1

Material Strength

In this chapter, we only briefly discuss beam theory elements such as beams, girders, or joists and their associated assumptions:

– The material is assumed to be homogeneous and isotropic. The deformations, because of external loads, are reversible and small (linear elasticity theory).

– The movements of the material points are negligible (first-order theory).

Two laws or principles therefore result from these assumptions:

– *Generalized Hooke's law*: it states that the relationships between external forces, stresses and deformations are linear and homogeneous.

– *The superposition principle*: a stress (or strain) produced by multiple applied loads is the superposition of the stresses produced by each of the loads assumed to act in isolation.

In the case of beams, there are two additional principles:

– *Saint-Venant's principle*: the stresses in a section Σ far from the application points of the external forces depend only on the stresses of the system built by forces applied on one side only of the sigma section Σ.

– *Navier–Bernoulli's principle*: when a beam deforms, the straight sections remain flat.

Therefore, to be able to apply the beam theory, it is necessary to ensure that these hypotheses are actually respected.

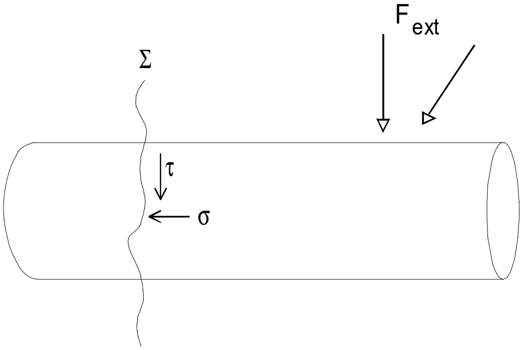

Figure 1.1. *Saint-Venant's principle*

It is generally considered that this theory gives reliable results if the following conditions are met:

– The width of the beam (cross-sectional dimension) is small compared with its length, that is, $\frac{1}{30} < \frac{h}{l} < \frac{1}{5}$ for a straight beam and $\frac{1}{100} < \frac{h}{l} < \frac{1}{5}$ for an arch, where h is the height of the beam and l is the length of the beam.

– The radius of curvature of the average fiber is greater than five times the height.

– For a beam with a variable cross-section, it must gradually vary along the mean fiber.

$\frac{1}{30} < \frac{h}{1} < \frac{1}{5}$ for a straight beam

$\frac{1}{100} < \frac{h}{1} < \frac{1}{5}$ for an arch

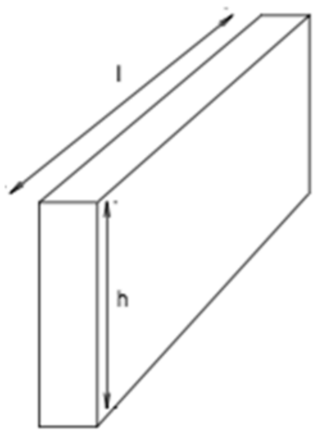

Figure 1.2. *Height/span ratio (h/l) for a beam and an arch*

The external loads applied to a beam are “actions” that produce “stresses” within the beam material. The most common stresses are as follows:

– the bending moment (bending of the beam);

– normal force (axial compression or traction);

– the shear force (beam shear).

In the following, we will study the different stresses that can be applied to beams.

NOTATIONS.–

M: bending moment

N: normal effort

T: shear force

t: shear force stress

E: modulus of elasticity

S: area of the section

σ_c: compressive stress

σ_t: tensile stress

1.1. Compression and traction

Consider a cross-section of any beam subjected to an external force perpendicular to the latter.

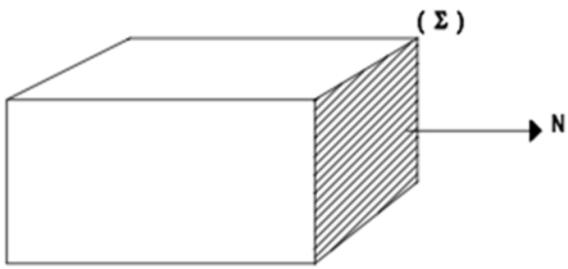

Figure 1.3. *Traction forces on a straight section*

Let S be the area of the section (Σ); this force causes a normal stress (perpendicular to (Σ)) on each element of the surface (Σ) to be constant over the entire section and equals:

$$\sigma_t = \frac{N}{S}$$

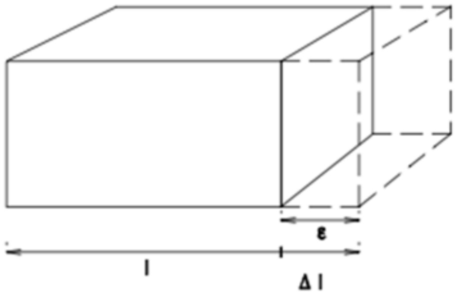

Figure 1.4. *Deformation of a prismatic element under stress by compression or traction*

