1

Introduction to Optimization in Mechanics

1.1. Introduction

The purpose of this chapter is to present a nonexhaustive state of the art of all of the fields that will be directly or indirectly covered in this book.

The first section of this chapter is dedicated to the general study of structural dynamics. This study will make it possible to define the notations essential to the calculation of dynamic responses, to the calculation of frequencies, eigenmodes and response functions. All of these aspects will be addressed in detail and through practical applications.

One of two conventional strategies can be used to solve the system of dynamical equilibrium equations of a structure [IMB 95]. The most common solution strategy in dynamic range is modal superposition, which is suitable for linear structures in which only the first modes are excited. On the other hand, direct solving methods use the integration of equations of motion, in order to deal with nonlinear structures. The latter can also be applied when the frequency content of the excitation covers a large number of modes of the mechanical structure under study.

The second section of this chapter is a presentation of a bibliographic study of structural optimization. The goal is to obtain the appropriate shapes for a part by minimizing a given criterion. In every area of structural mechanics, the impact of the proper design of a part is very important for its strength, service life and application use. This challenge is a daily one in high-tech sectors. The development of the art of engineering requires considerable effort to constantly improve structural design techniques. Optimization is essential in increasing performance and reducing the weight of aerospace and automotive vehicles, thus leading to substantial energy savings.

The last section of this chapter is dedicated to the description of the different tools for structural analysis with uncertain parameters. The taking into account of parameter uncertainty is a particularly critical problem in vibrational mechanics. However, the consideration of this effect can respond to different types of needs, where two main categories can be distinguished: analysis and design. In general, the modeled parts or structures must satisfy given specifications, which require, for example, that safety, reliability or comfort standards be observed.

During deterministic design, engineers try to find the best possible design among all of the potential solutions they have to study. This choice is based on cost as well as on improving product quality. In this case, the designer's objectives to get an optimal design are devised without worrying about the accuracy of the mechanical characteristics of the materials, the geometry and loading (uncertainty effects). The resulting optimal design may therefore represent an inadequate level of reliability. The objective of the approach integrating reliability analysis into the optimization problem and called RBDO (for Reliability Based Design Optimization) is to design structures while establishing the best compromise between cost and the proper functioning of the device.

1.2. Problems of general dynamics

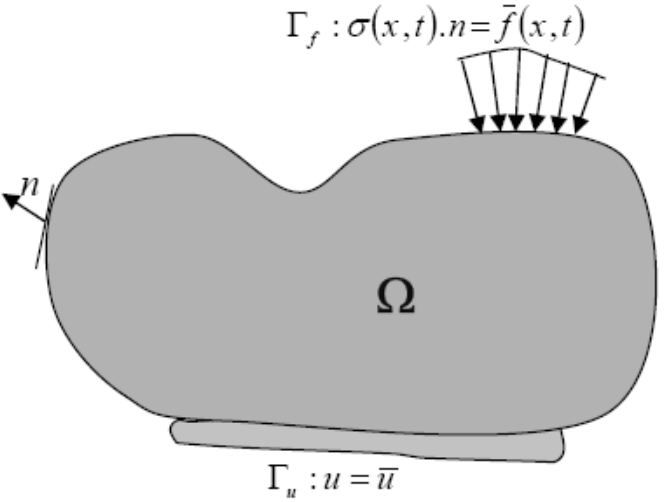

Figure 1.1. *System Ω subject to forces. For a color version of this figure, see www.iste.co.uk/elhami/multi-physics.zip*

