Chapter 1

Waveguides

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1.1. The different types of waveguides

Guided propagation involves channeling an electromagnetic wave within a space bounded by metallic or dielectric interfaces from a transmitter to a receiver. The main advantage of guided propagation is the transmission of the wave shielded from external disturbances (atmospheric and others); and, sometimes, with low attenuation. Wave guiding is achieved using devices called *waveguides*, whose structure varies depending on the frequency range of the waves:

– hollow waveguides with a rectangular cross-section for millimeter and centimeter waves;

– optical fibers for microwaves;

– coaxial cables for high frequencies.

Waveguides are generally composed of a vacuum or dielectric medium, bounded by metallic or dielectric walls. In such a medium, the waves must satisfy the Maxwell equations and the boundary conditions on the metallic or dielectric walls.

The most widely used waveguides are the following:

– *Metallic waveguides*: these are metal tubes with a circular or rectangular cross-section.

– *Dielectric waveguides and optical fibers*: a dielectric waveguide is a device that is made up of the juxtaposition of several dielectric materials with different refractive indices. The material with the highest refractive index is generally placed in the center of the guide. An electromagnetic wave that enters the guide essentially stays within this high-index material.

– *Coaxial cables*: a coaxial cable takes the form of an *inner conductor*, surrounded by an *insulation* and then a *braided and wrapped conductive shield*. All of this is wrapped within a *protective cover*. Coaxial cables will be studied in Chapter 2.

1.2. Description of a guided wave

1.2.1. *Propagation equation and field expressions*

We will see if it is possible to propagate a monochromatic wave with angular frequency ω, within the guide and along the Oz axis. The medium inside the guide, which will support the propagation of the electromagnetic wave, must have the lowest possible absorbance – it generally consists of air or a dielectric material. To simplify, let us assume that this propagation medium is an LIH (linear, isotropic, and homogeneous) dielectric, with electric permittivity ε and magnetic permeability μ (in the case of a vacuum, we have $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$). The behavior of an electromagnetic wave within the waveguide is studied by solving the Maxwell equations (in the propagation medium), taking into account the boundary conditions imposed on the wave (on the fields) on the internal faces delimiting the propagation medium.

The propagation medium is a perfect LIH dielectric material (i.e. without free charges and no free current). At any point in this medium, the Maxwell equations are written as:

$$\begin{aligned} &\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{D}} = 0 \\ &\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{B}} = 0 \\ &\overrightarrow{\nabla} \times \overrightarrow{\mathbf{H}} = \frac{\partial \overrightarrow{\mathbf{D}}}{\partial t} \\ &\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}} = -\mu \frac{\partial \overrightarrow{\mathbf{H}}}{\partial t} \end{aligned} \qquad [1.1]$$

where $\overrightarrow{\mathbf{D}} = \varepsilon \overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{B}} = \mu \overrightarrow{\mathbf{H}}$. In the case of a conducting medium, the free current term $\overrightarrow{\mathbf{J}}_{\text{free}}$ (which appears in $\overrightarrow{\nabla} \times \overrightarrow{\mathbf{H}} = \overrightarrow{\mathbf{J}}_{\text{free}} + \frac{\partial \overrightarrow{\mathbf{D}}}{\partial t}$) can be integrated into the displacement field, which becomes a generalized displacement field $\overrightarrow{\mathrm{D}}$, related to the electric field through the complex dielectric permittivity $\hat{\varepsilon}$. To obtain the propagation equation for the electromagnetic field, we apply the double curl to the electric and magnetic fields: $\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}}\right) = \overrightarrow{\nabla}\left(\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{E}}\right) - \Delta \overrightarrow{\mathbf{E}} = -\Delta \overrightarrow{\mathbf{E}}$, because $\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{E}} = 0$. The last equation in [1.1] also yields $\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}}\right) = -\mu \frac{\partial}{\partial t}\left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{H}}\right) = -\mu\varepsilon \frac{\partial^2 \overrightarrow{\mathbf{E}}}{\partial t^2}$. Thus, from this, we can derive the equation for the propagation of the electric field. The same procedure leads to the propagation equation for the magnetic excitation field in the medium. The propagation equations for the fields are written as:

$$\begin{aligned} &\Delta \overrightarrow{\mathbf{E}} - \mu\varepsilon \frac{\partial^2 \overrightarrow{\mathbf{E}}}{\partial t^2} = 0 \\ &\Delta \overrightarrow{\mathbf{H}} - \mu\varepsilon \frac{\partial^2 \overrightarrow{\mathbf{H}}}{\partial t^2} = 0 \end{aligned} \qquad [1.2]$$

To study propagation in a waveguide, we first need to choose an analytical expression for the electromagnetic field that satisfies both the propagation equations [4.2] and the boundary conditions imposed on the medium. In the case of a cylindrical guide, cylindrical coordinates should be chosen, while in the case of a rectangular guide, a Cartesian system is preferable. In both cases, the choice of plane-wave propagation is not possible (with the exception of certain special situations) because a plane wave cannot satisfy the multiple boundary conditions imposed (the plane-wave amplitude is assumed to be constant). To do this, the fields must be decomposed into longitudinal fields $\left(\overrightarrow{\mathbf{E}}_z, \overrightarrow{\mathbf{H}}_z\right)$ (parallel to the direction of the guide, chosen here as the Oz axis) and in transverse fields $\left(\overrightarrow{\mathbf{E}}_\perp, \overrightarrow{\mathbf{H}}_\perp\right)$ (perpendicular to Oz). In the expressions for $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$, the dependence in z (assumed to be sinusoidal) is separated from the dependence in (x, y).

We therefore consider an electromagnetic wave propagating along the Oz axis as a plane wave:

$$\begin{aligned}\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}}, t\right) &= \Re\left[\overrightarrow{\mathbf{E}}\left(x, y\right) e^{-\mathrm{i}\left(\omega t - \mathsf{k}_z z\right)}\right] \\ \overrightarrow{\mathbf{H}}\left(\overrightarrow{\mathbf{r}}, t\right) &= \Re\left[\overrightarrow{\mathbf{H}}\left(x, y\right) e^{-\mathrm{i}\left(\omega t - \mathsf{k}_z z\right)}\right]\end{aligned} \tag{1.3}$$

but with variable amplitude. Here, k_z is the *wavenumber* or the *propagation constant* in the Oz direction. The problem is therefore reduced to solving the Maxwell equations for the amplitudes $\overrightarrow{\mathbf{E}}\left(x, y\right)$ and $\overrightarrow{\mathbf{H}}\left(x, y\right)$, which depend on the transverse coordinates: the problem thus becomes a 2D problem. Let us start by decomposing the fields into transverse and longitudinal components:

$$\begin{aligned}\overrightarrow{\mathbf{E}} &= \overrightarrow{\mathbf{E}}_\perp + E_z \overrightarrow{\mathbf{k}} \\ \overrightarrow{\mathbf{H}} &= \overrightarrow{\mathbf{H}}_\perp + H_z \overrightarrow{\mathbf{k}}\end{aligned} \tag{1.4}$$

where $\overrightarrow{\mathbf{E}}_\perp$ and $\overrightarrow{\mathbf{H}}_\perp$ are perpendicular to Oz. We also define the transverse gradient $\overrightarrow{\nabla}_\perp = \overrightarrow{\mathbf{i}}\frac{\partial}{\partial x} + \overrightarrow{\mathbf{j}}\frac{\partial}{\partial y} = \overrightarrow{\nabla} - \overrightarrow{\mathbf{k}}\frac{\partial}{\partial z}$. We want to show here that the transverse field $\overrightarrow{\mathbf{E}}_\perp$ (and, consequently, $\overrightarrow{\mathbf{E}}$) is fully determined if we know the E_z components of the electric field and H_z of the magnetic field.

This result is also true for the transverse field $\overrightarrow{\mathbf{H}}_\perp$. First, let us observe that:

$$\begin{aligned}\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}} &= \left(\overrightarrow{\nabla}_\perp + \overrightarrow{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \left(\overrightarrow{\mathbf{E}}_\perp + E_z \overrightarrow{\mathbf{k}}\right) \\ &= \overrightarrow{\nabla}_\perp \times \overrightarrow{\mathbf{E}}_\perp + \overrightarrow{\mathbf{k}} \times \frac{\partial \overrightarrow{\mathbf{E}}_\perp}{\partial z} - \overrightarrow{\mathbf{k}} \times \overrightarrow{\nabla}_\perp E_z\end{aligned} \tag{1.5}$$

The first term is oriented along $\overrightarrow{\mathbf{k}}$, while the other two are transverse. Next, we write the Maxwell equations for the fields [1.3], separating the transverse and longitudinal components. For the Maxwell–Faraday equation, we obtain:

$$\begin{aligned}\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}} + \mu \frac{\partial \overrightarrow{\mathbf{H}}}{\partial t} &= \overrightarrow{\nabla}_\perp \times \overrightarrow{\mathbf{E}}_\perp + \mathrm{i}\mathsf{k}_z \overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{E}}_\perp - \overrightarrow{\mathbf{k}} \times \overrightarrow{\nabla}_\perp E_z \\ &\quad - \mathrm{i}\mu\omega \overrightarrow{\mathbf{H}}_\perp - \mathrm{i}\mu\omega H_z \overrightarrow{\mathbf{k}} = 0\end{aligned} \qquad [1.6]$$

which, after separating the components, gives:

$$\overrightarrow{\nabla}_\perp \times \overrightarrow{\mathbf{E}}_\perp - \mathrm{i}\mu\omega H_z \overrightarrow{\mathbf{k}} = 0 \qquad [1.7]$$

$$\overrightarrow{\mathbf{k}} \times \left(\mathrm{i}\mathsf{k}_z \overrightarrow{\mathbf{E}}_\perp - \overrightarrow{\nabla}_\perp E_z\right) - \mathrm{i}\mu\omega \overrightarrow{\mathbf{H}}_\perp = 0 \qquad [1.8]$$

Taking the cross product of equation [1.8] with $\overrightarrow{\mathbf{k}}$ yields:

$$\mathrm{i}\mathsf{k}_z \overrightarrow{\mathbf{E}}_\perp + \mathrm{i}\mu\omega \overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{H}}_\perp = \overrightarrow{\nabla}_\perp E_z \qquad [1.9]$$

Similarly, by taking the dot product of equation [1.7] with $\overrightarrow{\mathbf{k}}$, we find:

$$\overrightarrow{\mathbf{k}} \cdot \left(\overrightarrow{\nabla}_\perp \times \overrightarrow{\mathbf{E}}_\perp\right) = \mathrm{i}\mu\omega H_z \qquad [1.10]$$

Let us repeat the same procedure for the Maxwell–Ampère equation:

$$\begin{aligned}\overrightarrow{\nabla} \times \overrightarrow{\mathbf{H}} - \varepsilon \frac{\partial \overrightarrow{\mathbf{E}}}{\partial t} &= \overrightarrow{\nabla}_\perp \times \overrightarrow{\mathbf{H}}_\perp + \mathrm{i}\mathsf{k}_z \overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{H}}_\perp - \overrightarrow{\mathbf{k}} \times \overrightarrow{\nabla}_\perp H_z \\ &\quad + \mathrm{i}\varepsilon\omega \overrightarrow{\mathbf{E}}_\perp + \mathrm{i}\varepsilon\omega E_z \overrightarrow{\mathbf{k}} = 0\end{aligned} \qquad [1.11]$$

which leads to:

$$\overrightarrow{\nabla}_\perp \times \overrightarrow{\mathbf{H}}_\perp + \mathrm{i}\varepsilon\omega E_z \overrightarrow{\mathbf{k}} = 0 \qquad [1.12]$$

$$\overrightarrow{\mathbf{k}} \times \left(\mathrm{i}\mathsf{k}_z \overrightarrow{\mathbf{H}}_\perp - \overrightarrow{\nabla}_\perp H_z\right) + \mathrm{i}\varepsilon\omega \overrightarrow{\mathbf{E}}_\perp = 0 \qquad [1.13]$$

and consequently:

$$\mathrm{i}\mathsf{k}_z \overrightarrow{\mathbf{H}}_\perp - \mathrm{i}\varepsilon\omega \overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{E}}_\perp = \overrightarrow{\nabla}_\perp H_z \qquad [1.14]$$

$$\overrightarrow{\mathbf{k}} \cdot \left(\overrightarrow{\nabla}_\perp \times \overrightarrow{\mathbf{H}}_\perp\right) = -\mathrm{i}\varepsilon\omega E_z \qquad [1.15]$$

The other two Maxwell equations can be reduced to:

$$\overrightarrow{\nabla}_\perp \cdot \overrightarrow{\mathbf{E}}_\perp = -\mathsf{i}\mathsf{k}_z E_z \quad [1.16]$$

$$\overrightarrow{\nabla}_\perp \cdot \overrightarrow{\mathbf{H}}_\perp = -\mathsf{i}\mathsf{k}_z H_z \quad [1.17]$$

Equations [1.9–1.10] and [1.14–1.17] make it possible to express $\overrightarrow{\mathbf{E}}_\perp$ and $\overrightarrow{\mathbf{H}}_\perp$ in terms of E_z and H_z. To see this, take the cross product of $\overrightarrow{\mathbf{k}}$ with equation [1.9] and then substitute equation [1.14] into the result:

$$\frac{\mathsf{k}_z}{\omega\varepsilon}\left(\mathsf{i}\mathsf{k}_z\overrightarrow{\mathbf{H}}_\perp - \overrightarrow{\nabla}_\perp H_z\right) - \mathsf{i}\omega\mu\overrightarrow{\mathbf{H}}_\perp = \overrightarrow{\mathbf{k}} \times \overrightarrow{\nabla}_\perp E_z \quad [1.18]$$

which gives:

$$\overrightarrow{\mathbf{H}}_\perp = \frac{1}{\omega^2 - \mathsf{k}_z^2 v^2}\left[\frac{\mathsf{i}\omega}{\mu}\overrightarrow{\mathbf{k}} \times \overrightarrow{\nabla}_\perp E_z + \mathsf{i}\mathsf{k}_z v^2 \overrightarrow{\nabla}_\perp H_z\right] \quad [1.19]$$

where $v^2 = \frac{1}{\varepsilon\mu}$. Repeating the same procedure for [1.14] and taking into account [1.9], we get:

$$\frac{\mathsf{k}_z}{\omega\mu}\left(-\mathsf{i}\mathsf{k}_z\overrightarrow{\mathbf{E}}_\perp + \overrightarrow{\nabla}_\perp E_z\right) + \mathsf{i}\omega\varepsilon\overrightarrow{\mathbf{E}}_\perp = \overrightarrow{\mathbf{k}} \times \overrightarrow{\nabla}_\perp H_z \quad [1.20]$$

which allows us to derive $\overrightarrow{\mathbf{E}}_\perp$:

$$\overrightarrow{\mathbf{E}}_\perp = \frac{1}{\omega^2 - \mathsf{k}_z^2 v^2}\left[-\frac{\mathsf{i}\omega}{\varepsilon}\overrightarrow{\mathbf{k}} \times \overrightarrow{\nabla}_\perp H_z + \mathsf{i}\mathsf{k}_z v^2 \overrightarrow{\nabla}_\perp E_z\right] \quad [1.21]$$

Equations [1.19] and [1.21] therefore allow us to determine all transverse components $\overrightarrow{\mathbf{E}}_\perp$ and $\overrightarrow{\mathbf{H}}_\perp$, as soon as the two longitudinal components E_z and H_z are known. We can thus limit ourselves to solving the propagation equation only for E_z and H_z, and then deduce $\overrightarrow{\mathbf{E}}_\perp$ and $\overrightarrow{\mathbf{H}}_\perp$ using equations [1.19] and [1.21]. In particular, when $\mathsf{k}_z^2 \neq \frac{\omega^2}{v^2}$, the electromagnetic field is zero if the components E_z and H_z are simultaneously zero.

The components E_z and H_z satisfy the Helmholtz equation:

$$\left(\overrightarrow{\nabla}^2 + \varepsilon\mu\omega^2\right)\begin{pmatrix} E_z \\ H_z \end{pmatrix} = 0 \quad [1.22]$$

Substituting expressions [1.3] into [1.22], we find:

$$\left(\overrightarrow{\nabla}_{\perp}^{2}+\gamma^{2}\right)\begin{pmatrix}E_z\\H_z\end{pmatrix}=0 \qquad [1.23]$$

where:

$$\gamma^2 = \varepsilon\mu\omega^2 - \mathsf{k}_z^2 \qquad [1.24]$$

We must solve this equation, taking into account the boundary conditions imposed in the Oxy plane due to the presence of the waveguide walls (conductive walls or interfaces between two dielectric media).

1.2.2. *Classification of propagation modes*

Unlike waves propagating in vacuum, electromagnetic waves are not always transverse, that is, the electric and magnetic fields are not necessarily perpendicular to the direction of propagation (the Oz axis). Different propagation modes can be distinguished:

1) *TM (transverse magnetic) mode*: the field $\overrightarrow{\mathbf{H}}$ is perpendicular to the direction of propagation, but $E_z \neq 0$. We must therefore solve equation [1.23] for E_z. This mode is also called an **E** type wave (since E_z is non-zero).

2) *TE (transverse electric) wave*: the electric field $\overrightarrow{\mathbf{E}}$ is perpendicular to the direction of propagation, but $H_z \neq 0$. We must therefore solve equation [1.23] for H_z. This mode is also called an **H** type wave (since H_z is non-zero).

3) *TEM (transverse electric and magnetic) mode*: the fields $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ are perpendicular to the direction of propagation, as if the wave were propagating in a vacuum (i.e. $E_z = H_z = 0$).

According to [1.19] and [1.21], for a TEM wave to exist, it is required that $\overrightarrow{\mathbf{E}}_{\perp}\gamma^2 = 0$ and $\overrightarrow{\mathbf{H}}_{\perp}\gamma^2 = 0$. For $\overrightarrow{\mathbf{E}}_{\perp} \neq 0$ and $\overrightarrow{\mathbf{H}}_{\perp} \neq 0$, we must require that $\gamma^2 = 0$. The dispersion relation for TEM waves is therefore:

$$\mathsf{k}_z^2 = \varepsilon\mu\omega^2 \qquad [1.25]$$

(the propagation constant is fixed for TEM waves). We also see from [1.7] and [1.16] that the transverse electric field $\overrightarrow{\mathbf{E}}_{\perp}$ satisfies the following relations:

$$\begin{aligned}&\overrightarrow{\nabla}_{\perp}\times\overrightarrow{\mathbf{E}}_{\perp}=0\\&\overrightarrow{\nabla}_{\perp}\cdot\overrightarrow{\mathbf{E}}_{\perp}=0\end{aligned} \qquad [1.26]$$

These are the two-dimensional electrostatic equations in the absence of charges. We know that this implies that $\overrightarrow{\mathbf{E}}_\perp$ derives from a scalar potential, that is, there exists a function $V(x, y)$ such that $\overrightarrow{\mathbf{E}}_\perp = -\overrightarrow{\nabla}_\perp V$. Since $\overrightarrow{\nabla}_\perp \cdot \overrightarrow{\mathbf{E}}_\perp = 0$, this potential satisfies the 2D Laplace equation:

$$\Delta_\perp V = 0 \qquad [1.27]$$

Relations similar to [1.26] are satisfied by the transverse magnetic excitation field $\overrightarrow{\mathbf{H}}_\perp$. In fact, equations [1.12] and [1.17] yield (for TEM waves):

$$\begin{aligned} &\overrightarrow{\nabla}_\perp \times \overrightarrow{\mathbf{H}}_\perp = 0 \\ &\overrightarrow{\nabla}_\perp \cdot \overrightarrow{\mathbf{H}}_\perp = 0 \end{aligned} \qquad [1.28]$$

which also indicates that the magnetic excitation field $\overrightarrow{\mathbf{H}}_\perp$ derives from a potential $\Psi(x, y)$, satisfying the $2D$ Laplace equation:

$$\begin{aligned} &\overrightarrow{\mathbf{H}}_\perp = \overrightarrow{\nabla}_\perp \Psi \\ &\Delta_\perp \Psi = 0 \end{aligned} \qquad [1.29]$$

For TEM waves, the fields $\overrightarrow{\mathbf{E}}_\perp$ and $\overrightarrow{\mathbf{H}}_\perp$ are related (see [1.13–1.14]) in this way:

$$\begin{aligned} &\overrightarrow{\mathbf{H}}_\perp = \frac{\mathsf{k}_z}{\mu\omega} \overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{E}}_\perp = Z^{-1} \overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{E}}_\perp \\ &\overrightarrow{\mathbf{E}}_\perp = -Z \overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{H}}_\perp \end{aligned} \qquad [1.30]$$

with $Z = \sqrt{\frac{\mu}{\varepsilon}}$. In other words, the $\overrightarrow{\mathbf{H}}$ field lines are perpendicular to the $\overrightarrow{\mathbf{E}}$ field lines. A consequence of equations [1.30] is that a TEM wave cannot propagate inside a hollow waveguide surrounded by a conductor (as will be demonstrated explicitly later for rectangular waveguides) because the electrostatic field is always zero inside a conductor. This result is quite general and can be extended to any hollow waveguide with conductive walls and a cross-section of any shape. To see this, let us consider two closed contours C_1 and C_2 of the Oxy plane, surrounding the guide at positions z_1 and z_2, respectively. Let us choose the following surfaces: (1) S_1 the surface bounded by C_1 in the Oxy plane; (2) S_2 the surface bounded by C_2 in the Oxy plane; (3) S_{env} is the surface that envelopes the guide, bounded on one side by C_1 and on the other by C_2. The integrals $\int_{S_1} \Psi \overrightarrow{\mathbf{H}}_\perp \cdot d\overrightarrow{\mathbf{S}}_1$, $\int_{S_2} \Psi \overrightarrow{\mathbf{H}}_\perp \cdot d\overrightarrow{\mathbf{S}}_2$ and $\int_{S_{\text{env}}} \Psi \overrightarrow{\mathbf{H}}_\perp \cdot d\overrightarrow{\mathbf{S}}_{\text{env}}$ are zero, because $\overrightarrow{\mathbf{H}}_\perp$ is in the Oxy plane for the first two and zero on the $\overrightarrow{\mathbf{S}}_{\text{env}}$ surface for the third. This means that:

$$\int_{S_2} \Psi \overrightarrow{\mathbf{H}}_\perp \cdot d\overrightarrow{\mathbf{S}}_2 - \int_{S_1} \Psi \overrightarrow{\mathbf{H}}_\perp \cdot d\overrightarrow{\mathbf{S}}_1 + \int_{S_{\text{env}}} \Psi \overrightarrow{\mathbf{H}}_\perp \cdot d\overrightarrow{\mathbf{S}}_{\text{env}} = \oint_S \Psi \overrightarrow{\mathbf{H}}_\perp \cdot d\overrightarrow{\mathbf{S}} = 0 \quad [1.31]$$

This last integral can be transformed into a volume integral using the divergence theorem:

$$0 = \oint_S \Psi \vec{\mathbf{H}}_\perp \cdot d\vec{\mathbf{S}} = \int_\tau \vec{\nabla}_\perp \cdot \left(\Psi \vec{\mathbf{H}}_\perp\right) d\tau = \int_\tau \left(\vec{\mathbf{H}}_\perp\right)^2 d\tau \qquad [1.32]$$

which gives $\vec{\mathbf{H}}_\perp = 0$ and then $\vec{\mathbf{E}}_\perp = 0$ in τ. Thus, a waveguide made up of a hollow conductor with no other conductor on its inside cannot conduct a TEM electromagnetic wave. These modes can exist in the presence of two or more conductors.

1.3. TE and TM modes in a hollow conducting waveguide

Consider a hollow waveguide surrounded by a conducting wall. In this case, TEM modes do not exist and only TE and TM waves are possible. To minimize energy losses inside these guides during propagation, it is assumed that the interior is vacuum (or air). Therefore, we take $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ in what follows. Let $\vec{\mathbf{n}}$ be the normal to the guide's conductive wall (directed toward the interior). Across this wall, the tangential component of $\vec{\mathbf{E}}$ must be continuous. The same applies to the normal component of $\vec{\mathbf{H}}$. Since the wall is a perfect conductor, the fields must vanish inside the conductor. Hence, by continuity, $\left(\vec{\mathbf{E}} \times \vec{\mathbf{n}}\right)_S$ and $\left(\vec{\mathbf{H}} \cdot \vec{\mathbf{n}}\right)_S$ must be zero on the inner side of the wall. Taking into account the decomposition [1.4], we deduce that:

$$\left(\vec{\mathbf{E}}_\perp \times \vec{\mathbf{n}}\right)_S = 0 \qquad [1.33]$$

$$(E_z)_S = 0 \qquad [1.34]$$

$$\left(\vec{\mathbf{H}}_\perp \cdot \vec{\mathbf{n}}\right)_S = 0 \qquad [1.35]$$

$\left(\vec{\mathbf{E}}_\perp \times \vec{\mathbf{n}}\right)_S$ and $(E_z)_S\, \vec{\mathbf{k}} \times \vec{\mathbf{n}}$ represent the transverse and longitudinal components, respectively, of $\left(\vec{\mathbf{E}} \times \vec{\mathbf{n}}\right)_S$ and they must both vanish independently. If we now calculate the scalar product of equation [1.14] with $\vec{\mathbf{n}}$, we obtain:

$$\mathrm{i}k_z \left(\vec{\mathbf{H}}_\perp \cdot \vec{\mathbf{n}}\right)_S - \mathrm{i}\omega\varepsilon \left[\vec{\mathbf{n}} \cdot \left(\vec{\mathbf{k}} \times \vec{\mathbf{E}}_\perp\right)\right]_S = \left(\vec{\mathbf{n}} \cdot \vec{\nabla}_\perp H_z\right)_S \qquad [1.36]$$

According to [1.35], the first term is zero on the wall. The mixed product of the second term is written as $\left[\vec{\mathbf{k}} \cdot \left(\vec{\mathbf{E}}_\perp \times \vec{\mathbf{n}}\right)\right]_S$; it is zero according to [1.33].

Therefore, the right-hand side of equation [1.36], which is the derivative of H_z in the $\overrightarrow{\mathbf{n}}$ direction, must also be zero:

$$\left(\frac{\partial H_z}{\partial n}\right)_S = 0 \qquad [1.37]$$

In the same way, let us take the vector product of equation [1.9] and $\overrightarrow{\mathbf{n}}$:

$$\mathrm{i}\mathrm{k}_z\left(\overrightarrow{\mathbf{E}}_\perp \times \overrightarrow{\mathbf{n}}\right)_S + \mathrm{i}\omega\mu\left[\left(\overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{H}}_\perp\right) \times \overrightarrow{\mathbf{n}}\right]_S = \left(\overrightarrow{\nabla}_\perp E_z \times \overrightarrow{\mathbf{n}}\right)_S \qquad [1.38]$$

The two terms on the left in [1.38] are obviously zero, since $\left[\left(\overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{H}}_\perp\right) \times \overrightarrow{\mathbf{n}}\right]_S$ $= -\overrightarrow{\mathbf{k}}\left(\overrightarrow{\mathbf{n}} \cdot \overrightarrow{\mathbf{H}}_\perp\right)_S = 0$, which gives:

$$\left(\overrightarrow{\mathbf{n}} \times \overrightarrow{\nabla}_\perp E_z\right)_S = 0 \qquad [1.39]$$

The problem therefore boils down to solving the Helmholtz equation [1.23], taking into account the boundary conditions [1.34] and [1.37]. As the boundary conditions are different for E_z and H_z, the solutions and eigenvalues must be different between the TM and TE modes.

1.3.1. *Rectangular waveguide*

Consider a waveguide with a rectangular cross-section (with sides a and b). The inner region of the waveguide is defined by $0 \leq x \leq a$ and $0 \leq y \leq b$.

The length of the waveguide along Oz is assumed to be infinite, as shown in Figure 1.1.

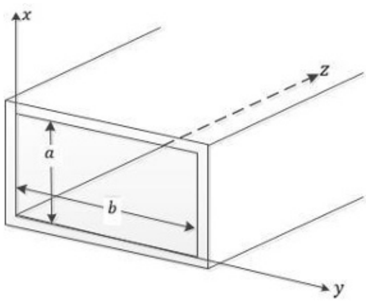

Figure 1.1. *Rectangular waveguide of height a and width b*

1.3.1.1. *TM waves ($H_z = 0$, $E_z \neq 0$)*

Let us first consider the TM modes. The Helmholtz equation is written as follows:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) E_z = 0 \qquad [1.40]$$

where $\gamma^2 = \frac{\omega^2}{c^2} - \mathsf{k}_z^2$. The boundary conditions are:

$$\begin{array}{l} E_z\,(x = 0, y) = E_z\,(x = a, y) = 0 \\ E_z\,(x, y = 0) = E_z\,(x, y = b) = 0 \end{array} \qquad [1.41]$$

The solution to [1.40] is obtained by separation of variables. We assume that the unknown function can be written in the form $E_z\,(x, y) = X\,(x)\,Y\,(y)$, where $X\,(x)$ (respectively, $Y\,(y)$) depends only on x (respectively, y). Substituting into the Helmholtz equation and dividing by XY, we obtain $\frac{X''}{X} + \frac{Y''}{Y} + \gamma^2 = 0$. The first term depends only on x, the second only on y and the third is constant. This equality can only hold if the first two terms are also constants, that is, $\frac{X''}{X} = -\mathsf{k}_x^2$, and $\frac{Y''}{Y} = -\mathsf{k}_y^2$, where the separation constants must satisfy the relation:

$$\mathsf{k}_x^2 + \mathsf{k}_y^2 = \gamma^2 \qquad \Longleftrightarrow \qquad \mathsf{k}_x^2 + \mathsf{k}_y^2 + \mathsf{k}_z^2 = \frac{\omega^2}{c^2} \qquad [1.42]$$

In particular, when $\omega = 0$, we obtain stationary solutions. If at least one of the wavenumbers is purely imaginary, the solutions are decreasing (exponential). If all three wavenumbers are real, there is pure propagation (sinusoidal solutions). The values of these wavenumbers are determined by the boundary conditions imposed on the walls in the Ox and Oy directions. The general solutions are given by:

$$\begin{array}{l} X\,(x) = A_1 \sin(\mathsf{k}_x x) + A_2 \cos(\mathsf{k}_x x) \\ Y\,(y) = B_1 \sin(\mathsf{k}_y y) + B_2 \cos(\mathsf{k}_y y) \end{array} \qquad [1.43]$$

and therefore:

$$E_z\,(x, y) = [A_1 \sin(\mathsf{k}_x x) + A_2 \cos(\mathsf{k}_x x)]\,[B_1 \sin(\mathsf{k}_y y) + B_2 \cos(\mathsf{k}_y y)] \quad [1.44]$$

where A_1, A_2, B_1 and B_2 are constants. Since the rectangular waveguide is composed of perfectly conducting walls, the longitudinal component of the electric field on these walls is zero (as already mentioned in [1.41]). In order to satisfy the boundary conditions at $x = 0$ and $y = 0$, it is essential that $A_2 = B_2 = 0$. Consequently, it follows that:

$$E_z\,(x, y) = E_0 \sin(\mathsf{k}_x x) \sin(\mathsf{k}_y y) \qquad [1.45]$$

where E_0 is the amplitude of the field. In addition, to satisfy the other two boundary conditions in $x = a$ and $y = b$, the wavenumbers k_x and k_y must be quantified as follows:

$$\begin{aligned} \mathsf{k}_x &= \frac{m\pi}{a} \quad m = 1, 2, 3, \ldots \\ \mathsf{k}_y &= \frac{n\pi}{b} \quad n = 1, 2, 3, \ldots \end{aligned} \qquad [1.46]$$

Note that if m or n is zero, the electric field E_z is zero. Therefore, the desired solution is:

$$E_z = \Re\left[E_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \qquad [1.47]$$

This represents what is called the *waveguide eigenmode*, denoted by TM_{mn}. Thus, the most general solution of the Helmholtz equation [1.22] is obtained as a linear combination of all possible eigenmodes. Since:

$$\mathsf{k}_z^2 = \frac{\omega^2}{c^2} - \mathsf{k}_x^2 - \mathsf{k}_y^2 = \frac{\omega^2}{c^2} - \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \qquad [1.48]$$

we observe the following:

1) k_z can only take discrete values.