Under the effect of this external force, the fibers of initial length l become elongated (case of traction) or shortened (case of compression) Δl such that:

$$\frac{\Delta l}{l} = \frac{N}{ES}$$

Indeed, according to the generalized Hooke's law, we know that the deformation equals:

$$\varepsilon_t = \frac{\sigma_t}{E} \text{ or } \varepsilon_t = \frac{\Delta l}{l} \text{ and } \sigma_t = \frac{N}{S}$$

Hence:

$$\frac{\Delta l}{l} = \frac{N}{ES}$$

The deformation therefore causes elongation for a traction effect and shortening for a compression force.

1.2. Pure flexion

Consider any beam and two straight sections of this beam.

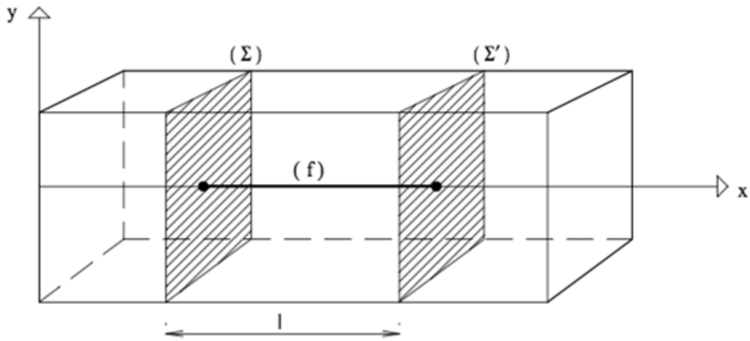

Figure 1.5. *Cross-sections of a beam being bent*

If we subject this beam to a system of forces generating a bending moment, according to the Navier–Bernoulli principle, the sections (Σ) and (Σ′) remain straight after deformation.

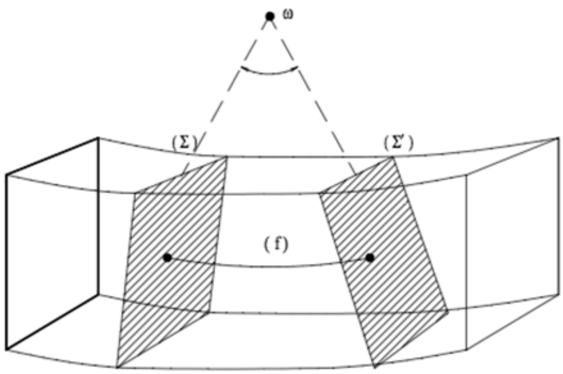

Figure 1.6. *Cross-sections after application of the bending moment*

The elongation of any fiber (f) between sections (Σ) and (Σ′) is a linear function of its coordinates in section (Σ).

The stress in the fiber (f) is then determined by considering Hooke’s law:

$$\sigma = a + by + cz$$

where the constants a, b and c are determined by the equivalence principle:

$$\iint_{(\Sigma)} \sigma dy\, dz = 0 \qquad \text{(force equilibrium)}$$

$$\iint_{(\Sigma)} \sigma .y.dy\, .dz = M \qquad \text{(moment equilibrium)}$$

$$\iint \sigma .z dy\, dz = 0 \qquad \text{(we assume M is directed along x)}$$

This ultimately leads to:

$$\sigma = \frac{M \quad y}{I}$$

where I is the inertia of the section relative to the main axis with the bending moment.