The formulation of a problem of dynamics in small perturbations on a domain Ω of boundary $\Gamma = \Gamma_u \cup \Gamma_f$ (see Figure 1.1) and in a time interval [0, T] is:

$$Div_x\ \sigma(x,t) + g(x)\ = \rho\ddot{u}(x,t) \qquad [1.1]$$

$$\varepsilon\ = \frac{1}{2}\left(\nabla u + \nabla^t u\right) \qquad [1.2]$$

Initial conditions:

$$u(x,0) = u_0(x)\ \ x \in \Omega \qquad [1.3]$$

$$\dot{u}(x,0) = \dot{u}_0(x)\ \ x \in \Omega \qquad [1.4]$$

Boundary condition:

$$u(x,t) = \overline{u}(x,t)\ \ (x,t) \in \Gamma_u \times [0,T] \qquad [1.5]$$

$$\sigma(x,t).n = \overline{f}(x,t)\ \ (x,t) \in \Gamma_f \times [0,T] \qquad [1.6]$$

u is the displacement vector, σ and ε are respectively the stress and strain tensor. ρ is the density. Vectors g, $\overline{f}$ and $\overline{u}$ respectively represent the volume force, the external force and the imposed displacement. $\vec{n}$ is the normal vector to the surface.

In the case of an isotropic elastic domain, the law of behavior is written as:

$$\sigma_{ij} = \lambda\,\varepsilon_{kk}\delta_{ij} + 2\mu\,\varepsilon_{ij} \qquad [1.7a]$$

λ and μ are the functions of Young's modulus and Poisson's ratio ν:

$$\lambda = \frac{E\,\nu}{(1+\nu).(1-2\nu)} \qquad [1.7b]$$

$$\mu = \frac{E}{2(1+\nu)} \qquad [1.8]$$

The dynamic problem presented above for this elastic example can be represented by the Navier equation as follows:

$$\mu\nabla^2 u_i(x,t)+(\lambda+\mu)\frac{\partial}{\partial x_i}\left(\nabla.u(x,t)\right)=\rho\ddot{u}(x,t) \qquad [1.9]$$

where ∇^2 denotes the Laplacian operator: $\nabla^2=\frac{\partial^2}{\partial^2 x_1}+\frac{\partial^2}{\partial^2 x_2}+\frac{\partial^2}{\partial^2 x_3}$ and "∇" is the notation of the divergence operator: $\nabla\cdot u=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}$.

1.2.1. *Finite element methods*

For structures with complex geometries, numerical methods such as the finite element method are used. In an elastodynamic problem, the displacements are generally expressed by a combination of vectors [GMÜ 97]:

$$u(x,t)=\left[B(x)\right]\left\{q(t)\right\} \qquad [1.10]$$

where $\left[B(x)\right]$ is the matrix of shape functions, and $\left\{q(t)\right\}$ is the vector of discrete real displacements, whose components are the discrete unknowns of the approximation.

In general, a structural mechanics problem, after discretization, is described by a system of second-order equations:

$$\begin{cases}[M]\left\{\ddot{q}(t)\right\}+[C]\left\{\dot{q}(t)\right\}+[K]\left\{q(t)\right\} = F(t)\\ \left\{q(0)\right\}=\left\{q_0\right\},\quad \left\{\dot{q}(0)\right\}=\left\{\dot{q}_0\right\}\end{cases} \qquad [1.11]$$

where N is the number of degrees of freedom of the system; $M(N\times N)$ is the positive definite symmetric mass matrix, $C(N\times N)$ and $K(N\times N)$ are the viscous damping and stiffness non-negative definite symmetric matrices respectively. F is the vector of the applied forces.

Mathematically, equation [1.11] represents a system of second-order differential equations that can be solved either by a direct integration method or by the mode superposition method.

1.2.2. *Modal assumption method*

We apply the following modal transformation to system [1.11]:

$$\{q\} = [\Phi]\{p\} \qquad [1.12]$$

$\{p\}$ is the vector of generalized coordinates; $[\Phi]$ is the modal matrix verifying the orthogonality properties: $[\Phi]^T [M][\Phi] = I$ and $[\Phi]^T [K][\Phi] = \left[w^2 \right]$ with: $\left[w^2 \right] = diag\left[w_1^2 \; w_2^2 \cdots \right]$, where w_i is the natural angular frequency, equation [1.12] reverts to:

$$\{\ddot{p}\} + [\Phi]^T [C][\Phi]\{\dot{p}\} + \left[w^2 \right]\{p\} = \{P\} \qquad [1.13]$$

where $\{P\} = [\Phi]^T \{F\}$ is the modal force vector.