2) k_z is real only if:

$$\omega \geq \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \qquad [1.49]$$

In other words, for a given value of ω, the (m, n) eigenmode can only propagate in the waveguide if condition [14.49] is satisfied. Otherwise, k_z becomes purely imaginary, that is, $\mathsf{k}_z = \mathrm{i}\delta$ with $\delta > 0$; this introduces the factor $e^{-\delta z}$ in the expression of E_z, preventing any propagation (the wave is therefore attenuated). Therefore, there is a lower limit for ω, called the *cutoff frequency* and denoted $\omega_{m,n}$, below which no propagation is possible.

For the TM_{mn} mode, we obtain:

$$\omega_{m,n} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \qquad [1.50]$$

Modes that are excited at a frequency lower than their cutoff frequency therefore do not propagate, but are still present over a certain distance near the excitation point of the waveguide.

Using the notation [1.50], equation [1.48] can also be written as:

$$\omega^2 = \omega_{m,n}^2 + c^2 \mathsf{k}_z^2 \qquad [1.51]$$

This is the *dispersion relation* of the waveguide; this formula resembles that of a plasma, as illustrated in Figure 1.2.

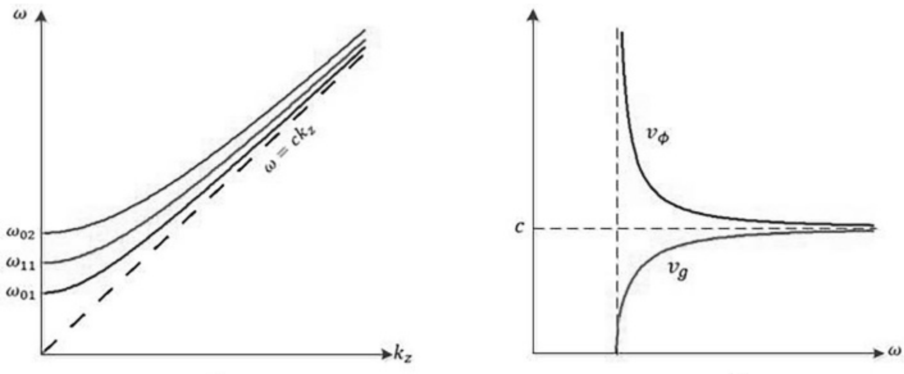

Figure 1.2. *(a) Dispersion relations for some modes of the rectangular waveguide. (b) Phase velocity and group velocity for the (m, n) mode. For a color version of this figure, see www.iste.co.uk/abdesselam/electrodynamics3.zip*

3) The phase and group velocities are given, respectively, by:

$$\begin{aligned} v_\phi &= \frac{\omega}{\mathsf{k}_z} = c\frac{\omega}{\sqrt{\omega^2 - \omega_{m,n}^2}} > c \\ v_g &= \frac{d\omega}{d\mathsf{k}_z} = c^2\frac{\mathsf{k}_z}{\omega} = \frac{c^2}{v_\phi} < c \end{aligned} \qquad [1.52]$$

As in a plasma, the phase velocity is greater than c, while the group velocity (the physical velocity) is less than c.

The other components of the fields $\overrightarrow{\mathbf{E}}_\perp$ and $\overrightarrow{\mathbf{H}}_\perp$ can be derived from the relations [1.19] and [1.21]:

$$\begin{aligned} \overrightarrow{\mathbf{E}}_\perp &= \frac{\mathsf{i}\mathsf{k}_z c^2}{\omega^2 - \mathsf{k}_z^2 c^2}\overrightarrow{\nabla}_\perp E_z \\ \overrightarrow{\mathbf{H}}_\perp &= \frac{\mathsf{i}\omega}{\mu\left(\omega^2 - \mathsf{k}_z^2 c^2\right)}\overrightarrow{\mathbf{k}} \times \overrightarrow{\nabla}_\perp E_z \end{aligned} \qquad [1.53]$$

Taking into account the expression [1.48] of the field E_z, we find:

$$\begin{cases} E_x = \Re\left[\dfrac{\mathrm{i}\mathsf{k}_x\mathsf{k}_z}{\mathsf{k}_x^2+\mathsf{k}_y^2}E_{mn}\cos\left(\mathsf{k}_x x\right)\sin\left(\mathsf{k}_y y\right)e^{-\mathrm{i}(\omega t-\mathsf{k}_z z)}\right] \\ E_y = \Re\left[\dfrac{\mathrm{i}\mathsf{k}_y\mathsf{k}_z}{\mathsf{k}_x^2+\mathsf{k}_y^2}E_{mn}\sin\left(\mathsf{k}_x x\right)\cos\left(\mathsf{k}_y y\right)e^{-\mathrm{i}(\omega t-\mathsf{k}_z z)}\right] \\ E_z = \Re\left[E_{mn}\sin\left(\mathsf{k}_x x\right)\sin\left(\mathsf{k}_y y\right)e^{-\mathrm{i}(\omega t-\mathsf{k}_z z)}\right] \end{cases} \quad [1.54]$$

$$\begin{cases} H_x = \Re\left[-\dfrac{\mathrm{i}\omega\mathsf{k}_y}{\mu c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)}E_{mn}\sin\left(\mathsf{k}_x x\right)\cos\left(\mathsf{k}_y y\right)e^{-\mathrm{i}(\omega t-\mathsf{k}_z z)}\right] \\ H_y = \Re\left[\dfrac{\mathrm{i}\omega\mathsf{k}_x}{\mu c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)}E_{mn}\cos\left(\mathsf{k}_x x\right)\sin\left(\mathsf{k}_y y\right)e^{-\mathrm{i}(\omega t-\mathsf{k}_z z)}\right] \\ H_z = 0 \end{cases} \quad [1.55]$$

We clearly see that the boundary conditions:

$$\begin{aligned} &E_x\left(y=0\right) = E_x\left(y=b\right) = 0 \\ &E_y\left(x=0\right) = E_y\left(x=a\right) = 0 \\ &H_x\left(x=0\right) = H_x\left(x=a\right) = 0 \\ &H_y\left(y=0\right) = H_y\left(y=b\right) = 0 \end{aligned} \quad [1.56]$$

are satisfied. The tangential components of the electric field $\overrightarrow{\mathbf{E}}$ and the normal components of the magnetic field $\overrightarrow{\mathbf{H}}$ are automatically zero on the walls. The solutions found in [1.54] and [1.55] can be interpreted as the result of the fields of four waves resulting from reflection on the waveguide walls. Indeed, let us rewrite the sine functions using De Moivre's formulas:

$$\begin{aligned} E_z = \Re\Big[&-\frac{1}{4}E_{mn}e^{-\mathrm{i}(\omega t-\mathsf{k}_x x-\mathsf{k}_y y-\mathsf{k}_z z)} + \frac{1}{4}E_{mn}e^{-\mathrm{i}(\omega t+\mathsf{k}_x x-\mathsf{k}_y y-\mathsf{k}_z z)} + \\ &\frac{1}{4}E_{mn}e^{-\mathrm{i}(\omega t-\mathsf{k}_x x+\mathsf{k}_y y-\mathsf{k}_z z)} - \frac{1}{4}E_{mn}e^{-\mathrm{i}(\omega t+\mathsf{k}_x x+\mathsf{k}_y y-\mathsf{k}_z z)}\Big] \end{aligned} \quad [1.57]$$

where $\mathsf{k}_x = \frac{m\pi}{a}$ and $\mathsf{k}_y = \frac{n\pi}{b}$. This expansion can be interpreted as follows: the first term corresponds to an incident wave propagating in the direction of the wave vector $\overrightarrow{\mathsf{k}} = (\mathsf{k}_x, \mathsf{k}_y, \mathsf{k}_z)$. The second term is the wave resulting from the reflection of this incident wave on the face parallel to the Oyz plane at $x = a$ (reflection coefficient equal to -1 and $\mathsf{k}_x \longrightarrow -\mathsf{k}_x$). The third term is the wave obtained after reflection of the incident wave on the face located at $y = b$. Finally, the last term comes from a double successive reflection on the faces at $x = a$ and $y = b$. This decomposition shows that the angles of incidence are quantized on the walls; only these values make it possible to achieve destructive interference on the boundaries. The propagation of the electromagnetic wave in the waveguide is accompanied by the propagation of currents and charges on the inner walls. The surface charge densities on the walls of the waveguide are obtained from the discontinuity equation of the normal component of $\overrightarrow{\mathbf{D}}$:

$$\begin{aligned}
\sigma(x=0) &= \varepsilon_0 E_x(x=0) = \Re\left[\frac{\mathsf{i}\mathsf{k}_x\mathsf{k}_z\varepsilon_0}{\mathsf{k}_x^2+\mathsf{k}_y^2}E_{mn}\sin(\mathsf{k}_y y)\,e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]\\
\sigma(x=a) &= -\varepsilon_0 E_x(x=a) = -\Re\left[\frac{\mathsf{i}\mathsf{k}_x\mathsf{k}_z\varepsilon_0(-1)^m}{\mathsf{k}_x^2+\mathsf{k}_y^2}E_{mn}\sin(\mathsf{k}_y y)\,e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]\\
\sigma(y=0) &= \varepsilon_0 E_y(y=0) = \Re\left[\frac{\mathsf{i}\mathsf{k}_y\mathsf{k}_z\varepsilon_0}{\mathsf{k}_x^2+\mathsf{k}_y^2}E_{mn}\sin(\mathsf{k}_x x)\,e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]\\
\sigma(y=b) &= -\varepsilon_0 E_y(y=b) = -\Re\left[\frac{\mathsf{i}\mathsf{k}_y\mathsf{k}_z\varepsilon_0(-1)^n}{\mathsf{k}_x^2+\mathsf{k}_y^2}E_{mn}\sin(\mathsf{k}_x x)\,e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]
\end{aligned} \qquad [1.58]$$

Similarly, the surface currents are found using the discontinuity equation for the tangential component of the field $\overrightarrow{\mathbf{H}}$:

$$\begin{aligned}
K_z(x=0) &= H_y(x=0) = \Re\left[\frac{\mathsf{i}\omega\mathsf{k}_x\varepsilon_0}{\mathsf{k}_x^2+\mathsf{k}_y^2}E_{mn}\sin(\mathsf{k}_y y)\,e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]\\
K_z(x=a) &= -H_y(x=a) = -\Re\left[\frac{\mathsf{i}\omega\mathsf{k}_x\varepsilon_0(-1)^m}{\mathsf{k}_x^2+\mathsf{k}_y^2}E_{mn}\sin(\mathsf{k}_y y)\,e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]\\
K_z(y=0) &= -H_x(y=0) = \Re\left[\frac{\mathsf{i}\omega\mathsf{k}_y\varepsilon_0}{\mathsf{k}_x^2+\mathsf{k}_y^2}E_{mn}\sin(\mathsf{k}_x x)\,e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]\\
K_z(y=b) &= H_x(y=b) = -\Re\left[\frac{\mathsf{i}\omega\mathsf{k}_y\varepsilon_0(-1)^n}{\mathsf{k}_x^2+\mathsf{k}_y^2}E_{mn}\sin(\mathsf{k}_x x)\,e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]
\end{aligned} \qquad [1.59]$$

The following are some important remarks:

1) The surface currents are purely directed along the Oz direction.

2) The charges $\sigma(x=0)$, $\sigma(x=a)$, $\sigma(y=0)$ and $\sigma(y=b)$ are, respectively, related to the currents $K_z(x=0)$, $K_z(x=a)$, $K_z(y=0)$ and $K_z(y=b)$ by charge conservation laws. For example, we can verify that:

$$\frac{\partial K_z(x=0)}{\partial z}+\frac{\partial\sigma(x=0)}{\partial t}=0 \iff K_z(x=0)=\frac{\omega}{\mathsf{k}_z}\sigma(x=0) \qquad [1.60]$$

etc.

3) If m is even (respectively, odd), the charges and currents on the opposite walls at $x=0$ and $x=a$ have opposite signs (respectively, the same sign).

4) Similarly, if n is even (respectively, odd), the opposite walls at $y = 0$ and $y = b$ carry charges of opposite signs (respectively, the same sign) and are traversed by currents flowing in opposite directions (respectively, in the same direction).

5) The field lines of $\overrightarrow{\mathbf{E}}$ in the Oxy plane are obtained from the following equation:

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{\mathsf{k}_y}{\mathsf{k}_x}\frac{\tan(\mathsf{k}_x x)}{\tan(\mathsf{k}_y y)} \implies \frac{[\cos(\mathsf{k}_x x)]^{(\mathsf{k}_y/\mathsf{k}_x)^2}}{\cos(\mathsf{k}_y y)} = \text{const.} \tag{1.61}$$

When the surface charge is positive, the electric field points outward from the wall, whereas it points toward the wall when the surface charge is negative.

6) For the field lines of $\overrightarrow{\mathbf{H}}$ in the Oxy plane, we must solve the equation:

$$\frac{dy}{dx} = \frac{H_y}{H_x} = -\frac{\mathsf{k}_x}{\mathsf{k}_y}\frac{\tan(\mathsf{k}_y y)}{\tan(\mathsf{k}_x x)} \implies \sin(\mathsf{k}_x x)\sin(\mathsf{k}_y y) = \text{const.} \tag{1.62}$$

The field configurations for the TM_{11}, TM_{12}, TM_{21} and TM_{22} modes are illustrated in Figure 1.3.

EXAMPLE 1.1.– The total charge $Q_{m,n}(z,t)$ per unit length of the waveguide carried by all walls at time t and position z for TM_{mn} mode is calculated as follows:

$$\begin{aligned} Q_{m,n}(z,t) &= \int_0^a [\sigma(y=0) + \sigma(y=b)]\,dx + \int_0^b [\sigma(x=0) + \sigma(x=a)]\,dy \\ &= \Re\left[\frac{i\mathsf{k}_z\varepsilon_0\,(1-(-1)^m)\,(1-(-1)^n)}{mn}\frac{ab}{\pi^2}E_{mn}e^{-i(\omega t - \mathsf{k}_z z)}\right] \end{aligned} \tag{1.63}$$

The total current intensity $I(z,t)$ on all walls at time t and position z is determined using a formula analogous to [1.63]:

$$\begin{aligned} I_{m,n}(z,t) &= \int_0^a [K_z(y=0) + K_z(y=b)]dx + \int_0^b [K_z(x=0) + K_z(x=a)]dy \\ &= \Re\left[\frac{i\omega\varepsilon_0\,(1-(-1)^m)\,(1-(-1)^n)}{mn}\frac{ab}{\pi^2}E_{mn}e^{-i(\omega t - \mathsf{k}_z z)}\right] \end{aligned} \tag{1.64}$$

It is easy to verify that the quantities $Q_{m,n}(z,t)$ and $I_{m,n}(z,t)$ are related by the continuity equation $\frac{\partial I_{m,n}}{\partial z} + \frac{\partial Q_{m,n}}{\partial t} = 0$.

We see that if m or n is even, then $Q_{m,n}(z,t)$ and $I_{m,n}(z,t)$ are zero. ■

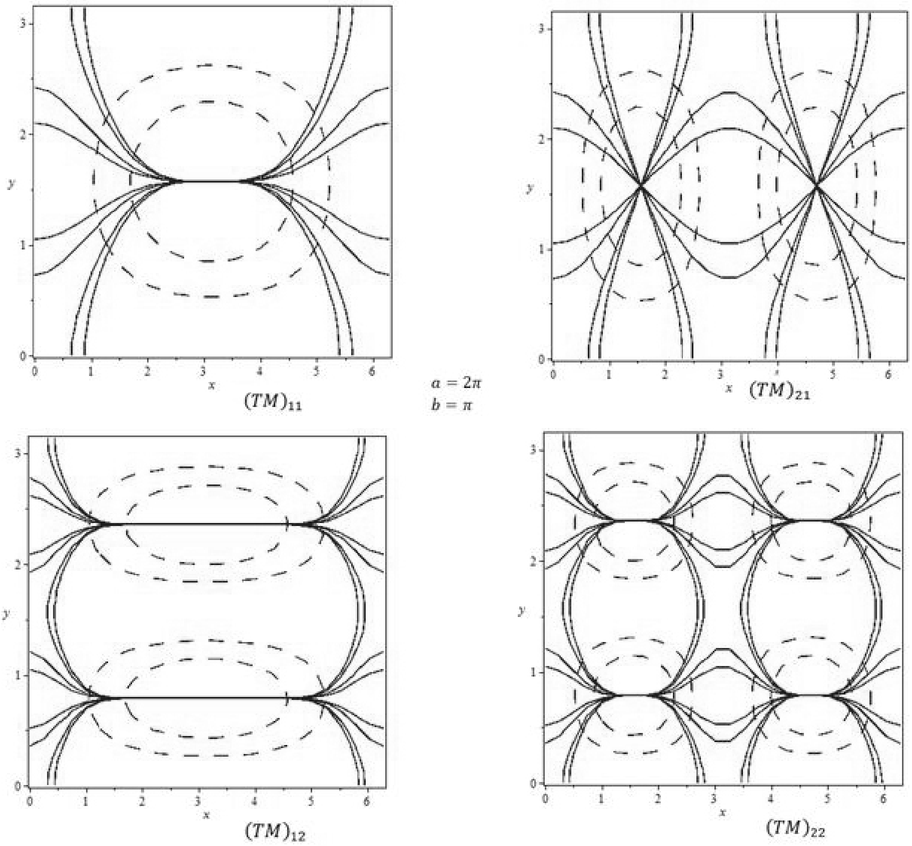

Figure 1.3. *Electric field lines (in blue) and magnetic field lines (dashed) for the TM$_{11}$, TM$_{12}$, TM$_{21}$ and TM$_{22}$= modes in the Oxy plane. For a color version of this figure, see www.iste.co.uk/abdesselam/electrodynamics3.zip*

Substituting previously found solutions [1.54] and [1.55] for fields $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ into the expression of the (time-averaged) Poynting vector gives:

$$\begin{aligned}\langle\overrightarrow{\mathbf{\Pi}}\rangle &= \frac{1}{2}\Re\left[\left(E_x\overrightarrow{\mathbf{i}} + E_y\overrightarrow{\mathbf{j}} + E_z\overrightarrow{\mathbf{k}}\right)e^{\mathsf{i}\mathsf{k}_z z} \times \left(H_x^\star\overrightarrow{\mathbf{i}} + H_y^\star\overrightarrow{\mathbf{j}}\right)e^{-\mathsf{i}\mathsf{k}_z^\star z}\right] \\ &= \frac{1}{2}\Re\left[\left(\left(E_xH_y^\star - E_yH_x^\star\right)\overrightarrow{\mathbf{k}} + E_z\left(H_x^\star\overrightarrow{\mathbf{j}} - H_y^\star\overrightarrow{\mathbf{i}}\right)\right)e^{\mathsf{i}(\mathsf{k}_z - \mathsf{k}_z^\star)z}\right]\end{aligned} \qquad [1.65]$$

If ω is below the cutoff frequency of a particular mode, k_z becomes purely imaginary, and the attenuation factor $e^{-|\mathsf{k}_z|z}$ then appears in the expression of the Poynting vector. For propagating modes where k_z is real, there is no dependence on z in [1.65]. In both cases, we see from [1.54] and [1.55] that the products of E_z, respectively, with $H_x^\star$ and $H_y^\star$ are purely imaginary, which means that the real parts of $E_z H_x^\star$ and $E_z H_y^\star$ are zero. Only the component of $\overrightarrow{\mathbf{\Pi}}$ directed along Oz can have a time average. The other two components (in the directions Ox and Oy) of the Poynting vector have zero time averages. Thus, we obtain:

$$\langle\overrightarrow{\mathbf{\Pi}}\rangle = \frac{\omega\varepsilon_0 \left|E_{mn}\right|^2}{2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)^2}\Re\left[\mathsf{k}_z e^{\mathsf{i}(\mathsf{k}_z-\mathsf{k}_z^\star)z}\left(\mathsf{k}_x^2\cos^2\left(\mathsf{k}_x x\right)\sin^2\left(\mathsf{k}_y y\right)\right.\right.$$
$$\left.\left.+\mathsf{k}_y^2\sin^2\left(\mathsf{k}_x x\right)\cos^2\left(\mathsf{k}_y y\right)\right)\right]\overrightarrow{\mathbf{k}} \qquad [1.66]$$

If k_z is purely imaginary, we obtain $\langle\Pi_z\rangle = 0$. On the other hand, if k_z is real, the average power flux over time is not equal to zero. The total power flux directed along the Oz direction is found by integrating [1.66] over the cross-sectional area of the waveguide (here, k_z is assumed to be real):

$$\langle P\rangle = \int_0^a\int_0^b \langle\Pi_z\rangle dxdy = \frac{\omega\varepsilon_0\mathsf{k}_z ab\left|E_{mn}\right|^2}{8\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)} \qquad [1.67]$$

since m and n are different from zero. To compute [1.67], the following identities were used:

$$\int_0^a\left\{\begin{matrix}\sin^2\left(\frac{m\pi x}{a}\right)\\ \cos^2\left(\frac{m\pi x}{a}\right)\end{matrix}\right\}dx = \frac{a}{m\pi}\left(\frac{m\pi x}{2a}\mp\frac{1}{4}\sin\left(\frac{2m\pi x}{a}\right)\right)\Bigg|_0^a = \frac{a}{2}$$
$$\int_0^b\left\{\begin{matrix}\sin^2\left(\frac{n\pi y}{b}\right)\\ \cos^2\left(\frac{n\pi y}{b}\right)\end{matrix}\right\}dy = \frac{b}{n\pi}\left(\frac{n\pi y}{2b}\mp\frac{1}{4}\sin\left(\frac{2n\pi y}{b}\right)\right)\Bigg|_0^b = \frac{b}{2} \qquad [1.68]$$

EXAMPLE 1.2. SCALAR AND VECTOR POTENTIALS OF A TM WAVE.– Using the relation between the magnetic field and its vector potential ($\overrightarrow{\mathbf{B}} = \overrightarrow{\nabla}\times\overrightarrow{\mathbf{A}}$), for a TM wave $(H_z = 0)$, we obtain:

$$\frac{\partial A_y}{\partial x} = \frac{\partial A_x}{\partial y} \qquad [1.69]$$

which means that there exists a scalar function $F(x, y, z, t)$ defined such that $A_x = \frac{\partial F}{\partial x}$ and $A_y = \frac{\partial F}{\partial y}$. Consequently, the components E_x and E_y of the electric field are written as:

$$E_x = -\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} = -\frac{\partial V}{\partial x} + \mathsf{i}\omega A_x = -\frac{\partial}{\partial x}\left(V - \mathsf{i}\omega F\right)$$
$$E_y = -\frac{\partial V}{\partial y} - \frac{\partial A_y}{\partial t} = -\frac{\partial V}{\partial y} + \mathsf{i}\omega A_y = -\frac{\partial}{\partial y}\left(V - \mathsf{i}\omega F\right) \qquad [1.70]$$

Let us redefine the electric potential:

$$V' = V - \mathsf{i}\omega F \qquad [1.71]$$

To preserve gauge invariance, the vector potential must also be redefined. Since:

$$E_z = -\mathsf{i}\mathsf{k}_z V + \mathsf{i}\omega A_z = -\mathsf{i}\mathsf{k}_z V' + \mathsf{i}\omega \left(A_z - \mathsf{i}\mathsf{k}_z F\right) \qquad [1.72]$$

we must set $A'_z = A_z - \mathsf{i}\mathsf{k}_z F$, which means that $E_z = -\mathsf{i}\mathsf{k}_z V' + \mathsf{i}\omega A'_z$. Let us take a function F of the form $F(x, y, z, t) = F(x, y)\, e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}$:

$$\begin{aligned} B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y} - \frac{\partial^2 F}{\partial y \partial z} = \frac{\partial}{\partial y}\left(A_z - \mathsf{i}\mathsf{k}_z F\right) = \frac{\partial A'_z}{\partial y} \\ B_y &= \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = \frac{\partial A_z}{\partial x} - \frac{\partial^2 F}{\partial x \partial z} = \frac{\partial}{\partial x}\left(A_z - \mathsf{i}\mathsf{k}_z F\right) = \frac{\partial A'_z}{\partial x} \end{aligned} \qquad [1.73]$$

We thus see that the new vector potential has only one component along Oz. Hence, we obtain:

$$\begin{aligned} &E_x = -\frac{\partial V'}{\partial x}, \quad E_y = -\frac{\partial V'}{\partial y}, \quad E_z = -\mathsf{i}\mathsf{k}_z V' + \mathsf{i}\omega A'_z \\ &B_x = \frac{\partial A'_z}{\partial x}, \qquad B_y = \frac{\partial A'_z}{\partial y} \end{aligned} \qquad [1.74]$$

We can furthermore impose the Lorenz gauge on the new potentials, $\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{A}}' + \frac{1}{c^2}\frac{\partial V'}{\partial t} = \mathsf{i}\mathsf{k}_z A'_z - \frac{\mathsf{i}\omega}{c^2} V' = 0$. This yields:

$$A'_z = \frac{\omega}{\mathsf{k}_z c^2} V' \qquad [1.75]$$

and consequently:

$$\begin{aligned} V' &= \frac{-\mathsf{i}\mathsf{k}_z}{\frac{\omega^2}{c^2} - \mathsf{k}_z^2} E_z \\ A'_z &= \frac{-\mathsf{i}\omega}{\omega^2 - c^2 \mathsf{k}_z^2} E_z \end{aligned} \qquad [1.76]$$

It is easy to verify that these potentials also satisfy the Helmholtz wave equation.

1.3.1.2. *TE waves ($E_z = 0$, $H_z \neq 0$)*

When the electric field is entirely in the Oxy plane (transverse field), we must first solve [1.22] for H_z. As in TM modes, we propose a solution of the form $H_z = \Re\left[H_z(x, y)\, e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right]$. This yields the following:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + \gamma^2\right) H_z = 0 \qquad [1.77]$$

In the same way, this equation is solved by separation of variables to obtain a solution of the same form as [1.44]:

$$H_z\left(x,y\right)=\left[A_1\sin\left(\mathsf{k}_x x\right)+A_2\cos\left(\mathsf{k}_x x\right)\right]\left[B_1\sin\left(\mathsf{k}_y y\right)+B_2\cos\left(\mathsf{k}_y y\right)\right] \qquad [1.78]$$

Since $\overrightarrow{\mathbf{H}}_\perp$ must be zero in the vicinity of the walls, we must require that, in accordance with [1.37]:

$$\begin{aligned}\left.\frac{\partial H_z}{\partial x}\right|_{x=0}&=\left.\frac{\partial H_z}{\partial x}\right|_{x=a}=0\\ \left.\frac{\partial H_z}{\partial y}\right|_{y=0}&=\left.\frac{\partial H_z}{\partial y}\right|_{y=b}=0\end{aligned} \qquad [1.79]$$

which implies that:

$$\begin{cases}\left[A_1\mathsf{k}_x\cos\left(\mathsf{k}_x x\right)-A_2\mathsf{k}_x\sin\left(\mathsf{k}_x x\right)\right]\big|_{x=0}=0\\ \left[B_1\mathsf{k}_y\cos\left(\mathsf{k}_y y\right)-B_2\mathsf{k}_y\sin\left(\mathsf{k}_y y\right)\right]\big|_{y=0}=0\end{cases}\Longrightarrow\begin{cases}A_1=0\\ B_1=0\end{cases} \qquad [1.80]$$

and (to obtain a non-trivial solution):

$$\begin{cases}-A_2\mathsf{k}_x\sin\left(\mathsf{k}_x a\right)=0\\ -B_2\mathsf{k}_y\sin\left(\mathsf{k}_y b\right)=0\end{cases}\Longrightarrow\begin{cases}\mathsf{k}_x=\dfrac{m\pi}{a}\\ \mathsf{k}_y=\dfrac{n\pi}{b}\end{cases} \qquad [1.81]$$

Thus, the desired solution (the *eigenmode of the waveguide denoted TE$_{mn}$*) is:

$$H_z=\Re\left[H_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\mathrm{i}(\omega t-\mathsf{k}_z z)}\right] \qquad [1.82]$$

We observe the following:

– As with the TM mode, k_x and k_y are quantized.

– k_z is given as a function of ω, k_x, and k_y by $\mathsf{k}_z=\sqrt{\frac{\omega^2}{c^2}-\mathsf{k}_x^2-\mathsf{k}_y^2}=\sqrt{\frac{\omega^2}{c^2}-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}$ (dispersion relation).

– k_z can be either real or purely imaginary. If k_z is real, the wave propagates purely. If k_z is purely imaginary, the wave is rapidly attenuated. The transition between pure propagation and attenuation occurs when $|\mathsf{k}_z|=0$, at the *cutoff frequency* $\omega_c=c\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$. This frequency varies for each mode (depending on m and n).