The elongation of the fiber (f) is then written as:

$$\varepsilon = \frac{\Delta l}{l} = \frac{M\ .\ y}{EI}$$

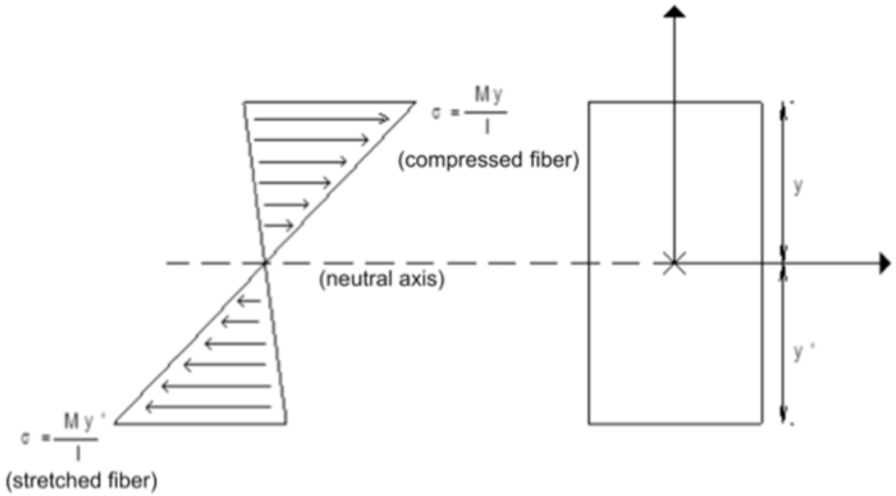

Figure 1.7. *Stress diagram of a cross-section of a beam subjected to a bending moment*

The relative movement of (Σ) and (Σ′) therefore results in a rotation:

$$\omega = \frac{M.L}{EI}$$

The constraints are represented in the diagram in Figure 1.7.

1.3. Shear strain

The effects of the shear force and bending moment usually occur simultaneously. The shear strain is given by:

$$t = \frac{T_m}{l_b}$$

where T_m is the static moment of the area above parallel to Gz. The maximum shear strain is given by:

$$t\ max = \frac{T}{bZ}$$

$$z = \frac{1}{\mu_0} \text{ internal torque lever arm}$$

The shear strain is then:

$$\gamma = \frac{T}{GS_I}$$

where S_I is the reduced section.

For:

– a rectangle: $S_I = 5/6\ S$;

– a diamond: $S_I = 30/31\ S$;

– a circle: $S_I = 9/10\ S$.

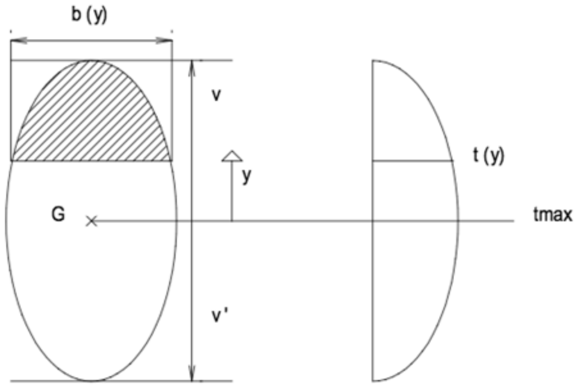

Figure 1.8. *Diagram of shear stress on a straight beam subjected to a bending moment*

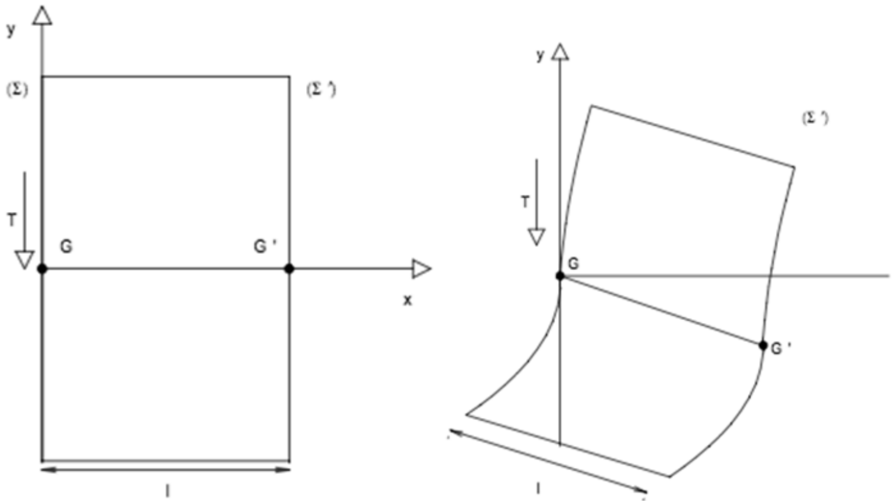

Figure 1.9. *Strain diagram*

1.4. Torsion

Studying torsion is relatively complex and is often done using the theory of elasticity, from which we retain the following elements:

– Torsional stresses are tangent stresses that are superimposed on the shear.