It can be assumed that the damping matrix is proportional to the mass and stiffness matrix. This assumption, known as the Rayleigh assumption, is quite commonly used in structural analysis. It therefore postulates that:

$$[C] = \alpha[M] + \beta[K] \qquad [1.14]$$

$$[\Phi]^T [C][\Phi] = \alpha[I] + \beta\left[w^2 \right] \qquad [1.15]$$

which amounts to writing that:

$$\{\ddot{p}\} + \left(\alpha[I] + \beta\left[w^2 \right] \right)\{\dot{p}\} + \left[w^2 \right]\{p\} = \{P\} \qquad [1.16]$$

The decoupled system becomes:

$$\ddot{p}_i + 2\zeta_i w_i \dot{p}_i + w_i^2 p_i = P_i \qquad [1.17]$$

$$2\zeta_i w_i = \alpha + \beta w_i^2, \quad i = 1, 2, \cdots\cdots, N \qquad [1.18]$$

The factor ζ_i is called the reduced damping coefficient for the i-th mode. α and β are the initially unknown values and are calculated from the reduced damping coefficient ζ_i.

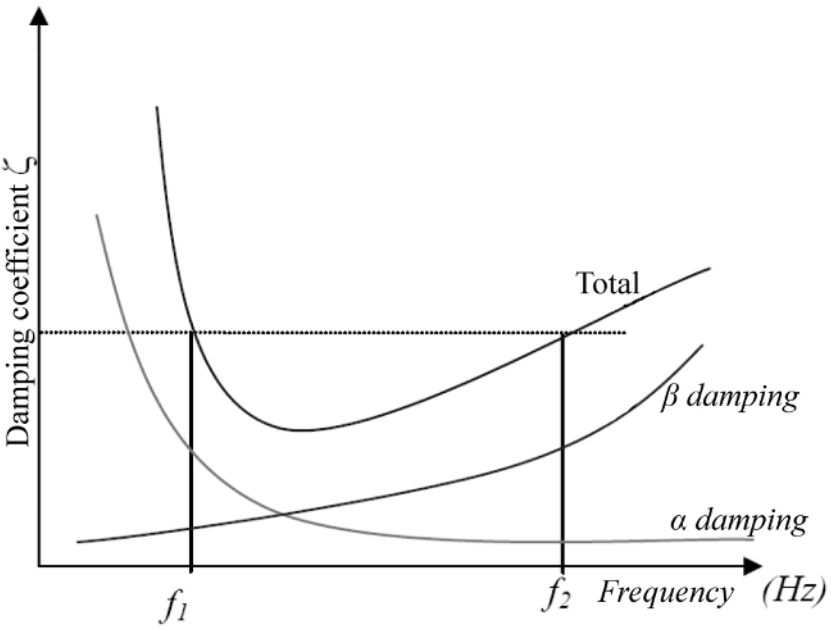

Figure 1.2. *Damping coefficient graph. For a color version of this figure, see www.iste.co.uk/elhami/multi-physics.zip*

In Figure 1.2, we graphically present the modal damping coefficient ζ; note that the sum of the two functions is almost constant at damping over the frequency range of interest. Thereby, given the modal damping (ζ) and a frequency interval (f_1 and f_2), the two equations can be simultaneously solved to determine α and β.

$$\frac{\alpha}{4\pi f_1} + \beta \pi f_1 = \zeta \qquad [1.19]$$

and

$$\frac{\alpha}{4\pi f_2} + \beta \pi f_2 = \zeta \qquad [1.20]$$

1.2.3. *Direct integration*

There are many methods to integrate differential equations. The general procedure consists of discretizing time and formulating what happens at time "t+Δt" as a function of what happens at time "t" based on Taylor expansions; in this part, we present the Newmark method and the Wilson method.