– For TE modes, either m or n (but not both) can be zero and still yield a non-trivial solution. When n equals zero, the fields do not vary with respect to y, and the electric field is purely directed along the Oy axis, while the magnetic excitation field is directed along Ox. If $a>b$, the lowest frequency occurs for the TE$_{10}$ mode, which is called the *dominant or fundamental mode*. No mode can propagate below this lowest critical frequency $\omega_{c0}=\frac{\pi c}{a}$.

– To calculate the other components of the fields $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$, we use relations [1.19] and [1.21] with the restriction $E_z = 0$:

$$\begin{aligned} \overrightarrow{\mathbf{E}}_\perp &= \frac{-\mathrm{i}\omega\mu_0}{\mathsf{k}_x^2 + \mathsf{k}_y^2} \overrightarrow{\mathbf{k}} \times \overrightarrow{\nabla}_\perp H_z \\ \overrightarrow{\mathbf{H}}_\perp &= \frac{\mathrm{i}\mathsf{k}_z}{\mathsf{k}_x^2 + \mathsf{k}_y^2} \overrightarrow{\nabla}_\perp H_z \end{aligned} \qquad [1.83]$$

which gives:

$$\begin{aligned} E_x &= \Re\left[-\frac{\mathrm{i}\omega\mu_0\mathsf{k}_y}{\mathsf{k}_x^2 + \mathsf{k}_y^2} H_{mn} \cos(\mathsf{k}_x x) \sin(\mathsf{k}_y y)\, e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\ E_y &= \Re\left[\frac{\mathrm{i}\omega\mu_0\mathsf{k}_x}{\mathsf{k}_x^2 + \mathsf{k}_y^2} H_{mn} \sin(\mathsf{k}_x x) \cos(\mathsf{k}_y y)\, e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\ E_z &= 0 \end{aligned} \qquad [1.84]$$

and:

$$\begin{aligned} H_x &= \Re\left[-\frac{\mathrm{i}\mathsf{k}_z\mathsf{k}_x}{\mathsf{k}_x^2 + \mathsf{k}_y^2} H_{mn} \sin(\mathsf{k}_x x) \cos(\mathsf{k}_y y)\, e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\ H_y &= \Re\left[-\frac{\mathrm{i}\mathsf{k}_z\mathsf{k}_y}{\mathsf{k}_x^2 + \mathsf{k}_y^2} H_{mn} \cos(\mathsf{k}_x x) \sin(\mathsf{k}_y y)\, e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\ H_z &= \Re\left[H_{mn} \cos(\mathsf{k}_x x) \cos(\mathsf{k}_y y)\, e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \end{aligned} \qquad [1.85]$$

As expected, we see that the tangential components of the electric field vanish at the walls.

– The electric field lines of $\overrightarrow{\mathbf{E}}$ in the Oxy plane:

$$\frac{dy}{dx} = \frac{E_y}{E_x} = -\frac{\mathsf{k}_x}{\mathsf{k}_y} \frac{\tan(\mathsf{k}_x x)}{\tan(\mathsf{k}_y y)} \implies \cos(\mathsf{k}_x x) \cos(\mathsf{k}_y y) = \text{const.} \qquad [1.86]$$

– The electric field lines of $\overrightarrow{\mathbf{H}}$ in the Oxy plane:

$$\frac{dy}{dx} = \frac{H_y}{H_x} = \frac{\mathsf{k}_y}{\mathsf{k}_x} \frac{\tan(\mathsf{k}_y y)}{\tan(\mathsf{k}_x x)} \implies \frac{[\sin(\mathsf{k}_x x)]^{(\mathsf{k}_y/\mathsf{k}_x)^2}}{\sin(\mathsf{k}_y y)} = \text{const.} \qquad [1.87]$$

– From these expressions, the surface charge and current on the walls are easily determined:

- face $x = 0$, $0 \leq y \leq b$:

$$\begin{aligned}
&\sigma_1 = \varepsilon_0 E_x\left(x=0\right) = \Re\left[-\frac{\mathsf{i}\omega\mathsf{k}_y}{c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)}H_{mn}\sin\left(\mathsf{k}_y y\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&K_{1,z} = H_y\left(x=0\right) = \Re\left[-\frac{\mathsf{i}\mathsf{k}_y\mathsf{k}_z}{\mathsf{k}_x^2+\mathsf{k}_y^2}H_{mn}\sin\left(\mathsf{k}_y y\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&K_{1,y} = -H_z\left(x=0\right) = \Re\left[-H_{mn}\cos\left(\mathsf{k}_y y\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&\frac{\partial K_{1,y}}{\partial y}+\frac{\partial K_{1,z}}{\partial z}+\frac{\partial \sigma_1}{\partial t}=0
\end{aligned} \qquad [1.88]$$

- face $x = a$, $0 \leq y \leq b$:

$$\begin{aligned}
&\sigma_2 = -\varepsilon_0 E_x\left(x=a\right) = \Re\left[\frac{\mathsf{i}\omega\mathsf{k}_y\left(-1\right)^m}{c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)}H_{mn}\sin\left(\mathsf{k}_y y\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&K_{2,z} = -H_y\left(x=a\right) = \Re\left[\frac{\mathsf{i}\mathsf{k}_y\mathsf{k}_z\left(-1\right)^m}{\mathsf{k}_x^2+\mathsf{k}_y^2}H_{mn}\sin\left(\mathsf{k}_y y\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&K_{2,y} = H_z\left(x=a\right) = \Re\left[\left(-1\right)^m H_{mn}\cos\left(\mathsf{k}_y y\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&\frac{\partial K_{2,y}}{\partial y}+\frac{\partial K_{2,z}}{\partial z}+\frac{\partial \sigma_2}{\partial t}=0
\end{aligned} \qquad [1.89]$$

- face $y = 0$, $0 \leq x \leq a$:

$$\begin{aligned}
&\sigma_3 = \varepsilon_0 E_y\left(y=0\right) = \Re\left[\frac{\mathsf{i}\omega\mathsf{k}_x}{c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)}H_{mn}\sin\left(\mathsf{k}_x x\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&K_{3,z} = -H_x\left(y=0\right) = \Re\left[\frac{\mathsf{i}\mathsf{k}_x\mathsf{k}_z}{\mathsf{k}_x^2+\mathsf{k}_y^2}H_{mn}\sin\left(\mathsf{k}_x x\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&K_{3,x} = H_z\left(y=0\right) = \Re\left[H_{mn}\cos\left(\mathsf{k}_x x\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&\frac{\partial K_{3,x}}{\partial x}+\frac{\partial K_{3,z}}{\partial z}+\frac{\partial \sigma_3}{\partial t}=0
\end{aligned} \qquad [1.90]$$

- face $y = b$, $0 \leq x \leq a$:

$$\begin{aligned}
&\sigma_4 = -\varepsilon_0 E_y\left(y=b\right) = \Re\left[-\frac{\mathsf{i}\omega\mathsf{k}_x\left(-1\right)^n}{c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)}H_{mn}\sin\left(\mathsf{k}_x x\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&K_{4,z} = H_x\left(y=b\right) = \Re\left[-\frac{\mathsf{i}\mathsf{k}_x\mathsf{k}_z\left(-1\right)^n}{\mathsf{k}_x^2+\mathsf{k}_y^2}H_{mn}\sin\left(\mathsf{k}_x x\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&K_{4,x} = -H_z\left(y=b\right) = \Re\left[-\left(-1\right)^n H_{mn}\cos\left(\mathsf{k}_x x\right)e^{-\mathsf{i}\left(\omega t-\mathsf{k}_z z\right)}\right]\\
&\frac{\partial K_{4,x}}{\partial x}+\frac{\partial K_{4,z}}{\partial z}+\frac{\partial \sigma_4}{\partial t}=0
\end{aligned} \qquad [1.91]$$

– The charge $Q_{12}(z,t)$ per unit length of the waveguide carried by the walls located at $x=0$ and $x=a$, at time t and position z, is equal to:

$$\begin{aligned} Q_{12}(z,t) &= \int_0^b [\sigma_1+\sigma_2]\, dy \\ &= \frac{\omega H_{mn}\left(1-(-1)^m\right)\left(1-(-1)^n\right)}{c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)} \sin\left(\omega t-\mathsf{k}_z z\right) \end{aligned} \qquad [1.92]$$

while the charge $Q_{34}(z,t)$, carried by the walls located at $y=0$ and $y=b$, is the opposite. Indeed, we have:

$$\begin{aligned} Q_{34}(z,t) &= \int_0^a [\sigma_3+\sigma_4]\, dx \\ &= -\frac{\omega H_{mn}\left(1-(-1)^m\right)\left(1-(-1)^n\right)}{c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)} \sin\left(\omega t-\mathsf{k}_z z\right) \end{aligned} \qquad [1.93]$$

– The total current $I_{12}(z,t)$ (in the direction of propagation) on the faces $x=0$ and $x=a$ is given by:

$$\begin{aligned} I_{12}(z,t) &= \int_0^b [K_{1,z}+K_{2,z}]\, dy \\ &= \frac{\mathsf{k}_z H_{mn}\left(1-(-1)^m\right)\left(1-(-1)^n\right)}{\mathsf{k}_x^2+\mathsf{k}_y^2} \sin\left(\omega t-\mathsf{k}_z z\right) \\ &= \frac{\mathsf{k}_z c^2}{\omega} Q_{12}(z,t) \end{aligned} \qquad [1.94]$$

and the total current $I_{34}(z,t)=\frac{\mathsf{k}_z c^2}{\omega}Q_{34}(z,t)$ on the faces $y=0$ and $y=b$ is the opposite, $I_{34}(z,t)=-I_{12}(z,t)$. The total current on the faces $x=0$ and $x=a$ (respectively, $y=0$ and $y=b$) along Oy (respectively, Ox) is zero.

– The time average over a period of the Poynting vector for the TE wave is given by:

$$\begin{aligned} \langle\overrightarrow{\mathbf{\Pi}}\rangle &= \frac{1}{2}\Re\left[\left(E_x\overrightarrow{\mathbf{i}}+E_y\overrightarrow{\mathbf{j}}\right)e^{\mathsf{i}\mathsf{k}_z z}\times\left(H_x^\star\overrightarrow{\mathbf{i}}+H_y^\star\overrightarrow{\mathbf{j}}+H_z^\star\overrightarrow{\mathbf{k}}\right)e^{-\mathsf{i}\mathsf{k}_z^\star z}\right] \\ &= \frac{1}{2}\Re\left[\left(\left(E_xH_y^\star-E_yH_x^\star\right)\overrightarrow{\mathbf{k}}+H_z^\star\left(E_y\overrightarrow{\mathbf{i}}-E_x\overrightarrow{\mathbf{j}}\right)\right)e^{\mathsf{i}(\mathsf{k}_z-\mathsf{k}_z^\star)z}\right] \end{aligned} \qquad [1.95]$$

The products $H_z^\star E_y$ and $H_z^\star E_x$ are purely imaginary; this implies that the time-averaged components of the Poynting vector in the Ox and Oy directions are zero. It therefore follows that:

$$\langle \Pi_z \rangle = \frac{\omega\mu_0 |H_{mn}|^2}{2\left(\mathsf{k}_x^2 + \mathsf{k}_y^2\right)^2} \Re\left[\mathsf{k}_z e^{\mathrm{i}(\mathsf{k}_z - \mathsf{k}_z^\star)z} \left(\mathsf{k}_x^2 \sin^2(\mathsf{k}_x x)\cos^2(\mathsf{k}_y y) + \mathsf{k}_y^2 \cos^2(\mathsf{k}_x x)\sin^2(\mathsf{k}_y y)\right)\right] \quad [1.96]$$

It is clear that the average $\langle \Pi_z \rangle$ is zero when k_z is purely imaginary (attenuation mode). For the propagation mode, we have $\langle \overrightarrow{\mathbf{\Pi}} \rangle \neq 0$. The power flux through the cross-sectional area of the waveguide is:

$$\langle P \rangle = \int_0^a \int_0^v \langle \Pi_z \rangle dx dy = \begin{cases} \dfrac{\omega\mu_0 \mathsf{k}_z |H_{mn}|^2 ab}{8\left(\mathsf{k}_x^2 + \mathsf{k}_y^2\right)} & m, n \neq 0 \\ \dfrac{\omega\mu_0 \mathsf{k}_z |H_{mn}|^2 ab}{4\left(\mathsf{k}_x^2 + \mathsf{k}_y^2\right)} & m \,\text{or}\, n = 0 \end{cases} \quad [1.97]$$

where the identities [1.68] were used once again. Note the factor of 2 in [1.97] for the TE_{m0} and TE_{0n} modes. The two indices m and n cannot both be zero at the same time because the TE_{00} mode reduces to a trivial magnetic field directed along Oz that is constant in space.

EXAMPLE 1.3. TEM WAVE INSIDE A RECTANGULAR WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS.– We will see below that a TEM wave cannot propagate inside a hollow waveguide. We have already seen that for a TEM wave to exist, it is necessary that $\gamma^2 = 0$, that is, $\mathsf{k}_z^2 = \varepsilon_0\mu_0\omega^2 = \frac{\omega^2}{c^2}$, and that the transverse fields $\overrightarrow{\mathbf{E}}_\perp$ and $\overrightarrow{\mathbf{H}}_\perp$ satisfy equations [1.26], [1.28] and [1.30]. From equation $\overrightarrow{\nabla}_\perp \times \overrightarrow{\mathbf{E}}_\perp = 0$, we can assert that there must exist a function $V(x, y)$ (actually, the electric potential) such that $\overrightarrow{\mathbf{E}}_\perp = -\overrightarrow{\nabla}_\perp V$. Furthermore, this potential must satisfy the 2D Laplace equation: $\Delta_\perp V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, since $\overrightarrow{\nabla}_\perp \cdot \overrightarrow{\mathbf{E}}_\perp = 0$. In the same way, this equation can be solved using the method of separation of variables, that is, taking $V(x, y) = X(x)\, Y(y)$, which gives $\frac{X''}{X} = -\frac{Y''}{Y} = -\alpha^2$, where α can, in principle, be complex. We immediately deduce the form of $V(x, y)$:

$$V(x, y) = \begin{cases} (A_1 \sin\alpha x + A_2 \cos\alpha x)\left(B_1 e^{\alpha y} + B_2 e^{-\alpha y}\right) & \alpha \neq 0 \\ (A_1' x + A_2')(B_1 y + B_2) & \alpha = 0 \end{cases} \quad [1.98]$$

The constants A_1, A_2, B_1, B_2, A_1', A_2', B_1' and B_2' must be determined from the boundary conditions (satisfied by the field):

$$\begin{aligned} \left.\frac{\partial V}{\partial x}\right|_{x,y=0} &= \left.\frac{\partial V}{\partial x}\right|_{x,y=b} = 0 \\ \left.\frac{\partial V}{\partial y}\right|_{x=0,y} &= \left.\frac{\partial V}{\partial y}\right|_{x=a,y} = 0 \end{aligned} \quad [1.99]$$

which leads to $B_1 = -B_2$, $B_1 \left(e^{\alpha b} - e^{-\alpha b}\right) = 0$, $A_2 = 0$, $A_1 \sin \alpha a = 0$, $A_1' = 0$, $A_2' = 0$, $B_1' = 0$ and $B_2' = 0$. We observe that for $\alpha = 0$, the solution is identically zero. To obtain a non-trivial solution when $\alpha \neq 0$, the parameter α must satisfy the constraints $e^{2\alpha b} = 1$ and $\sin \alpha a = 0$. These two constraints cannot be verified simultaneously, which means that the constants A_1, A_2, B_1 and B_2 must be equal to zero. In both cases, the result is a zero field: a TEM wave cannot propagate inside this waveguide.

1.3.2. *Attenuation coefficient in a lossy waveguide*

In real conductors with high but finite conductivity, the fields no longer become zero instantaneously upon entering the conducting walls but instead decay over a distance δ (called *skin depth*).

To approximately estimate the fields inside the walls, we proceed as follows (the subscript c will be used to denote the fields inside the walls): (1) inside the waveguide, the fields are assumed to be described by the mode expressions obtained for perfectly conducting walls; (2) on the boundaries (the walls), the tangential component of the magnetic field $\overrightarrow{\mathbf{H}}_{||}$ is continuous (due to the absence of surface currents); (3) inside the walls, the magnetic field $\overrightarrow{\mathbf{H}}_c$ decays exponentially in the direction perpendicular to the boundary and attenuates completely over a distance on the order of $\delta = \sqrt{\frac{2}{\gamma_c \mu_0 \omega}}$. Thus, the field $\overrightarrow{\mathbf{H}}_c$ is given by:

$$\overrightarrow{\mathbf{H}}_c = \left(\overrightarrow{\mathbf{H}}_{||}\right)_S e^{-\left|\overrightarrow{\mathbf{n}} \cdot \left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}_S\right)\right|/\delta} \tag{1.100}$$

where $\overrightarrow{\mathbf{n}}$ is the normal to the boundary surface S; (4) inside the walls, the electric field $\overrightarrow{\mathbf{E}}_c$ is calculated using a formula analogous to equation [7.134] (Volume 2):

$$\overrightarrow{\mathbf{E}}_c = (1 - \mathsf{i}) \sqrt{\frac{\omega \mu_0}{2\gamma_c}} \overrightarrow{\mathbf{n}} \times \overrightarrow{\mathbf{H}}_c = (1 - \mathsf{i}) \sqrt{\frac{\omega \mu_0}{2\gamma_c}} \left(\overrightarrow{\mathbf{n}} \times \overrightarrow{\mathbf{H}}_{||}\right)_S e^{-\left|\overrightarrow{\mathbf{n}} \cdot \left(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}_S\right)\right|/\delta} \tag{1.101}$$

where γ_c is the conductivity. Near the surface, the electric field $\overrightarrow{\mathbf{E}}_c$ is related to the current density $\overrightarrow{\mathbf{J}}$ by $\overrightarrow{\mathbf{J}} = \gamma_c \overrightarrow{\mathbf{E}}_c$. This leads to a power loss (averaged over one period) in the volume τ:

$$\langle P_d \rangle = \frac{1}{2} \int_\tau \overrightarrow{\mathbf{E}}_c \cdot \overrightarrow{\mathbf{J}}^\star d\tau = \frac{1}{2} \int_\tau \gamma_c \left|\overrightarrow{\mathbf{E}}_c\right|^2 d\tau \tag{1.102}$$

Since $\gamma_c \left|\overrightarrow{\mathbf{E}}_c\right|^2 = \mu_0 \omega \left|\overrightarrow{\mathbf{H}}_c\right|^2$, the power loss integral [1.102] can also be rewritten in the form:

$$\langle P_d \rangle = \frac{\mu_0 \omega}{2} \int_\tau \left|\overrightarrow{\mathbf{H}}_c\right|^2 d\tau \qquad [1.103]$$

From the real part of the complex Poynting theorem derived in section 7.2.4 of Volume 2, the divergence of the time-averaged electromagnetic power flux can be related to the time-averaged dissipated power as follows:

$$\overrightarrow{\nabla} \cdot \langle \overrightarrow{\mathbf{\Pi}} \rangle = -\langle P_d \rangle \qquad [1.104]$$

Integrating over the portion with thickness Δz of the waveguide, as shown in Figure 1.4(a), and then using the divergence theorem, we obtain:

$$\int_\tau \overrightarrow{\nabla} \cdot \langle \overrightarrow{\mathbf{\Pi}} \rangle d\tau = \oint_S \langle \overrightarrow{\mathbf{\Pi}} \rangle \cdot d\overrightarrow{\mathbf{S}} = \int_{z+\Delta z} \langle \Pi_z \left(z + \Delta z\right) \rangle dS - \int_z \langle \Pi_z \left(z\right) \rangle dS \quad [1.105]$$

Since $\langle P\left(z\right) \rangle = \int_z \langle \Pi_z \left(z\right) \rangle dS$ (the total power passing through a cross-section of the guide located at position z), equation [1.105] becomes:

$$\langle P\left(z + \Delta z\right) \rangle - \langle P\left(z\right) \rangle = -\int_\tau \langle p_d \rangle dxdydz \qquad [1.106]$$

Dividing by Δz and then taking the limit as $\Delta z \longrightarrow 0$, we obtain:

$$\lim_{\Delta z \longrightarrow 0} \frac{\langle P\left(z + \Delta z\right) \rangle - \langle P\left(z\right) \rangle}{\Delta z} = \frac{d\langle P \rangle}{dz} = -\int_\tau \langle p_d \rangle dxdy = -\langle P_{dL} \rangle \qquad [1.107]$$

where $\langle P_{dL} \rangle$ is the power dissipated per unit length. Since the fields vary as $e^{-\alpha z}$ (where α is called the *attenuation coefficient* along the waveguide), the power flux, which is proportional to the square of the fields, must vary as $e^{-2\alpha z}$; this implies that:

$$\frac{d\langle P \rangle}{dz} = -2\alpha \langle P \rangle = -\langle P_{dL} \rangle \qquad [1.108]$$

We see that α is proportional to the ratio between the power dissipated per unit length and the total electromagnetic power propagating inside the waveguide:

$$\alpha = -\frac{1}{2\langle P \rangle} \frac{d\langle P \rangle}{dz} = \frac{1}{2} \frac{\langle P_{dL} \rangle}{\langle P \rangle} \qquad [1.109]$$

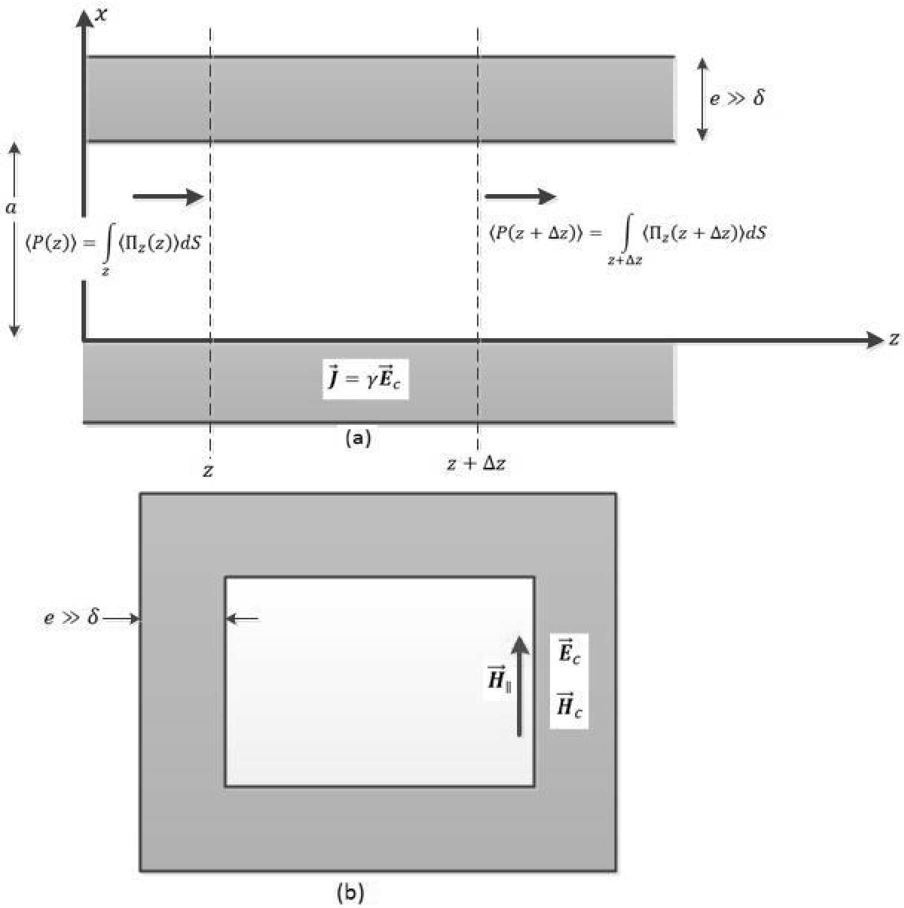

Figure 1.4. *Attenuation of a guided wave in a rectangular lossy waveguide*

Calculate $\langle P_{dL}\rangle$ for a rectangular waveguide with wall thickness e, as illustrated in Figure 1.4(b). We assume that $e \gg \delta$. The surface integral $\int_z \langle P_{dL}\rangle dxdy$ can be divided into four distinct integrals, one for each side.

Since the skin depth is much smaller than the dimensions of the waveguide, the contribution from the corners is neglected. Examine the contribution to the dissipated power of the wall at $x = a$. Inside this wall, the magnetic excitation field $\overrightarrow{\mathbf{H}}_c$ takes the form $\overrightarrow{\mathbf{H}}_c = \overrightarrow{\mathbf{H}}_\parallel (x = a)\, e^{-(x-a)/\delta}$. Consequently, the power dissipated at this wall $(x = a)$, assuming $e \gg \delta$, is given by:

$$\langle P_{dL}\rangle|_{x=a} = \frac{\mu_0\omega}{2}\int_{y=0}^{b}\int_{x=a}^{a+e}\left|\overrightarrow{\mathbf{H}}_{\parallel}(x=a)\right|^2 e^{-2(x-a)/\delta}dxdy \qquad [1.110]$$

$$\simeq \frac{\mu_0\omega}{2}\int_{y=0}^{b}\int_{x=a}^{\infty}\left|\overrightarrow{\mathbf{H}}_{\parallel}(x=a)\right|^2 e^{-2(x-a)/\delta}dxdy = \frac{\mu_0\omega\delta}{4}\int_{y=0}^{b}\left|\overrightarrow{\mathbf{H}}_{\parallel}(x=a)\right|^2 dy$$

Each of the four sides contributes similarly to [1.110]. Consequently, the total (time-averaged) power dissipated per unit length is:

$$\langle P_{dL}\rangle = \frac{\mu_0\omega\delta}{4}\oint\left|\overrightarrow{\mathbf{H}}_{\parallel}\right|^2 dl \qquad [1.111]$$

where the integral is taken around the perimeter of the waveguide. The field $\overrightarrow{\mathbf{H}}_{\parallel}$ can be replaced in the previous equation by the surface current $\overrightarrow{\mathbf{K}}$ of the perfect conductor using the relation $\left|\overrightarrow{\mathbf{K}}\right|^2 = \left|\overrightarrow{\mathbf{n}}\times\overrightarrow{\mathbf{H}}_{\parallel}\right|^2 = \left|\overrightarrow{\mathbf{H}}_{\parallel}\right|^2$. By making this substitution, we obtain:

$$\langle P_{dL}\rangle = \frac{\mu_0\omega\delta}{4}\oint\left|\overrightarrow{\mathbf{K}}\right|^2 dl \qquad [1.112]$$

This result has a simple interpretation: assuming that the surface current $\overrightarrow{\mathbf{K}}$ is uniformly distributed over a strip of thickness δ (and zero elsewhere), we write $\overrightarrow{\mathbf{J}} = \frac{\overrightarrow{\mathbf{K}}}{\delta}$. Thus, the total power dissipated per unit length can then be calculated using a formula such as the following:

$$\langle P_{dL}\rangle = \frac{1}{2}\oint\overrightarrow{\mathbf{E}}\cdot\overrightarrow{\mathbf{J}}^{\star}\delta dl = \frac{1}{2\gamma_c}\oint\left|\overrightarrow{\mathbf{J}}\right|^2\delta dl = \frac{1}{2\gamma_c\delta}\oint\left|\overrightarrow{\mathbf{K}}\right|^2 dl \qquad [1.113]$$

which is consistent with the result [1.112]. The above results also apply to circular waveguides or waveguides of any cross-section. $R_S = \frac{1}{\gamma_c\delta}$ is called *surface resistance*. Here are the main results:

– *waves TM$_{mn}$*: by substituting results [1.59] into [1.113], we obtain:

$$\begin{aligned}\langle P_{dL}\rangle &= \frac{1}{\gamma_c\delta}\int_0^b\frac{\omega^2\mathsf{k}_x^2\varepsilon_0^2\left|E_{mn}\right|^2}{\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)^2}\sin^2(\mathsf{k}_y y)\,dy \\ &\quad+\frac{1}{\gamma_c\delta}\int_0^a\frac{\omega^2\mathsf{k}_y^2\varepsilon_0^2\left|E_{mn}\right|^2}{\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)^2}\sin^2(\mathsf{k}_x x)\,dx \\ &= \frac{\left|E_{mn}\right|^2}{2\gamma_c\delta}\frac{\omega^2\varepsilon_0^2\left(\mathsf{k}_x^2 b+\mathsf{k}_y^2 a\right)}{\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)^2}\end{aligned} \qquad [1.114]$$

where we have taken into account the identities [1.68]. Consequently, the attenuation coefficient of the TM_{mn} wave is equal to:

$$\alpha = \frac{\langle P_{dL}\rangle}{2\langle P\rangle} = \frac{2\varepsilon_0\omega}{\gamma_c\delta \mathsf{k}_z ab}\left(\frac{m^2b^3+n^2a^3}{m^2b^2+n^2a^2}\right) \tag{1.115}$$

– *waves TE*$_{mn}$ $(m, n \neq 0)$: taking into account the results [1.88]–[1.91], we find:

$$\begin{aligned}\langle P_{dL}\rangle &= \frac{1}{\gamma_c\delta}\int_0^b\left\{\frac{\mathsf{k}_y^2\mathsf{k}_z^2\left|H_{mn}\right|^2}{\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)^2}\sin^2\left(\mathsf{k}_y y\right)+\left|H_{mn}\right|^2\cos^2\left(\mathsf{k}_y y\right)\right\}dy + \\ &\quad \frac{1}{\gamma_c\delta}\int_0^a\left\{\frac{\mathsf{k}_x^2\mathsf{k}_z^2\left|H_{mn}\right|^2}{\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)^2}\sin^2\left(\mathsf{k}_x x\right)+\left|H_{mn}\right|^2\cos^2\left(\mathsf{k}_x x\right)\right\}dx \\ &= \frac{\left|H_{mn}\right|^2}{2\gamma_c\delta}\left[\frac{\mathsf{k}_z^2\left(\mathsf{k}_x^2 a+\mathsf{k}_y^2 b\right)}{\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)^2}+a+b\right]\end{aligned} \tag{1.116}$$

and consequently:

$$\alpha = \frac{\langle P_{dL}\rangle}{2\langle P\rangle} = \frac{2}{\gamma_c\delta\mu_0\omega\mathsf{k}_z ab}\left[\frac{\mathsf{k}_z^2\left(m^2b+n^2a\right)ab}{m^2b^2+n^2a^2}+\pi^2\frac{(a+b)\left(m^2b^2+n^2a^2\right)}{a^2b^2}\right] \tag{1.117}$$

– *waves TE*$_{m0}$ $(m \neq 0)$:

$$\begin{aligned}\langle P_{dL}\rangle &= \frac{1}{\gamma_c\delta}\left[\int_0^b\left|H_{m0}\right|^2 dy+\int_0^a\left\{\frac{\mathsf{k}_z^2\left|H_{m0}\right|^2}{\mathsf{k}_x^2}\sin^2\left(\mathsf{k}_x x\right)\right.\right. \\ &\quad \left.\left.+\left|H_{m0}\right|^2\cos^2\left(\mathsf{k}_x x\right)\right\}dx\right] = \frac{\left|H_{m0}\right|^2}{\gamma_c\delta}\left[b+\left(\frac{\omega}{c\mathsf{k}_x}\right)^2\frac{a}{2}\right]\end{aligned} \tag{1.118}$$

$$\alpha = \frac{\langle P_{dL}\rangle}{2\langle P\rangle} = \frac{2\left(\frac{m\pi}{a}\right)^2\left[b+\left(\frac{\omega a}{m\pi c}\right)^2\frac{a}{2}\right]}{\gamma_c\delta\mu_0\omega\mathsf{k}_z ab} \tag{1.119}$$

– *waves TE*$_{0n}$ $(n \neq 0)$:

$$\langle P_{dL}\rangle = \frac{\left|H_{0n}\right|^2}{\gamma_c\delta}\left[a+\left(\frac{\omega}{c\mathsf{k}_y}\right)^2\frac{b}{2}\right] \tag{1.120}$$

$$\alpha = \frac{\langle P_{dL}\rangle}{2\langle P\rangle} = \frac{2\left(\frac{n\pi}{b}\right)^2\left[a+\left(\frac{\omega b}{n\pi c}\right)^2\frac{b}{2}\right]}{\gamma_c\delta\mu_0\omega\mathsf{k}_z ab} \tag{1.121}$$

1.3.3. *Propagation between two parallel planes*

Consider two parallel plate electrodes (perfect conductors) separated by a small distance b and enclosing an HLI medium with permittivity ε_0 and permeability μ_0 (vacuum for simplicity), as illustrated in Figure 1.5(a). Since this spacing is much smaller than the width l and length L of the electrodes, the edge effects are neglected. In other words, the two conducting planes can be considered infinite. To obtain solutions for the guided wave propagation solutions of TM and TE waves between these two planes (taken parallel to the Oxz plane), it suffices to take $a \longrightarrow \infty$ from the previous results.

