{\mathbf{S}}$; this is a positive contribution to the flux. These contributions are equal in magnitude but opposite in sign, so the total net flux is zero.

2) If the point-like charge q is now inside the surface, as shown in Figure 1.19(b), the electric field points outward when the charge q is positive, with a normal component in the direction of $d\vec{\mathbf{S}}$ at any point on S, so the flux is positive. On the contrary, if the charge q is negative, $\vec{\mathbf{E}}$ and $d\vec{\mathbf{S}}$ are oriented in opposite directions at any point of S, so that the flux is now negative. In both cases, the total net flux must therefore be non-zero. The function $\frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}$ becomes singular when $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}'$ coincide (because the charge is inside the surface). Consequently, the relation $\Delta \left(\frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \right) = 0$ is no longer true when $\vec{\mathbf{r}} = \vec{\mathbf{r}}'$ and must be replaced by the Dirac distribution $-.4\pi\delta \left(\vec{\mathbf{r}} - \vec{\mathbf{r}}' \right)$ (see Appendix A.1, which will be available online at www.iste.co.uk/abdesselam/electrodynamics1.zip):

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{q}{4\pi\varepsilon_0} \int_\tau 4\pi\delta \left(\vec{\mathbf{r}} - \vec{\mathbf{r}}' \right) d\tau = \frac{q}{\varepsilon_0} \qquad [1.115]$$

It can clearly be seen that the sign of the flux depends on the polarity of the point-like charge. This result can also be obtained in the following way: since $\Delta \left(\frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \right)$ is equal to zero everywhere, except when $\vec{\mathbf{r}} = \vec{\mathbf{r}}'$, it is possible to reduce the surface S to a small spherical surface S' with radius R surrounding the point-like charge (i.e. centered on the point $\overrightarrow{\mathbf{r}}'$). In the rest of the volume, we have $\overrightarrow{\mathbf{r}} \neq \overrightarrow{\mathbf{r}}'$ and, consequently, $\Delta\left(\frac{1}{\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}'\right|}\right) = 0$:

$$\Phi = \oint_S \frac{q\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)}{4\pi\varepsilon_0\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} \cdot d\overrightarrow{\mathbf{S}} = \oint_{\text{Sphere}S'} \frac{q\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)}{4\pi\varepsilon_0\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} \cdot d\overrightarrow{\mathbf{S}}' \quad [1.116]$$

It is known that the electric field is purely radial over this sphere, that is, it is in the same direction as $d\overrightarrow{\mathbf{S}}'$. By performing the change of variable $\overrightarrow{\mathbf{R}} = \overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'$ and noting that $\overrightarrow{\mathbf{R}} \cdot d\overrightarrow{\mathbf{S}}' = R^3 d\Omega = R^3 \sin\theta d\theta d\varphi$, we obtain:

$$\oint_{S'} \frac{q\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)}{4\pi\varepsilon_0\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} \cdot d\overrightarrow{\mathbf{S}}' = \frac{q}{4\pi\varepsilon_0}\int_0^{\pi}\int_0^{2\pi} \frac{R^3 \sin\theta d\theta d\varphi}{R^3} = \frac{q}{\varepsilon_0} \quad [1.117]$$

Thus, the flux across a closed surface S of the electric field produced by a point-like charge located within this surface is:

$$\Phi = \oint_S \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = \frac{q}{\varepsilon_0} \quad [1.118]$$

If there are multiple point-like charges within the surface S, each charge q_i gives rise to a flux $\frac{q_i}{\varepsilon_0}$. It thus follows that the net flux of $\overrightarrow{\mathbf{E}}$ across a closed surface is equal to the product of $\frac{1}{\varepsilon_0}$ into the net charge enclosed by the surface:

$$\oint_S \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = \frac{1}{\varepsilon_0}\sum_{q_i \text{ within } S} q_i \quad [1.119]$$

Charges located outside S do not contribute to the flux.

For a continuous distribution of charges, the flux, across any closed surface S, of the differential electric field $d\overrightarrow{\mathbf{E}}$ created by the differential element dq has the value:

$$d\Phi = \oint_S d\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = \oint_S \frac{dq}{4\pi\varepsilon_0}\frac{\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|^3} \cdot d\overrightarrow{\mathbf{S}} = \begin{cases} 0 & dq \text{ outside } S \\ \frac{dq}{\varepsilon_0} & dq \text{ inside } S \end{cases} \quad [1.120]$$

By integrating over the entire distribution, we obtain:

$$\Phi = \int_D \oint_S d\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = \oint_S \left(\int_D d\overrightarrow{\mathbf{E}}\right) \cdot d\overrightarrow{\mathbf{S}} = \oint_S \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = \frac{1}{\varepsilon_0}\int_{D\cap\tau} dq \quad [1.121]$$

where Dist. $\cap$ τ indicates the part of the distribution located within the surface S. The quantity $\int_{\text{Dist.}\cap\tau} dq$ represents the total charge contained within S, as seen in Figure 1.20. We will use the notation:

$$Q_S = \int_{\text{Dist.}\cap\tau} dq \qquad [1.122]$$

Equation [1.127] therefore takes the form:

$$\oint_S \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = \frac{Q_S}{\varepsilon_0} \qquad [1.123]$$

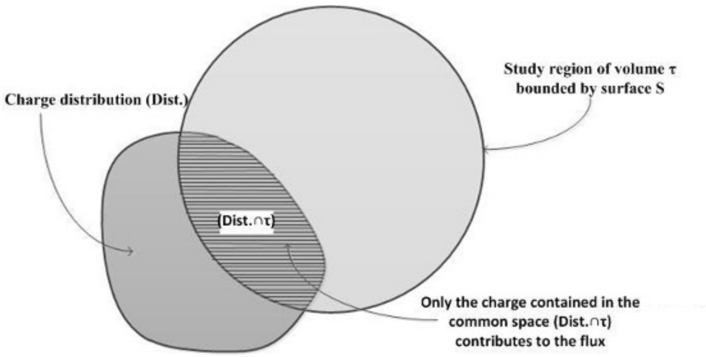

Figure 1.20. *Only the charge Q_S contained in the overlapping region between τ and Dist. contributes to the flux of the electric field across the surface S. The charge outside τ has no effect*

This is the *integral expression of Gauss's theorem* (or *Gauss's law*): *the electric flux across a closed surface, oriented from the interior to the exterior, is equal to the product of $\frac{1}{\varepsilon_0}$ multiplied by the total charge contained in the closed surface.* The surface S is sometimes called a *Gaussian surface*. Q_S includes all enclosed differential charge elements, so the total charge enclosed in this surface can be a line integral, a surface integral and/or a volume integral in addition to the sum of the point-like charges:

$$\begin{aligned}\oint_S \varepsilon_0 \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} &= \sum_{q_i \text{ within } S} q_i + \int_{q \text{ within } S} dq \\ &= \left(\sum q_i + \int_{\mathcal{C}} \lambda dl + \int_S \sigma dS + \int_\tau \rho d\tau\right)_{\text{charge within } S}\end{aligned} \qquad [1.124]$$

Charges outside the volume do not contribute to the total flux across the surface S. Note that the quantities $\sum q_i$, $\int_C \lambda dl$ and $\int_S \sigma dS$ can also be expressed by triple integrals using Dirac distributions. Gauss' theorem [1.124] can be used to significantly simplify the calculation of the field for charges distributed in a spatially symmetric manner. The idea is to find a surface S with sections that are tangents to the electric field, so that the scalar product is zero, and sections perpendicular to the field and over which the field is constant, so that the scalar product and integration become simple multiplications. If such an appropriate surface is found, the surface integral becomes very easy to evaluate. Knowing the direction of the electric field often suggests the appropriate surface on which the integration can be performed [1.124].

1.6.2. *Differential form*

If a volume distribution of charges with density ρ and volume τ' is completely enclosed by a closed surface S (volume contained in S is greater than tau'), Gauss' theorem makes it possible to write $\oint_S \varepsilon_0 \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = Q_S = \int_{\tau'} \rho d\tau'$. Since ρ is zero outside the volume τ', the integral on the right can be extended over the whole of τ without affecting the result. That is:

$$\oint_S \varepsilon_0 \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = \int_\tau \rho d\tau \qquad [1.125]$$

The left-hand side of [1.125] can be transformed into a volume integral using the divergence theorem:

$$\int_\tau \overrightarrow{\nabla} \cdot \left(\varepsilon_0 \overrightarrow{\mathbf{E}}\right) d\tau = \int_\tau \rho d\tau \qquad [1.126]$$

Since [1.126] must be true for any volume τ, the quantities under the integral must therefore be equal at all points in space, that is, $\overrightarrow{\nabla} \cdot \left(\varepsilon_0 \overrightarrow{\mathbf{E}}\right) = \rho$, which gives:

$$\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{E}} = \frac{\rho}{\varepsilon_0} \qquad [1.127]$$

(As the permittivity of the vacuum, ε_0 is constant, it is possible to move it outside the divergence operator). This is the *differential expression for Gauss' theorem, the local form, or the Maxwell–Gauss equation*. This local form is not enough to determine $\overrightarrow{\mathbf{E}}$ from ρ, as we can add any vector function $\overrightarrow{\mathbf{A}}$ with zero divergence and obtain another solution for [1.127]. Considerations of symmetry can allow us to give more information on E, but in general, this local form of Gauss' theorem must be complemented by other constraints. The first constraint can be provided by equation [1.72], but this is not always sufficient to uniquely determine the electrostatic field because if $\overrightarrow{\mathbf{E}}$ satisfies equations [1.127] and [1.72], then $\overrightarrow{\mathbf{E}} + \overrightarrow{\mathbf{A}}$, where $\overrightarrow{\mathbf{A}}$ is constant, also satisfies these equations.