1.2.3.1. *Newmark method*

Newmark proposed a method where the velocity and displacements at t+Δt are estimated with respect to $\{\ddot{q}_t\},\{\dot{q}_t\},\{q\}$ and to accelerations $\{\ddot{q}_{t+\Delta t}\}$. In addition, the displacement and velocity are developed in the Taylor series using the two independent parameters β and γ as well as the time step [KLE 92]:

$$\{q_{n+1}\}=\{q_n\}+\{q_n\}\Delta t+\left\{\left(\frac{1}{2}+\beta\right)\ddot{q}_n+\beta\,\ddot{q}_{n+1}\right\}\Delta t^2 \qquad [1.21]$$

$$\{\dot{q}_{n+1}\}=\{\dot{q}_n\}+\{(1-\gamma)\,\ddot{q}_n+\gamma\,\ddot{q}_n\}\,\Delta t \qquad [1.22]$$

where $\{q_n\}$, $\{\dot{q}_n\}$ and $\{\ddot{q}_n\}$ are the approximations of $\{q(t_n)\}$, $\{\dot{q}(t_n)\}$ and $\{\ddot{q}(t_n)\}$ respectively, and $t_{n+1} = t_n + \Delta t$, with Δt being the time steps, the two independent parameters β and γ ensure the accuracy and stability of the solution. When $\gamma \geq 1/2$ $\beta \geq (\gamma + 0.5)/4$.

By carrying these equations forward into the equation of motion, the following matrix relation is obtained:

$$\left(M+\Delta t\,\gamma C+\Delta t^2\,\beta K\right)\ddot{q}_{n+1}=C\,\dot{q}_{n+1}+K\,q_{n+1}-F_{n+1} \qquad [1.23]$$

with $\dot{q}_{n+1}=\dot{q}_n+(1-\gamma)\Delta t\,\ddot{q}_n$ and

$$q_{n+1}=q_n+\Delta t\,\dot{q}_n+\Delta t^2\left(\frac{1}{2}-\beta\right)\ddot{q}_n \qquad [1.24]$$

The acceleration at time t = 0 is provided by the equilibrium conditions and the initial conditions on $\{q\}$ and $\{\dot{q}\}$. The solution of equation [1.23] requires that a linear system be solved at each time step.

1.2.3.2. *The Wilson-θ method*

This is an extension of a method in which the acceleration is assumed to linearly vary over the interval $\left[n\Delta t, (n+1)\Delta t \right]$; Wilson [KLE 92] assumes that this linear variation occurs over the interval $\left[n\Delta t, (n+1)\Delta t \right]$. θ is a parameter whose value recommended by Wilson is 1.4.

If τ denotes time in the interval $[0, \theta\Delta t]$, then the acceleration in the interval $[t, t+\theta\Delta t]$ is written [KLE 92] as:

$$\ddot{q}_{t+\tau} = \ddot{q}_t + \frac{\tau}{\theta\Delta t}\left(\ddot{q}_{t+\theta\Delta t} - \ddot{q}_t\right) \qquad [1.25]$$

Speed and displacement are obtained by successive integrations

$$\dot{q}_{t+\tau} = \dot{q}_t + \tau\ddot{q}_t + \frac{t^2}{2\theta\Delta t}\left(\ddot{q}_{t+\theta\Delta t} - \ddot{q}_t\right) \qquad [1.26]$$

$$q_{t+\tau} = q_t + \dot{q}\,\tau + \frac{1}{2}\ddot{u}_t\tau^2\,\frac{\tau^3}{6\theta\Delta t}\ddot{q}_{t+\theta\Delta t} - \ddot{q}_t \qquad [1.27]$$

These basic equations were generalized by Hughes [TRO 87]. They are given for time $t = \theta\Delta t$ and with the notation $q(n\Delta t) = q_n$:

$$M\,\ddot{q}_{n+\theta} + C\,\dot{q}_{n+\theta} + K\,q_{n+\theta} = F_{n+\theta} \qquad [1.28]$$

$$\ddot{q}_{n+\theta} = (1-\theta)\ddot{q}_n + \theta\ddot{q}_{n+1} \qquad [1.29]$$

$$\dot{q}_{n+\theta} = \dot{q}_n + \theta\Delta t\left[(1-\gamma)\ddot{q}_n + 2\beta\ddot{q}_{n+\theta}\right] \qquad [1.30]$$

$$q_{n+\theta} = q_n + \theta\Delta t\,\dot{q}_n + \frac{(\theta\Delta t)^2}{2}\left[(1-2\beta)\ddot{q}_n + 2\beta\ddot{q}_{n+\theta}\right] \qquad [1.31]$$

$$F_{n+\theta} = (1-\theta)F_n + \theta F_{n+1} \qquad [1.32]$$

These equations are equal to the previous ones for the values $\beta = 1/6$ and $\gamma = 1/2$.