Since the system here is invariant under translation along the Ox direction, it follows that the components E_z (for the TM wave) or H_z (for the TE wave) must be independent of x. Consequently, all other components determined from [1.19] and [1.21] must also be independent of x. As this system contains two conductors, it can also support a TEM mode. This topic will be addressed in the following sections.

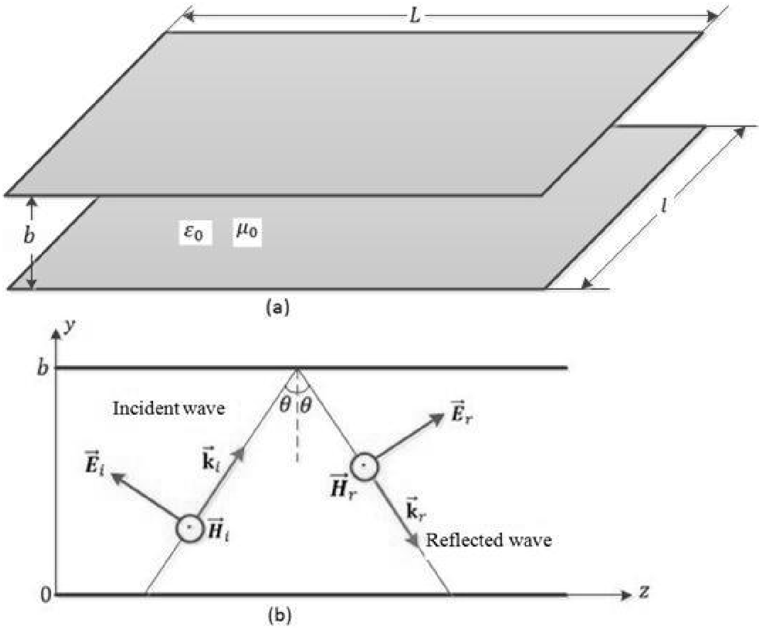

Figure 1.5. *Propagation of an EM wave between two infinite planes. For a color version of this figure, see www.iste.co.uk/abdesselam/electrodynamics3.zip*

1.3.3.1. *waves TM ($H_z = 0$, $E_z \neq$)*

To obtain a non-trivial solution for E_z when taking the limit $a \longrightarrow \infty$ in expression [1.47], we simply perform the substitution $E_{mn} \sin\left(\frac{m\pi x}{a}\right) \longrightarrow E_n$, which gives:

$$E_z = \Re\left[E_n \sin\left(\mathsf{k}_y y\right) e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \qquad [1.122]$$

where n is an integer, $sfk_y = \frac{n\pi}{b}$ and $sfk_z = \sqrt{\frac{\omega^2}{c^2} - \frac{n^2\pi^2}{b^2}}$. In fact, [1.122] is the solution of the Helmholtz equation $\left(\frac{\partial^2}{\partial y^2} + \gamma^2\right) E_z = 0$, with boundary conditions $E_z\left(y = 0\right) = E_z\left(y = b\right) = 0$. We can see that $\omega_{c,n} = \frac{cn\pi}{b}$ appears here as the cut-off frequency for the n mode. If $\omega > \omega_{c,n}$, we have an unattenuated propagation. On the other hand, if $\omega < \omega_{c,n}$, we have an evanescent wave. For a given ω, all modes with a lower frequency cutoff will pass, but not the others. This is a high-pass filter.

The other components of the fields $\overrightarrow{\mathbf{E}}_\perp$ and $\overrightarrow{\mathbf{H}}_\perp$ are obtained from [1.19] and [1.21]:

$$\begin{aligned} &E_x = H_y = 0 \\ &E_y = \Re\left[\frac{\mathrm{i}\mathsf{k}_z}{\mathsf{k}_y} E_n \cos\left(\mathsf{k}_y y\right) e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\ &H_x = \Re\left[-\frac{\mathrm{i}\omega}{\mu_0 c^2 \mathsf{k}_y} E_n \cos\left(\mathsf{k}_y y\right) e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \end{aligned} \qquad [1.123]$$

This wave appears as the result of two waves originating from reflections on the planes $y = 0$ and $y = b$, as illustrated in Figure 1.5(b). In fact, let us define $\mathsf{k}_y = \mathsf{k}\cos\theta = \frac{\omega}{c}\cos\theta$, $\mathsf{k}_z = \mathsf{k}\cos\theta = \frac{\omega}{c}\sin\theta$, $\overrightarrow{\mathsf{k}}_i = \mathsf{k}_y \overrightarrow{\mathbf{j}} + \mathsf{k}_z \overrightarrow{\mathbf{k}}$, $\overrightarrow{\mathsf{k}}_r = -\mathsf{k}_y \overrightarrow{\mathbf{j}} + \mathsf{k}_z \overrightarrow{\mathbf{k}}$, $E_n = \frac{2}{i}\cos\theta E_i$, $E_r = E_i$:

$$\begin{aligned} &E_x = H_y = H_z = 0 \\ &E_y = \Re\left[E_i \sin\theta e^{-\mathrm{i}\left(\omega t - \overrightarrow{\mathsf{k}}_i \cdot \overrightarrow{\mathbf{r}}\right)}\right] + \Re\left[E_r \sin\theta e^{-\mathrm{i}\left(\omega t - \overrightarrow{\mathsf{k}}_r \cdot \overrightarrow{\mathbf{r}}\right)}\right] \\ &E_z = \Re\left[-E_i \cos\theta e^{-\mathrm{i}\left(\omega t - \overrightarrow{\mathsf{k}}_i \cdot \overrightarrow{\mathbf{r}}\right)}\right] + \Re\left[E_r \cos\theta e^{-\mathrm{i}\left(\omega t - \overrightarrow{\mathsf{k}}_r \cdot \overrightarrow{\mathbf{r}}\right)}\right] \\ &H_x = \Re\left[-\sqrt{\frac{\varepsilon_0}{\mu_0}} E_i e^{-\mathrm{i}\left(\omega t - \overrightarrow{\mathsf{k}}_i \cdot \overrightarrow{\mathbf{r}}\right)}\right] + \Re\left[-\sqrt{\frac{\varepsilon_0}{\mu_0}} E_r e^{-\mathrm{i}\left(\omega t - \overrightarrow{\mathsf{k}}_r \cdot \overrightarrow{\mathbf{r}}\right)}\right] \end{aligned} \qquad [1.124]$$

The first terms on the right-hand side of equations [1.124] represent the field components of the incident wave, while the second terms correspond to those of the reflected wave. The boundary conditions are satisfied at the planes $y = 0$ and $y = b$.

$$\begin{cases} E_z\left(y = 0\right) = 0 \Longrightarrow E_i = E_r \\ E_z\left(y = b\right) = 0 \Longrightarrow E_i e^{\mathrm{i}\mathsf{k}_y b} = E_r e^{-\mathrm{i}\mathsf{k}_y b} \Longrightarrow e^{2\mathrm{i}\mathsf{k}_y b} = 1 \Longrightarrow \mathsf{k}_y = \dfrac{n\pi}{b} \end{cases} \qquad [1.125]$$

The results are consistent with those previously obtained. The angle of incidence is therefore indeed quantized.

$$\cos\theta = \frac{cn\pi}{\omega b} \qquad [1.126]$$

We therefore conclude that only discrete values of the angle of incidence on the walls enable destructive interference on the boundaries. The following are the main results:

1) The surface charges and currents observed, respectively, on the planes $y = 0$ and $y = b$:

$$\begin{aligned}
&\sigma(y=0) = \varepsilon_0 E_y(y=0) = \Re\left[\frac{\mathsf{i}\mathsf{k}_z\varepsilon_0}{\mathsf{k}_y} E_n e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] \\
&K_z(y=0) = -H_x(y=0) = \Re\left[\frac{\mathsf{i}\omega\varepsilon_0}{\mathsf{k}_y} E_n e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] = \frac{\omega}{\mathsf{k}_z}\sigma(y=0) \\
&\sigma(y=b) = -\varepsilon_0 E_y(y=b) = \Re\left[-\frac{\mathsf{i}\mathsf{k}_z\varepsilon_0(-1)^n}{\mathsf{k}_y} E_n e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] \\
&K_z(y=b) = H_x(y=b) = \Re\left[-\frac{\mathsf{i}\omega\varepsilon_0(-1)^n}{\mathsf{k}_y} E_n e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] = \frac{\omega}{\mathsf{k}_z}\sigma(y=b)
\end{aligned} \qquad [1.127]$$

2) The time-averaged complex Poynting vector:

$$\langle\overrightarrow{\mathbf{\Pi}}\rangle = \frac{1}{2}\Re\left[E_z H_x^\star\right]\overrightarrow{\mathbf{j}} - \frac{1}{2}\Re\left[E_y H_x^\star\right]\overrightarrow{\mathbf{k}} = \frac{\mathsf{k}_z\varepsilon_0\omega\left|E_n\right|^2}{2\mathsf{k}_y^2}\cos^2(\mathsf{k}_y y)\,\overrightarrow{\mathbf{k}} \qquad [1.128]$$

3) The average power transmitted between the two planes:

$$\langle P\rangle = \int_{x=0}^{l}\int_{y=0}^{b}\langle\Pi_z\rangle dx dy = \frac{\mathsf{k}_z\varepsilon_0\omega\left|E_n\right|^2 lb}{4\mathsf{k}_y^2} \qquad [1.129]$$

4) The average power dissipated per unit length for real conductors with high conductivity:

$$\langle P_{dL}\rangle = \frac{1}{2\gamma_c\delta}\oint\left|\overrightarrow{\mathbf{K}}\right|^2 dl = \frac{\varepsilon_0^2\omega^2\left|E_n\right|^2 l}{\gamma_c\delta\mathsf{k}_y^2} \qquad [1.130]$$

5) The attenuation coefficient of the TM modes:

$$\alpha = \frac{\langle P_{dL}\rangle}{2\langle P\rangle} = \frac{2\varepsilon_0\omega}{\gamma_c\delta\mathsf{k}_z b} \qquad [1.131]$$

1.3.3.2. *Waves TE ($E_z = 0$, $H_z \neq 0$)*

For the TE modes, we must make the substitution $H_{mn} \cos(\mathsf{k}_x x) \longrightarrow H_n$ in [1.82], then calculate the other components using equations [1.19] and [1.21]:

$$\begin{aligned} H_y &= \Re\left[-\frac{\mathsf{i}\mathsf{k}_z}{\mathsf{k}_y} H_n \sin(\mathsf{k}_y y)\, e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] \\ H_z &= \Re\left[H_n \cos(\mathsf{k}_y y)\, e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] \\ H_x &= E_y = E_z = 0 \\ E_x &= \Re\left[-\frac{\mathsf{i}\omega\mu_0}{\mathsf{k}_y} H_n \sin(\mathsf{k}_y y)\, e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] \end{aligned} \quad [1.132]$$

The following are the essential results:

1) The surface charges and surface currents experienced, respectively, on the planes $y = 0$ and $y = b$:

$$\begin{aligned} &\sigma(y=0) = \sigma(y=b) = 0 \\ &K_z(y=0) = K_z(y=b) = 0 \\ &K_x(y=0) = H_z(y=0) = \Re\left[H_n e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] \\ &K_x(y=b) = H_z(y=b) = \Re\left[(-1)^n H_n e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] \end{aligned} \quad [1.133]$$

2) The time-average of the complex Poynting vector:

$$\langle \overrightarrow{\mathbf{\Pi}} \rangle = -\frac{1}{2}\Re\left[E_x H_z^\star\right] \overrightarrow{\mathbf{j}} + \frac{1}{2}\Re\left[E_x H_y^\star\right] \overrightarrow{\mathbf{k}} = \frac{\mathsf{k}_z \mu_0 \omega \left|H_n\right|^2}{2\mathsf{k}_y^2} \sin^2(\mathsf{k}_y y)\, \overrightarrow{\mathbf{k}} \quad [1.134]$$

3) The average power flowing between the two planes:

$$\langle P \rangle = \int_{x=0}^{l} \int_{y=0}^{b} \langle \Pi_z \rangle dx dy = \frac{\mathsf{k}_z \mu_0 \omega \left|H_n\right|^2 lb}{4\mathsf{k}_y^2} \quad [1.135]$$

4) The average power dissipated per unit length for real conductors with high conductivity:

$$\langle P_{dL} \rangle = \frac{1}{2\gamma_c \delta} \oint \left|\overrightarrow{\mathbf{K}}\right|^2 dl = \frac{\left|H_n\right|^2 l}{\gamma \delta} \quad [1.136]$$

5) The attenuation coefficient of the TM modes:

$$\alpha = \frac{\langle P_{dL} \rangle}{2\langle P \rangle} = \frac{2\mathsf{k}_y^2}{\gamma_c \delta \mathsf{k}_z \mu_0 \omega b} \quad [1.137]$$

1.3.4. *Cylindrical waveguide*

Consider a hollow, perfectly conductive waveguide with a circular cross-section of radius of a. It is assumed that inside the waveguide $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$. The position of a point inside the waveguide is identified using cylindrical coordinates (r, φ, z).

1.3.4.1. *TM modes ($H_z = 0$, $E_z \neq 0$)*

Let us first study the TM modes. The Helmholtz equation is written as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + \gamma^2 E_z = 0 \qquad [1.138]$$

where $\gamma^2 = \frac{\omega^2}{c^2} - \mathrm{k}_z^2$. The boundary condition is given as:

$$E_z\left(r = a, \varphi\right) = 0 \qquad [1.139]$$

The solutions of [1.138] are obtained using the method of separation of variables and involve Bessel functions. By setting $E_z\left(r, \varphi\right) = R\left(r\right)\Phi\left(\varphi\right)$, equation [1.138] becomes $\Phi\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{R}{r^2}\frac{d^2\Phi}{d\varphi^2} + \gamma^2 R\Phi = 0$, or, equivalently, $\frac{r^2}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \gamma^2 r^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}$. The first term depends on r, while the second depends on φ. This equality can only hold if both sides are constant. Thus, we find:

$$\frac{d^2\Phi}{d\varphi^2} + \nu^2\Phi = 0 \qquad [1.140]$$

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \gamma^2 r^2 = \nu^2 \qquad [1.141]$$

The solution of [1.140] is written as $\Phi\left(\varphi\right) = C_0 e^{\mathrm{i}\nu\varphi}$, where C_0 is a constant. Φ must also be periodic, that is, $\Phi\left(\varphi + 2\pi\right) = \Phi\left(\varphi\right)$, which gives $e^{2\pi\mathrm{i}\nu} = 1$. We therefore conclude that ν must be an integer. We set $\nu = m$ with $m = 0, \pm 1, \pm 2, \ldots$. Substituting $\nu = m$ into equation [1.141], we obtain $\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\gamma^2 - \frac{m^2}{r^2}\right)R = 0$. The general solutions are of the form $R_m\left(r\right) = A_m J_m\left(\gamma r\right) + B_m N_m\left(\gamma r\right)$, where J_m and N_m are Bessel functions of the first and second kind, respectively. Since N_m is singular near $r = 0$, we must take $B_m = 0$. Therefore, the physically acceptable solution is $R_m\left(r\right) = A_m J_m\left(\gamma r\right)$. Hence, the Helmholtz equation [1.138] provides the following solution:

$$\hat{E}_z\left(r, \varphi\right) = \Re\left[A_m J_m\left(\gamma r\right) e^{\pm \mathrm{i} m\varphi}\right] \qquad [1.142]$$

The boundary condition [1.139] then amounts to $J_m\left(\gamma a\right) = 0$, which gives $\gamma = \frac{\rho_{mn}}{a}$, where ρ_{mn} are the roots of the Bessel function J_m, that is, $J_m\left(\rho_{mn}\right) = 0$.

We therefore conclude that each root ρ_{mn} of the Bessel functions corresponds to a propagation mode. The dispersion relation for this mode is written as:

$$\omega^2 = \omega_{mn}^2 + c^2 \mathsf{k}_z^2 = \frac{c^2 \rho_{mn}^2}{a^2} + c^2 \mathsf{k}_z^2 \, (m \geq 0, n \geq 1) \quad [1.143]$$

It therefore follows that:

$$E_z(r, \varphi, z) = \Re\left[E_{mn} J_m\left(\frac{\rho_{mn} r}{a}\right) e^{\pm \mathrm{i} m \varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \quad [1.144]$$

where E_{mn} is a complex amplitude. The other components of $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ are calculated from relations [1.19] and [1.21]:

$$\begin{aligned}
E_r &= \Re\left[\frac{\mathrm{i}\mathsf{k}_z a^2}{\rho_{mn}^2} \frac{\partial E_z}{\partial r}\right] = \Re\left[\frac{\mathrm{i}\mathsf{k}_z a E_{mn}}{\rho_{mn}} J_m'\left(\frac{\rho_{mn} r}{a}\right) e^{\pm \mathrm{i} m \varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
E_\varphi &= \Re\left[\frac{\mathrm{i}\mathsf{k}_z a^2}{\rho_{mn}^2} \frac{1}{r} \frac{\partial E_z}{\partial \varphi}\right] = \Re\left[\mp \frac{m \mathsf{k}_z a^2 E_{mn}}{\rho_{mn}^2 r} J_m\left(\frac{\rho_{mn} r}{a}\right) e^{\pm \mathrm{i} m \varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
H_r &= \Re\left[-\frac{\mathrm{i}\omega \varepsilon_0 a^2}{\rho_{mn}^2} \frac{1}{r} \frac{\partial E_z}{\partial \varphi}\right] = \Re\left[\pm \frac{m \omega a^2 \varepsilon_0 E_{mn}}{\rho_{mn}^2 r} J_m\left(\frac{\rho_{mn} r}{a}\right) e^{\pm \mathrm{i} m \varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
H_\varphi &= \Re\left[\frac{\mathrm{i}\omega \varepsilon_0 a^2}{\rho_{mn}^2} \frac{\partial E_z}{\partial r}\right] = \Re\left[\frac{\mathrm{i}\omega a \varepsilon_0 E_{mn}}{\rho_{mn}} J_m'\left(\frac{\rho_{mn} r}{a}\right) e^{\pm \mathrm{i} m \varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right]
\end{aligned} \quad [1.145]$$

We observe the following:

1) When ω becomes less than $\frac{c\rho_{mn}}{a}$, k_z becomes imaginary and the wave no longer propagates.

2) A double degeneracy for each mode when $m > 0$. We can consider modes in $\cos m\varphi$ or modes in $\sin m\varphi$, depending on the nature of the amplitude E_{mn}.

3) The charge accumulated on the surface of the waveguide is:

$$\begin{aligned}
\sigma(r = a) &= -\varepsilon_0 E_r(r = a) \\
&= \Re\left[-\frac{\mathrm{i}\mathsf{k}_z \varepsilon_0 a E_{mn}}{\rho_{mn}} J_m'(\rho_{mn}) e^{\pm \mathrm{i} m \varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right]
\end{aligned} \quad [1.146]$$

4) The current flowing on the surface of the waveguide is:

$$\begin{aligned}
K_z(r = a) &= -H_\varphi(r = a) = \Re\left[-\frac{\mathrm{i}\omega a \varepsilon_0 E_{mn}}{\rho_{mn}} J_m'(\rho_{mn}) e^{\pm \mathrm{i} m \varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
&= \frac{\omega}{\mathsf{k}_z} \sigma(r = a)
\end{aligned} \quad [1.147]$$

5) The time-averaged Poynting vector is given by:

$$\begin{aligned}\langle\overrightarrow{\mathbf{\Pi}}\rangle &= \frac{1}{2}\Re\left[\overrightarrow{\mathbf{E}}\times\overrightarrow{\mathbf{H}}^{\star}\right]\\ &= \frac{1}{2}\Re\left[-E_zH_\varphi^\star\overrightarrow{\mathbf{e}}_r + E_zH_r^\star\overrightarrow{\mathbf{e}}_\varphi + \left(E_rH_\varphi^\star - E_\varphi H_r^\star\right)\overrightarrow{\mathbf{k}}\right]\end{aligned} \qquad [1.148]$$

The term $\Re\left[E_zH_\varphi^\star\right]$ is zero: no energy propagates in the transverse $\overrightarrow{\mathbf{e}}_r$ direction. In the $\overrightarrow{\mathbf{e}}_\varphi$ and $\overrightarrow{\mathbf{k}}$ directions, we have:

$$\begin{aligned}\langle\Pi_\varphi\rangle &= \pm\frac{m\varepsilon_0\omega a^2\left|E_{mn}\right|^2 J_m^2\left(\frac{\rho_{mn}r}{a}\right)}{2\rho_{mn}^2 r}\\ \langle\Pi_z\rangle &= \frac{\mathsf{k}_z\varepsilon_0\omega a^2\left|E_{mn}\right|^2}{2\rho_{mn}^2}\left[J_m'^2\left(\frac{\rho_{mn}r}{a}\right) + \frac{m^2a^2}{\rho_{mn}^2r^2}J_m^2\left(\frac{\rho_{mn}r}{a}\right)\right]\end{aligned} \qquad [1.149]$$

In a circular waveguide, energy propagates in the longitudinal direction and in the azimuthal direction (in a loop).

6) For the TM_{01} mode, we have $\rho_{01} \simeq 2.405$. Using the identity $J_0' = -J_1$, the field components are written as:

$$\begin{aligned}E_r &= \Re\left[-\frac{\mathsf{i}\mathsf{k}_z aE_{01}}{\rho_{01}}J_1\left(\frac{\rho_{01}r}{a}\right)e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]\\ E_z &= \Re\left[E_{01}J_0\left(\frac{\rho_{01}r}{a}\right)e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]\\ E_\varphi &= H_r = H_z = 0\\ H_\varphi &= \Re\left[-\frac{\mathsf{i}\omega a\varepsilon_0E_{01}}{\rho_{01}}J_1\left(\frac{\rho_{01}r}{a}\right)e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]\end{aligned} \qquad [1.150]$$

and consequently:

$$\begin{aligned}\langle\Pi_\varphi\rangle &= 0\\ \langle\Pi_z\rangle &= \frac{\mathsf{k}_z\varepsilon_0\omega a^2\left|E_{01}\right|^2 J_0'^2\left(\frac{\rho_{01}r}{a}\right)}{2\rho_{01}^2}\end{aligned} \qquad [1.151]$$

For this mode, $\omega_c = \frac{c\rho_{01}}{a}$. The cutoff wavelength is equal to $\lambda_c = \frac{2\pi c}{\omega_c} = \frac{2\pi a}{\rho_{01}} \simeq 2.61a$.

1.3.4.2. *TE modes ($E_z = 0$, $H_z \neq 0$)*

For TE modes, the Helmholtz equation for H_z is written in the same way as before. In this case, we have $E_z = 0$ and $H_z = \Re\left[H_0J_m(\gamma r)\,e^{\pm\mathsf{i}m\varphi}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right]$ with the boundary condition $-\frac{\partial H_z}{\partial r}_{r=a} = 0$, which implies that $J_m'(\gamma a) = 0$. Therefore, γa

must be a root of J'_m, which we denote here as ρ'_{mn}. The component H_z is then written as:

$$H_z = \Re\left[H_{mn} J_m\left(\frac{\rho'_{mn} r}{a}\right) e^{\pm im\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \qquad [1.152]$$

which leads to the transverse fields:

$$\begin{aligned}
H_r &= \Re\left[\frac{\mathrm{i}\mathsf{k}_z a^2}{\rho'^2_{mn}} \frac{\partial H_z}{\partial r}\right] = \Re\left[\frac{\mathrm{i}\mathsf{k}_z a H_{mn}}{\rho'_{mn}} J'_m\left(\frac{\rho'_{mn} r}{a}\right) e^{\pm \mathrm{i}m\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
H_\varphi &= \Re\left[\frac{\mathrm{i}\mathsf{k}_z a^2}{\rho'^2_{mn}} \frac{1}{r} \frac{\partial H_z}{\partial \varphi}\right] = \Re\left[\mp\frac{m\mathsf{k}_z a^2 H_{mn}}{\rho'^2_{mn} r} J_m\left(\frac{\rho'_{mn} r}{a}\right) e^{\pm \mathrm{i}m\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
E_r &= \Re\left[\frac{\mathrm{i}\omega\mu_0 a^2}{\rho'^2_{mn}} \frac{1}{r} \frac{\partial H_z}{\partial \varphi}\right] = \Re\left[\mp\frac{m\omega a^2 \mu_0 H_{mn}}{\rho'^2_{mn} r} J_m\left(\frac{\rho'_{mn} r}{a}\right) e^{\pm \mathrm{i}m\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
E_\varphi &= \Re\left[-\frac{\mathrm{i}\omega\mu_0 a^2}{\rho'^2_{mn}} \frac{\partial H_z}{\partial r}\right] = \Re\left[-\frac{\mathrm{i}\omega a \mu_0 H_{mn}}{\rho'_{mn}} J'_m\left(\frac{\rho'_{mn} r}{a}\right) e^{\pm \mathrm{i}m\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right]
\end{aligned} \qquad [1.153]$$

The following are some essential remarks:

1) The average energy flux through the waveguide is given as:

$$\begin{aligned}
\langle\vec{\mathbf{\Pi}}\rangle &= \frac{1}{2}\Re\left[E_\varphi H_z^\star \vec{\mathbf{e}}_r - E_r H_z^\star \vec{\mathbf{e}}_\varphi + \left(E_r H_\varphi^\star - E_\varphi H_r^\star\right)\vec{\mathbf{k}}\right] \\
&= \pm\frac{m\omega a^2 \mu_0 \left|H_{mn}\right|^2 J_m^2\left(\frac{\rho'_{mn} r}{a}\right)}{2\rho'^2_{mn} r} \vec{\mathbf{e}}_\varphi + \\
&\quad \frac{\mathsf{k}_z \omega a^2 \mu_0 \left|H_{mn}\right|^2}{2\rho'^2_{mn}}\left[J'^2_m\left(\frac{\rho'_{mn} r}{a}\right) + \frac{m^2 a^2}{\rho'^2_{mn} r^2} J_m^2\left(\frac{\rho'_{mn} r}{a}\right)\right]\vec{\mathbf{k}}
\end{aligned} \qquad [1.154]$$

2) The charge and current experienced at the surface of the waveguide are given by:

$$\begin{aligned}
\sigma(r=a) &= -\varepsilon_0 E_r(r=a) = \Re\left[\pm\frac{m\omega a H_{mn}}{c^2 \rho'^2_{mn}} J_m\left(\rho'_{mn}\right) e^{\pm \mathrm{i}m\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
K_z(r=a) &= -H_\varphi(r=a) = \Re\left[\pm\frac{m\mathsf{k}_z a H_{mn}}{\rho'^2_{mn}} J_m\left(\rho'_{mn}\right) e^{\pm \mathrm{i}m\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
K_\varphi(r=a) &= H_z(r=a) = \Re\left[H_{mn} J_m\left(\rho'_{mn}\right) e^{\pm \mathrm{i}m\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right]
\end{aligned} \qquad [1.155]$$

3) The smallest root among all Bessel functions J_m and their derivatives J'_m is the first root of J'_1; it is approximately $\rho'_{11} \simeq 1.84118$. Therefore, the mode with the lowest cutoff frequency is the TE_{11} mode (dominant mode). For this mode, we have:

$$
\begin{aligned}
E_r &= \Re\left[\mp\frac{m\omega a^2\mu_0 H_{11}}{\rho_{11}'^2 r} J_1\left(\frac{\rho_{11}' r}{a}\right) e^{\pm \mathrm{i}\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
E_\varphi &= \Re\left[-\frac{\mathrm{i}\omega a\mu_0 H_{11}}{\rho_{11}'} J_1'\left(\frac{\rho_{11}' r}{a}\right) e^{\pm \mathrm{i}\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
E_z &= 0 \\
H_r &= \Re\left[\frac{\mathrm{i}\mathsf{k}_z a H_{11}}{\rho_{11}'} J_1'\left(\frac{\rho_{11}' r}{a}\right) e^{\pm \mathrm{i}\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
H_\varphi &= \Re\left[\mp\frac{\mathsf{k}_z a^2 H_{11}}{\rho_{11}'^2 r} J_1\left(\frac{\rho_{11}' r}{a}\right) e^{\pm \mathrm{i}\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] \\
H_z &= \Re\left[H_{11} J_1\left(\frac{\rho_{11}' r}{a}\right) e^{\pm \mathrm{i}\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right]
\end{aligned} \tag{1.156}
$$

For this mode, $\omega_c = \frac{c\rho_{11}'}{a}$. The cutoff wavelength is $\lambda_c = \frac{2\pi c}{\omega_c} = \frac{2\pi a}{\rho_{11}'} \simeq 3.41a$.