1.6.3. *The Helmholtz–Hodge theorem*

In order to uniquely determine the field $\overrightarrow{\mathbf{E}}$, we must use constraints related to the asymptotic behavior of the field. This can be done with the Helmholtz–Hodge theorem, which stipulates that any vector field can be separated into a longitudinal (*irrotational*, curl-free) component and a transverse (*solenoidal*, divergence-free) component, or the sum of the gradient of a scalar field and the curl of a vector field.

Suppose there exists a vector field $\overrightarrow{\mathbf{E}}$ whose curl $\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}}$ and divergence $\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{E}}$ we know. Using the Dirac distribution, $\overrightarrow{\mathbf{E}}$ can be expressed as follows:

$$\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) = \int_{\tau'} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)\delta\left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right) d\tau' = -\frac{1}{4\pi}\int_{\tau'} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)\Delta\left(\frac{1}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|}\right) d\tau' \quad [1.128]$$

Further, since Δ acts only on $\overrightarrow{\mathbf{r}}$, we can derive the following (for simplicity's sake, we will write $R = \left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|$):

$$\begin{aligned}
\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) &= -\frac{1}{4\pi}\Delta\int_{\tau'} \frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{R} d\tau' = -\frac{1}{4\pi}\left[\overrightarrow{\nabla}\left(\overrightarrow{\nabla} \cdot \int_{\tau'} \frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{R} d\tau'\right) - \right. \\
&\left. \overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \int_{\tau'} \frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{R} d\tau'\right)\right] = -\frac{1}{4\pi}\left[\overrightarrow{\nabla}\left(\int_{\tau'} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right) \cdot \overrightarrow{\nabla}\left(\frac{1}{R}\right) d\tau'\right) + \right. \\
&\left. \overrightarrow{\nabla} \times \left(\int_{\tau'} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right) \times \overrightarrow{\nabla}\left(\frac{1}{R}\right) d\tau'\right)\right] = -\frac{1}{4\pi}\left[-\overrightarrow{\nabla}\left(\int_{\tau'} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right) \cdot \overrightarrow{\nabla}'\left(\frac{1}{R}\right) d\tau'\right)\right. \\
&\left. -\overrightarrow{\nabla} \times \left(\int_{\tau'} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right) \times \overrightarrow{\nabla}'\left(\frac{1}{R}\right) d\tau'\right)\right] \qquad [1.129]
\end{aligned}$$

where the following relations have been used: $\Delta\overrightarrow{\mathbf{A}} = \overrightarrow{\nabla}\left(\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{A}}\right) - \overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{A}}\right)$ and $\overrightarrow{\nabla}\left(\frac{1}{R}\right) = -\overrightarrow{\nabla}'\left(\frac{1}{R}\right)$. Now, by virtue of the identities $\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\nabla}\psi = -\psi\left(\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{E}}\right) +$

$\overrightarrow{\nabla} \cdot \left(\psi \overrightarrow{\mathbf{A}}\right)$ and $\overrightarrow{\mathbf{A}} \times \overrightarrow{\nabla} \psi = \psi \left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}}\right) - \overrightarrow{\nabla} \times \left(\psi \overrightarrow{\mathbf{A}}\right)$, we find that:

$$\begin{aligned} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) = -\frac{1}{4\pi}\Bigg[& -\overrightarrow{\nabla}\left(-\int_{\tau'} \frac{\overrightarrow{\nabla}' \cdot \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} d\tau' + \int_{\tau'} \overrightarrow{\nabla}' \cdot \left(\frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|}\right) d\tau'\right) \\ & -\overrightarrow{\nabla} \times \left(\int_{\tau'} \frac{\overrightarrow{\nabla}' \times \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} d\tau' - \int_{\tau'} \overrightarrow{\nabla}' \times \left(\frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|}\right) d\tau'\right)\Bigg] \end{aligned} \quad [1.130]$$

Using the divergence theorem. the preceding equation can also be rewritten in the form:

$$\begin{aligned} & \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) = -\frac{1}{4\pi}\Bigg[-\overrightarrow{\nabla}\left(-\int_{\tau'} \frac{\overrightarrow{\nabla}' \cdot \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} d\tau' + \oint_{S'} \overrightarrow{\mathbf{n}}' \cdot \frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} dS'\right) \\ & -\overrightarrow{\nabla} \times \left(\int_{\tau'} \frac{\overrightarrow{\nabla}' \times \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} d\tau' - \oint_{S'} \overrightarrow{\mathbf{n}}' \times \frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} dS'\right)\Bigg] \\ & = -\overrightarrow{\nabla}\left[\frac{1}{4\pi}\int_{\tau'} \frac{\overrightarrow{\nabla}' \cdot \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} d\tau' - \frac{1}{4\pi}\oint_{S'} \overrightarrow{\mathbf{n}}' \cdot \frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} dS'\right] \\ & +\overrightarrow{\nabla} \times \left[\frac{1}{4\pi}\int_{\tau'} \frac{\overrightarrow{\nabla}' \times \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} d\tau' - \frac{1}{4\pi}\oint_{S'} \overrightarrow{\mathbf{n}}' \times \frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} dS'\right] \end{aligned} \quad [1.131]$$

By defining:

$$\begin{aligned} V\left(\overrightarrow{\mathbf{r}}\right) &= \frac{1}{4\pi}\int_{\tau'} \frac{\overrightarrow{\nabla}' \cdot \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} d\tau' - \frac{1}{4\pi}\oint_{S'} \overrightarrow{\mathbf{n}}' \cdot \frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} dS', \\ \overrightarrow{\mathbf{A}}\left(\overrightarrow{\mathbf{r}}\right) &= \frac{1}{4\pi}\int_{\tau'} \frac{\overrightarrow{\nabla}' \times \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} d\tau' - \frac{1}{4\pi}\oint_{S'} \overrightarrow{\mathbf{n}}' \times \frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\right|} dS' \end{aligned} \quad [1.132]$$

it finally comes that:

$$\overrightarrow{\mathbf{E}} = -\overrightarrow{\nabla} V + \overrightarrow{\nabla} \times \overrightarrow{\mathbf{A}} \quad [1.133]$$

This is the so-called *Helmholtz–Hodge decomposition theorem*. In particular, if we assume that E satisfies the constraints [1.72] and [1.127], and we extend the previous calculations to the whole space (i.e. we tend S to infinity), the expressions [1.132] reduce to (as X is zero outside Y, the first term on the right remains the same):

$$\begin{aligned} V\left(\overrightarrow{\mathbf{r}}\right) &= \frac{1}{4\pi\varepsilon_0}\int_{\tau'} \frac{\rho\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}'\right|}d\tau' - \frac{1}{4\pi}\lim_{S'\longrightarrow\infty}\oint_{S'}\overrightarrow{\mathbf{n}}'\cdot\frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}'\right|}dS' \\ \overrightarrow{\mathbf{A}}\left(\overrightarrow{\mathbf{r}}\right) &= -\frac{1}{4\pi}\lim_{S'\longrightarrow\infty}\oint_{S'}\overrightarrow{\mathbf{n}}'\times\frac{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}'\right)}{\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}'\right|}dS' \end{aligned} \qquad [1.134]$$

It is clearly seen that if $\overrightarrow{\mathbf{E}}$ is zero at infinity, equations [1.72] and [1.127] admit a unique solution, given by:

$$\begin{aligned} \overrightarrow{\mathbf{E}} &= -\overrightarrow{\nabla}V \\ V\left(\overrightarrow{\mathbf{r}}\right) &= \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\frac{\rho\left(\overrightarrow{\mathbf{r}}'\right)d\tau'}{\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}\right|} \end{aligned} \qquad [1.135]$$

In fact, $\overrightarrow{\mathbf{E}}$ cannot be zero at infinity unless the charge Q is finite. The Helmholtz–Hodge theorem can also be stated as follows: if ρ tends to zero faster than $\frac{1}{r^2}$ (i.e. ρ is spatially bounded, $\int_{\text{space}}\rho d\tau \leq M$, where M is a finite constant), there exists a unique vector function $\overrightarrow{\mathbf{E}}$ of the position, which satisfies (1) $\overrightarrow{\nabla}\cdot\overrightarrow{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$; (2) $\overrightarrow{\nabla}\times\overrightarrow{\mathbf{E}} = 0$ (this equation, valid only in electrostatics, is at the origin of the continuity equations for the electric field at the interface of an electrically charged sheet); and (3) $\lim_{r\longrightarrow\infty}\overrightarrow{\mathbf{E}} = 0$. This solution is given by [1.135].

By replacing $\overrightarrow{\mathbf{E}}$ with $-\overrightarrow{\nabla}V$ in [1.127], we obtain the *Poisson formula*:

$$\Delta V = -\frac{1}{\varepsilon_0}\rho \qquad [1.136]$$

In a region where the charge density ρ is zero, the previous equation reduces to:

$$\Delta V = 0 \qquad [1.137]$$

This is the *Laplace equation*. It appears in many areas of physics. These solutions obey the extreme value theorem: V has no extrema in any region where $\Delta V = 0$. A consequence of this theorem is the uniqueness theorem: the potential V is uniquely determined by the charge distribution in a volume τ and by the value of V or $\frac{\partial V}{\partial n} = \overrightarrow{\nabla}V\cdot\overrightarrow{\mathbf{n}}$ at any point on the surface surrounding this volume, where $\overrightarrow{\mathbf{n}}$ is the normal to the surface at the point under consideration. We will return to these results in detail.