– These constraints are perpendicular to the radius vector coming from the center of torsion.

– For a circle, these constraints are proportional to this radius.

1.4.1. *Circular section*

When the deformation of a section due to torsion is analyzed, point M′ becomes M'_1 and therefore:

$$M'M'_1 = \gamma\ dx = \tau/G\ dx = \rho\ d\Theta$$

where $\rho = GM = G'M'$.

That is, $\tau = G\rho\ d\Theta/dx$.

The elemental force $\tau\ d\varpi$ then produces an elementary moment $\rho\tau\ d\varpi$. The sum of these elementary moments must balance the torsional moment Mt, hence:

$$Mt = GI_p\ d\Theta/dx = \tau/\rho\ I_p$$

where I_p is the polar moment of inertia.

We deduce that:

$$\tau = \rho\ Mt/I_p \text{ and } d\Theta/dx = Mt/(GI_p)$$

1.4.2. *Applying this to a circle*

Applying the polar inertia calculation to a circle gives the following result:

$$Ip = Ix + Iy = 2\ \pi\ R^4/4 = \pi\ R^4/2$$

That is, $\tau_{max} = 0.637\ Mt/R^3$, and $d\Theta/dx = 0.637\ Mt/(GR^4)$.

1.4.3. *Rectangular section*

It is calculated from a series development and a formula of the same type as for the circular section:

$$d\Theta/dx = Mt/(G*J)$$

where J is the torsion modulus, $J = Kab^3$.

We deduce that:

$d\Theta/dx = Mt/(GKab^3)$ and $\tau_{max} = K' Mt/(ab^2)$.

Caquot's approximation is used to determine parameters K and K′.

If $m = a/b$:

$1/K = (1 + 1/m^2)[0.225 - 0.035 ((m - 1)/(m + 1))^2]$

$K' = 0.601 - 0.226 (m - 1)/(m^2 + 1)^{0.5}$

1.5. Spherical shell theory

Considering a spherical shell of radius r, opening radius b in horizontal projection and deflection f, the radius of the sphere is then:

$r = (b^2 + f^2)/2f$

The angle at the vertex is equal to $\varphi = \pi/2 - \alpha$ according to the diagram in Figure 1.10.

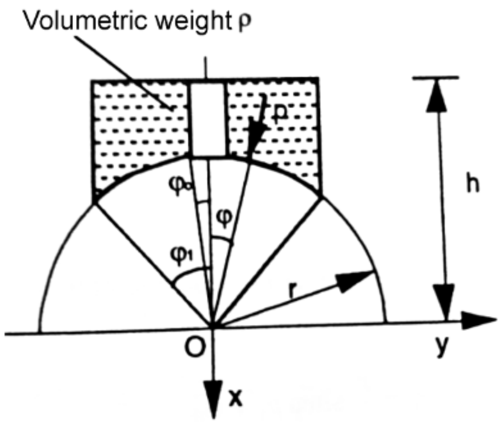

Figure 1.10. *Spherical shell diagram*

With these notations, $x = -r \cos\varphi$ and $dx = r \sin\varphi \, d\varphi$; $y = r \sin\varphi$ and $dy = r \cos\varphi \, d\varphi$; and $ds = r \, d\varphi$.

The equation is $d(\sigma\ r\ \sin^2\varphi)/r\ d\varphi = r \sin\varphi\ p_x/e$, where p_x is the component according to x of the load p applied to the shell and e is the thickness of the shell.

The normal stress on the lower edge parallel to the meridian is σ.

The second equation is then $d(\sigma\ r \sin\varphi \cos\varphi)/r\ d\varphi = r \sin\varphi\ p_y/e + \sigma_\theta$, where p_y is the component according to y of the external load.

The normal stress on the lower edge along a parallel equals σ_θ.

Both equations give $\sigma_\theta = r/e\ (p_x \cos\varphi - p_y \sin\varphi) - \sigma$.