4) Since $J_0' = -J_1$, the zeros of J_0' coincide with those of J_1; this implies a degeneracy between the TM_{1n} and TE_{0n} modes.

1.4. Dielectric waveguides

1.4.1. *Propagation between two parallel dielectric planes*

Let us briefly study the propagation modes TE and TM between two identical parallel planes (each with a thickness e, separated by a distance b) composed of a dielectric LIH material with permittivity ε_2 and permeability μ_0. Between the two plates, there is another dielectric medium with permittivity $\varepsilon_1 > \varepsilon_2$. We seek solutions where the fields are essentially confined within medium 1, as illustrated in Figure 1.6.

In other words, the solutions must be exponential inside the plates to prevent energy loss from the guide by radiation. The solutions are oscillatory only in medium 1 to ensure guided propagation.

1.4.1.1. *TM solutions*

The solutions are independent of x. For TM waves propagating in the Oz direction, the component E_z must be of the form:

$$
E_z = \begin{cases} \Re\left[Ae^{-\alpha(y-b)} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[\left(B\sin\left(\mathsf{k}_y y\right) + C\cos\left(\mathsf{k}_y y\right)\right) e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[De^{\alpha y} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases} \tag{1.157}
$$

The solutions decrease exponentially to remain mainly localized in the plates. sfk_y, sfk_z and the attenuation coefficient α obey the following restrictions (obtained from the Helmholtz equation):

$$\begin{aligned} \mathsf{k}_y^2 + \mathsf{k}_z^2 &= \frac{\omega^2 \mathsf{n}_1^2}{c^2} \\ -\alpha^2 + \mathsf{k}_z^2 &= \frac{\omega^2 \mathsf{n}_2^2}{c^2} \end{aligned} \qquad [1.158]$$

where $\varepsilon_{1,2} = \varepsilon_0 \mathsf{n}_{1,2}^2$. The component k_z must be the same everywhere in order to satisfy the boundary conditions at each interface. It must also satisfy the following constraint:

$$\frac{\omega^2 \mathsf{n}_2^2}{c^2} \le \mathsf{k}_z^2 \le \frac{\omega^2 \mathsf{n}_1^2}{c^2} \qquad [1.159]$$

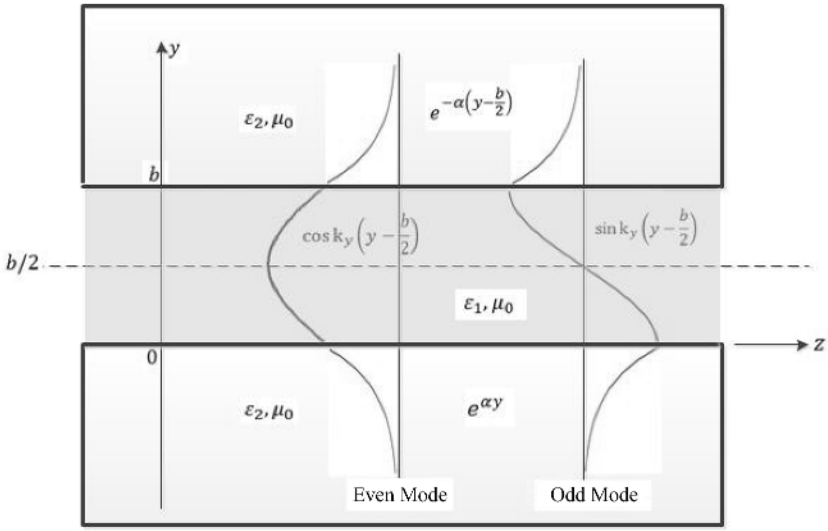

Figure 1.6. *Propagation of TM and TE modes in a dielectric medium with permittivity ε_1 placed between two parallel dielectric plates with permittivity $\varepsilon_2 < \varepsilon_1$*

The other components of the fields $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ can be found from [1.19] and [1.21]:

$$E_x = H_y = 0$$

$$E_y = \begin{cases} \Re\left[\dfrac{\mathsf{ik}_z}{\alpha} A e^{-\alpha(y-b)} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[\dfrac{\mathsf{ik}_z}{\mathsf{k}_y}\left(B\cos\left(\mathsf{k}_y y\right) - C\sin\left(\mathsf{k}_y y\right)\right) e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[-\dfrac{\mathsf{ik}_z}{\alpha} D e^{\alpha y} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases}$$

$$H_x = \begin{cases} \Re\left[-\dfrac{\mathsf{i}\omega\varepsilon_2}{\alpha} A e^{-\alpha(y-b)} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[-\dfrac{\mathsf{i}\omega\varepsilon_1}{\mathsf{k}_y}\left(B\cos\left(\mathsf{k}_y y\right) - C\sin\left(\mathsf{k}_y y\right)\right) e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[\dfrac{\mathsf{i}\omega\varepsilon_2}{\alpha} D e^{\alpha y} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases} \quad [1.160]$$

At the interfaces $y = 0$ and $y = b$, the tangential components of the fields $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ are continuous:

$$\begin{aligned} E_z\left(y = 0^-\right) = E_z\left(y = 0^+\right) &\Longrightarrow C - D = 0 \\ E_z\left(y = b^-\right) = E_z\left(y = b^+\right) &\Longrightarrow A - B\sin\left(\mathsf{k}_y b\right) \\ &\qquad - C\cos\left(\mathsf{k}_y b\right) = 0 \\ H_x\left(y = b^-\right) = H_x\left(y = b^+\right) &\Longrightarrow \frac{\alpha\varepsilon_1}{\mathsf{k}_y\varepsilon_2} B + D = 0 \\ E_z\left(y = b^-\right) = E_z\left(y = b^+\right) &\Longrightarrow A - \frac{\alpha\varepsilon_1}{\mathsf{k}_y\varepsilon_2} B\cos\left(\mathsf{k}_y b\right) \\ &\qquad + \frac{\alpha\varepsilon_1}{\mathsf{k}_y\varepsilon_2} C\sin\left(\mathsf{k}_y b\right) = 0 \end{aligned} \quad [1.161]$$

The system of equations [1.161] admits a non-trivial solution if:

$$\det\begin{vmatrix} 0 & 0 & 1 & -1 \\ 1 & -\sin\left(\mathsf{k}_y b\right) & -\cos\left(\mathsf{k}_y b\right) & 0 \\ 0 & \frac{\alpha\varepsilon_1}{\mathsf{k}_y\varepsilon_2} & 0 & 1 \\ 1 & -\frac{\alpha\varepsilon_1}{\mathsf{k}_y\varepsilon_2}\cos\left(\mathsf{k}_y b\right) & \frac{\alpha\varepsilon_1}{\mathsf{k}_y\varepsilon_2}\sin\left(\mathsf{k}_y b\right) & 0 \end{vmatrix} = 0$$

which gives:

$$\alpha = \begin{cases} \dfrac{\mathsf{k}_y\varepsilon_2}{\varepsilon_1}\tan\dfrac{\mathsf{k}_y b}{2} \\ -\dfrac{\mathsf{k}_y\varepsilon_2}{\varepsilon_1}\dfrac{1}{\tan\frac{\mathsf{k}_y b}{2}} \end{cases} \quad [1.162]$$

To obtain the allowed values of α and k_z, we must solve equations [1.158] and [1.162] simultaneously using graphical methods, as illustrated in Figure 1.7.

The solutions correspond to the intersection points of the curves (the black points). The following can be seen graphically:

1) For a guided wave to occur, it is necessary that $n\pi \leq \frac{\mathsf{k}_y b}{2} < \left(n + \frac{1}{2}\right)\pi$ for the first case, and $\left(n + \frac{1}{2}\right)\pi \leq \frac{\mathsf{k}_y b}{2} < (n+1)\pi$ for the second case.

2) First case: each time that k_y exceeds the critical value $\frac{2n\pi}{b}$, a new intersection point appears corresponding to a new propagation mode. In other words, each time the frequency ω exceeds the critical value.

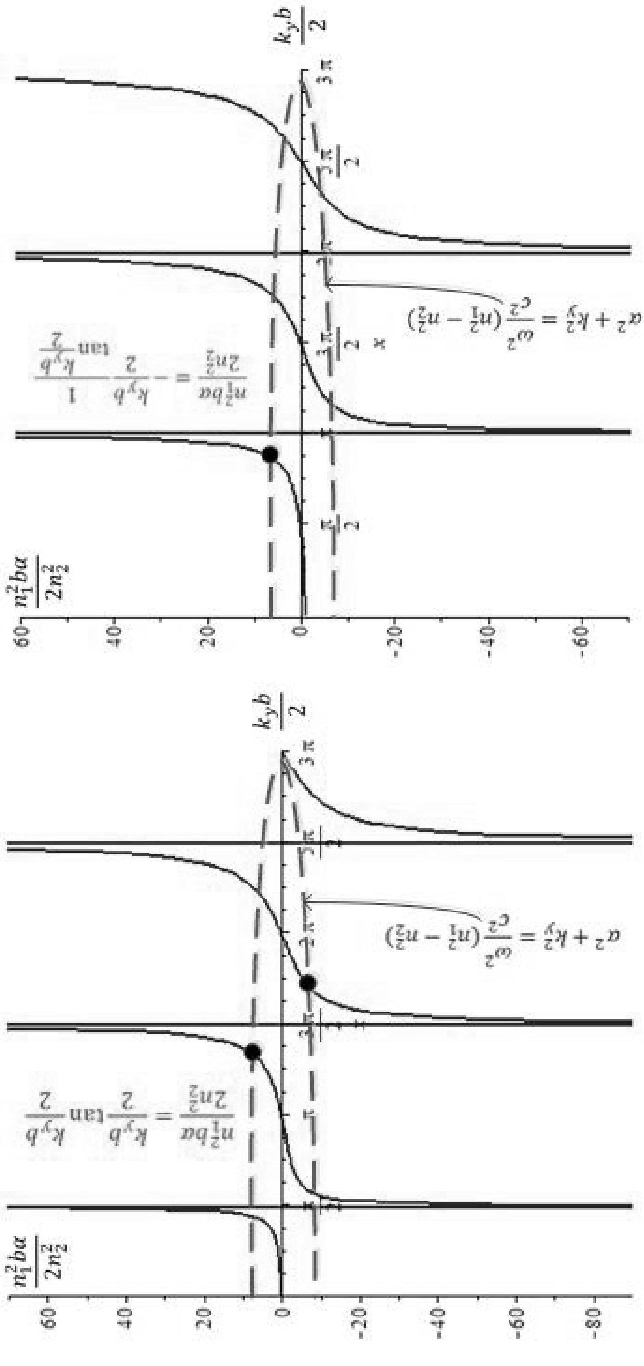

Figure 1.7. *Graphical solution of equations [1.158] and [1.162]. For a color version of this figure, see www.iste.co.uk/abdesselam/electrodynamics3.zip*

$$\omega_n = \frac{2n\pi c}{b\sqrt{\mathsf{n}_1^2 - \mathsf{n}_2^2}} \qquad [1.163]$$

a new propagation mode will appear; ω_n is the *cutoff frequency of this mode*.

3) Second case: each time the frequency ω exceeds the critical value:

$$\omega_n = \frac{(2n+1)\,\pi c}{b\sqrt{\mathsf{n}_1^2 - \mathsf{n}_2^2}} \qquad [1.164]$$

a new propagation mode is added to the others.

4) Odd solution ($\alpha = \frac{\mathsf{k}_y \varepsilon_2}{\varepsilon_1} \tan \frac{\mathsf{k}_y b}{2}$, $B = \frac{A}{\tan \frac{\mathsf{k}_y b}{2}}$, $C = -A$ and $D = -A$):

$$\begin{aligned}
&E_x = H_y = H_z = 0\\
&E_y = \begin{cases} \Re\left[A\dfrac{\mathsf{ik}_z}{\alpha} e^{-\alpha(y-b)} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[A\dfrac{\mathsf{ik}_z}{\mathsf{k}_y}\dfrac{\cos \mathsf{k}_y\left(y - \frac{b}{2}\right)}{\sin \frac{\mathsf{k}_y b}{2}} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[A\dfrac{\mathsf{ik}_z}{\alpha} e^{\alpha y} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases}\\
&E_z = \begin{cases} \Re\left[A e^{-\alpha(y-b)} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[A\dfrac{\sin \mathsf{k}_y\left(y - \frac{b}{2}\right)}{\sin \frac{\mathsf{k}_y b}{2}} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[-A e^{\alpha y} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases}\\
&H_x = \begin{cases} \Re\left[-A\dfrac{\mathsf{i}\omega\varepsilon_2}{\alpha} e^{-\alpha(y-b)} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[-A\dfrac{\mathsf{i}\omega\varepsilon_1}{\mathsf{k}_y}\dfrac{\cos \mathsf{k}_y\left(y - \frac{b}{2}\right)}{\sin \frac{\mathsf{k}_y b}{2}} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[-A\dfrac{\mathsf{i}\omega\varepsilon_2}{\alpha} e^{\alpha y} e^{-\mathsf{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases}
\end{aligned} \qquad [1.165]$$

This solution is called *odd* because E_z is directed oppositely in the two half-planes above and below the line $y = \frac{b}{2}$.

5) Even solution ($\alpha = -\frac{\mathsf{k}_y \varepsilon_2}{\varepsilon_1} \frac{1}{\tan \frac{\mathsf{k}_y b}{2}}$, $B = A \tan \frac{\mathsf{k}_y b}{2}$, $C = A$ and $D = A$):

$$E_x = H_y = H_z = 0$$

$$E_y = \begin{cases} \Re\left[A\dfrac{\mathsf{i}\mathsf{k}_z}{\alpha}e^{-\alpha(y-b)}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y > b \\ \Re\left[-A\dfrac{\mathsf{i}\mathsf{k}_z}{\mathsf{k}_y}\dfrac{\sin \mathsf{k}_y\left(y-\frac{b}{2}\right)}{\cos\frac{\mathsf{k}_y b}{2}}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[-A\dfrac{\mathsf{i}\mathsf{k}_z}{\alpha}e^{\alpha y}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y < 0 \end{cases}$$
$$E_z = \begin{cases} \Re\left[Ae^{-\alpha(y-b)}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y > b \\ \Re\left[A\dfrac{\cos \mathsf{k}_y\left(y-\frac{b}{2}\right)}{\cos\frac{\mathsf{k}_y b}{2}}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[Ae^{\alpha y}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y < 0 \end{cases} \quad [1.166]$$
$$H_x = \begin{cases} \Re\left[-A\dfrac{\mathsf{i}\omega\varepsilon_2}{\alpha}e^{-\alpha(y-b)}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y > b \\ \Re\left[A\dfrac{\mathsf{i}\omega\varepsilon_1}{\mathsf{k}_y}\dfrac{\sin \mathsf{k}_y\left(y-\frac{b}{2}\right)}{\cos\frac{\mathsf{k}_y b}{2}}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[A\dfrac{\mathsf{i}\omega\varepsilon_2}{\alpha}e^{\alpha y}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y < 0 \end{cases}$$

This solution is called *even* because E_z is directed in the same direction in the two half-planes above and below the line $y = \frac{b}{2}$.

1.4.1.2. *TE solutions*

We follow the same procedure for the TE solutions. Let us start with:

$$E_z = E_y = H_x = 0$$
$$H_z = \begin{cases} \Re\left[A'e^{-\alpha(y-b)}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y > b \\ \Re\left[\left(B'\sin\left(\mathsf{k}_y y\right) + C'\cos\left(\mathsf{k}_y y\right)\right)e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[D'e^{\alpha y}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y < 0 \end{cases}$$
$$E_x = \begin{cases} \Re\left[A'\dfrac{\mathsf{i}\omega}{\alpha}e^{-\alpha(y-b)}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y > b \\ \Re\left[\dfrac{\mathsf{i}\omega}{\mathsf{k}_y}\left(B'\cos\left(\mathsf{k}_y y\right) - C'\sin\left(\mathsf{k}_y y\right)\right)e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[-\dfrac{\mathsf{i}\omega}{\alpha}D'e^{\alpha y}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y < 0 \end{cases} \quad [1.167]$$
$$H_y = \begin{cases} \Re\left[A'\dfrac{\mathsf{i}\mathsf{k}_z}{\alpha}e^{-\alpha(y-b)}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y > b \\ \Re\left[\dfrac{\mathsf{i}\mathsf{k}_z}{\mathsf{k}_y}\left(B'\cos\left(\mathsf{k}_y y\right) - C'\sin\left(\mathsf{k}_y y\right)\right)e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[-\dfrac{\mathsf{i}\mathsf{k}_z}{\alpha}D'e^{\alpha y}e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)}\right] & y < 0 \end{cases}$$

where E_x E_y, H_x and H_y are obtained using [1.19] and [1.21]. The components H_z and E_x are continuous at the interfaces $y = 0$ and $y = b$:

$$\begin{cases} H_z\left(y=0^-\right) = H_z\left(y=0^+\right) \Longrightarrow C' - D' = 0 \\ H_z\left(y=b^-\right) = H_z\left(y=b^+\right) \Longrightarrow A' - B'\sin\left(\mathsf{k}_y b\right) - C'\cos\left(\mathsf{k}_y b\right) = 0 \\ E_x\left(y=0^-\right) = E_x\left(y=0^+\right) \Longrightarrow D' + \dfrac{\alpha}{\mathsf{k}_y}B' = 0 \\ E_x\left(y=b^-\right) = E_x\left(y=b^+\right) \Longrightarrow A' - \dfrac{\alpha}{\mathsf{k}_y}B'\cos\left(\mathsf{k}_y b\right) \\ \qquad\qquad + \dfrac{\alpha}{\mathsf{k}_y}C'\sin\left(\mathsf{k}_y b\right) = 0 \end{cases} \qquad [1.168]$$

H_y imposes the same constraints. Equations [1.168] have a solution unless $\alpha = \mathsf{k}_y \tan\frac{\mathsf{k}_y b}{2}$ or $\alpha = -\mathsf{k}_y \frac{1}{\tan\frac{\mathsf{k}_y b}{2}}$.

1) Odd solution $\left(\alpha = \mathsf{k}_y \tan\frac{\mathsf{k}_y b}{2}\right)$ $\left(B' = \frac{A'}{\tan\frac{\mathsf{k}_y b}{2}}, C' = -A', D' = -A'\right)$:

$$E_z = E_y = H_x = 0$$

$$H_z = \begin{cases} \Re\left[A' e^{-\alpha(y-b)} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[A' \dfrac{\sin \mathsf{k}_y\left(y-\frac{b}{2}\right)}{\sin\frac{\mathsf{k}_y b}{2}} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[-A' e^{\alpha y} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases}$$

$$E_x = \begin{cases} \Re\left[A' \dfrac{\mathrm{i}\omega}{\alpha} e^{-\alpha(y-b)} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[A' \dfrac{\mathrm{i}\omega}{\mathsf{k}_y} \dfrac{\cos \mathsf{k}_y\left(y-\frac{b}{2}\right)}{\sin\frac{\mathsf{k}_y b}{2}} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[A' \dfrac{\mathrm{i}\omega}{\alpha} e^{\alpha y} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases} \qquad [1.169]$$

$$H_y = \begin{cases} \Re\left[A' \dfrac{\mathrm{i}\mathsf{k}_z}{\alpha} e^{-\alpha(y-b)} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[A' \dfrac{\mathrm{i}\mathsf{k}_z}{\mathsf{k}_y} \dfrac{\cos \mathsf{k}_y\left(y-\frac{b}{2}\right)}{\sin\frac{\mathsf{k}_y b}{2}} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[A' \dfrac{\mathrm{i}\mathsf{k}_z}{\alpha} e^{\alpha y} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases}$$

2) Even solution $\left(\alpha = -\frac{\mathsf{k}_y}{\tan\frac{\mathsf{k}_y b}{2}}\right)$ $\left(B' = A'\tan\frac{\mathsf{k}_y b}{2}, C' = A', D' = A'\right)$:

$$
\begin{aligned}
&E_z = E_y = H_x = 0\\
&H_z = \begin{cases} \Re\left[A' e^{-\alpha(y-b)} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[A' \dfrac{\cos \mathsf{k}_y\left(y-\frac{b}{2}\right)}{\cos\frac{\mathsf{k}_y b}{2}} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[A' e^{\alpha y} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases}\\
&E_x = \begin{cases} \Re\left[A' \dfrac{\mathrm{i}\omega}{\alpha} e^{-\alpha(y-b)} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[-A' \dfrac{\mathrm{i}\omega}{\mathsf{k}_y} \dfrac{\sin \mathsf{k}_y\left(y-\frac{b}{2}\right)}{\cos\frac{\mathsf{k}_y b}{2}} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[-A' \dfrac{\mathrm{i}\omega}{\alpha} e^{\alpha y} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases}\\
&H_y = \begin{cases} \Re\left[A' \dfrac{\mathrm{i}\mathsf{k}_z}{\alpha} e^{-\alpha(y-b)} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y > b \\ \Re\left[-A' \dfrac{\mathrm{i}\mathsf{k}_z}{\mathsf{k}_y} \dfrac{\sin \mathsf{k}_y\left(y-\frac{b}{2}\right)}{\cos\frac{\mathsf{k}_y b}{2}} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & 0 < y < b \\ \Re\left[-A' \dfrac{\mathrm{i}\mathsf{k}_z}{\alpha} e^{\alpha y} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}\right] & y < 0 \end{cases}
\end{aligned}
\qquad [1.170]
$$

1.4.2. *Step index optical fiber*

In this section, we examine propagation modes in a circular waveguide composed entirely of dielectric materials. Consider an infinite cylinder of radius a made of a dielectric material of dielectric constant ε_1 (of index n_1) and surrounded by another dielectric medium with permittivity $\varepsilon_2 < \varepsilon_1$ (i.e. $\mathsf{n}_2 < \mathsf{n}_1$) that extends over the region $a < r < b$, as illustrated in Figure 1.8. These media are supposed to be non-magnetic $\mu = \mu_0$. This system is the simplest model of an *step index optical fiber*. The first medium (ε_1) is the *core* of the fiber, and the second medium (ε_2) is the *cladding*. Since the fields decrease exponentially in the cladding as a function of r over a short distance, we can assume that the cladding is infinite.

Unlike wave guides that are wrapped in a conductor, we consider the fields both within the core of the waveguide and in the cladding and then apply continuity conditions to the core-cladding interface of: (1) the tangential components of $\overrightarrow{\mathbf{H}}$; (2) the tangential components of $\overrightarrow{\mathbf{E}}$ (since $\overrightarrow{\mathbf{J}}_{\text{free}} = 0$); (3) the normal component of $\overrightarrow{\mathbf{H}}$ ($\overrightarrow{\mathbf{B}}_{\text{normal}} = \mu_0 H_r \overrightarrow{\mathbf{e}}_r$); and (4) the normal component of $\overrightarrow{\mathbf{D}}$ (since $\sigma_{\text{free}} = 0$). These continuity conditions are more complex than for a conducting wall and, therefore, the TM and TE modes generally do not exist separately, but are coupled in what are called *HE and EH modes* (in the first case, H_z is dominant, while in the second case, E_z is dominant). Only modes that are not dependent on φ (modes with azimuthal symmetry) can be separated into TE and TM modes. In each of these media, E_z and H_z satisfy the Helmholtz equation:

$$\begin{aligned}&\left(\overrightarrow{\nabla}_{\perp}^{2}+\left[\frac{\mathsf{n}_1^2\omega^2}{c^2}-\mathsf{k}_z^2\right]\right)\begin{pmatrix}E_z\\H_z\end{pmatrix}=0,\ r<a\\&\left(\overrightarrow{\nabla}_{\perp}^{2}+\left[\frac{\mathsf{n}_2^2\omega^2}{c^2}-\mathsf{k}_z^2\right]\right)\begin{pmatrix}E_z\\H_z\end{pmatrix}=0,\ r>a\end{aligned}\qquad [1.171]$$

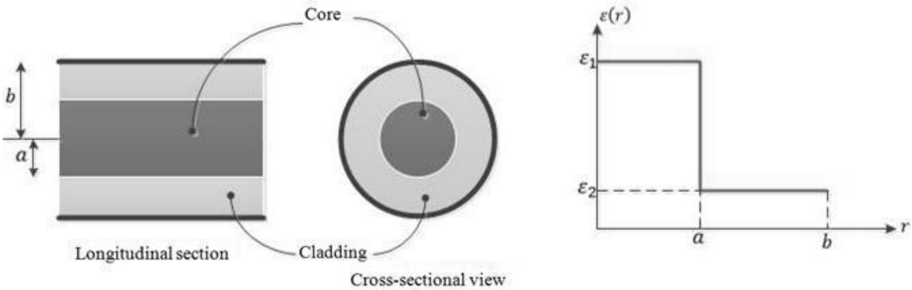

Figure 1.8. *Structure of a step index optical fiber structure*

We define $\gamma^2=\frac{\mathsf{n}_1^2\omega^2}{c^2}-\mathsf{k}_z^2$ and, to avoid an oscillatory solution (and therefore an energy loss from the fiber due to radiation) in the cladding, we take $\beta^2=-\frac{\mathsf{n}_2^2\omega^2}{c^2}+\mathsf{k}_z^2$. From these expressions, we see the necessity of having $\varepsilon_1>\varepsilon_2$ ($\mathsf{n}_1>\mathsf{n}_2$). Moreover, the propagation constant k_z must satisfy the condition $\frac{\mathsf{n}_2^2\omega^2}{c^2}<\mathsf{k}_z^2<\frac{\mathsf{n}_1^2\omega^2}{c^2}$. In order for the fields to remain finite as $r\longrightarrow 0$ and $r\longrightarrow\infty$, the solutions to equations [1.171] must be of the form:

$$\begin{aligned}E_z&=\begin{cases}A_mJ_m(\gamma r)\,e^{\pm\mathrm{i}m\varphi} & r<a\\B_mK_m(\beta r)\,e^{\pm\mathrm{i}m\varphi} & r>a\end{cases}\\H_z&=\begin{cases}C_mJ_m(\gamma r)\,e^{\pm\mathrm{i}m\varphi} & r<a\\D_mK_m(\beta r)\,e^{\pm\mathrm{i}m\varphi} & r>a\end{cases}\end{aligned}\qquad [1.172]$$

The function K_m does not diverge at infinity. This choice ensures that no energy propagates outward from the fiber, as the fields decay exponentially with r in the cladding.

1.4.3. *Modes with azimuthal symmetry*

Let us first consider the propagation modes without dependence on φ ($m=0$). The solutions of the Helmholtz equation (E_z and H_z) and the other field components obtained from the relations [1.19] and [1.21] are equal to (we have used the identities $J_0' = -J_1$ and $K_0' = -K_1$):

$$r < a : \begin{cases} E_z = A_0 J_0(\gamma r) \\ E_r = -\dfrac{\mathsf{i}\mathsf{k}_z}{\gamma} A_0 J_1(\gamma r) \\ E_\varphi = \dfrac{\mathsf{i}\omega\mu_0}{\gamma} C_0 J_1(\gamma r) \end{cases} \quad \begin{cases} H_z = C_0 J_0(\gamma r) \\ H_r = -\dfrac{\mathsf{i}\mathsf{k}_z}{\gamma} C_0 J_1(\gamma r) \\ H_\varphi = -\dfrac{\mathsf{i}\omega\varepsilon_1}{\gamma} A_0 J_1(\gamma r) \end{cases}$$

$$r > a : \begin{cases} E_z = B_0 K_0(\beta r) \\ E_r = \dfrac{\mathsf{i}\mathsf{k}_z}{\beta} B_0 K_1(\beta r) \\ E_\varphi = -\dfrac{\mathsf{i}\omega\mu_0}{\beta} D_0 K_1(\beta r) \end{cases} \quad \begin{cases} H_z = D_0 K_0(\beta r) \\ H_r = \dfrac{\mathsf{i}\mathsf{k}_z}{\beta} D_0 K_1(\beta r) \\ H_\varphi = \dfrac{\mathsf{i}\omega\varepsilon_2}{\beta} B_0 K_1(\beta r) \end{cases} \qquad [1.173]$$

It is clear that the two modes TE and TM are distinct in this case.