1.3. Structural optimization

Structural optimization is not a recent concern. If we search the archives, we find a calculation made by Galileo that gives the law of thickness of a free embedded beam assuming a constant stress distribution [TRO 87]. The most remarkable thing is that the result of this calculation is entirely confirmed by current theories. Obviously since Galileo, thousands of publications have been produced, and if we only consider the modern era of optimization, it is found that since 1960, Schmit has introduced the idea of coupling structural analysis by finite elements and nonlinear mathematical programming to search for the different automated optimal designs. As a matter of fact, he proved that structural design could be formulated by a mathematical programming problem [SCH 84].

The determination of the proper shape of structural components is a major engineering issue. In every area of structural mechanics, the impact of the proper design of a part is very important for its strength, service life and operation use. This can be seen as a daily challenge in the cutting-edge sectors of space research, aeronautics, automotive, naval competition, fine mechanics, precision mechanics or civil engineering structures, etc. The development of the art of engineering requires considerable effort to constantly improve structural design techniques. Optimization plays a key role in increasing performance and reducing the weight of aerospace and automotive vehicles, resulting in substantial energy savings.

The constant development of computer-aided design techniques and optimization strategies has to be viewed in this context. Structural optimization has been the focus of great interest for more than 20 years. Still not applied enough to traditional techniques of engineering departments, it is gradually being integrated into them as its reliability increases. Starting from the simplest problems, the field of application of structural optimization is now expanding to new and more interesting challenges.

To illustrate the evolution of structural optimization techniques, we can arbitrarily divide structure optimization into three main families: design optimization, shape optimization and topology optimization. Historically, each has been approached in increasing order of difficulty and generality [DUY 96].

1.3.1. *Design optimization*

This enables the improvement of a structural model while following the available resources (called constraints or limitations). The automated sizing of structures is the special case in which only the cross-section or cross-sectional thickness of the components of a structure whose shape and topology are preset can be modified [AZI 02], [BOU 01a]. No modification of the geometry model is possible.

1.3.2. *Shape optimization*

Shape optimization allows for shape changes that are compatible with a predetermined topology. In its conventional form, optimization modifies the parametric representation of the boundaries of the domain [ZHA 92, AFO 02]. By moving the boundaries of the domains, a better solution can be looked for among all of the structures obtained by homeomorphic transformation of the original structure (see Figure 1.3). In this case, it is clear that a change in the transverse dimensions as well as a modification of the configuration of the structure are possible, but altering the connectivity or the nature of the structural members is certainly not admissible.

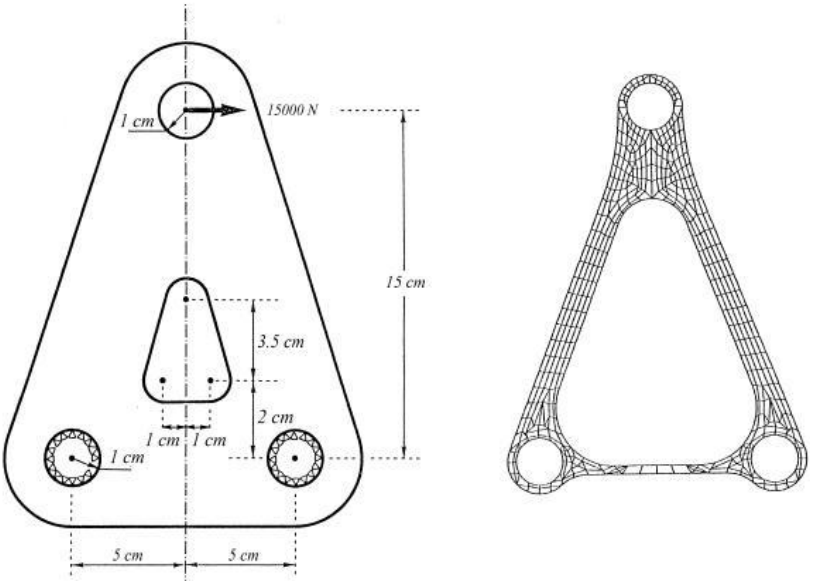

Figure 1.3. *Shape optimization: the initial model and the solution of the problem after five iterations [ZHA 92]*