1.6.4. *Applications*

The electric field is calculated as follows:

1) Using Gauss' theorem, when the charge distribution is symmetric so that the electric field is constant on certain hypothetical surfaces. This is generally the simplest method.

2) Find a solution to Poisson's equation or Laplace's equation, if the boundary conditions are known, to determine the integration constants.

EXAMPLE 1.22. (Solid Sphere Uniformly Charged in Volume).– Consider a sphere of radius R and center O, uniformly charged in volume (the total charge in the entire sphere $Q = \frac{4}{3}\pi\rho R^3$, where ρ is the charge density). Gauss' theorem is the simplest method here to calculate the electric field. The most appropriate Gaussian surface is a sphere centered on O and passing through the point of study M (this may be inside or outside the source).

– $\overrightarrow{\mathbf{E}}_{ext}$ *calculation*: the Gaussian surface in Figure 1.21(a), for $r > R$, contains the total charge Q. Moreover, by symmetry, we know that $\overrightarrow{\mathbf{E}}_{hboxext}$ must be radial and constant on this surface. Consequently, Gauss' theorem allows us to write $\oint_{S_{\text{ext}}} \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = 4\pi r^2 \left|\overrightarrow{\mathbf{E}}_{\text{ext}}\right| = \frac{Q}{\varepsilon_0}$, which results in:

$$\overrightarrow{\mathbf{E}}_{\text{ext}} = \frac{Q}{4\pi\varepsilon_0}\frac{\overrightarrow{\mathbf{r}}}{r^3} \qquad [1.138]$$

– $\overrightarrow{\mathbf{E}}_{int}$ *calculation*: let us now calculate the electric field at a point M located inside the charged sphere. The Gaussian surface, passing through this point, has a radius that is smaller than R. It therefore contains a fraction of the total charge, namely, $\frac{4}{3}\pi\rho r^3 = \frac{Qr^3}{R^3}$, as shown in Figure 1.21(b). Similarly, the electric field observed on this surface is also radial and constant. In this case, Gauss' theorem gives us: $\oint_{S_{\text{int}}} \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = 4\pi r^2 \left|\overrightarrow{\mathbf{E}}_{\text{int}}\right| = \frac{1}{\varepsilon_0}\rho\left(\frac{4}{3}\pi r^3\right)$. Therefore, it follows from this that:

$$\overrightarrow{\mathbf{E}}_{\text{int}} = \frac{\rho}{3\varepsilon_0}\overrightarrow{\mathbf{r}} = \frac{Q}{4\pi\varepsilon_0 R^3}\overrightarrow{\mathbf{r}} \qquad [1.139]$$

Note that if the charge distribution is not spherically symmetrical, the $\overrightarrow{\mathbf{E}}_{\text{ext}}$ and $\overrightarrow{\mathbf{E}}_{\text{int}}$ fields will also depend on the variables θ and φ, that is, they are no longer constant on the Gaussian surface. Consequently, Gauss' theorem will only give the mean value of the radial components of $\overrightarrow{\mathbf{E}}_{\text{ext}}$ and $\overrightarrow{\mathbf{E}}_{\text{int}}$ on this surface. Let us now try to obtain results [1.138] and [1.139] using Poisson's equation [1.136]. Given the spherical symmetry, we know that the potential can only depend on the radial coordinate r. Consequently, Poisson's equation is written as (in spherical coordinates):

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = \begin{cases} -\dfrac{\rho}{\varepsilon_0} & r \leq R \\ 0 & r > R \end{cases} \quad [1.140]$$

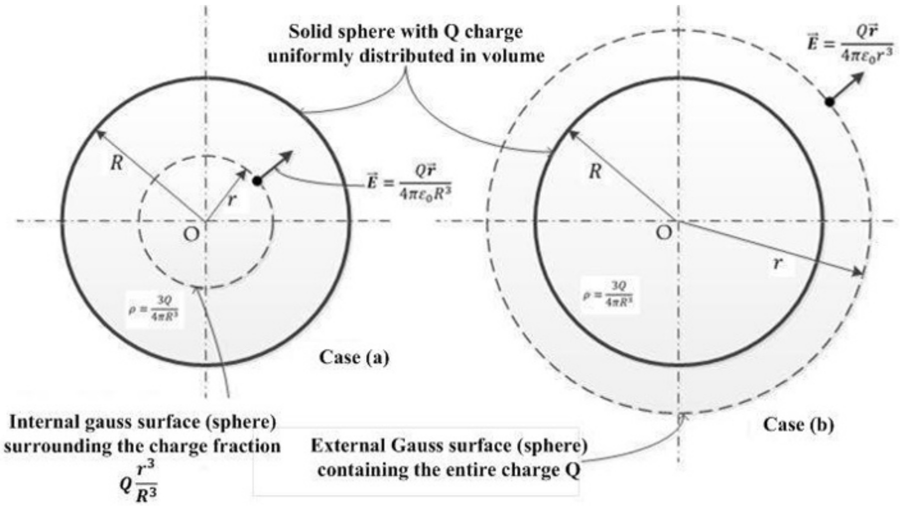

Figure 1.21. *The Gaussian surfaces for a sphere that is uniformly charged in its volume shows that the electric field outside it is equivalent to the field created by a point-like charge Q placed at the center O. Within the sphere, the electric field is linear, as the internal Gaussian surface contains only a fraction of the charge, proportional to r^3*

Let us begin by finding the form of the potential outside the sphere. By integrating twice with respect to r, it comes that:

$$V_{\text{ext}}(r) = \frac{A}{r} + B \quad [1.141]$$

where A and B are integration constants. As the potential must be zero at infinity, we must choose $B = 0$. The asymptotic behavior of the potential must be used to determine A. If we place ourselves very far from the charged sphere, we will see it essentially as a point-like charge. It is therefore necessary to have:

$$\lim_{r \longrightarrow \infty} V_{\text{ext}}(r) = \frac{Q}{4\pi\varepsilon_0 r} = \frac{\rho R^3}{3\varepsilon_0 r} \quad [1.142]$$

By identification, it can be derived from this that $A = \frac{Q}{4\pi\varepsilon_0} = \frac{\rho R^3}{3\varepsilon_0}$. Within the sphere, the Poisson equation is written as:

$$\Delta V_{\text{int}} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV_{\text{int}}}{dr}\right) = -\frac{\rho}{\varepsilon_0} \qquad [1.143]$$

The first integration gives $r^2\frac{dV}{dr} = -\frac{\rho r^3}{3\varepsilon_0} + C$, or again, $\frac{dV}{dr} = -\frac{\rho r}{3\varepsilon_0} + \frac{C}{r^2}$ (C is an integration constant). By integrating again, we get:

$$V_{\text{int}} = -\frac{\rho r^2}{6\varepsilon_0} - \frac{C}{r} + D \qquad [1.144]$$

where D is a second integration constant. As the potential V_{int} must not diverge at $r = 0$, the constant C is necessarily zero. To determine D, let us use the continuity of the potential at $r = R$:

$$\lim_{r\longrightarrow R^-} V_{\text{int}}(r) = \lim_{r\longrightarrow R^+} V_{\text{ext}}(r) \qquad [1.145]$$

which shows that $-\frac{\rho R^2}{6\varepsilon_0} + D = \frac{\rho R^2}{3\varepsilon_0}$ and consequently, $D = \frac{\rho R^2}{2\varepsilon_0}$. Finally, the potential is written as:

$$V(r) = \begin{cases} \dfrac{\rho}{6\varepsilon_0}\left(3R^2 - r^2\right) = \dfrac{Q}{8\pi\varepsilon_0 R}\left[3 - \left(\dfrac{r}{R}\right)^2\right] & r \leq R \\ \dfrac{\rho R^3}{3\varepsilon_0 r} = \dfrac{Q}{4\pi\varepsilon_0 r} & r > R \end{cases} \qquad [1.146]$$

Now, to determine the $\overrightarrow{\mathbf{E}}$ field, we need to apply equation [1.73]. The same results are clearly obtained as in [1.39] and [1.40].

EXAMPLE 1.23.– If the sphere has a charge density with spherical symmetry described by $\rho(r') = \rho_0\left(1 - \frac{r'^2}{R^2}\right)$ for $r' \leq R$ and 0 for $r' > R$, the Gaussian surface for $r > R$ always encloses the total charge of the sphere $Q_{\text{tot}} = \int_0^R \rho(r')\,dr'$, while the Gaussian surface located at $r \leq R$ contains a part of the total charge, namely, $Q_r = \int_0^r \rho(r')\,dr'$. Since the field $\overrightarrow{\mathbf{E}}$ is radial and constant on these surfaces, Gauss' theorem is written as:

$$\varepsilon_0 \oint_S \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = \varepsilon_0 \left|\overrightarrow{\mathbf{E}}\right| 4\pi r^2$$
$$= \begin{cases} \displaystyle\int_0^r \rho_0\left(1 - \frac{r'^2}{R^2}\right) 4\pi r'^2 dr' = 4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^5}{5R^2}\right) & r \leq R \\ \displaystyle\int_0^R \rho_0\left(1 - \frac{r'^2}{R^2}\right) 4\pi r'^2 dr' = \frac{8\pi\rho_0 R^3}{15} & r > R \end{cases}$$

and consequently:

$$\overrightarrow{\mathbf{E}}(r) = \begin{cases} \dfrac{\rho_0}{\varepsilon_0}\left[\dfrac{1}{3} - \dfrac{1}{5}\left(\dfrac{r}{R}\right)^2\right]\overrightarrow{\mathbf{r}} & r \leq R \\ \dfrac{2\rho_0 R^3}{15\varepsilon_0}\dfrac{\overrightarrow{\mathbf{r}}}{r^3} & r > R \end{cases} \qquad [1.147]$$