To that end, we treat the two modes separately:

– *TE modes*$(E_z = 0)$: for the TE modes, the previous equations reduce to the following (fixing $A_0 = B_0 = 0$):

$$r < a : \begin{cases} E_r = E_z = H_\varphi = 0 \\ E_\varphi = \dfrac{\mathsf{i}\omega\mu_0}{\gamma} C_0 J_1(\gamma r) \\ H_r = -\dfrac{\mathsf{i}\mathsf{k}_z}{\gamma} C_0 J_1(\gamma r) \\ H_z = C_0 J_0(\gamma r) \end{cases} \quad r > a : \begin{cases} E_r = E_z = H_\varphi = 0 \\ E_\varphi = -\dfrac{\mathsf{i}\omega\mu_0}{\beta} D_0 K_1(\beta r) \\ H_r = \dfrac{\mathsf{i}\mathsf{k}_z}{\beta} D_0 K_1(\beta r) \\ H_z = D_0 K_0(\beta r) \end{cases} \qquad [1.174]$$

The continuity equations for H_r, H_z and E_φ, at the interface $r = a$, lead, respectively, to the following coupled equations:

$$\begin{aligned} &-\frac{\mathsf{i}\mathsf{k}_z}{\gamma} C_0 J_1(\gamma a) = \frac{\mathsf{i}\mathsf{k}_z}{\beta} D_0 K_1(\beta a) \\ &C_0 J_0(\gamma a) = D_0 J K_0(\beta a) \\ &\frac{\mathsf{i}\omega\mu_0}{\gamma} C_0 J_1(\gamma a) = -\frac{\mathsf{i}\omega\mu_0}{\beta} D_0 K_1(\beta a) \end{aligned} \qquad [1.175]$$

It can be seen that H_r and E_φ yield the same constraints. Eliminating the ratio $\frac{C_0}{D_0}$ between the first two equations, we obtain:

$$-\frac{J_1(\gamma a)}{\gamma a J_0(\gamma a)} = \frac{K_1(\beta a)}{\beta a K_0(\beta a)} \qquad [1.176]$$

– *TM modes* $(H_z = 0)$: for the TM modes (choosing $C_0 = D_0 = 0$), we get:

$$r < a : \begin{cases} E_\varphi = H_r = H_z = 0 \\ E_r = -\dfrac{\mathsf{i}\mathsf{k}_z}{\gamma} A_0 J_1(\gamma r) \\ E_z = A_0 J_0(\gamma r) \\ H_\varphi = -\dfrac{\mathsf{i}\omega\varepsilon_1}{\gamma} A_0 J_1(\gamma r) \end{cases} \qquad r > a : \begin{cases} E_\varphi = H_r = H_r = 0 \\ E_r = \dfrac{\mathsf{i}\mathsf{k}_z}{\beta} B_0 K_1(\beta r) \\ E_z = B_0 K_0(\beta r) \\ H_\varphi = \dfrac{\mathsf{i}\omega\varepsilon_2}{\beta} B_0 K_1(\beta r) \end{cases} \qquad [1.177]$$

The continuity of D_r, E_z and H_φ, at the interface $r = a$, enforces the following (the components of D_r and H_φ yield the same constraints):

$$\begin{aligned} &-\frac{\mathsf{i}\mathsf{k}_z\varepsilon_1}{\gamma} A_0 J_1(\gamma r) = \frac{\mathsf{i}\mathsf{k}_z\varepsilon_2}{\beta} B_0 K_1(\beta r) \\ &A_0 J_0(\gamma r) = B_0 K_0(\beta r) \\ &-\frac{\mathsf{i}\omega\varepsilon_1}{\gamma} A_0 J_1(\gamma r) = \frac{\mathsf{i}\omega\varepsilon_2}{\beta} B_0 K_1(\beta r) \end{aligned} \qquad [1.178]$$

and consequently, by eliminating the ratio $\frac{A_0}{B_0}$ between these equations:

$$-\frac{\varepsilon_1 J_1(\gamma a)}{\gamma a J_0(\gamma a)} = \frac{\varepsilon_2 K_1(\beta a)}{\beta a K_0(\beta a)} \qquad [1.179]$$

Equations [1.176] and [1.179] can be solved graphically. Noting that:

$$\gamma^2 + \beta^2 = (\varepsilon_1 - \varepsilon_2)\,\mu_0\omega^2 = \left(\mathsf{n}_1^2 - \mathsf{n}_2^2\right)\frac{\omega^2}{c^2} \qquad [1.180]$$

we observe that $|\gamma|$ and $|\beta|$ are each less than $\frac{\omega}{c}\sqrt{\mathsf{n}_1^2 - \mathsf{n}_2^2}$. Let us first plot the functions $-\frac{J_1(\gamma a)}{\gamma a J_0(\gamma a)}$ and $\frac{K_1(\beta a)}{\beta a K_0(\beta a)}$ as functions of their respective arguments γa and βb, as illustrated in Figures 1.9(a) and (b). From these graphs, we observe that $\frac{J_1}{J_0}$ diverges and changes sign at the level of the roots of J_0. Moreover, since $J_0(x) \xrightarrow[x\to 0]{} 1$ and $J_1(x) \xrightarrow[x\to 0]{} \frac{x}{2}$, the ratio $\frac{J_1(x)}{xJ_0(x)} \xrightarrow[x\to 0]{} \frac{1}{2}$. Similarly, given that $K_{0,1}(x) \xrightarrow[x\to\infty]{} \sqrt{\frac{\pi}{2x}}e^{-x}$, $K_0(x) \xrightarrow[x\to 0]{} -\ln\frac{x}{2}$ and $K_1(x) \xrightarrow[x\to 0]{} \frac{1}{x}$, we deduce $\frac{K_1(x)}{xK_0(x)} \xrightarrow[x\to\infty]{} \frac{1}{x} \longrightarrow 0$ and $\frac{K_1(x)}{xK_0(x)} \xrightarrow[x\to 0]{} -\frac{\ln\frac{x}{2}}{x^2} \longrightarrow +\infty$. The graphical solution proceeds as follows in the case of TE modes (the same procedure applies for the TM modes): in a coordinate system, we plot the function $\frac{J_1\left(\sqrt{x}\right)}{\sqrt{x}J_0\left(\sqrt{x}\right)}$ as a function of x. On the same graph, we plot the function $\frac{K_1\left(\sqrt{y}\right)}{\sqrt{y}K_0\left(\sqrt{y}\right)}$ as a function of $y = \frac{\omega a}{c}\sqrt{\mathsf{n}_1^2 - \mathsf{n}_2^2} - x$. The intersections (the black points) of the two curves correspond to the solutions of equation [1.176], as shown in Figure 1.9(c).

If $\frac{\omega a}{c}\sqrt{\mathsf{n}_1^2-\mathsf{n}_2^2} < x_{01} = 2.405$, no intersection is possible. The first intersection occurs when $\frac{\omega a}{c}\sqrt{\mathsf{n}_1^2-\mathsf{n}_2^2} = x_{01}$, which gives $\omega_{01} = \frac{c x_{01}}{a\sqrt{\mathsf{n}_1^2-\mathsf{n}_2^2}}$. At this frequency, we have $\beta = 0$, and consequently $\mathsf{k}_z = \frac{\mathsf{n}_2\omega_{01}}{c}$: the wave propagates in the cladding as if there were no guiding. Below this frequency, β becomes imaginary, which turns K_0 and K_1 into Bessel functions that oscillate radially and decay as $\frac{1}{\sqrt{r}}$. The fiber no longer acts as a waveguide but rather as an antenna. When ω increases beyond ω_{01}, β grows more rapidly than γ if $\mathsf{n}_1 \gg \mathsf{n}_2$. This means that the wave decays rapidly with r in the cladding. ω_{01} thus appears as the first cutoff frequency. Each time x crosses a root of J_0, a new mode appears, each with its own cutoff frequency:

$$\omega_{0n} = \frac{c x_{0n}}{a\sqrt{\mathsf{n}_1^2-\mathsf{n}_2^2}} \qquad [1.181]$$

1.4.4. *Modes with azimuthal dependence*

The most general modes (with azimuthal dependence) propagating along dielectrics are called *HE modes* and *EH modes*. They have both $E_z \neq 0$ and $H_z \neq 0$. In particular, it can be shown that the HE_{11} mode is dominant and has no cutoff frequency. Let us now examine all of this in detail. In general, the components of the fields $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ take the form:

$$r < a : \begin{cases} E_z = A_m J_m(\gamma r)\, e^{\pm i m\varphi} \\ E_r = \left[\mp\dfrac{m\omega\mu_0}{\gamma^2 r} C_m J_m(\gamma r) + \dfrac{\mathrm{i}\mathsf{k}_z}{\gamma} A_m J_m'(\gamma r)\right] e^{\pm \mathrm{i} m\varphi} \\ E_\varphi = \left[\mp\dfrac{m\mathsf{k}_z}{\gamma^2 r} A_m J_m(\gamma r) - \dfrac{\mathrm{i}\omega\mu_0}{\gamma} C_m J_m'(\gamma r)\right] e^{\pm \mathrm{i} m\varphi} \\ H_z = C_m J_m(\gamma r)\, e^{\pm \mathrm{i} m\varphi} \\ H_r = \left[\pm\dfrac{m\omega\varepsilon_1}{\gamma^2 r} A_m J_m(\gamma r) + \dfrac{\mathrm{i}\mathsf{k}_z}{\gamma} C_m J_m'(\gamma r)\right] e^{\pm \mathrm{i} m\varphi} \\ H_\varphi = \left[\mp\dfrac{m\mathsf{k}_z}{\gamma^2 r} C_m J_m(\gamma r) + \dfrac{\mathrm{i}\omega\varepsilon_1}{\gamma} A_m J_m'(\gamma r)\right] e^{\pm \mathrm{i} m\varphi} \end{cases} \qquad [1.182]$$

and:

$$r > a : \begin{cases} E_z = B_m K_m(\beta r)\, e^{\pm \mathrm{i} m\varphi} \\ E_r = \left[\pm\dfrac{m\omega\mu_0}{\beta^2 r} D_m K_m(\beta r) - \dfrac{\mathrm{i}\mathsf{k}_z}{\beta} B_m K_m'(\beta r)\right] e^{\pm \mathrm{i} m\varphi} \\ E_\varphi = \left[\pm\dfrac{m\mathsf{k}_z}{\beta^2 r} B_m K_m(\beta r) + \dfrac{\mathrm{i}\omega\mu_0}{\beta} D_m K_m'(\beta r)\right] e^{\pm \mathrm{i} m\varphi} \\ H_z = D_m K_m(\beta r)\, e^{\pm \mathrm{i} m\varphi} \\ H_r = \left[\mp\dfrac{m\omega\varepsilon_2}{\beta^2 r} B_m K_m(\beta r) - \dfrac{\mathrm{i}\mathsf{k}_z}{\beta} D_m K_m'(\beta r)\right] e^{\pm \mathrm{i} m\varphi} \\ H_\varphi = \left[\pm\dfrac{m\mathsf{k}_z}{\beta^2 r} D_m K_m(\beta r) - \dfrac{\mathrm{i}\omega\varepsilon_2}{\beta} B_m K_m'(\beta r)\right] e^{\pm \mathrm{i} m\varphi} \end{cases} \qquad [1.183]$$

The continuity of the components E_z, H_z, E_φ, H_φ, D_r and B_r, at the interface $r = a$, imposes the following equalities, respectively:

$$\begin{aligned}
&A_m J_m = B_m K_m && \text{(i)}\\
&C_m J_m = D_m K_m && \text{(ii)}\\
&\mp\frac{m\mathsf{k}_z}{\gamma^2 a}A_m J_m - \frac{\mathsf{i}\omega\mu_0}{\gamma}C_m J'_m = \pm\frac{m\mathsf{k}_z}{\beta^2 a}B_m K_m + \frac{\mathsf{i}\omega\mu_0}{\beta}D_m K'_m && \text{(iii)}\\
&\mp\frac{m\mathsf{k}_z}{\gamma^2 a}C_m J_m + \frac{\mathsf{i}\omega\varepsilon_1}{\gamma}A_m J'_m = \pm\frac{m\mathsf{k}_z}{\beta^2 a}D_m K_m - \frac{\mathsf{i}\omega\varepsilon_2}{\beta}B_m K'_m && \text{(iv)}\\
&\varepsilon_1\left(\mp\frac{m\omega\mu_0}{\gamma^2 a}C_m J_m + \frac{\mathsf{i}\mathsf{k}_z}{\gamma}A_m J'_m\right) = \varepsilon_2\left(\pm\frac{m\omega\mu_0}{\beta^2 a}D_m K_m - \frac{\mathsf{i}\mathsf{k}_z}{\beta}B_m K'_m\right) && \text{(v)}\\
&\pm\frac{m\omega\varepsilon_1}{\gamma^2 a}A_m J_m + \frac{\mathsf{i}\mathsf{k}_z}{\gamma}C_m J'_m = \mp\frac{m\omega\varepsilon_2}{\beta^2 a}B_m K_m - \frac{\mathsf{i}\mathsf{k}_z}{\beta}D_m K'_m && \text{(vi)}
\end{aligned} \quad [1.184]$$

where we have used the notation $J_m = J_m(\gamma a)$, $J'_m = J'_m(\gamma a)$, $K_m = K_m(\gamma a)$ and $K'_m = K'_m(\gamma a)$. Let us rewrite equation [1.184iii] in the following form:

$$\mp\frac{m\mathsf{k}_z}{a}\left(\frac{A_m J_m}{\gamma^2} + \frac{B_m K_m}{\beta^2}\right) = \mathsf{i}\omega\mu_0\left(\frac{C_m J'_m}{\gamma} + \frac{D_m K'_m}{\beta}\right) \quad [1.185]$$

Substituting into [1.185] the coefficients B_m and D_m obtained from equations [1.184i] and [1.184ii], we obtain:

$$\mp\frac{m\mathsf{k}_z A_m}{a}\left(\frac{1}{\gamma^2} + \frac{1}{\beta^2}\right) = \mathsf{i}\omega\mu_0 C_m\left(\frac{J'_m}{\gamma J_m} + \frac{K'_m}{\beta K_m}\right) \quad [1.186]$$

Similarly, the same procedure can be applied to equations [1.184iv], [1.184v] and [1.184vi]:

$$\begin{aligned}
&\pm\frac{m\mathsf{k}_z}{a}\left(\frac{C_m J_m}{\gamma^2} + \frac{D_m K_m}{\beta^2}\right) = \mathsf{i}\omega\left(\frac{\varepsilon_1 A_m J'_m}{\gamma} + \frac{\varepsilon_2 B_m K'_m}{\beta}\right) && \text{(i)}\\
&\pm\frac{m\omega\mu_0}{a}\left(\frac{\varepsilon_1 C_m J_m}{\gamma^2} + \frac{\varepsilon_2 D_m K_m}{\beta^2}\right) = \mathsf{i}\mathsf{k}_z\left(\frac{\varepsilon_1 A_m J'_m}{\gamma} + \frac{\varepsilon_2 B_m K'_m}{\beta}\right) && \text{(ii)}\\
&\mp\frac{m\omega}{a}\left(\frac{\varepsilon_1 A_m J_m}{\gamma^2} + \frac{\varepsilon_2 B_m K_m}{\beta^2}\right) = \mathsf{i}\mathsf{k}_z\left(\frac{C_m J'_m}{\gamma} + \frac{D_m K'_m}{\beta}\right) && \text{(iii)}
\end{aligned} \quad [1.187]$$

which yields:

$$\begin{aligned}
&\pm\frac{m\mathsf{k}_z C_m}{a}\left(\frac{1}{\gamma^2} + \frac{1}{\beta^2}\right) = \mathsf{i}\omega A_m\left(\frac{\varepsilon_1 J'_m}{\gamma J_m} + \frac{\varepsilon_2 K'_m}{\beta K_m}\right) && \text{(i)}\\
&\pm\frac{m\omega\mu_0 C_m}{a}\left(\frac{\varepsilon_1}{\gamma^2} + \frac{\varepsilon_2}{\beta^2}\right) = \mathsf{i}\mathsf{k}_z A_m\left(\frac{\varepsilon_1 J'_m}{\gamma J_m} + \frac{\varepsilon_2 K'_m}{\beta K_m}\right) && \text{(ii)}\\
&\mp\frac{m\omega A_m}{a}\left(\frac{\varepsilon_1}{\gamma^2} + \frac{\varepsilon_2}{\beta^2}\right) = \mathsf{i}\mathsf{k}_z C_m\left(\frac{J'_m}{\gamma J_m} + \frac{K'_m}{\beta K_m}\right) && \text{(iii)}
\end{aligned} \quad [1.188]$$

The constants A_m and C_m can now be eliminated. Multiplying [1.186] by [1.188i] and then simplifying by $A_m C_m$, we obtain:

$$\left(\frac{J'_m}{\gamma J_m}+\frac{K'_m}{\beta K_m}\right)\left(\frac{\varepsilon_1 J'_m}{\gamma J_m}+\frac{\varepsilon_2 K'_m}{\beta K_m}\right)=\frac{m^2\mathsf{k}_z^2}{\mu_0 a^2\omega^2}\left(\frac{1}{\gamma^2}+\frac{1}{\beta^2}\right)^2$$
$$=\frac{m^2\mathsf{k}_z^2\omega^2}{\mu_0 a^2 c^4\gamma^2\beta^2}\left(\mathsf{n}_1^2-\mathsf{n}_2^2\right)^2 \qquad [1.189]$$

Similarly, proceeding in the same way with formulas [1.188ii] and [1.188iii], we find:

$$\left(\frac{J'_m}{\gamma J_m}+\frac{K'_m}{\beta K_m}\right)\left(\frac{\varepsilon_1 J'_m}{\gamma J_m}+\frac{\varepsilon_2 K'_m}{\beta K_m}\right)=\frac{m^2\omega^2\mu_0}{a^2\mathsf{k}_z^2}\left(\frac{\varepsilon_1}{\gamma^2}+\frac{\varepsilon_2}{\beta^2}\right)^2 \qquad [1.190]$$

The expressions [1.189] and [1.190] are identical. In fact, dividing [1.186] by [1.188iii] (respectively, [1.188i] by [1.188iii]), we deduce that:

$$\left(\frac{\varepsilon_1}{\gamma^2}+\frac{\varepsilon_2}{\beta^2}\right)^2=\frac{\mathsf{k}_z^2}{\mu_0\omega^2}\left(\frac{1}{\gamma^2}+\frac{1}{\beta^2}\right)^2 \qquad [1.191]$$

which show that [1.189] and [1.190] are indeed the same. Equation [1.189] provides the relationship between the constant k_z and the frequency ω. For a given frequency ω and a given value of m, it allows us to determine the propagation constant k_z. In other words, the solutions of [1.189] yield the dispersion relations of the possible modes. The relation [1.189] is called the *dispersion relation* of the step index optical fiber or the *characteristic equation* of the propagation modes. In fact, for each value of m, this equation admits a finite number of solutions that satisfy the guiding condition $\frac{\mathsf{n}_2^2\omega^2}{c^2} < \mathsf{k}_z^2 < \frac{\mathsf{n}_1^2\omega^2}{c^2}$. This multiplicity of solutions results from the oscillatory nature of the Bessel functions J_m. Thus, each solution of [1.189] can be characterized by two indices (m, n), where m characterizes the azimuthal dependence of the electromagnetic field and n corresponds to the *n*th solution of [1.189] associated with the given value of m. It should be noted that the total number of modes that can propagate depends on the frequency of the incident wave in the fiber. In general, the number of modes increases with frequency. Once the solutions of [1.189] are obtained, the constants B_m, C_m and D_m can be expressed in terms of A_m, and the explicit expressions of the field components for each propagation mode can then be determined. From [1.184i], [1.184ii] and [1.186], we deduce that:

$$\begin{aligned}
B_m &= \frac{J_m(\gamma a)}{K_m(\beta a)}A_m\\
C_m &= \pm\frac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega}PA_m\\
D_m &= \pm\frac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega}\frac{J_m(\gamma a)}{K_m(\beta a)}PA_m
\end{aligned} \qquad [1.192]$$

where:

$$P = \frac{m\left(\dfrac{1}{\gamma^2 a^2}+\dfrac{1}{\beta^2 a^2}\right)}{\dfrac{J'_m(\gamma a)}{\gamma a J_m(\gamma a)}+\dfrac{K'_m(\beta a)}{\beta a K_m(\beta a)}} \qquad [1.193]$$

When the azimuthal number m is equal to zero, the constraints [1.176] and [1.179] are satisfied, since the right-hand side of [1.189] vanishes. This implies that one or the other of the two factors on the left must cancel. If the term $\frac{J'_m}{\gamma J_m}+\frac{K'_m}{\beta K_m}$ is zero, this corresponds to the TE modes. On the other hand, if the term $\frac{\varepsilon_1 J'_m}{\gamma J_m}+\frac{\varepsilon_2 K'_m}{\beta K_m}$ is zero, this corresponds to the modes TM. We conclude that when $m=0$, the TE and TM modes are not coupled. The right-hand side of equation [1.189] therefore acts to couple the TM and TE solutions, resulting in HE and EH mode types. When the two refractive indices n_1 and n_2 are very close, the right-hand side of [1.189] can be neglected as a first approximation. Consequently, we obtain equations similar to [1.176] and [1.179], namely, $-\frac{J'_m(\gamma a)}{\gamma a J_m(\gamma a)}=\frac{K'_m(\beta a)}{\beta a K_m(\beta a)}$ or $-\frac{\varepsilon_1 J'_m(\gamma a)}{\gamma a J_m(\gamma a)}=\frac{\varepsilon_2 K'_m(\beta a)}{\beta a K_m(\beta a)}$. The graphical solutions of these equations show no cutoff frequency for $m \geq 1$.

Taking into account equations [1.192], the longitudinal fields take the following form:

$$E_z = \begin{cases} A_m J_m(\gamma r)\, e^{\pm \mathrm{i} m\varphi} e^{-\mathrm{i}(\omega t-\mathsf{k}_z z)} & r<a \\ A_m \frac{J_m(\gamma a)}{K_m(\beta a)} K_m(\beta r)\, e^{\pm \mathrm{i} m\varphi} e^{-\mathrm{i}(\omega t-\mathsf{k}_z z)} & r>a \end{cases}$$
$$H_z = \begin{cases} \pm\frac{\mathrm{i}\mathsf{k}_z P}{\mu_0\omega} A_m J_m(\gamma r)\, e^{\pm \mathrm{i} m\varphi} e^{-\mathrm{i}(\omega t-\mathsf{k}_z z)} & r<a \\ \pm\frac{\mathrm{i}\mathsf{k}_z P}{\mu_0\omega} A_m \frac{J_m(\gamma a)}{K_m(\beta a)} K_m(\beta r)\, e^{\pm \mathrm{i} m\varphi} e^{-\mathrm{i}(\omega t-\mathsf{k}_z z)} & r>a \end{cases} \qquad [1.194]$$

and, consequently:

$$\left|\frac{H_z}{E_z}\right| = \frac{\mathsf{k}_z P}{\mu_0\omega} \qquad [1.195]$$

If $\left|\frac{H_z}{E_z}\right| > 1$, the field is denoted EH, while for $\left|\frac{H_z}{E_z}\right| < 1$, the field is denoted HE. The transverse components of the electromagnetic field, given by equations [1.182] and [1.183], involve derivatives of Bessel functions. These derivatives can be eliminated using the recurrence relations: $J'_m(x) = \frac{m}{x}J_m(x) - J_{m+1}(x) = -\frac{m}{x}J_m(x) + J_{m-1}(x)$ and $K'_m(x) = \frac{m}{x}K_m(x) - K_{m+1}(x) = -\frac{m}{x}K_m(x) - K_{m-1}(x)$. Here are the results:

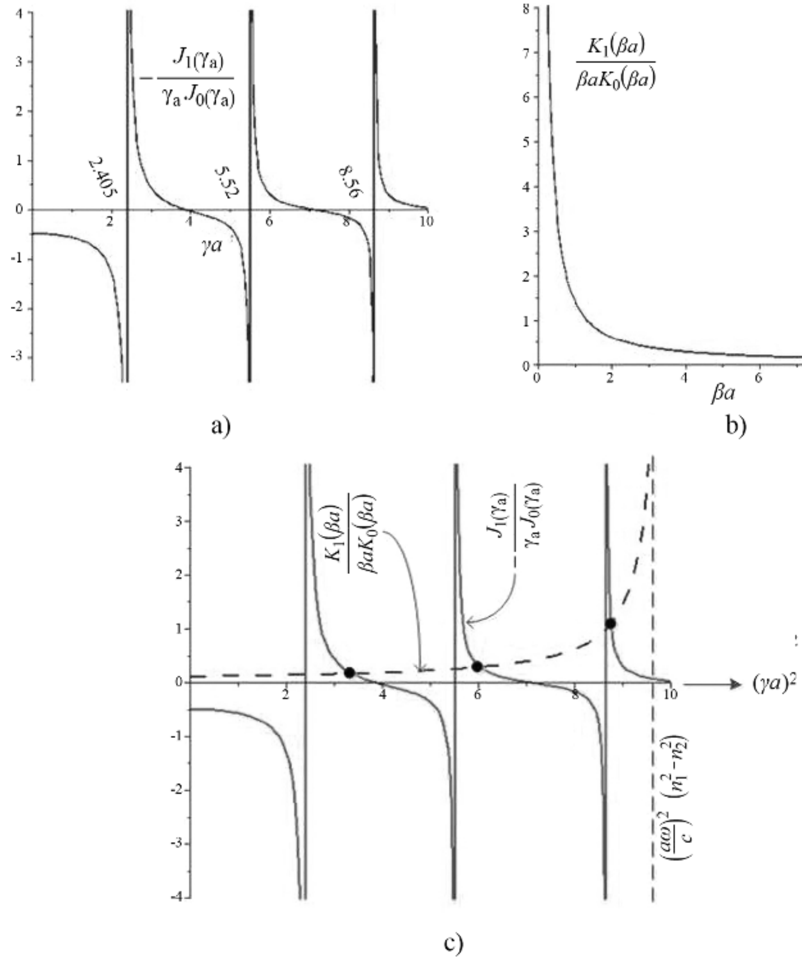

Figure 1.9. *(a and b) Behavior of the functions* $\frac{J_1(\gamma a)}{\gamma a J_0(\gamma a)}$ *and* $\frac{K_1(\beta a)}{\beta a K_0(\beta a)}$ *as functions of* γa *and* βa*, respectively. (c) Graphical solutions of equation [1.176]. The function* $\frac{J_1(\sqrt{x})}{\sqrt{x} J_1(\sqrt{x})}$ *appears in red, and the function* $\frac{K_1(\sqrt{y})}{\sqrt{y} K_1(\sqrt{y})}$ *in blue (the graph of* $\frac{K_1(\sqrt{y})}{\sqrt{y} K_1(\sqrt{y})}$ *is oriented to the left and plotted over the interval* $\left[0, \frac{\omega a}{c}\sqrt{\mathsf{n}_1^2 - \mathsf{n}_2^2}\right]$*). The solutions (intersection points) are marked with a black dot. For a color version of this figure, see www.iste.co.uk/abdesselam/electrodynamics3.zip*

– *Modes EH*$_{mn}$: knowing that $J'_m(\gamma r) = \frac{m}{\gamma r} J_m(\gamma r) - J_{m+1}(\gamma r)$ and $K'_m(\beta r) = \frac{m}{\beta r} K_m(\beta r) - K_{m+1}(\beta r)$, the fields for the EH$_{mn}$ modes can be expressed as follows:

- In the core $(r < a)$, we have (where $\mathcal{F}_{\pm m} = e^{\pm \mathrm{i} m\varphi} e^{-\mathrm{i}(\omega t - \mathsf{k}_z z)}$ is the phase term):

$$\begin{cases} E_r = \dfrac{\mathrm{i}\mathsf{k}_z}{\gamma} A_m \left[\dfrac{m}{\gamma r} (1-P) J_m(\gamma r) - J_{m+1}(\gamma r) \right] \mathcal{F}_{\pm m} \\ E_\varphi = \mp \dfrac{\mathsf{k}_z}{\gamma} A_m \left[\dfrac{m}{\gamma r} (1-P) J_m(\gamma r) + P J_{m+1}(\gamma r) \right] \mathcal{F}_{\pm m} \\ H_r = \pm \dfrac{\omega \varepsilon_1}{\gamma} A_m \left[\dfrac{m}{\gamma r} \left(1 - \dfrac{\mathsf{k}_z^2 c^2}{\omega^2 n_1^2} P \right) J_m(\gamma r) + \dfrac{\mathsf{k}_z^2 c^2}{\omega^2 n_1^2} P J_{m+1}(\gamma r) \right] \mathcal{F}_{\pm m} \\ H_\varphi = \dfrac{\mathrm{i}\omega \varepsilon_1}{\gamma} A_m \left[\dfrac{m}{\gamma r} \left(1 - \dfrac{\mathsf{k}_z^2 c^2}{\omega^2 n_1^2} P \right) J_m(\gamma r) - J_{m+1}(\gamma r) \right] \mathcal{F}_{\pm m} \end{cases}$$

[1.196]

- In the cladding $(r > a)$, we have:

$$\begin{cases} E_r = -\dfrac{\mathrm{i}\mathsf{k}_z}{\beta} \dfrac{J_m(\gamma a)}{K_m(\beta a)} A_m \left[\dfrac{m}{\beta r} (1-P) K_m(\beta r) - K_{m+1}(\beta r) \right] \mathcal{F}_{\pm m} \\ E_\varphi = \pm \dfrac{\mathsf{k}_z}{\beta} \dfrac{J_m(\gamma a)}{K_m(\beta a)} A_m \left[\dfrac{m}{\beta r} (1-P) K_m(\beta r) + P K_{m+1}(\beta r) \right] \mathcal{F}_{\pm m} \\ H_r = \mp \dfrac{\omega \varepsilon_2}{\beta} \dfrac{J_m(\gamma a)}{K_m(\beta a)} A_m \left[\dfrac{m}{\beta r} \left(1 - \dfrac{\mathsf{k}_z^2 c^2}{\omega^2 n_2^2} P \right) K_m(\beta r) \right. \\ \left. + \dfrac{\mathsf{k}_z^2 c^2}{\omega^2 n_2^2} P K_{m+1}(\beta r) \right] \mathcal{F}_{\pm m} \\ H_\varphi = -\dfrac{\mathrm{i}\omega \varepsilon_2}{\beta} \dfrac{J_m(\gamma a)}{K_m(\beta a)} A_m \left[\dfrac{m}{\beta r} \left(1 - \dfrac{\mathsf{k}_z^2 c^2}{\omega^2 n_2^2} P \right) K_m(\beta r) - K_{m+1}(\beta r) \right] \mathcal{F}_{\pm m} \end{cases}$$

[1.197]

– *Modes HE*$_{mn}$: since $J'_m(\gamma r) = -\frac{m}{\gamma r} J_m(\gamma r) + J_{m-1}(\gamma r)$ and $K'_m(\beta r) = -\frac{m}{\beta r} K_m(\beta r) - K_{m-1}(\beta r)$, the fields for the modes HE$_{mn}$ can be expressed as follows:

- In the core $(r < a)$, we have:

$$\begin{cases} E_r = \dfrac{\mathrm{i}\mathsf{k}_z}{\gamma} A_m \left[-\dfrac{m}{\gamma r} (1+P) J_m(\gamma r) + J_{m-1}(\gamma r) \right] \mathcal{F}_{\pm m} \\ E_\varphi = \mp \dfrac{\mathsf{k}_z}{\gamma} A_m \left[\dfrac{m}{\gamma r} (1+P) J_m(\gamma r) - P J_{m-1}(\gamma r) \right] \mathcal{F}_{\pm m} \\ H_r = \pm \dfrac{\omega \varepsilon_1}{\gamma} A_m \left[\dfrac{m}{\gamma r} \left(1 + \dfrac{\mathsf{k}_z^2 c^2}{\omega^2 n_1^2} P \right) J_m(\gamma r) - \dfrac{\mathsf{k}_z^2 c^2}{\omega^2 n_1^2} P J_{m-1}(\gamma r) \right] \mathcal{F}_{\pm m} \\ H_\varphi = \dfrac{\mathrm{i}\omega \varepsilon_1}{\gamma} A_m \left[-\dfrac{m}{\gamma r} \left(1 + \dfrac{\mathsf{k}_z^2 c^2}{\omega^2 n_1^2} P \right) J_m(\gamma r) + J_{m-1}(\gamma r) \right] \mathcal{F}_{\pm m} \end{cases}$$