1.3.3. *Topological optimization*

Topological optimization makes more fundamental modifications possible to the nature of the structure. This time, the geometry of the part is considered without any presumptions about the connectivity of the domains or structural members present in the solution. Optimizing the topology naturally leads to determining the optimal shape or transverse dimensions of the structure in a certain way, so that some authors [ROZ 93] also refer to it as generalized shape optimization. The final

structure must satisfy the stresses defined beforehand by the user (usually related to the restriction of the maximal von Mises stress) [KIM 00].

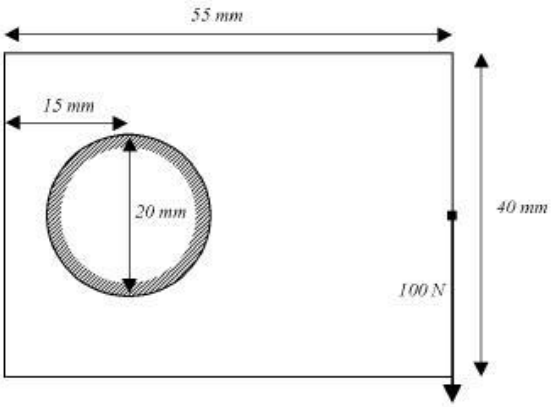

Figure 1.4. *Definition of the lattice problem by Michell [REY 99]*

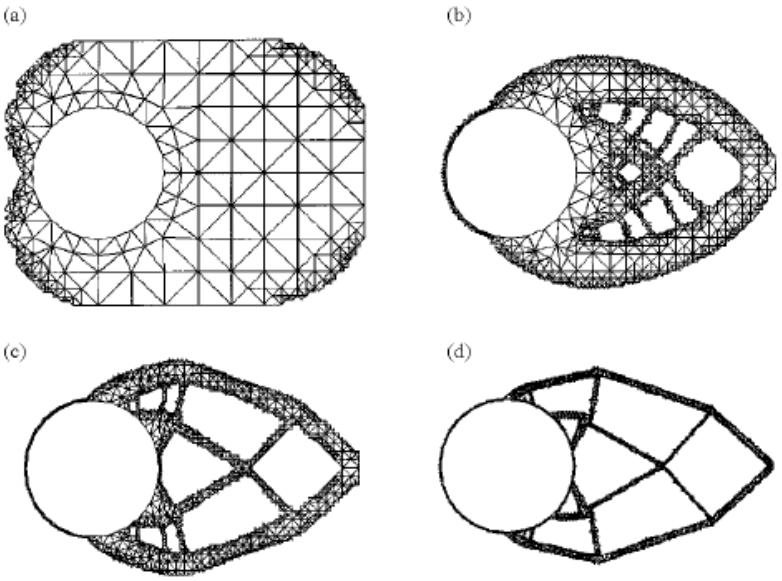

Figure 1.5. *Topology optimization solution applied to the Michell lattice [REY 99]*

Described in Figure 1.4, the Michell lattice problem (the rectangular domain is loaded in its lower-right corner, whereas all on the left side are fixed) is a classical

reference in topological optimization. [DEB 99] solved the problem by an adaptive change technique, which consists of the following: once the initial finite element problem is established, the method begins with a mesh refinement by the subdivision of elements of regions where the stress (von Mises) is minimal. Then, the subdivided elements have a minimal stress and will be eliminated, etc., until convergence.

The solution of the structures obtained after 6, 42, 75 and 120 iterations of this process respectively is represented in Figure 1.4. At the 120th iteration, only 8.8% of the entire (initial) sector remains.