In the same way, $\overrightarrow{\mathbf{E}}$ can be calculated from the expression for the potential, starting from the Poisson equation: $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = \begin{cases} \frac{\rho_0}{\varepsilon_0}\left(1 - \frac{r^2}{R^2}\right) & r \leq R \\ 0 & r > R \end{cases}$; after integration, this gives:

$$V(r) = \begin{cases} -\dfrac{\rho_0}{\varepsilon_0}\left(\dfrac{r^2}{6} - \dfrac{r^4}{20R^2}\right) - \dfrac{C}{r} + D & r \leq R \\ \dfrac{A}{r} + B & r > R \end{cases} \qquad [1.148]$$

Since the potential at infinity must be zero and does not diverge at the center of the sphere, the constants C and B must be chosen to be zero. To determine A, we must use the asymptotic behavior of the potential $V(r)$ at infinity. In fact, we must have $\lim_{r \longrightarrow \infty} V(r) = \frac{Q_{\text{tot}}}{4\pi\varepsilon_0 r} = \frac{2\rho R^3}{15\varepsilon_0 r}$ and, consequently, $A = \frac{2\rho R^3}{15\varepsilon_0}$. In addition, we must use the property of continuity of the potential on the surface of the sphere to calculate D: $\lim_{r \longrightarrow R^-} V_{\text{int}}(r) = \lim_{r \longrightarrow R^+} V_{\text{ext}}(r)$. It comes that: $-\frac{7\rho R^2}{60\varepsilon_0} + D = \frac{2\rho R^2}{15\varepsilon_0}$ and therefore, $D = \frac{\rho R^2}{4\varepsilon_0}$. Finally, the potential is written as:

$$V(r) = \begin{cases} \dfrac{\rho_0 R^2}{4\varepsilon_0}\left[1 - \dfrac{2}{3}\left(\dfrac{r}{R}\right)^2 + \dfrac{1}{5}\left(\dfrac{r}{R}\right)^4\right] & (r \leq R) \\ \dfrac{2\rho_0 R^3}{15\varepsilon_0 r} & (r > R) \end{cases} \qquad [1.149]$$

Using [1.73], we arrive again at expression [1.147].

EXAMPLE 1.24. (Hollow Sphere Charged on Surface).– Let us now assume that the total charge Q is uniformly distributed on the external surface of the sphere. Realizing from symmetry that the field $\overrightarrow{\mathbf{E}}$ is purely radial and depends only on r and not on θ and φ, let us start by drawing Gaussian spheres of radius r, as shown in Figure 1.22, inside the charged sphere $(r < R)$ and outside the charged sphere $(r > R)$. The Gaussian sphere inside the sphere encloses no charge, while the external sphere encloses the full charge $Q = \sigma 4\pi R^2$, that is, $\varepsilon_0 \oint_S \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = \varepsilon_0 \left|\overrightarrow{\mathbf{E}}\right| 4\pi r^2 = \begin{cases} 0 & r < R \\ \sigma 4\pi R^2 & r > R \end{cases}$ and therefore, $\left|\overrightarrow{\mathbf{E}}\right| = \begin{cases} 0 & r < R \\ \frac{\sigma R^2}{\varepsilon_0 r^2} & r > R \end{cases}$. Thus, we find:

$$\overrightarrow{\mathbf{E}}(r) = \begin{cases} 0 & r < R \\ \dfrac{\sigma R^2 \overrightarrow{\mathbf{r}}}{\varepsilon_0 r^3} \equiv \dfrac{Q\overrightarrow{\mathbf{r}}}{4\pi\varepsilon_0 r^3} & r > R \end{cases} \qquad [1.150]$$

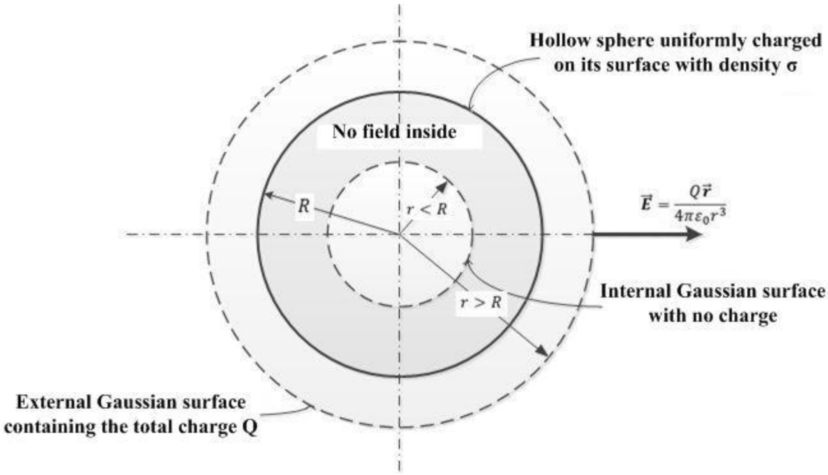

Figure 1.22. *Applying Gauss' theorem to concentric, spherical (Gaussian) surfaces inside* $(r < R)$ *and outside* $(r > R)$ *the charged sphere shows that the electric field inside is zero, while outside, it is as though the entire charge* $Q = \sigma 4\pi R^2$ *was concentrated into a point-like charge at the origin*

Note that the Poisson equation gives the solution $V(r) = A$ for $r < R$ and $\frac{B}{r}$ for $r > R$. The asymptotic behavior of the potential at infinity dictates that $B = \frac{Q}{4\pi\varepsilon_0} = \frac{\sigma R^2}{\varepsilon_0}$. The value of A is obtained using the continuity property of $V(r)$ on the boundary of the sphere, $\lim_{r\longrightarrow R^-} V(r) = \lim_{r\longrightarrow R^+} V(r)$, which shows that $A = \frac{B}{R} = \frac{Q}{4\pi\varepsilon_0 R} = \frac{\sigma R}{\varepsilon_0}$.

EXAMPLE 1.25. (Uniformly Charged Cylinder).–

1) An infinitely long cylinder with axis Oz and radius R carries the uniform charge density ρ within its volume, as seen in Figure 1.23. Let M be the point of coordinates (r, φ, z) (inside or outside the cylinder). For reasons of symmetry, the

electric field is purely radial and depends only on the distance r between M and the axis Oz. This can be demonstrated as follows: the plane (P_1) passing through M and containing the axis Oz is a plane of symmetry. The same applies to a plane(P_2) passing through M and perpendicular to Oz. The field must therefore be contained in both (P_1) and (P_2); therefore, it acts in the direction common to (P_1) and (P_2), that is, it is radial: $\overrightarrow{\mathbf{E}}(M) = E(r, \varphi, z)\,\frac{\overrightarrow{\mathbf{r}}}{r}$. Since the distribution of charges is translation invariant along Oz and rotation invariant around Oz, it can be concluded that the modulus of $\overrightarrow{\mathbf{E}}$ can only depend on the distance r, between M and the axis Oz, that is, $\overrightarrow{\mathbf{E}}(M) = E(r)\,\frac{\overrightarrow{\mathbf{r}}}{r}$. Let us apply Gauss' theorem to a cylindrical surface of height L, with axis Oz and radius r. The outgoing flux of $\overrightarrow{\mathbf{E}}$ across the bases of the Gaussian surface is zero, because $\overrightarrow{\mathbf{E}}$ is perpendicular to $d\overrightarrow{\mathbf{S}}_B$. At every point on the lateral section of the Gaussian surface, $\overrightarrow{\mathbf{E}}$ and $d\overrightarrow{\mathbf{S}}$ are collinear. Further, $E(r)$ is constant over this lateral part. The flux of $\overrightarrow{\mathbf{E}}$ across this lateral surface is therefore equal to $2\pi r L E(r)$. For a Gaussian surface located outside $(r > R)$, the charge enclosed within this surface is $Q = \pi R^2 L\rho$ (Figure 1.23(a)).

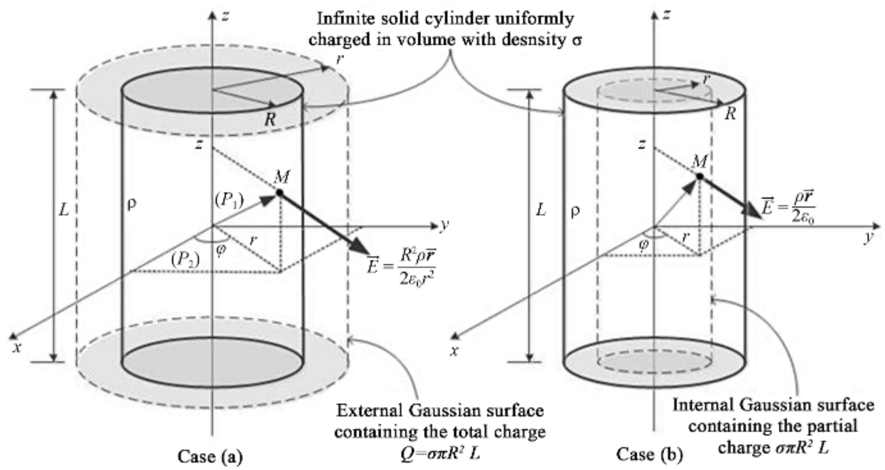

Figure 1.23. *Applying Gauss' theorem to concentric, cylindrical Gaussian surfaces shows that the external electric field behaves as though the charge on the cylinder was linearly distributed along the Oz axis, with the (linear) charge density $\lambda = \pi R^2 \rho$. Inside the cylinder, the electric field is proportional to the distance, r, between M and the axis Oz*