[1.198]

- In the cladding $(r > a)$, we have:

$$\begin{cases} E_r = \frac{\mathrm{i}\mathsf{k}_z}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)}A_m\left[\frac{m}{\beta r}(1+P)K_m(\beta r)+K_{m-1}(\beta r)\right]\mathcal{F}_{\pm m} \\ E_\varphi = \pm\frac{\mathsf{k}_z}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)}A_m\left[\frac{m}{\beta r}(1+P)K_m(\beta r)+PK_{m-1}(\beta r)\right]\mathcal{F}_{\pm m} \\ H_r = \mp\frac{\omega\varepsilon_2}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)}A_m\left[\frac{m}{\beta r}\left(1+\frac{\mathsf{k}_z^2c^2}{\omega^2n_2^2}P\right)K_m(\beta r)\right. \\ \left.+\frac{\mathsf{k}_z^2c^2}{\omega^2n_2^2}PK_{m-1}(\beta r)\right]\mathcal{F}_{\pm m} \\ H_\varphi = \frac{\mathrm{i}\omega\varepsilon_2}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)}A_m\left[\frac{m}{\beta r}\left(1+\frac{\mathsf{k}_z^2c^2}{\omega^2n_2^2}P\right)K_m(\beta r)\right. \\ \left.+K_{m-1}(\beta r)\right]\mathcal{F}_{\pm m} \end{cases} \quad [1.199]$$

1.4.5. *Propagation modes in weakly guiding fibers*

1.4.5.1. *Dispersion relation*

When the difference between the indices n_1 and n_2 is very small, we can write in the first approximation that $\mathsf{n}_1 \simeq \mathsf{n}_2$ and consequently we must have $\mathsf{k}_z \simeq \frac{\mathsf{n}_1\omega}{c} \simeq \frac{\mathsf{n}_2\omega}{c}$, given the guiding condition $\frac{\mathsf{n}_2\omega}{c} < \mathsf{k}_z < \frac{\mathsf{n}_1\omega}{c}$. Under this approximation, the dispersion relation [1.189] becomes:

$$\left(\frac{J_m'(\gamma a)}{\gamma a J_m(\gamma a)}+\frac{K_m'(\beta a)}{\beta a K_m(\beta a)}\right)^2 \simeq m^2\left(\frac{1}{(\gamma a)^2}+\frac{1}{(\beta a)^2}\right)^2 \quad [1.200]$$

The latter is divided into two relations:

$$\frac{J_m'(\gamma a)}{\gamma a J_m(\gamma a)}+\frac{K_m'(\beta a)}{\beta a K_m(\beta a)} = m\left(\frac{1}{\gamma^2a^2}+\frac{1}{\beta^2a^2}\right) \quad [1.201]$$

or:

$$\frac{J_m'(\gamma a)}{\gamma a J_m(\gamma a)}+\frac{K_m'(\beta a)}{\beta a K_m(\beta a)} = -m\left(\frac{1}{\gamma^2a^2}+\frac{1}{\beta^2a^2}\right) \quad [1.202]$$

These two dispersion relations correspond, respectively, to the EH and HE modes. In this weak-guidance approximation, the expression for P defined by [1.193] reduces to $P = \pm 1$. By eliminating the derivative functions J_m' and K_m' from the relation [1.201] using the identities $J_m'(x) = \frac{m}{x}J_m(x) - J_{m+1}(x)$ and $K_m'(x) = \frac{m}{x}K_m(x) - K_{m+1}(x)$, we obtain:

$$\frac{\gamma a J_m(\gamma a)}{J_{m+1}(\gamma a)}+\frac{\beta a K_m(\beta a)}{K_{m+1}(\beta a)} = 0 \quad [1.203]$$

The expression [1.203] is the characteristic equation of the EH$_{mn}$ modes. The index n designates the nth root of equation [1.203]. This equation corresponds to $P = 1$. Similarly, we can eliminate the derivative functions J'_m and K'_m from [1.202] using the formulas $J'_m(x) = -\frac{m}{x}J_m(x) + J_{m-1}(x)$ and $K'_m(x) = -\frac{m}{x}K_m(x) - K_{m-1}(x)$. The relation [1.202] then becomes:

$$\frac{\gamma a J_m(\gamma a)}{J_{m-1}(\gamma a)} - \frac{\beta a K_m(\beta a)}{K_{m-1}(\beta a)} = 0 \qquad [1.204]$$

This is the characteristic equation of the modes HE$_{mn}$ and is associated with $P = -1$.

1.4.5.2. *Degeneracy of the modes*

The dispersion relation of HE$_{m+1,n}$, which follows directly from [1.204], is written as:

$$\frac{\gamma a J_{m+1}(\gamma a)}{J_m(\gamma a)} - \frac{\beta a K_{m+1}(\beta a)}{K_m(\beta a)} = 0 \qquad [1.205]$$

By substituting into this relation the expressions $J_{m+1}(x) = \frac{2m}{x}J_m(x) - J_{m-1}(x)$ and $K_{m+1}(x) = \frac{2m}{x}K_m(x) + K_{m-1}(x)$, we obtain:

$$\frac{\gamma a J_{m-1}(\gamma a)}{J_m(\gamma a)} + \frac{\beta a K_{m-1}(\beta a)}{K_m(\beta a)} = 0 \qquad [1.206]$$

This last relation is exactly the same as [1.203] for the EH modes, but with the index $m-1$. In other words, the dispersion relation of the HE$_{m+1,n}$ modes is identical to that of the EH$_{m-1,n}$ modes. This means that these two modes have exactly the same value of k_z. This is called the degeneracy of the HE$_{m+1,n}$ and EH$_{m-1,n}$ modes.

1.4.5.3. *Expressions of the field components*

Let us now give the expressions for the field components in the weak-guidance approximation in cylindrical coordinates:

$$E_r = -\frac{\mathsf{ik}_z}{\gamma}A_m J_{m+1}(\gamma r)\mathcal{F}_{\pm m} \qquad E_r = \frac{\mathsf{ik}_z}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)}A_m K_{m+1}(\beta r)\mathcal{F}_{\pm m}$$

$$E_\varphi = \mp\frac{\mathsf{k}_z}{\gamma}A_m J_{m+1}(\gamma r)\mathcal{F}_{\pm m} \qquad E_\varphi = \pm\frac{\mathsf{k}_z}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)}A_m K_{m+1}(\beta r)\mathcal{F}_{\pm m}$$

$$E_z = A_m J_m(\gamma r)\mathcal{F}_{\pm m} \qquad E_z = A_m\frac{J_m(\gamma a)}{K_m(\beta a)}K_m(\beta r)\mathcal{F}_{\pm m}$$

$$H_r = \pm\tfrac{\omega\varepsilon_1}{\gamma}A_m J_{m+1}(\gamma r)\mathcal{F}_{\pm m} \qquad H_r = \mp\frac{\omega\varepsilon_2}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)}A_m K_{m+1}(\beta r)\mathcal{F}_{\pm m}$$

$$\underbrace{\begin{aligned} H_\varphi &= -\frac{\mathsf{i}\omega\varepsilon_1}{\gamma} A_m J_{m+1}(\gamma r)\,\mathcal{F}_{\pm m} \\ H_z &= \pm\frac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega} A_m J_m(\gamma r)\,\mathcal{F}_{\pm m} \end{aligned}}_{EH_{mn}\ modes\ (r<a)} \qquad \underbrace{\begin{aligned} H_\varphi &= \frac{\mathsf{i}\omega\varepsilon_2}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m+1}(\beta r)\,\mathcal{F}_{\pm m} \\ H_z &= \pm\frac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega} A_m \frac{J_m(\gamma a)}{K_m(\beta a)} K_m(\beta r)\,\mathcal{F}_{\pm m} \end{aligned}}_{EH_{mn}\ modes\ (r<a)}$$

[1.207]

$$\underbrace{\begin{aligned} E_r &= \frac{\mathsf{i}\mathsf{k}_z}{\gamma} A_m J_{m-1}(\gamma r)\,\mathcal{F}_{\pm m} \\ E_\varphi &= \mp\frac{\mathsf{k}_z}{\gamma} A_m J_{m-1}(\gamma r)\,\mathcal{F}_{\pm m} \\ E_z &= A_m J_m(\gamma r)\,\mathcal{F}_{\pm m} \\ H_r &= \pm\frac{\omega\varepsilon_1}{\gamma} A_m J_{m-1}(\gamma r)\,\mathcal{F}_{\pm m} \\ H_\varphi &= \frac{\mathsf{i}\omega\varepsilon_1}{\gamma} A_m J_{m-1}(\gamma r)\,\mathcal{F}_{\pm m} \\ H_z &= \mp\frac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega} A_m J_m(\gamma r)\,\mathcal{F}_{\pm m} \end{aligned}}_{HE_{mn}\ modes\ (r<a)} \qquad \underbrace{\begin{aligned} E_r &= \frac{\mathsf{i}\mathsf{k}_z}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m-1}(\beta r)\,\mathcal{F}_{\pm m} \\ E_\varphi &= \mp\frac{\mathsf{k}_z}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m-1}(\beta r)\,\mathcal{F}_{\pm m} \\ E_z &= A_m \frac{J_m(\gamma a)}{K_m(\beta a)} K_m(\beta r)\,\mathcal{F}_{\pm m} \\ H_r &= \pm\frac{\omega\varepsilon_2}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m-1}(\beta r)\,\mathcal{F}_{\pm m} \\ H_\varphi &= \frac{\mathsf{i}\omega\varepsilon_2}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m-1}(\beta r)\,\mathcal{F}_{\pm m} \\ E_z &= \mp\frac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega} A_m \frac{J_m(\gamma a)}{K_m(\beta a)} K_m(\beta r)\,\mathcal{F}_{\pm m} \end{aligned}}_{HE_{mn}\ modes\ (r>a)}$$

[1.208]

In system $Oxyz$, we have $E_x = E_r\cos\varphi - E_\varphi\sin\varphi = \frac{1}{2}(E_r + \mathsf{i}E_\varphi)\,e^{\mathsf{i}\varphi} + \frac{1}{2}(E_r - \mathsf{i}E_\varphi)\,e^{-\mathsf{i}\varphi}$, $E_y = E_r\cos\varphi + E_\varphi\sin\varphi = \frac{1}{2}(E_\varphi - \mathsf{i}E_r)e^{\mathsf{i}\varphi} + \frac{1}{2}(E_\varphi + \mathsf{i}E_r)e^{-\mathsf{i}\varphi}$, and $E_x = E_z$ (the same expressions apply for H_x, H_y and H_z).

We thus obtain the following expressions:

$$\underbrace{\begin{aligned} E_x &= -\frac{\mathsf{i}\mathsf{k}_z}{\gamma} A_m J_{m+1}(\gamma r)\,\mathcal{F}_{\pm(m+1)} \\ E_y &= \mp\frac{\mathsf{k}_z}{\gamma} A_m J_{m+1}(\gamma r)\,\mathcal{F}_{\pm(m+1)} \\ E_z &= A_m J_m(\gamma r)\,\mathcal{F}_{\pm m} \\ H_x &= \pm\frac{\omega\varepsilon_1}{\gamma} A_m J_{m+1}(\gamma r)\,\mathcal{F}_{\pm(m+1)} \\ H_y &= -\frac{\mathsf{i}\omega\varepsilon_1}{\gamma} A_m J_{m+1}(\gamma r)\,\mathcal{F}_{\pm(m+1)} \\ H_z &= \pm\frac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega} A_m J_m(\gamma r)\,\mathcal{F}_{\pm m} \end{aligned}}_{EH_{mn}\ modes\ (r<a)} \qquad \underbrace{\begin{aligned} E_x &= \frac{\mathsf{i}\mathsf{k}_z}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m+1}(\beta r)\,\mathcal{F}_{\pm(m+1)} \\ E_y &= \pm\frac{\mathsf{k}_z}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m+1}(\beta r)\,\mathcal{F}_{\pm(m+1)} \\ E_z &= A_m \frac{J_m(\gamma a)}{K_m(\beta a)} K_m(\beta r)\,\mathcal{F}_{\pm m} \\ H_x &= \mp\frac{\omega\varepsilon_2}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m+1}(\beta r)\,\mathcal{F}_{\pm(m+1)} \\ H_y &= \frac{\mathsf{i}\omega\varepsilon_2}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m+1}(\beta r)\,\mathcal{F}_{\pm(m+1)} \\ H_z &= \pm\frac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega} A_m \frac{J_m(\gamma a)}{K_m(\beta a)} K_m(\beta r)\,\mathcal{F}_{\pm m} \end{aligned}}_{EH_{mn}\ modes\ (r>a)}$$

[1.209]

$$\underbrace{\begin{aligned} E_x &= \frac{\mathsf{i}\mathsf{k}_z}{\gamma} A_m J_{m-1}(\gamma r)\,\mathcal{F}_{\pm(m-1)} \\ E_y &= \mp\frac{\mathsf{k}_z}{\gamma} A_m J_{m-1}(\gamma r)\,\mathcal{F}_{\pm(m-1)} \\ E_z &= A_m J_m(\gamma r)\,\mathcal{F}_{\pm m} \\ H_x &= \pm\frac{\omega\varepsilon_1}{\gamma} A_m J_{m-1}(\gamma r)\,\mathcal{F}_{\pm(m-1)} \\ H_y &= \frac{\mathsf{i}\omega\varepsilon_1}{\gamma} A_m J_{m-1}(\gamma r)\,\mathcal{F}_{\pm(m-1)} \\ H_z &= \mp\frac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega} A_m J_m(\gamma r)\,\mathcal{F}_{\pm m} \end{aligned}}_{HE_{mn}\ modes\ (r<a)} \quad \underbrace{\begin{aligned} E_x &= \frac{\mathsf{i}\mathsf{k}_z}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m-1}(\beta r)\,\mathcal{F}_{\pm(m-1)} \\ E_y &= \mp\frac{\mathsf{k}_z}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m-1}(\beta r)\,\mathcal{F}_{\pm(m-1)} \\ E_z &= A_m\frac{J_m(\gamma a)}{K_m(\beta a)} K_m(\beta r)\,\mathcal{F}_{\pm m} \\ H_x &= \pm\frac{\omega\varepsilon_2}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m-1}(\beta r)\,\mathcal{F}_{\pm(m-1)} \\ H_y &= \frac{\mathsf{i}\omega\varepsilon_2}{\beta}\frac{J_m(\gamma a)}{K_m(\beta a)} A_m K_{m-1}(\beta r)\,\mathcal{F}_{\pm(m-1)} \\ H_z &= \mp\frac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega} A_m\frac{J_m(\gamma a)}{K_m(\beta a)} K_m(\beta r)\,\mathcal{F}_{\pm m} \end{aligned}}_{HE_{mn}\ modes\ (r>a)} \tag{1.210}$$

1.4.5.4. *Construction of linearly polarized LP$_{mn}$ modes*

When two or more modes are degenerate, it is impossible to propagate each mode individually within the guide. In practice, the degenerate modes propagate simultaneously and at the same speed. As a result, they interfere with each other. At the fiber outlet, the observed signal results from the superposition of all degenerate modes with the same propagation constant. Equations [1.209] and [1.210] (obtained under weak guiding conditions) show that the modes EH$_{m-1,n}$ and HE$_{m+1,n}$ are degenerate. Consequently, these two modes interfere, leading to a polarized mode along the Ox or Oy axis, denoted as LP$_{mn}$.

By adding the EH$_{m-1,n}$ and HE$_{m+1,n}$ modes (assuming $A_{m-1} = A_{m+1} = A$), we find:

$$r<a\ \begin{cases} E_x = H_y = 0 \\ E_y = \mp\dfrac{2\mathsf{k}_z}{\gamma} A J_m(\gamma r)\,\mathcal{F}_{\pm m} \\ E_z = A\left[J_{m-1}(\gamma r)\,e^{\mp\mathsf{i}\varphi} + J_{m+1}(\gamma r)\,e^{\pm\mathsf{i}\varphi}\right]\mathcal{F}_{\pm m} \\ H_x = \pm\dfrac{2\omega\varepsilon_1}{\gamma} A J_m(\gamma r)\,\mathcal{F}_{\pm m} \\ H_z = \pm\dfrac{\mathsf{i}\mathsf{k}_z}{\mu_0\omega} A\left[J_{m-1}(\gamma r)\,e^{\mp\mathsf{i}\varphi} - J_{m+1}(\gamma r)\,e^{\pm\mathsf{i}\varphi}\right]\mathcal{F}_{\pm m} \end{cases} \tag{1.211}$$

$$r > a \begin{cases} E_x = \frac{\mathsf{i}\mathsf{k}_z}{\beta} A \left[\frac{J_{m-1}(\gamma a)}{K_{m-1}(\beta a)} + \frac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} \right] K_m(\beta r)\, \mathcal{F}_{\pm m} \\ E_y = \pm \frac{\mathsf{k}_z}{\beta} A \left[\frac{J_{m-1}(\gamma a)}{K_{m-1}(\beta a)} - \frac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} \right] K_m(\beta r)\, \mathcal{F}_{\pm m} \\ E_z = A_m \left[\frac{J_{m-1}(\gamma a)}{K_{m-1}(\beta a)} K_{m-1}(\beta r)\, e^{\mp \mathsf{i}\varphi} \right. \\ \left. + \frac{J_{m+1}(\gamma a)}{K_{m-1}(\beta a)} K_{m+1}(\beta r)\, e^{\pm \mathsf{i}\varphi} \right] \mathcal{F}_{\pm m} \\ H_x = \mp \frac{\omega \varepsilon_2}{\beta} A \left[\frac{J_{m-1}(\gamma a)}{K_{m-1}(\beta a)} - \frac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} \right] K_m(\beta r)\, \mathcal{F}_{\pm m} \\ H_y = \frac{\mathsf{i}\omega \varepsilon_2}{\beta} A \left[\frac{J_{m-1}(\gamma a)}{K_{m-1}(\beta a)} + \frac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} \right] K_m(\beta r)\, \mathcal{F}_{\pm m} \\ H_z = \pm \frac{\mathsf{i}\mathsf{k}_z}{\mu_0 \omega} A \left[\frac{J_{m-1}(\gamma a)}{K_{m-1}(\beta a)} K_{m-1}(\beta r)\, e^{\mp \mathsf{i}\varphi} \right. \\ \left. - \frac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} K_{m+1}(\beta r)\, e^{\pm \mathsf{i}\varphi} \right] \mathcal{F}_{\pm m} \end{cases} \qquad [1.212]$$

We demonstrate from equations [1.205] and [1.206] that:

$$\frac{J_{m-1}(\gamma a)}{K_{m-1}(\gamma a)} + \frac{J_{m+1}(\gamma a)}{K_{m+1}(\gamma a)} = 0 \qquad [1.213]$$

Thus, formulas [1.212] reduce to:

$$r > a \begin{cases} E_x = H_y = 0 \\ E_y = \mp \frac{2\mathsf{k}_z}{\beta} A \frac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} K_m(\beta r)\, \mathcal{F}_{\pm m} \\ E_z = A_m \left[-K_{m-1}(\beta r)\, e^{\mp \mathsf{i}\varphi} + K_{m+1}(\beta r)\, e^{\pm \mathsf{i}\varphi} \right] \frac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} \mathcal{F}_{\pm m} \\ H_x = \pm \frac{2\omega \varepsilon_2}{\beta} A \frac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} K_m(\beta r)\, \mathcal{F}_{\pm m} \\ H_z = \mp \frac{\mathsf{i}\mathsf{k}_z}{\mu_0 \omega} A \left[K_{m-1}(\beta r)\, e^{\mp \mathsf{i}\varphi} + K_{m+1}(\beta r)\, e^{\pm \mathsf{i}\varphi} \right] \frac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} \mathcal{F}_{\pm m} \end{cases}$$

[1.214]

We clearly see that the addition of the EH$m-1, n$ and HE$m+1, n$ modes leads to a linearly polarized LPmn mode along the Oy axis (i.e. with $E_x = 0$), described by relations [1.211] and [1.214]. When subtracting the EH$m-1, n$ and HE$m+1, n$ modes, we obtain a linearly polarized mode along the Ox axis (i.e. with $E_y = 0$). In fact, the EH$m-1, n +$ HE$m+1, n$ and EH$m-1, n -$ HE$_{m+1,n}$ modes are identical in the sense that one can be transformed into the other simply by interchanging Ox and Oy.

In real notation, the electric and magnetic fields (of the LP$_{mn}$ mode) are written as:

$$r < a \begin{cases} E_x = H_y = 0 \\ E_y = \mp \dfrac{2\mathsf{k}_z}{\gamma} A J_m(\gamma r) \cos(\omega t - \mathsf{k}_z z \mp m\varphi) \\ E_z = A\left[J_{m-1}(\gamma r)\cos\left[\omega t - \mathsf{k}_z z \mp (m-1)\varphi\right] + \right. \\ \qquad \left. J_{m+1}(\gamma r)\cos\left[\omega t - \mathsf{k}_z z \mp (m+1)\varphi\right]\right] \\ H_x = \pm \dfrac{2\omega\varepsilon_1}{\gamma} A J_m(\gamma r)\cos(\omega t - \mathsf{k}_z z \mp m\varphi) \\ H_z = \pm \dfrac{\mathsf{k}_z}{\mu_0 \omega} A\left[J_{m-1}(\gamma r)\sin\left[\omega t - \mathsf{k}_z z \mp (m-1)\varphi\right] - \right. \\ \qquad \left. J_{m+1}(\gamma r)\sin\left[\omega t - \mathsf{k}_z z \mp (m+1)\varphi\right]\right] \end{cases} \quad [1.215]$$

$$r > a \begin{cases} E_x = H_y = 0 \\ E_y = \mp \dfrac{2\mathsf{k}_z}{\beta} A \dfrac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} K_m(\beta r)\cos(\omega t - \mathsf{k}_z z \mp m\varphi) \\ E_z = A_m\left[-K_{m-1}(\beta r)\cos\left[\omega t - \mathsf{k}_z z \mp (m-1)\varphi\right] + \right. \\ \qquad \left. K_{m+1}(\beta r)\cos\left[\omega t - \mathsf{k}_z z \mp (m+1)\varphi\right]\right] \dfrac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} \\ H_x = \pm \dfrac{2\omega\varepsilon_2}{\beta} A \dfrac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} K_m(\beta r)\cos(\omega t - \mathsf{k}_z z \mp m\varphi) \\ H_z = \mp \dfrac{\mathsf{k}_z}{\mu_0\omega} A\left[K_{m-1}(\beta r)\sin\left[\omega t - \mathsf{k}_z z \mp (m-1)\varphi\right] + \right. \\ \qquad \left. K_{m+1}(\beta r)\sin\left[\omega t - \mathsf{k}_z z \mp (m+1)\varphi\right]\right] \dfrac{J_{m+1}(\gamma a)}{K_{m+1}(\beta a)} \end{cases} \quad [1.216]$$

The previous reasoning does not apply to the $m = 0$ case. In fact, the HE$_{1n}$ mode cannot be associated with the EH$_{-1,n}$ mode, since there are no modes with negative m indices. The field components of the LP$_{1n}$ mode are therefore obtained directly from those of the mode HE$_{1n}$, so its dispersion relation is that of the HE$_{1n}$ mode:

$$\frac{\gamma a J_1(\gamma a)}{J_0(\gamma a)} - \frac{\beta a K_1(\beta a)}{K_0(\beta a)} = 0 \quad [1.217]$$

1.5. Resonant cavities

Instead of guiding a wave, it is also possible to trap the energy of an electromagnetic wave inside a conductor that is closed on all sides (an *electromagnetic cavity*). The boundary conditions imposed by the walls now show that monochromatic traveling waves cannot propagate inside: only standing waves can exist, with discrete, so-called *quantized frequencies*. Our aim is therefore to explicitly find the cavity's natural frequencies, as well as the configuration of the fields describing each oscillation mode.

1.5.1. *Rectangular metallic resonant cavity*

A rectangular metallic waveguide becomes a resonant cavity by adding flat walls at each end, specifically at $z = 0$ and $z = l$. For a given frequency, two progressive waves propagating in the $+Oz$ and $-Oz$ directions must overlap to form a standing wave that satisfies the boundary conditions at the borders of the walls located at $z = 0$ and $z = l$. These boundary conditions are $H_z = 0$ and $E_x = E_y = 0$ on the new walls. The interior of the cavity is assumed to be empty.

Let us first consider the TE modes. Using equations [1.84] and [1.85], we obtain the following:

$$\begin{aligned}
E_x &= \Re\left[-\frac{\mathsf{i}\omega\mu_0\mathsf{k}_y}{\mathsf{k}_x^2+\mathsf{k}_y^2}\cos\left(\mathsf{k}_x x\right)\sin\left(\mathsf{k}_y y\right)\left(H_{mn}^+ e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)} + H_{mn}^- e^{-\mathsf{i}(\omega t+\mathsf{k}_z z)}\right)\right]\\
E_y &= \Re\left[\frac{\mathsf{i}\omega\mu_0\mathsf{k}_x}{\mathsf{k}_x^2+\mathsf{k}_y^2}\sin\left(\mathsf{k}_x x\right)\cos\left(\mathsf{k}_y y\right)\left(H_{mn}^+ e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)} + H_{mn}^- e^{-\mathsf{i}(\omega t+\mathsf{k}_z z)}\right)\right]\\
E_z &= 0\\
H_x &= \Re\left[-\frac{\mathsf{i}\mathsf{k}_z\mathsf{k}_x}{\mathsf{k}_x^2+\mathsf{k}_y^2}\sin\left(\mathsf{k}_x x\right)\cos\left(\mathsf{k}_y y\right)\left(H_{mn}^+ e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)} - H_{mn}^- e^{-\mathsf{i}(\omega t+\mathsf{k}_z z)}\right)\right]\\
H_y &= \Re\left[-\frac{\mathsf{i}\mathsf{k}_z\mathsf{k}_y}{\mathsf{k}_x^2+\mathsf{k}_y^2}\cos\left(\mathsf{k}_x x\right)\sin\left(\mathsf{k}_y y\right)\left(H_{mn}^+ e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)} - H_{mn}^- e^{-\mathsf{i}(\omega t+\mathsf{k}_z z)}\right)\right]\\
H_z &= \Re\left[\cos\left(\mathsf{k}_x x\right)\cos\left(\mathsf{k}_y y\right)\left(H_{mn}^+ e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)} + H_{mn}^- e^{-\mathsf{i}(\omega t+\mathsf{k}_z z)}\right)\right]
\end{aligned} \tag{1.218}$$

where $\mathsf{k}_x = \frac{m\pi}{a}$ and $\mathsf{k}_y = \frac{n\pi}{b}$. H_{mn}^+ and H_{mn}^- are the amplitudes of waves propagating in the $+Oz$ and $-Oz$ directions, respectively. For H_z to cancel at $z = 0$, choose $H_{mn}^- = -H_{mn}^+$, and for it to cancel at $z = l$, quantize k_z: $\mathsf{k}_z = \frac{p\pi}{l}$, where p is a positive integer. Thus:

$$\begin{aligned}
E_x &= \frac{2\omega_{mnp}\mu_0\mathsf{k}_y}{\mathsf{k}_x^2+\mathsf{k}_y^2}H_{mn}^+\cos\left(\mathsf{k}_x x\right)\sin\left(\mathsf{k}_y y\right)\sin\left(\mathsf{k}_z z\right)\cos\left(\omega_{mnp}t\right)\\
E_y &= -\frac{2\omega_{mnp}\mu_0\mathsf{k}_x}{\mathsf{k}_x^2+\mathsf{k}_y^2}H_{mn}^+\sin\left(\mathsf{k}_x x\right)\cos\left(\mathsf{k}_y y\right)\sin\left(\mathsf{k}_z z\right)\cos\left(\omega_{mnp}t\right)\\
E_z &= 0
\end{aligned}$$

$$
\begin{aligned}
H_x &= -\frac{2\mathsf{k}_z\mathsf{k}_x}{\mathsf{k}_x^2+\mathsf{k}_y^2} H_{mn}^+ \sin(\mathsf{k}_x x)\cos(\mathsf{k}_y y)\cos(\mathsf{k}_z z)\sin(\omega_{mnp} t) \\
H_y &= -\frac{2\mathsf{k}_z\mathsf{k}_y}{\mathsf{k}_x^2+\mathsf{k}_y^2} H_{mn}^+ \cos(\mathsf{k}_x x)\sin(\mathsf{k}_y y)\cos(\mathsf{k}_z z)\sin(\omega_{mnp} t) \\
H_z &= 2H_{mn}^+ \cos(\mathsf{k}_x x)\cos(\mathsf{k}_y y)\sin(\mathsf{k}_z z)\sin(\omega_{mnp} t)
\end{aligned}
\qquad [1.219]
$$

We can easily verify that E_x and E_y vanish in $z=0$ and $z=l$. This corresponds to the TE_{mnp} mode. The index p must be non-null in order for H_z to be non-null as well. The indices m or n (but not both) may be zero.