Topological optimization can also be performed for lattices. For example, [DEB 99] described a method to find the optimal cross-sections and topology of a lattice for 2D and 3D using a genetic algorithm. The aim was to reduce mass to a minimum, under constraints of forces and displacements that have predefined values. Figure 1.6 shows an example of a 3D lattice where the genetic algorithm converged to a nine-element lattice (from a 39-element configuration).

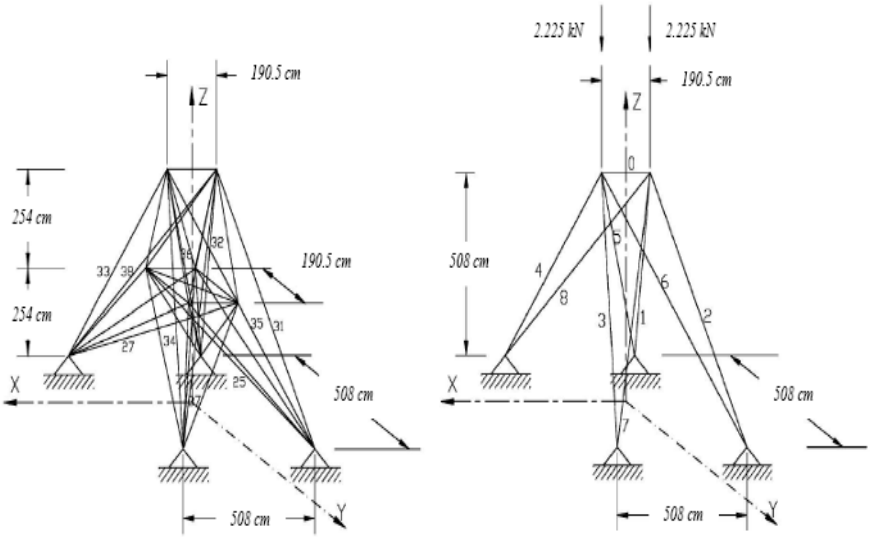

Figure 1.6. *Example of topology optimization [DEB 99]*

Recent developments in the coupling of shape and topological optimization methods by the finite element method can be found in [MAC 03].

1.3.4. *Definitions and formulation of an optimization problem*

An optimization problem is usually formulated as a minimization problem and written as:

$$\begin{cases} \min\limits_{x} f(x) \\ such\,that \\ g_i(x) \le 0,\ i = 1 \cdots\cdots, m, \\ h_j(x) = 0,\ j = 1 \cdots\cdots, p, \\ x \in S \subset \Re^n, \end{cases} \qquad [1.33]$$

where f is the scalar function to be minimized, called *the cost function* or *objective function*, x is *the vector of the optimization variables*, g_i are the inequality *constraints* and h_j are the equality constraints, and S is *the space of variables* (also called the search space). S indicates what type of variables are considered: real, integer, mixed (real and integer in the same problem), discrete, bounded continuous, etc.

A point x_A is called an *admissible point* if $x_A \in S$ and if the optimization constraints are satisfied: $g_i(x_A) \le 0, i = 1 \cdots\cdots, m$ and $h_j(x_A) = 0,\ j = 1 \cdots\cdots, p.$ The solution of (I.40) is the set of optima $\left\{x^*\right\}$.

x^* is a global minimum of f if and only if $f(x^*) \le f(x)\ \forall\ x \in S.$

x^* is a local minimum of f if and only if $f(x^*) \le f(x)\ \forall\ x \in S / \left\| x - x^* \right\| \le \varepsilon,$ $\varepsilon > 0.$

Figure 1.7 shows an example of a single-variable function, with local minima and a global minimum. Among the local minima, the one with the smallest value of f is the global minimum.

A multimodal function exhibits several (local) minima, and a unimodal function has only one minimum, the global minimum. Figure 1.8 shows a multimodal function with two variables.