Consequently, Gauss' theorem makes it possible to write that $2\pi r L E(r) = \frac{\pi R^2 L \rho}{\varepsilon_0}$; this gives $E(r) = \frac{R^2 \rho}{2r\varepsilon_0}$ and consequently:

$$\vec{\mathbf{E}}(r) = \frac{R^2 \rho \vec{\mathbf{r}}}{2r^2 \varepsilon_0} \qquad [1.151]$$

The electric field [1.151] is derived from the potential:

$$V(r) = -\frac{R^2 \rho_0 \ln r}{2\varepsilon_0} \qquad [1.152]$$

For a Gaussian surface inside the cylinder, the enclosed charge is $Q_r = \pi r^2 L \rho$ (Figure 1.23(b)). It follows that $2\pi r L E(r) = \frac{\pi r^2 L \rho_0}{\varepsilon_0}$ and therefore:

$$\vec{\mathbf{E}}(r) = \frac{\rho_0 \vec{\mathbf{r}}}{2\varepsilon_0} \qquad [1.153]$$

This field is derived from the potential:

$$V(r) = -\frac{\rho_0 r^2}{4\varepsilon_0} \qquad [1.154]$$

2) If the charge is now uniformly distributed on the lateral surface of the cylinder, we obtain $\varepsilon_0 \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \varepsilon_0 E_r 2\pi r L = \begin{cases} 0 & r < R \\ \sigma 2\pi R L & r > R \end{cases}$. This means that:

$$\vec{\mathbf{E}}(r) = \begin{cases} 0 & r < R \\ \dfrac{\sigma R \vec{\mathbf{r}}}{\varepsilon_0 r^2} & r > R \end{cases}$$
$$V(r) = \begin{cases} -\dfrac{\sigma R \ln R}{\varepsilon_0} & r < R \\ -\dfrac{\sigma R \ln r}{\varepsilon_0} & r > R \end{cases} \qquad [1.155]$$

Outside of the cylinder, the electric field resembles the electric field created by an infinitely long wire with charge density $\lambda = 2\pi R \sigma$.

EXAMPLE 1.26. (Uniformly Charged Plane).–

1) Let us calculate the field created at any point in space by a uniformly charged plane (charge density σ). The plane is assumed to be very large (infinitely large) to neglect edge effects. The axis Oz is oriented perpendicular to the plane as shown in Figure 1.24(a). For reasons of symmetry, the electric field $\vec{\mathbf{E}}$ can only be perpendicular to the plane, and its modulus can only depend on the distance between

the point M and the plane, that is, on z. In fact, $\overrightarrow{\mathbf{E}}$ is translation invariant along the axes Ox and Oy (i.e. it is independent of the variables x and y): $\overrightarrow{\mathbf{E}} = \overrightarrow{\mathbf{E}}(0, 0, z)$. In addition, the Oxz and Oyz planes are symmetry planes. This implies that the field lies at their intersection, that is, along the Oz axis: $\overrightarrow{\mathbf{E}} = E(0, 0, z)\,\overrightarrow{\mathbf{k}}$. y symmetry, $\overrightarrow{\mathbf{E}}$ is equal and opposite at distance $-z$. Given the symmetry of the problem, Gauss' theorem appears to be the best method for calculating $\overrightarrow{\mathbf{E}}$. An ideal Gaussian surface is a surface over which $\overrightarrow{\mathbf{E}}$ is constant and perpendicular or, on the contrary, $\overrightarrow{\mathbf{E}}$ is parallel. In this case, a cylinder whose lateral surface is perpendicular to the plane can be chosen as the Gaussian surface (i.e. parallel to $\overrightarrow{\mathbf{E}}$) and whose bases are parallel to the plane and equidistant from z on either side. On this basis, the field will be perpendicular to the surface of constant magnitude and outgoing. The total flux of $\overrightarrow{\mathbf{E}}$ across the cylinder is $\Phi = \Phi_1 + \Phi_{\text{lat.}} + \Phi_2$. On the lateral surface, we have $\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = 0$, that is, $\Phi_{\text{lat.}} = 0$. On the surface S_1, we have $\Phi_1 = \int_{S_1} E(z)\, dS = E(z)\, S$. On the surface S_2, we have $\overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}} = -E(-z)\, dS$. Moreover, the charged plane is a symmetry plane, $E(-z) = -E(z)$, and therefore $\Phi_2 = \int_{S_2} E(z)\, dS = E(z)\, S$. We can therefore deduce that $\Phi = 2E(z)\, S$. The charge contained inside this Gaussian surface is σS. Applying Gauss' theorem, we write $\Phi = \frac{Q_{\text{int.}}}{\varepsilon_0}$ and, consequently, $2E(z)\, S = \frac{\sigma S}{\varepsilon_0}$. So, we obtain:

$$\overrightarrow{\mathbf{E}} = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\overrightarrow{\mathbf{k}} & \text{If } z > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\overrightarrow{\mathbf{k}} & \text{If } z < 0 \end{cases} \qquad [1.156]$$

2) If we now have two parallel planes A and B that are uniformly charged, with opposite polarities $-\sigma$ and $+\sigma$, and located, respectively, at $z_A = \frac{d}{2}$ and $z_B = -\frac{d}{2}$, we can draw Gaussian surfaces (i) passing through the two planes A and B, (ii) passing only through the A plane and (iii) passing only through the B plane, as shown in Figure 1.24(b). It can be seen that the plane A produces the field $-\frac{\sigma}{2\varepsilon_0}\overrightarrow{\mathbf{k}}$ in the subspace $z > \frac{d}{2}$ and $\frac{\sigma}{2\varepsilon_0}\overrightarrow{\mathbf{k}}$ in the subspace $z < \frac{d}{2}$. Similarly, B produces the field $\frac{\sigma}{2\varepsilon_0}\overrightarrow{\mathbf{k}}$ in the subspace $z > -\frac{d}{2}$ and $-\frac{\sigma}{2\varepsilon_0}\overrightarrow{\mathbf{k}}$ in the subspace $z < -\frac{d}{2}$. Using the principle of superposition, it can be concluded that the system of two planes produces the field $\frac{\sigma}{\varepsilon_0}\overrightarrow{\mathbf{k}}$ in the subspace $-\frac{d}{2} \leq z \leq \frac{d}{2}$ and 0 everywhere. It is verified that the electric field is indeed zero on the Gaussian surface (i). This electric field can be derived from the potential as follows:

$$V(z) = \begin{cases} -\dfrac{\sigma z}{\varepsilon_0} & \text{If } -\frac{d}{2} \leq z \leq \frac{d}{2} \\ -\dfrac{\sigma d\,|z|}{2\varepsilon_0 z} & \text{If } |z| > \frac{d}{2} \end{cases} \qquad [1.157]$$

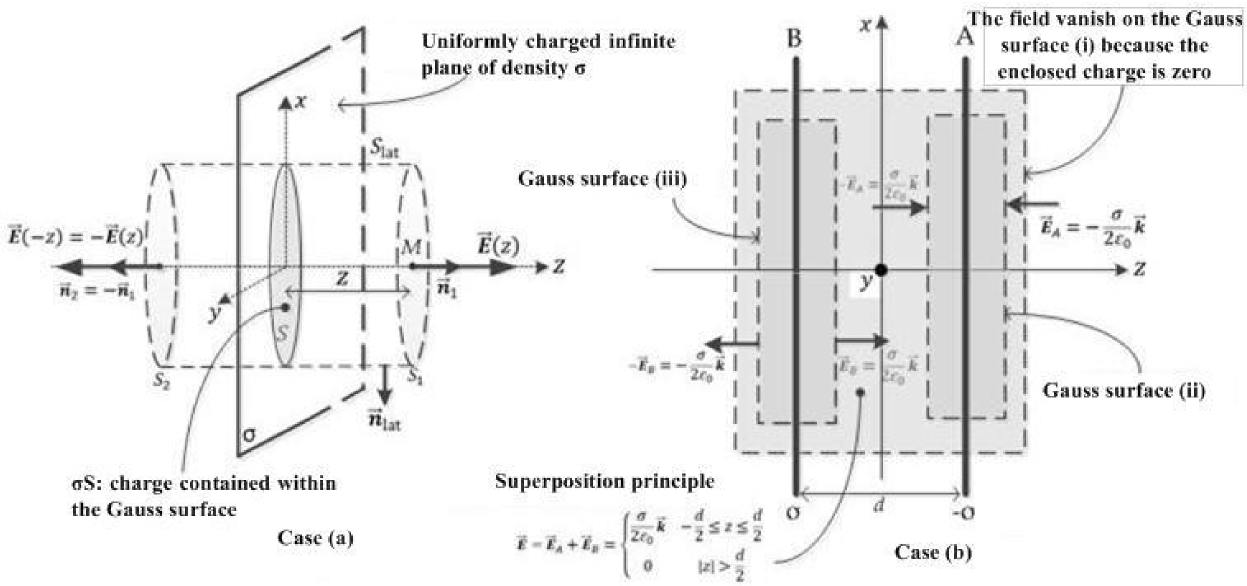

Figure 1.24. *Gauss theorem applied to a uniformly charged plane shows that the electric field is constant (on either side of the plane), perpendicular and outwardly directed (respectively, inwardly directed) if $\sigma > 0$ (respectively, $\sigma < 0$). For a double plane with opposite polarities, the field is constant between the planes (directed from the positive plate to the negative plate) and the external field is zero*

1.6.5. *Maximum and minimum potentials*

In a space with no electric charges, the potential does not admit any extreme values, that is, it does not go through a maximum or minimum. This is the *extreme value theorem*. In fact, if there is a maximum (respectively, a minimum) value for the potential, at a point M, the potential V would decrease (respectively, increase) in all directions around M. The electric field $\overrightarrow{\mathbf{E}} = -\overrightarrow{\nabla} V$ would therefore be directed from M outward all around M (respectively, from the outside toward M). Consequently, there would be a non-zero flux of $\overrightarrow{\mathbf{E}}$ across a closed surface surrounding M and Gauss' theorem would lead to the existence of a positive (respectively, negative) charge at this point, which contradicts the initial hypotheses. This result will be reviewed later. A direct consequence of what we have just seen is that a field line can only start from a place where charges exist, since the start of a field line corresponds to a maximum potential.