For TM waves, we must take the following superposition (see [1.54] and [1.55]):

$$
\begin{aligned}
E_x &= \Re\left[-\frac{\mathsf{i}\mathsf{k}_x\mathsf{k}_z}{\mathsf{k}_x^2+\mathsf{k}_y^2}\cos(\mathsf{k}_x x)\sin(\mathsf{k}_y y)\left(E_{mn}^+ e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)} - E_{mn}^- e^{-\mathsf{i}(\omega t+\mathsf{k}_z z)}\right)\right] \\
E_y &= \Re\left[\frac{\mathsf{i}\mathsf{k}_y\mathsf{k}_z}{\mathsf{k}_x^2+\mathsf{k}_y^2}\sin(\mathsf{k}_x x)\cos(\mathsf{k}_y y)\left(E_{mn}^+ e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)} - E_{mn}^- e^{-\mathsf{i}(\omega t+\mathsf{k}_z z)}\right)\right] \\
E_z &= \Re\left[\sin(\mathsf{k}_x x)\sin(\mathsf{k}_y y)\left(E_{mn}^+ e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)} + E_{mn}^- e^{-\mathsf{i}(\omega t+\mathsf{k}_z z)}\right)\right] \\
H_x &= \Re\left[-\frac{\mathsf{i}\omega\mathsf{k}_y}{\mu_0 c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)}\sin(\mathsf{k}_x x)\cos(\mathsf{k}_y y)\right. \\
&\left.\left(E_{mn}^+ e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)} + E_{mn}^- e^{-\mathsf{i}(\omega t+\mathsf{k}_z z)}\right)\right] \\
H_y &= \Re\left[\frac{\mathsf{i}\omega\mathsf{k}_x}{\mu_0 c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)}\cos(\mathsf{k}_x x)\sin(\mathsf{k}_y y)\right. \\
&\left.\left(E_{mn}^+ e^{-\mathsf{i}(\omega t-\mathsf{k}_z z)} + E_{mn}^- e^{-\mathsf{i}(\omega t+\mathsf{k}_z z)}\right)\right] \\
H_z &= 0
\end{aligned}
\qquad [1.220]
$$

To satisfy the constraints $E_x(z=0) = E_y(z=0) = 0$ and $E_x(z=l) = E_y(z=l) = 0$, we must have $E_{mn}^- = E_{mn}^+$ and $\mathsf{k}_z = \frac{n\pi}{l}$ (i.e. k_z is quantized). It therefore follows that:

$$
\begin{aligned}
E_x &= -\frac{2\mathsf{k}_x\mathsf{k}_z}{\mathsf{k}_x^2+\mathsf{k}_y^2} E_{mn}^+ \cos(\mathsf{k}_x x)\sin(\mathsf{k}_y y)\sin(\mathsf{k}_z z)\cos(\omega_{mnp} t) \\
E_y &= -\frac{2\mathsf{k}_y\mathsf{k}_z}{\mathsf{k}_x^2+\mathsf{k}_y^2} E_{mn}^+ \sin(\mathsf{k}_x x)\cos(\mathsf{k}_y y)\sin(\mathsf{k}_z z)\cos(\omega_{mnp} t) \\
E_z &= 2E_{mn}^+ \sin(\mathsf{k}_x x)\sin(\mathsf{k}_y y)\cos(\mathsf{k}_z z)\cos(\omega_{mnp} t) \\
H_x &= -\frac{2\omega_{mnp}\mathsf{k}_y}{\mu_0 c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)} E_{mn}^+ \sin(\mathsf{k}_x x)\cos(\mathsf{k}_y y)\cos(\mathsf{k}_z z)\sin(\omega_{mnp} t) \\
H_y &= \frac{2\omega_{mnp}\mathsf{k}_x}{\mu_0 c^2\left(\mathsf{k}_x^2+\mathsf{k}_y^2\right)} E_{mn}^+ \cos(\mathsf{k}_x x)\sin(\mathsf{k}_y y)\cos(\mathsf{k}_z z)\sin(\omega_{mnp} t) \\
H_z &= 0
\end{aligned}
\qquad [1.221]
$$

where m and n must both be non-null in order for E_z to be non-null and for the solution to be non-trivial. This is the mode TM_{mnp}. In both cases, the oscillation frequencies are now quantized. The allowed values are given as:

$$\omega_{mnp} = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{l}\right)^2} \qquad [1.222]$$

Obviously, the distinction between the TE and TM modes is somewhat ambiguous in a rectangular cavity, since any of the three Cartesian axes can be chosen as the waveguide axis. The simplest modes are TE_{101}, TE_{011} and TM_{110}. In each case, only one component of the electric field $\overrightarrow{\mathbf{E}}$ is non-null (respectively, E_z, E_y and E_x), and the corresponding component of the magnetic field $\overrightarrow{\mathbf{H}}$ vanishes.

EXAMPLE 1.4. *TE_{101}* MODE.– In the TE_{101} mode, the fields are written as:

$$\begin{aligned}
\overrightarrow{\mathbf{E}} &= -\frac{2\sqrt{a^2+l^2}}{l} Z_0 H_{10}^{+} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi z}{l}\right) \cos\left(\omega_0 t\right) \overrightarrow{\mathbf{j}} \\
\overrightarrow{\mathbf{H}} &= 2H_{10}^{+} \left[-\frac{a}{l} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi z}{l}\right) \overrightarrow{\mathbf{i}} + \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi z}{l}\right) \overrightarrow{\mathbf{k}}\right] \sin\left(\omega_0 t\right)
\end{aligned} \qquad [1.223]$$

where $\omega_0 = \frac{c\pi\sqrt{a^2+l^2}}{al}$ is the natural frequency of the mode and $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$.

1) Since the electric field is aligned with the Oy axis, electric charges accumulate only on the faces $y = 0$ and $y = b$, with the respective surface charge densities:

$$\begin{aligned}
\sigma\left(y=0\right) &= \varepsilon_0 \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{E}}\Big|_{y=0^+} = -\frac{2\sqrt{a^2+l^2}}{cl} H_{10}^{+} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi z}{l}\right) \cos\left(\omega_0 t\right) \\
\sigma\left(y=b\right) &= -\varepsilon_0 \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{E}}\Big|_{y=b^-} = -\frac{2\sqrt{a^2+l^2}}{cl} H_{10}^{+} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi z}{l}\right) \cos\left(\omega_0 t\right)
\end{aligned} \qquad [1.224]$$

It is clear that the total charge is zero.

2) The discontinuity of the tangential component of $\overrightarrow{\mathbf{H}}$ at the metal – vacuum interface allows us to calculate the surface currents accumulated on the various faces of the cavity:

$$\begin{aligned}
\overrightarrow{\mathbf{K}}\Big|_{x=0} &= \overrightarrow{\mathbf{i}} \times \overrightarrow{\mathbf{H}}\Big|_{x=0^+} = -2H_{10}^{+} \sin\left(\frac{\pi z}{l}\right) \sin\left(\omega_0 t\right) \overrightarrow{\mathbf{j}} \\
\overrightarrow{\mathbf{K}}\Big|_{x=a} &= -\overrightarrow{\mathbf{i}} \times \overrightarrow{\mathbf{H}}\Big|_{x=a^-} = -2H_{10}^{+} \sin\left(\frac{\pi z}{l}\right) \sin\left(\omega_0 t\right) \overrightarrow{\mathbf{j}}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\mathbf{K}}\Big|_{y=0} &= \overrightarrow{\mathbf{j}} \times \overrightarrow{\mathbf{H}}\Big|_{y=0^+} = 2H_{10}^{+}\left[\frac{a}{l}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{l}\right)\overrightarrow{\mathbf{k}}\right.\\
&\left.+\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{l}\right)\overrightarrow{\mathbf{i}}\right]\sin\left(\omega_0 t\right)\\
\overrightarrow{\mathbf{K}}\Big|_{y=b} &= -\overrightarrow{\mathbf{j}} \times \overrightarrow{\mathbf{H}}\Big|_{y=b^-} = -2H_{10}^{+}\left[\frac{a}{l}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{l}\right)\overrightarrow{\mathbf{k}}\right.\\
&\left.+\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{l}\right)\overrightarrow{\mathbf{i}}\right]\sin\left(\omega_0 t\right)\\
\overrightarrow{\mathbf{K}}\Big|_{z=0} &= \overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{H}}\Big|_{z=0^+} = -2H_{10}^{+}\frac{a}{l}\sin\left(\frac{\pi x}{a}\right)\sin\left(\omega_0 t\right)\overrightarrow{\mathbf{j}}\\
\overrightarrow{\mathbf{K}}\Big|_{z=l} &= -\overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{H}}\Big|_{z=l^-} = -2H_{10}^{+}\frac{a}{l}\sin\left(\frac{\pi x}{a}\right)\sin\left(\omega_0 t\right)\overrightarrow{\mathbf{j}}
\end{aligned} \qquad [1.225]$$

3) For TE_{101} mode, the electromagnetic energy stored in the cavity is equal to:

$$\begin{aligned}
W &= \frac{1}{2}\int_{\text{cavity}}\left(\varepsilon_0\overrightarrow{\mathbf{E}}^2+\mu_0\overrightarrow{\mathbf{H}}^2\right)dxdydz\\
&= 2\mu_0 H_{10}^{+2}\int_0^a\int_0^b\int_0^l\left\{\frac{\left(a^2+l^2\right)}{l^2}\sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi z}{l}\right)\cos^2\left(\omega_0 t\right)+\right.\\
&\left.\left[\frac{a^2}{l^2}\sin^2\left(\frac{\pi x}{a}\right)\cos^2\left(\frac{\pi z}{l}\right)+\cos^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi z}{l}\right)\right]\sin^2\left(\omega_0 t\right)\right\}dxdydz\\
&= \frac{\mu_0 ab\left(a^2+l^2\right)H_{10}^{+2}}{2l} = \xi_1 H_{10}^{+2}
\end{aligned} \qquad [1.226]$$

where $\xi_1 = \frac{\mu_0 ab\left(a^2+l^2\right)}{2l}$ (identities [1.68] were used).

4) In reality, the metal used to make the cavity is not a perfect conductor. It has a finite conductivity γ_c. In this case, the current is not a surface current but flows within a layer of thickness δ (skin effect), which leads to energy dissipation due to Joule heating. The instantaneous power dissipated by each surface element dS of the cavity is given by the expression $dP_d = \frac{\overrightarrow{\mathbf{K}}^2}{\gamma_c\delta}dS$. Thus, the total power dissipated by the entire cavity is given as:

$$\begin{aligned}
P_d &= \frac{1}{\gamma_c\delta}\left\{\int_0^a\int_0^l\left[\overrightarrow{\mathbf{K}}^2\left(y=0\right)+\overrightarrow{\mathbf{K}}^2\left(y=b\right)\right]dxdz\right.\\
&+\int_0^b\int_0^l\left[\overrightarrow{\mathbf{K}}^2\left(x=0\right)+\right.\\
&\left.\left.\overrightarrow{\mathbf{K}}^2\left(x=a\right)\right]dydz+\int_0^a\int_0^b\left[\overrightarrow{\mathbf{K}}^2\left(z=0\right)+\overrightarrow{\mathbf{K}}^2\left(z=l\right)\right]dxdy\right\}
\end{aligned}$$

$$= \frac{2}{\gamma_c \delta}\left\{\int_0^a \int_0^l \overrightarrow{\mathbf{K}}^2 (y=0)\, dxdz + \int_0^b \int_0^l \overrightarrow{\mathbf{K}}^2 (x=0)\, dydz \right.$$
$$\left. + \int_0^a \int_0^b \overrightarrow{\mathbf{K}}^2 (z=0)\, dxdy \right\}$$
$$= \frac{4a^2 H_{10}^{+2}}{\gamma_c \delta}\left[\frac{ab}{l^2} + \frac{bl}{a^2} + \frac{1}{2}\left(\frac{a}{l} + \frac{l}{a}\right)\right] \sin^2(\omega_0 t) \quad [1.227]$$

5) The power dissipated (averaged over one period) is equal to:

$$\langle P_d \rangle = \frac{4a^2 H_{10}^{+2}}{\gamma_c \delta}\left[\frac{ab}{l^2} + \frac{bl}{a^2} + \frac{1}{2}\left(\frac{a}{l} + \frac{l}{a}\right)\right] \langle \sin^2(\omega_0 t)\rangle = \xi_2 H_{10}^{+2} \quad [1.228]$$

where $\xi_2 = \frac{2a^2}{\gamma_c \delta}\left[\frac{ab}{l^2} + \frac{bl}{a^2} + \frac{1}{2}\left(\frac{a}{l} + \frac{l}{a}\right)\right]$ (since $\langle \sin^2(\omega_0 t)\rangle = \frac{1}{2}$).

6) In the absence of sources compensating for the losses, the variation in total energy over time is only due to the dissipation of energy in the walls. Therefore, we write that:

$$\frac{dW_{\text{em}}}{dt} = -\langle P_d \rangle \implies \frac{dH_{10}^{+}}{dt} + \frac{\xi_2}{2\xi_1} H_{10}^{+} = 0 \quad [1.229]$$

7) The *quality factor* Q of a cavity is defined as the ratio of the electromagnetic energy stored in the cavity to the energy loss per oscillation cycle:

$$Q = 2\pi \frac{W_{\text{em}}}{T_0 \langle P_d \rangle} = \frac{\omega_0 W_{\text{em}}}{\langle P_d \rangle} \quad [1.230]$$

where ω_0 is the frequency of the oscillation mode under consideration. It follows, for mode TE_{101}, that:

$$Q = \frac{\omega_0 \xi_1}{\xi_2} = \frac{\pi \gamma_c \delta Z_0}{4} \frac{2b\left(a^2 + l^2\right)^{3/2}}{al\left(a^2 + l^2\right) + 2b\left(a^3 + l^3\right)} \quad [1.231]$$

Since both the energy W_{em} stored in the cavity and the power dissipated (averaged over one period) are proportional to the square of the field amplitudes, we see that the quality factor defined in this way is independent of the amplitude and depends only on the shape of the cavity, the oscillation mode under consideration and the conductivity of the walls.

8) Equation [1.229] admits the following solution:

$$H_{10}^{+}(t) = H_{10}^{+}(0)\, e^{-\frac{\xi_2 t}{2\xi_1}} \equiv H_{10}^{+}(0)\, e^{-\frac{\omega_0 t}{2Q}} \quad [1.232]$$

We can see that the variations in electric and magnetic fields [1.238] are no longer purely oscillatory over time, but that the fields decrease exponentially:

$$\begin{aligned}
&\overrightarrow{\mathbf{E}} = -\frac{2\sqrt{a^2+l^2}}{l}\sqrt{\frac{\mu_0}{\varepsilon_0}}H_{10}^{+}(0)\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{l}\right)\cos(\omega_0 t)\,e^{-\frac{\omega_0 t}{2Q}}\overrightarrow{\mathbf{j}}\\
&\overrightarrow{\mathbf{H}} = 2H_{10}^{+}(0)\left[-\frac{a}{l}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{l}\right)\overrightarrow{\mathbf{i}}\right.\\
&\left.+\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{l}\right)\overrightarrow{\mathbf{k}}\right]\sin(\omega_0 t)\,e^{-\frac{\omega_0 t}{2Q}}
\end{aligned} \qquad [1.233]$$

The higher the quality factor, the closer the damping term $e^{-\frac{\omega_0 t}{2Q}}$ is to 1.

1.5.2. *Cylindrical resonant cavity*

By closing the cylindrical waveguide with circular walls at $z = 0$ and $z = l$, a standing wave is established inside. By superimposing two traveling waves of the same frequency propagating in opposite directions along $\pm Oz$, we obtain for the TM mode:

$$\begin{aligned}
E_r &= \Re\left[\frac{\mathsf{i}\mathsf{k}_z a}{\rho_{mn}}J_m'\left(\frac{\rho_{mn}r}{a}\right)e^{\pm\mathsf{i}m\varphi}\left(E_{mn}^{+}e^{\mathsf{i}\mathsf{k}_z z}-E_{mn}^{-}e^{-\mathsf{i}\mathsf{k}_z z}\right)e^{-\mathsf{i}\omega t}\right]\\
E_\varphi &= \Re\left[\mp\frac{m\mathsf{k}_z a^2}{\rho_{mn}^2 r}J_m\left(\frac{\rho_{mn}r}{a}\right)e^{\pm\mathsf{i}m\varphi}\left(E_{mn}^{+}e^{\mathsf{i}\mathsf{k}_z z}-E_{mn}^{-}e^{-\mathsf{i}\mathsf{k}_z z}\right)e^{-\mathsf{i}\omega t}\right]\\
E_z &= \Re\left[J_m\left(\frac{\rho_{mn}r}{a}\right)e^{\pm\mathsf{i}m\varphi}\left(E_{mn}^{+}e^{\mathsf{i}\mathsf{k}_z z}+E_{mn}^{-}e^{-\mathsf{i}\mathsf{k}_z z}\right)e^{-\mathsf{i}\omega t}\right]\\
H_r &= \Re\left[\pm\frac{m\omega a^2\varepsilon_0}{\rho_{mn}^2 r}J_m\left(\frac{\rho_{mn}r}{a}\right)e^{\pm\mathsf{i}m\varphi}\left(E_{mn}^{+}e^{\mathsf{i}\mathsf{k}_z z}+E_{mn}^{-}e^{-\mathsf{i}\mathsf{k}_z z}\right)e^{-\mathsf{i}\omega t}\right]\\
H_\varphi &= \Re\left[\frac{\mathsf{i}\omega a\varepsilon_0}{\rho_{mn}}J_m'\left(\frac{\rho_{mn}r}{a}\right)e^{\pm\mathsf{i}m\varphi}\left(E_{mn}^{+}e^{\mathsf{i}\mathsf{k}_z z}+E_{mn}^{-}e^{-\mathsf{i}\mathsf{k}_z z}\right)e^{-\mathsf{i}\omega t}\right]\\
H_z &= 0
\end{aligned} \qquad [1.234]$$

For E_r and E_φ to be canceled on the new boundaries, we must have $E_{mn}^{-} = E_{mn}^{+}$ and $\mathsf{k}_z = \frac{p\pi}{l}$, where p is an integer. It therefore follows that:

$$\begin{aligned}
E_r &= -\frac{2\mathsf{k}_z a}{\rho_{mn}}J_m'\left(\frac{\rho_{mn}r}{a}\right)e^{\pm\mathsf{i}m\varphi}E_{mn}^{+}\sin(\mathsf{k}_z z)\cos(\omega_{mnp}t)\\
E_\varphi &= \mp\frac{2\mathsf{i}m\mathsf{k}_z a^2}{\rho_{mn}^2 r}J_m\left(\frac{\rho_{mn}r}{a}\right)e^{\pm\mathsf{i}m\varphi}E_{mn}^{+}\sin(\mathsf{k}_z z)\cos(\omega_{mnp}t)\\
E_z &= 2J_m\left(\frac{\rho_{mn}r}{a}\right)e^{\pm\mathsf{i}m\varphi}E_{mn}^{+}\cos(\mathsf{k}_z z)\cos(\omega_{mnp}t)\\
H_r &= \pm\frac{2m\omega_{mnp}a^2\varepsilon_0}{\rho_{mn}^2 r}J_m\left(\frac{\rho_{mn}r}{a}\right)e^{\pm\mathsf{i}m\varphi}E_{mn}^{+}\cos(\mathsf{k}_z z)\cos(\omega_{mnp}t)\\
H_\varphi &= \frac{2\omega_{mnp}a\varepsilon_0}{\rho_{mn}}J_m'\left(\frac{\rho_{mn}r}{a}\right)e^{\pm\mathsf{i}m\varphi}E_{mn}^{+}\cos(\mathsf{k}_z z)\sin(\omega_{mnp}t)\\
H_z &= 0
\end{aligned} \qquad [1.235]$$

Repeating the same procedure for the TE mode, we find the following:

$$
\begin{aligned}
E_r &= \mp\frac{2m\omega_{mnp}a^2\mu_0}{\rho'^2_{mn}r}J_m\left(\frac{\rho'_{mn}r}{a}\right)e^{\pm \mathrm{i}m\varphi}H^+_{mn}\sin\left(\mathsf{k}_z z\right)\sin\left(\omega_{mnp}t\right)\\
E_\varphi &= \frac{2\omega_{mnp}a\mu_0}{\rho'_{mn}}J'_m\left(\frac{\rho'_{mn}r}{a}\right)e^{\pm \mathrm{i}m\varphi}H^+_{mn}\sin\left(\mathsf{k}_z z\right)\cos\left(\omega_{mnp}t\right)\\
E_z &= 0\\
H_r &= \frac{2\mathsf{k}_z a}{\rho'_{mn}}J'_m\left(\frac{\rho'_{mn}r}{a}\right)e^{\pm \mathrm{i}m\varphi}H^+_{mn}\cos\left(\mathsf{k}_z z\right)\sin\left(\omega_{mnp}t\right)\\
H_\varphi &= \mp\frac{2m\mathsf{k}_z a^2}{\rho'^2_{mn}r}J_m\left(\frac{\rho'_{mn}r}{a}\right)e^{\pm \mathrm{i}m\varphi}H^+_{mn}\cos\left(\mathsf{k}_z z\right)\cos\left(\omega_{mnp}t\right)\\
H_z &= 2J_m\left(\frac{\rho'_{mn}r}{a}\right)e^{\pm im\varphi}H^+_{mn}\sin\left(\mathsf{k}_z z\right)\sin\left(\omega_{mnp}t\right)
\end{aligned}
\qquad [1.236]
$$

We clearly see that H_z, E_r and E_φ vanish at $z = 0$ and $z = l$. In both cases, the frequencies are quantized as follows:

$$
\omega_{mnp} = c\sqrt{\frac{\rho^2_{mn}}{a^2} + \left(\frac{p\pi}{l}\right)^2} \qquad [1.237]
$$

EXAMPLE 1.5. *TM*$_{010}$ MODE.– In the *TM*$_{010}$ mode, the fields are expressed as follows:

$$
\begin{aligned}
\overrightarrow{\mathbf{E}} &= 2E^+_{01}J_0\left(\frac{\rho_{01}r}{a}\right)\cos\left(\omega_0 t\right)\overrightarrow{\mathbf{k}}\\
\overrightarrow{\mathbf{H}} &= 2Z_0^{-1}E^+_{01}J_0\left(\frac{\rho_{01}r}{a}\right)\sin\left(\omega_0 t\right)\overrightarrow{\mathbf{e}}_\varphi
\end{aligned}
\qquad [1.238]
$$

where $\omega_0 = \frac{c\rho_{01}}{a}$ is the *natural frequency of the mode* and $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$.

1) Now using the boundary conditions for a metal–vacuum interface, we give the expressions for the surface charge density σ and the surface current density $\overrightarrow{\mathbf{K}}$ on the walls of the cavity. The surface charge density on the lateral surface of the cylinder is zero for this mode, since the electric field is collinear with the Oz axis. On the flat walls of the cylinder (at $z = 0$ and $z = l$), the discontinuity of the normal component of the electric field requires that:

$$
\begin{aligned}
\sigma\left(z=0\right) &= \varepsilon_0 E_z|_{z=0^+} = 2\varepsilon_0 E^+_{01}J_0\left(\frac{\rho_{01}r}{a}\right)cos\left(\omega_0 t\right)\\
\sigma\left(z=l\right) &= -\,\varepsilon_0 E_z|_{z=l^-} = -2\varepsilon_0 E^+_{01}J_0\left(\frac{\rho_{01}r}{a}\right)cos\left(\omega_0 t\right)
\end{aligned}
\qquad [1.239]
$$

Note that the total charge sums to zero (the system is thus electrically neutral). For the calculation of the surface current density $\overrightarrow{\mathbf{K}}$, all walls carry the currents:

$$\begin{aligned}\overrightarrow{\mathbf{K}}(r=a) &= -\overrightarrow{\mathbf{e}}_r \times \overrightarrow{\mathbf{H}}\Big|_{r=a^-} = -2Z_0^{-1}J_0'(\rho_{01})\,E_{01}^+\sin(\omega_0 t)\,\overrightarrow{\mathbf{k}}\\ \overrightarrow{\mathbf{K}}(z=0) &= \overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{H}}\Big|_{z=0^+} = -2Z_0^{-1}J_0'\left(\frac{\rho_{01}r}{a}\right)E_{01}^+\sin(\omega_0 t))\overrightarrow{\mathbf{e}}_r\\ \overrightarrow{\mathbf{K}}(z=l) &= -\overrightarrow{\mathbf{k}} \times \overrightarrow{\mathbf{H}}\Big|_{z=l^-} = 2Z_0^{-1}J_0'\left(\frac{\rho_{01}r}{a}\right)E_{01}^+\sin(\omega_0 t))\overrightarrow{\mathbf{e}}_r\end{aligned} \quad [1.240]$$

2) Let us calculate the electromagnetic energy stored in the cavity for this fundamental mode TM_{010}:

$$\begin{aligned}W_{\text{em}} &= \int_{r=0}^{a}\int_{\varphi=0}^{2\pi}\int_{z=0}^{l}\frac{1}{2}\left(\varepsilon_0\overrightarrow{\mathbf{E}}^2+\mu_0\overrightarrow{\mathbf{H}}^2\right)rdrd\varphi dz\\ &= 4\pi\varepsilon_0 lE_{01}^2\left[\cos^2(\omega_0 t)\int_0^a J_0^2\left(\frac{\rho_{01}r}{a}\right)rdr+\sin^2(\omega_0 t)\int_0^a J'^2_0\left(\frac{\rho_{01}r}{a}\right)rdr\right]\\ &= 4\pi\varepsilon_0 lE_{01}^2\frac{a^2}{\rho_{01}^2}\left[\cos^2(\omega_0 t)\int_0^{\rho_{01}} J_0^2(x)\,xdx+\sin^2(\omega_0 t)\int_0^{\rho_{01}} J'^2_0(x)\,xdx\right]\end{aligned}$$

[1.241]

Since:

$$\int_0^{\rho_{01}} xJ_0^2(x)\,dx = \int_0^{\rho_{01}} xJ'^2_0(x)\,dx = I \quad [1.242]$$

(to be verified by integration by parts), we deduce that:

$$W_{\text{em}} = \pi a^2 l \times 4\varepsilon_0 E_{01}^2 \times \frac{I}{\rho_{01}^2} \equiv \xi_1 E_{01}^{+2} \quad [1.243]$$

where:

$$\xi_1 = 4\pi\varepsilon_0 a^2 l\frac{I}{\rho_{01}^2} \quad [1.244]$$

3) In the case of a real conductor, the instantaneous power dissipated by the Joule effect in each surface element dS of the wall is equal to $dP_d = \frac{1}{\gamma_c\delta}\overrightarrow{\mathbf{K}}^2 dS$. At the level of the lateral wall, the dissipated power is equal to:

$$P_d|_{r=a} = \int_0^l\int_0^{2\pi}\frac{\overrightarrow{\mathbf{K}}^2(r=a)}{\gamma_c\delta}ad\varphi dz = \frac{8\pi alE_{01}^{+2}}{Z_0^2\gamma_c\delta}J_0'^2(\rho_{01})\sin^2(\omega_0 t) \quad [1.245]$$

Similarly, the powers dissipated by the two flat surfaces are equal, respectively, to:

$$
\begin{aligned}
P_d|_{z=0} &= \int_0^a \int_0^{2\pi} \frac{\overrightarrow{\mathbf{K}}^2(z=0)}{\gamma_c\delta} r dr d\varphi = \frac{8\pi E_{01}^{+2}}{Z_0^2\gamma_c\delta}\sin^2(\omega_0 t)\int_0^a J_0'^2\left(\frac{\rho_{01}r}{a}\right) r dr \\
&= \frac{8\pi a^2 I E_{01}^{+2}}{Z_0^2\gamma_c\delta\rho_{01}^2}\sin^2(\omega_0 t) \\
P_d|_{z=l} &= \frac{8\pi a^2 I E_{01}^{+2}}{Z_0^2\gamma_c\delta\rho_{01}^2}\sin^2(\omega_0 t)
\end{aligned}
\quad [1.246]
$$

4) The powers dissipated (averaged over a period) are:

$$
\begin{aligned}
\langle P_d\rangle_{r=a} &= \frac{8\pi a l E_{01}^{+2}}{Z_0^2\gamma_c\delta} J_0'^2(\rho_{01})\langle\sin^2(\omega_0 t)\rangle = \frac{4\pi a l E_{01}^{+2}}{Z_0^2\gamma_c\delta} J_0'^2(\rho_{01}) = \xi_2 E_{01}^{+2} \\
\langle P_d\rangle_{z=0} &= \langle P_d\rangle_{z=l} = \frac{4\pi a^2 I E_{01}^{+2}}{Z_0^2\gamma_c\delta\rho_{01}^2} = \xi_3 E_{01}^{+2}
\end{aligned}
\quad [1.247]
$$

where:

$$
\begin{aligned}
\xi_2 &= \frac{4\pi a l}{Z_0^2\gamma_c\delta} J_0'^2(\rho_{01}) \\
\xi_3 &= \frac{4\pi a^2 I}{Z_0^2\gamma_c\delta\rho_{01}^2}
\end{aligned}
\quad [1.248]
$$

Thus, the power (averaged over a period) dissipated by the entire cavity is equal to:

$$
\langle P_d\rangle_{\text{total}} = \langle P_d\rangle_{r=a} + \langle P_d\rangle_{z=0} + \langle P_d\rangle_{z=l} = (\xi_2 + 2\xi_3) E_{01}^{+2} \quad [1.249]
$$

5) The quality factor of the cylindrical cavity for this mode is given by:

$$
Q = \frac{\omega_0 W_{\text{em}}}{\langle P_d\rangle_{\text{total}}} = \frac{\omega_0\xi_1}{\xi_2 + 2\xi_3} = \frac{l I Z_0\gamma_c\delta}{l\rho_{01}^2 J_0'(\rho_{01}) + 2aI} \quad [1.250]
$$

6) The variation in electromagnetic energy is mainly due to the dissipation of energy through the walls:

$$
\frac{dW_{\text{em}}}{dt} = -\langle P_d\rangle_{\text{total}} \implies \frac{dE_{01}^+}{dt} = -\frac{\xi_2 + 2\xi_3}{2\xi_1} E_{01}^+ \quad [1.251]
$$

Therefore, we obtain:

$$E_{01}^{+}(t) = E_{01}^{+}(0)\, e^{-\frac{(\xi_2+2\xi_3)t}{2\xi_1}} \equiv E_{01}^{+}(0)\, e^{-\frac{\omega_0 t}{2Q}} \qquad [1.252]$$

Also, in this case, we see that the electric and magnetic fields [1.239] no longer vary purely oscillatory over time but decrease exponentially:

$$\begin{aligned} \overrightarrow{\mathbf{E}} &= 2E_{01}^{+}(0)\, J_0\left(\frac{\rho_{01} r}{a}\right) \cos(\omega_0 t)\, e^{-\frac{\omega_0 t}{2Q}}\, \overrightarrow{\mathbf{k}} \\ \overrightarrow{\mathbf{H}} &= 2Z_0^{-1} E_{01}^{+}(0)\, J_0\left(\frac{\rho_{01} r}{a}\right) \sin(\omega_0 t)\, e^{-\frac{\omega_0 t}{2Q}}\, \overrightarrow{\mathbf{e}}_{\varphi} \end{aligned} \qquad [1.253]$$