– If we apply a self-weight load g, the formula becomes the following, using the notations in Figure 1.10:

$$\sigma = \sigma_0 \sin^2\varphi_0/\sin^2\varphi + gr/e\ (\cos\varphi_0 - \cos\varphi)/\sin^2\varphi$$

$$\sigma_\theta = gr/e \cos\varphi - \sigma$$

In the case where there is no opening at the top of the dome, then $\varphi_0 = 0$, and the equations become:

$$\sigma = gr/e\ (1 + \cos\varphi) \text{ and } \sigma_\theta = gr/e\ (\cos\varphi - 1/(1 + \cos\varphi))$$

– If we apply a load p per horizontal m² (snow, for example), the formula becomes the following, using the notations in Figure 1.10 and $p_x = p \cos\varphi$:

$$\sigma = \sigma_0 \sin^2\varphi_0/\sin^2\varphi + pr/e\ (1 - \sin^2\varphi_0/\sin^2\varphi)$$

$$\sigma_\theta = pr/e \cos^2\varphi - \sigma$$

In the case where there is no opening at the top of the dome, then $\varphi_0 = 0$, and the equations become:

$$\sigma = pr/e \text{ and } \sigma_\theta = pr/e\ (\cos^2\varphi - 1/2)$$

– If we apply a charge q per vertical m² (live load, for example), the formula becomes the following, using the notations in Figure 1.10 and $p_y = -q \sin\varphi$:

$$\sigma = \sigma_0 \sin^2\varphi_0/\sin^2\varphi$$

$$\sigma_\theta = qr/e \sin^2\varphi - \sigma$$

In the case where there is no opening at the top of the dome, then $\varphi_0 = 0$, and the equations become:

$$\sigma = 0 \text{ and } \sigma_\theta = qr/e \sin^2\varphi$$

– If we apply a uniform pressure p (gas pressure, for example), the formula becomes the following, using the notations in Figure 1.10 and $p_x = p \cos\varphi$ and $p_y = -p \sin\varphi$:

$$\sigma = \sigma_0 \sin^2\varphi_0/\sin^2\varphi + pr\,(1 - \sin^2\varphi_0/\sin^2\varphi)/2e$$

$$\sigma_\theta = pr/e - \sigma$$

In the case where there is no opening at the top of the dome, then $\varphi_0 = 0$, and the equations become:

$$\sigma = pr/2e \text{ and } \sigma_\theta = pr/2e$$

– If a hydraulic load with density ρ is applied (water load on the bottom dome of a reservoir, for example), the formula becomes the following, using the notations in Figure 1.10 and $p_x = p \cos\varphi$ and $p_y = -p \sin\varphi$ and $p = \rho(h + x) = \rho(h - r \cos\varphi)$:

$$\sigma = \sigma_0 \sin^2\varphi_0/\sin^2\varphi + \rho rh/2e\,(1 - \sin^2\varphi_0/\sin^2\varphi) + \rho r^2/3e\,(\cos^3\varphi - \cos^3\varphi_0)/\sin^2\varphi$$

$$\sigma_\theta = \rho r/e\,(h - r \cos\varphi) - \sigma$$

In the case where there is no opening at the top of the dome, then $\varphi_0 = 0$, and the equations become:

$$\sigma = \rho rh/2e + \rho r^2/3e\,(\cos^3\varphi - 1)/\sin^2\varphi \text{ and } \sigma_\theta = \rho r/e\,(h - r \cos\varphi) - \sigma$$

1.6. Cylindrical shell theory

Taking a cylindrical shell of radius r and thickness e, the shell equations are simplified by considering that:

$$y = r;\ dx/ds = 1 \text{ and } dy/ds = 0$$

The equations become $\mathbf{d\sigma/dx = p_x/e}$ and $\boldsymbol{\sigma_\theta = -r\, p_y/e}$.

– If we apply a self-weight load g, the formula becomes:

$\sigma = \sigma 0 + gx/e$, where x denotes the shell height above the calculated point

$\sigma_\theta = 0$

– If a hydraulic load with density ρ is applied (water load in a reservoir, for example), the formulas become, with $p_x = 0$ and $p_y = q$:

$\sigma = \sigma_0$

$\sigma_\theta = qr/e$