1.7. Electric field across a charged sheet

In the previous examples, we saw that (a) in the case of a volumetric charge distribution, the electrostatic field $\overrightarrow{\mathbf{E}}$ and the electrostatic potential V are continuous at all points, and (b) in the case of a surface distribution, there is a discontinuity in the field $\overrightarrow{\mathbf{E}}$ as it passes through the charged surface. This result is general. In fact, the tangential component of $\overrightarrow{\mathbf{E}}$ is continuous as it passes through the charged surface, whereas the normal component of $\overrightarrow{\mathbf{E}}$ is discontinuous; (c) the electrostatic potential V is always continuous.

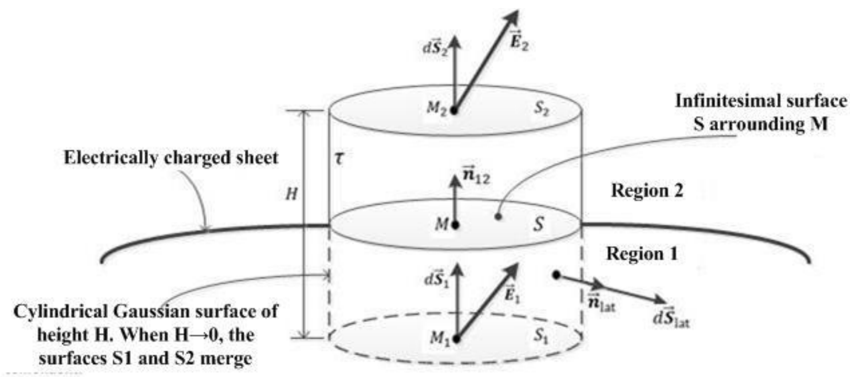

Figure 1.25. *Application of Gauss' theorem to an infinitely small Gaussian cylindrical surface containing a certain surface charge shows that the normal component $\varepsilon_0 \overrightarrow{\mathbf{E}}$ is discontinuous at the interface of a charged sheet*

To demonstrate result (b), consider an electrically charged sheet, that is, a surface distribution of charges σ, dividing the space into two regions 1 and 2. Let us draw a Gaussian surface, around a point M located on the sheet. This surface is infinitesimally small, cylindrical and closed, enclosing a volume τ and traversing the sheet, as shown in Figure 1.25. The surfaces S_1, S_2 and S are chosen parallel to the plane that is tangent to the sheet at the point M and are very small, so that $\overrightarrow{\mathbf{E}}$ and σ do not vary much over these planes. By applying Gauss' theorem:

$$\int_{S_2} \overrightarrow{\mathbf{E}}_2\Big|_{M_2} \cdot d\overrightarrow{\mathbf{S}}_2 + \int_{S_1} \overrightarrow{\mathbf{E}}_1\Big|_{M_1} \cdot d\overrightarrow{\mathbf{S}}_1 + \int_{S_{\text{lat}}} \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{S}}_{\text{lat}} = \frac{1}{\varepsilon_0} \int_S \sigma (M)\, dS \tag{1.158}$$

where S_{lat} is the lateral surface of the cylinder. For a flattened cylinder, that is, when H tends toward zero, it is possible to eliminate the flow through the lateral surface from the previous expression, since S_{lat} becomes zero. Since, in addition, $d\overrightarrow{\mathbf{S}}_2 = \overrightarrow{\mathbf{n}}_{12} dS$ and $d\overrightarrow{\mathbf{S}}_2 = -\overrightarrow{\mathbf{n}}_{12} dS$, we obtain $\frac{1}{\varepsilon_0} \int_S \sigma (M)\, dS = \lim_{H \longrightarrow 0} \int_S \left[\overrightarrow{\mathbf{E}}\,(M_2) - \overrightarrow{\mathbf{E}}\,(M_1)\right] \cdot \overrightarrow{\mathbf{n}}_{12} dS$. This gives:

$$\frac{\sigma (M)\, S}{\varepsilon_0} = \left[\overrightarrow{\mathbf{E}}_2\,(M) - \overrightarrow{\mathbf{E}}_1\,(M)\right] \cdot \overrightarrow{\mathbf{n}}_{12} S \tag{1.159}$$

because the fields $\overrightarrow{\mathbf{E}}_1$ and $\overrightarrow{\mathbf{E}}_2$ are almost constants over the surfaces S_1 and S_2, respectively. $\overrightarrow{\mathbf{n}}_{12}$ here represents the normal to the sheet directed from medium 1 to medium 2. Upon simplifying by S, we finally find:

$$\frac{\sigma (M)}{\varepsilon_0} = \left[\overrightarrow{\mathbf{E}}_2\,(M) - \overrightarrow{\mathbf{E}}_1\,(M)\right] \cdot \overrightarrow{\mathbf{n}}_{12} = E_{n_{12}}\,(M)\big|_2 - E_{n_{12}}\,(M)\big|_1 \tag{1.160}$$

If we change the orientation, that is, direct it from medium 2 to medium 1, the above equation takes the form:

$$\frac{\sigma (M)}{\varepsilon_0} = \left[\overrightarrow{\mathbf{E}}_1\,(M) - \overrightarrow{\mathbf{E}}_2\,(M)\right] \cdot \overrightarrow{\mathbf{n}}_{21} = E_{n_{21}}\,(M)\big|_1 - E_{n_{21}}\,(M)\big|_2 \tag{1.161}$$

where $\overrightarrow{\mathbf{n}}_{21} = -\overrightarrow{\mathbf{n}}_{12}$ is the normal directed from medium 2 to medium 1. The normal component, E_n, of the electric field $\overrightarrow{\mathbf{E}}$ is therefore not continuous as it crosses a charged sheet.

Let us now draw an infinitesimal rectangular contour $ABCD$ around the point M on the charged sheet and passing across the sheet. The sides AB and CD are parallel to the plane tangent to the sheet at point M, while the sides BC and DA are perpendicular to this plane. The electric field $\overrightarrow{\mathbf{E}}$ is constant along the sides of the rectangle, as seen in Figure 1.26. Since $\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}} = 0$, the circulation of the field $\overrightarrow{\mathbf{E}}$ along the contour $ABCD$ is zero:

$$0 = \int_{AB} \overrightarrow{\mathbf{E}}\,(M_2) \cdot d\,\overrightarrow{\mathbf{l}} + \int_{BC} \overrightarrow{\mathbf{E}}\,(M_3) \cdot d\,\overrightarrow{\mathbf{l}} + \int_{CD} \overrightarrow{\mathbf{E}}\,(M_1) \cdot d\,\overrightarrow{\mathbf{l}} + \int_{DA} \overrightarrow{\mathbf{E}}\,(M_4) \cdot d\,\overrightarrow{\mathbf{l}} \qquad [1.162]$$

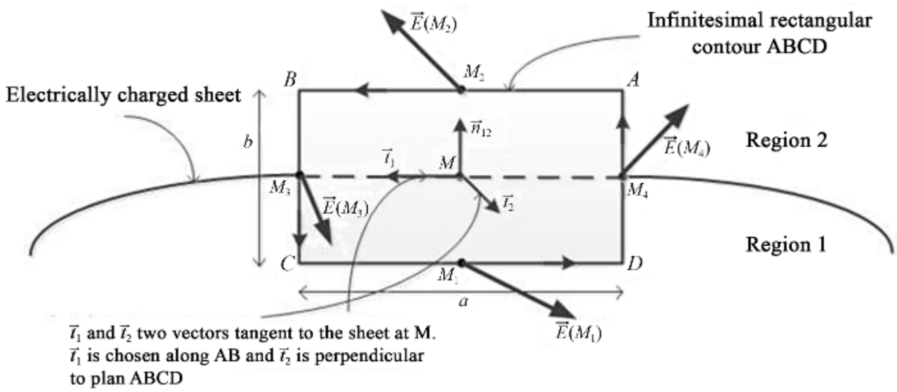

Figure 1.26. *Stokes' theorem, applied to an infinitesimal, rectangular contour and traversing a charge sheet separating two regions, 1 and 2, shows that the tangential component of $\overrightarrow{\mathbf{E}}$ is continuous along the interface*

By moving points B and C (respectively, A and D) to M_3 (respectively, M_4), that is, by applying the limit $b \longrightarrow 0$ (flattened contour) and noting that the circulation integrals $\lim_{b \longrightarrow 0} \int_{BC} \overrightarrow{bfE}\,(M_3) \cdot d\,\overrightarrow{\mathbf{l}}$ and $\lim_{b \longrightarrow 0} \int_{DA} \overrightarrow{\mathbf{E}}\,(M_4) \cdot d\,\overrightarrow{\mathbf{l}}$ become equal to zero, we obtain:

$$0 = \int_{AB} \left[\overrightarrow{\mathbf{E}}_2\,(M) - \overrightarrow{\mathbf{E}}_1\,(M) \right] \cdot \overrightarrow{\mathbf{t}}_1 dl \qquad [1.163]$$

where $\overrightarrow{\mathbf{t}}_1$ is a unit vector oriented in the direction of AB. Since $\overrightarrow{\mathbf{AB}}$ is any vector, we deduce that:

$$\left[\overrightarrow{\mathbf{E}}_2\,(M) - \overrightarrow{\mathbf{E}}_1\,(M) \right] \cdot \overrightarrow{\mathbf{t}}_1 = 0 \qquad [1.164]$$

By orienting the contour in the perpendicular direction, that is, toward $\overrightarrow{\mathbf{t}}_2$, we also find:

$$\left[\overrightarrow{\mathbf{E}}_2(M) - \overrightarrow{\mathbf{E}}_1(M)\right] \cdot \overrightarrow{\mathbf{t}}_2 = 0 \qquad [1.165]$$

We can therefore conclude that the projections of $\overrightarrow{\mathbf{E}}_2(M) - \overrightarrow{\mathbf{E}}_1(M)$ along $\overrightarrow{\mathbf{t}}_1$ and $\overrightarrow{\mathbf{t}}_2$ are zero. Consequently, the tangential part of $\overrightarrow{\mathbf{E}}_2(M) - \overrightarrow{\mathbf{E}}_1(M)$ is effectively zero. Equations [1.164] and [1.165] can be combined into a single equation:

$$\left[\overrightarrow{\mathbf{E}}_2(M) - \overrightarrow{\mathbf{E}}_1(M)\right] \times \overrightarrow{\mathbf{n}}_{12} = E_t(M)\big|_2 - E_t(M)\big|_1 = 0 \qquad [1.166]$$

The tangential component of the electric field is continuous across a charged sheet. Equations [1.160] and [1.166] are sometimes called *continuity equations of the electric field* at the interface of a charged sheet. They can be combined into a single equation of the form:

$$\overrightarrow{\mathbf{E}}_2(M) - \overrightarrow{\mathbf{E}}_1(M) = \frac{\sigma(M)}{\varepsilon_0}\overrightarrow{\mathbf{n}}_{12} \qquad [1.167]$$

where $\sigma(M)$ is the surface charge density at a point M on a charged sheet, $\overrightarrow{\mathbf{E}}_1(M)$ and $\overrightarrow{\mathbf{E}}_2(M)$ are fields in the neighborhood of M in the regions 1 and 2 separated by the sheet and $\overrightarrow{\mathbf{n}}_{12}$ is a unit vector normal to the sheet at M and oriented from 1 toward 2.

EXAMPLE 1.27. (Cylindrical and Spherical Sheets).–

(1) Let us write the continuity equations for the scalar potential V and the electric field $\overrightarrow{\mathbf{E}}$ through a sheet in the shape of an infinitely long cylinder, with axis Oz and radius R. The sheet carries a surface charge density $\sigma(\varphi, z)$, as seen in Figure 1.27(a). The scalar potentials (respectively, electric fields) outside and inside the cylinder are denoted by $V^{\text{ext}}(r, \varphi, z)$ (respectively, $\overrightarrow{\mathbf{E}}^{\text{ext}}(r, \varphi, z)$) and $V^{\text{int}}(r, \varphi, z)$ (respectively, $\overrightarrow{\mathbf{E}}^{\text{int}}(r, \varphi, z)$).

It has already been seen that the potential is continuous over the cylindrical interface when σ is finite, that is:

$$V^{\text{ext}}(R, \varphi, z) = V^{\text{int}}(R, \varphi, z) \qquad [1.168]$$

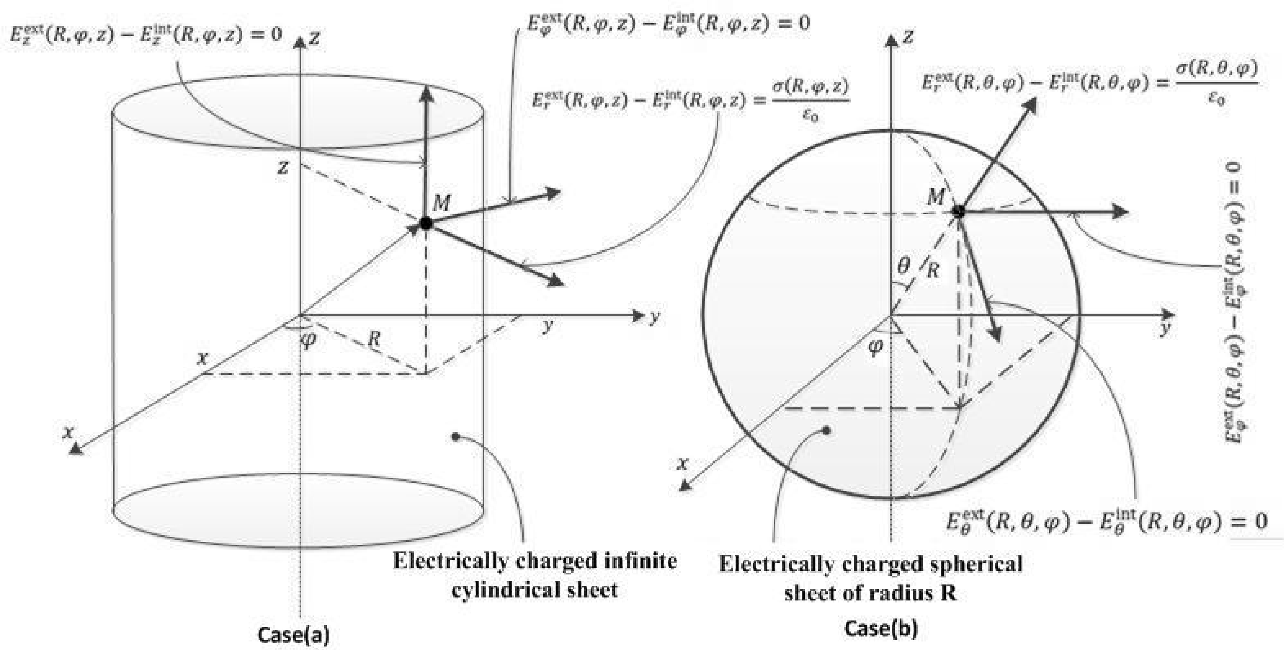

Figure 1.27. *Continuity equations for a cylindrical sheet (infinite, with axis Oz and radius R) and for a spherical sheet (radius R). Only the normal components are discontinuous, while the tangential components are continuous at the interfaces*

However, the situation is different for the electric field. According to the continuity equation [1.167], the normal (i.e. radial) component of the field across the interface obeys:

$$E_r^{\text{ext}}(R,\varphi,z) - E_r^{\text{int}}(R,\varphi,z) = \frac{\sigma(\varphi,z)}{\varepsilon_0} \qquad [1.169]$$

while the two other tangential components verify the constraints:

$$E_\varphi^{\text{ext}}(R,\varphi,z) - E_\varphi^{\text{int}}(R,\varphi,z) = 0 \qquad [1.170]$$

$$E_z^{\text{ext}}(R,\varphi,z) - E_z^{\text{int}}(R,\varphi,z) = 0 \qquad [1.171]$$

In terms of the scalar potentials $V^{\text{ext}}(r,\varphi,z)$ and $V^{\text{int}}(r,\varphi,z)$, equations [1.169]–[1.171] are written as:

$$\begin{aligned} \left.\frac{\partial V^{\text{int}}}{\partial r}\right|_{r=R} - \left.\frac{\partial V^{\text{ext}}}{\partial r}\right|_{r=R} &= \frac{\sigma(\varphi,z)}{\varepsilon_0} \\ \left.\frac{\partial V^{\text{int}}}{\partial \varphi}\right|_{r=R} - \left.\frac{\partial V^{\text{ext}}}{\partial \varphi}\right|_{r=R} &= 0 \\ \left.\frac{\partial V^{\text{int}}}{\partial z}\right|_{r=R} - \left.\frac{\partial V^{\text{ext}}}{\partial z}\right|_{r=R} &= 0 \end{aligned} \qquad [1.172]$$

(2) For a spherical sheet with radius R, separating two regions 1 (internal) and 2 (external) and bearing a surface charge density $\sigma(\theta,\varphi)$, as in Figure 1.27(b), the continuity equations for the electric potential and the electric field are written as:

$$V^{\text{ext}}(R,\theta,\varphi) = V^{\text{int}}(R,\theta,\varphi) \qquad [1.173]$$

and:

$$\begin{aligned} E^{\text{ext}}(R,\theta,\varphi) - E^{\text{int}}(R,\theta,\varphi) &= \left.\frac{\partial V^{\text{int}}}{\partial r}\right|_{r=R} - \left.\frac{\partial V^{\text{ext}}}{\partial r}\right|_{r=R} = \frac{\sigma(\theta,\varphi)}{\varepsilon_0} \\ E^{\text{ext}}(R,\theta,\varphi) - E^{\text{int}}(R,\theta,\varphi) &= \frac{1}{R}\left(\left.\frac{\partial V^{\text{int}}}{\partial \theta}\right|_{r=R} - \left.\frac{\partial V^{\text{ext}}}{\partial \theta}\right|_{r=R}\right) = 0 \\ E^{\text{ext}}(R,\theta,\varphi) - E^{\text{int}}(R,\theta,\varphi) &= \frac{1}{R}\left(\left.\frac{\partial V^{\text{int}}}{\partial \varphi}\right|_{r=R} - \left.\frac{\partial V^{\text{ext}}}{\partial \varphi}\right|_{r=R}\right) = 0 \end{aligned} \qquad [1.174]$$

1.8. Model of the hydrogen atom

Unlike Thomson's model, the atom can also be modeled by a fixed, point-like positive charge (the nucleus) and an electron cloud, that is, a zone surrounding the nucleus. This is the zone in which negatively charged particles are found: electrons, with charge $-e$. These electrons are distributed with a volume density ρ_e with spherical symmetry. To simplify, we shall restrict ourselves to the hydrogen atom. To calculate the electric field at a point M, we must take into account all charges found within the atom, that is, the point-like charge of the proton, $+e$, and the charge distributed in the electron cloud. For symmetry considerations, the field and electrostatic potential are actually functions of r, that is, $\overrightarrow{\mathbf{E}}(M) = E(r)\overrightarrow{\mathbf{e}}_r \equiv E(r)\frac{\overrightarrow{\mathbf{r}}}{r}$ and $V(M) = V(r)$. The field and potential resulting from the proton are given as:

$$\overrightarrow{\mathbf{E}}^+(M) = \frac{e}{4\pi\varepsilon_0 r^2}\overrightarrow{\mathbf{e}}_r \equiv \frac{e\overrightarrow{\mathbf{r}}}{4\pi\varepsilon_0 r^3} \qquad [1.175]$$

and:

$$V^+(M) = \frac{e}{4\pi\varepsilon_0 r} \qquad [1.176]$$

In addition, the electron density within the cloud can be chosen as $\rho_e(r) = \alpha e^{-\frac{r}{\beta}}$, where α is a negative constant with the dimension of a volume charge and β is a positive constant, so that the volume density converges to 0 when $r \to \infty$. β has the dimension of length. The elementary charge contained between two spheres with center O and having radii r and $r + dr$ will correspond to the product of $\rho_e(r)$ and the volume, $4\pi r^2 dr$, of the space between the two spheres. That is, $dq = 4\pi r^2 \alpha e^{-\frac{r}{\beta}} dr$; this corresponds to the radial density: $\lambda(r) = 4\pi r^2 \alpha e^{-\frac{r}{\beta}}$.

1) The radial density $\lambda(r)$ admits an extreme value at the point $r_0 = 2\beta$. In effect: $\lambda'(r) = 4\pi\alpha e^{-\frac{r}{\beta}}\left(2r - \frac{r^2}{\beta}\right)$; this shows that $\lambda'(r_0) = 0$.

2) The electron cloud contains a single electron: $\int_{r=0}^{\infty}\lambda(r)\,dr = \int_{r=0}^{\infty} e^{-\frac{r}{\beta}} \times 4\pi r^2 \alpha dr = 8\pi\alpha\beta^3 = -e$. Therefore, α and β can be expressed in terms of r_0 and $-e$: $\beta = \frac{r_0}{2}$ et $\alpha = -\frac{e}{\pi r_0^3}$. It becomes:

$$\lambda(r) = -\frac{e}{r_0}\left(\frac{2r}{r_0}\right)^2 e^{-\frac{2r}{r_0}} \qquad [1.177]$$

3) If $r \ll r_0$, we obtain $\lambda(r) \simeq -\frac{e}{r_0}\left(\frac{2r}{r_0}\right)^2$. In the neighborhood of $r = 0$, the curve is a parabola with a concave pointing downward. Similarly, we have

$\lim_{r\to\infty}\lambda(r)=0$. Consequently, the function $\lambda(r)$ reaches its minimum at the point r_0, $\lambda(r_0)=-\frac{4e}{r_0}e^{-2}$.

The electric field associated with the electron cloud can be obtained using Gauss' theorem (the distribution has spherical symmetry), $\oint_S \overrightarrow{\mathbf{E}}^-\left(\overrightarrow{\mathbf{r}}\right)\cdot d\overrightarrow{\mathbf{S}}=\frac{Q(r)}{\varepsilon_0}$; this gives $4\pi r^2E^-(r)=\frac{Q(r)}{\varepsilon_0}$, where $Q(r)=\int_{u=0}^{r}\lambda(u)\,du=-e+e\left[2\left(\frac{r}{r_0}\right)^2+\frac{2r}{r_0}+1\right]\exp\left(-\frac{2r}{r_0}\right)$. We thus find:

$$\overrightarrow{\mathbf{E}}^-\left(\overrightarrow{\mathbf{r}}\right)=-\frac{e}{4\pi\varepsilon_0}\left[\frac{1}{r^2}-\left(\frac{2}{r_0^2}+\frac{2}{r_0r}+\frac{1}{r^2}\right)\exp\left(-\frac{2r}{r_0}\right)\right]\overrightarrow{\mathbf{e}}_r \qquad [1.178]$$

If $r\ll r_0$, we have the approximation $\overrightarrow{\mathbf{E}}^-\left(\overrightarrow{\mathbf{r}}\right)\sim-\frac{er}{3\pi\varepsilon_0r_0^3}\overrightarrow{\mathbf{e}}_r$. The electron charge appears as though it was uniformly distributed in an imaginary sphere with an effective radius $R=\left(\frac{3}{4}\right)^{1/3}r_0$. In the same way, if $r\gg r_0$, then $\overrightarrow{\mathbf{E}}^-\left(\overrightarrow{\mathbf{r}}\right)\sim-\frac{e}{4\pi\varepsilon_0r^2}\overrightarrow{\mathbf{e}}_r$. At the point r_0, we find $\overrightarrow{\mathbf{E}}^-(r_0)=-\frac{e[1-5\exp(-2)]}{4\pi\varepsilon_0r_0^2}\overrightarrow{\mathbf{e}}_r$.

To obtain the potential $V^-\left(\overrightarrow{\mathbf{r}}\right)$ associated with $\overrightarrow{\mathbf{E}}^-\left(\overrightarrow{\mathbf{r}}\right)$, we write:

$$V^-\left(\overrightarrow{\mathbf{r}}\right)=-\frac{e}{4\pi\varepsilon_0}\left[\frac{1}{r}+\left(f(r)-\frac{1}{r}-\frac{1}{r_0}\right)\exp\left(-\frac{2r}{r_0}\right)\right] \qquad [1.179]$$

where $f(r)$ is a function that must be determined. By calculating $\overrightarrow{\nabla}V^-\left(\overrightarrow{\mathbf{r}}\right)$ and then comparing with the expression of $\overrightarrow{\mathbf{E}}^-\left(\overrightarrow{\mathbf{r}}\right)$, we obtain the differential equation $f'-\frac{2}{r_0}f=0$, which admits the solution $f=A\exp\left(\frac{2r}{r_0}\right)$, where A is a constant. We can derive from this that $V^-\left(\overrightarrow{\mathbf{r}}\right)=-\frac{e}{4\pi\varepsilon_0}\left[\frac{1}{r}+A-\left(\frac{1}{r}+\frac{1}{r_0}\right)\exp\left(-\frac{2r}{r_0}\right)\right]$. Since $V^-\left(\overrightarrow{\mathbf{r}}\right)$ must be zero for $r\to\infty$, we will choose $A=0$. The potential $V^-\left(\overrightarrow{\mathbf{r}}\right)$ is therefore:

$$V^-\left(\overrightarrow{\mathbf{r}}\right)=-\frac{e}{4\pi\varepsilon_0}\left[\frac{1}{r}-\left(\frac{1}{r}+\frac{1}{r_0}\right)\exp\left(-\frac{2r}{r_0}\right)\right] \qquad [1.180]$$

For $r\ll r_0: V^-\left(\overrightarrow{\mathbf{r}}\right)\sim\frac{e}{4\pi\varepsilon_0}\left(-\frac{1}{r_0}+\frac{2r^2}{3r_0^3}\right)$. For $r\gg r_0: V^-\left(\overrightarrow{\mathbf{r}}\right)\sim-\frac{e}{4\pi\varepsilon_0r}$.

By superposition, the field $\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right)$ and the associated potential $V\left(\overrightarrow{\mathbf{r}}\right)$, created by the hydrogen atom, are:

$$\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right)=\frac{e}{4\pi\varepsilon_0}\left(\frac{2}{r_0^2}+\frac{2}{r_0r}+\frac{1}{r^2}\right)\exp\left(-\frac{2r}{r_0}\right)\overrightarrow{\mathbf{e}}_r \qquad [1.181]$$

and:

$$V\left(\overrightarrow{\mathbf{r}}\right) = \frac{e}{4\pi\varepsilon_0}\left(\frac{1}{r} + \frac{1}{r_0}\right)\exp\left(-\frac{2r}{r_0}\right) \qquad [1.182]$$

For $r \ll r_0$, we find (first-order expansion): $\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) \sim \frac{e}{4\pi\varepsilon_0 r^2}\overrightarrow{\mathbf{e}}_r$ and $V\left(\overrightarrow{\mathbf{r}}\right) \sim \frac{e}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_0}\right)$. For $r \gg r_0$: $\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}\right) \sim 0$ and $V\left(\overrightarrow{\mathbf{r}}\right) \sim 0$. A discontinuity in the electrostatic field is therefore observed at $r = r_0$. The system behaves exactly as if the electron charge was uniformly distributed across the surface of a sphere of radius r_0, whose area is $4\pi r_0^2$. In effect, it has been verified that $E_r^{\text{ext.}} - E_r^{\text{int.}} = \frac{\sigma}{\varepsilon_0}$, which gives:

$$\sigma = -\frac{e}{4\pi r_0^2} \qquad [1.183]$$