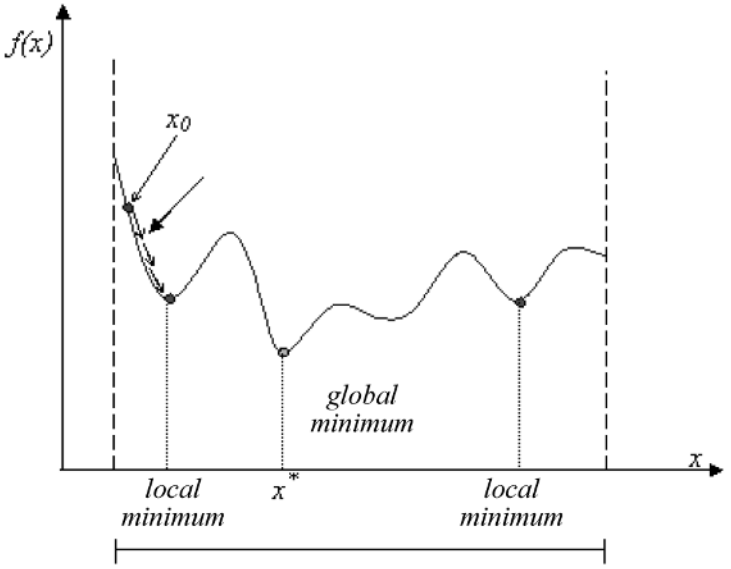

Figure 1.7. *Local minima and global minimum of a function with one variable. For a color version of this figure, see www.iste.co.uk/elhami/multi-physics.zip*

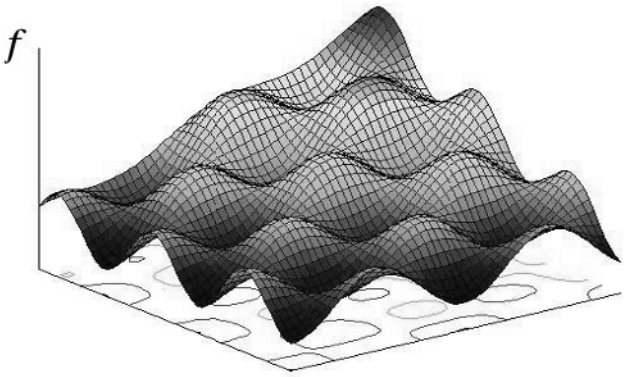

Figure 1.8. *A multimodal function with two variables. For a color version of this figure, see www.iste.co.uk/elhami/multi-physics.zip*

A local (or algorithm or search) method is that which converges to a local minimum. Local research usually starts from an initial point χ_0 with an initial step Δ_0. These parameters will condition the descent of one of the valleys of the function (see diagrams in Figure 1.7). There are many existing local methods. The

oldest and most widely used are the methods where the direction of descent is deduced from the derivatives of the function (steepest slope method, Newton's method, conjugate guardian method, quasi-Newtonian methods [MIN 86]).

The aim of global methods is to achieve one or more global optima. Typically, the machine cost of an optimization is conditioned by the number of evaluations of the objective function [MOH 04]. If, for example, a mechanical system modeled by finite elements is to be optimized, the computation time will mainly be the sum of the time of the simulations used by the optimizer.

1.4. Structures with uncertain parameters

Accounting for uncertainties when studying a structure must take into account the two types of sources of uncertainty: those concerning excitation and those concerning structure. Dessombz [DES 00] classified uncertainties into four main categories based on Prasthofer's work: random parameters, poorly known parameters, variable parameters and model uncertainties:

a) *Random parameters*. These are, for example, the dimensions of a part whose tolerance is known, or Young's modulus or the density of a material such as steel.

b) *Poorly known parameters*. The case of boundary conditions is a typical problem. An embedment corresponds to a stiffness with a very high value, but of which only an order of magnitude is known at best. Similarly, the different types of assembly, such as welding, gluing, etc., are difficult to model, and the deterministic values used to represent these phenomena appear to be largely insufficient.

c) *Variable parameters*. We can mention the parameters that can be variable over time, and that cannot be controlled or not very well (typically material degradation or aging). We can also acknowledge parameters whose value can be known at a given time. In this case, it may be an excitation force (a train crossing over a bridge), a quantity of fuel in a tank or even special settings.

d) *Model uncertainties*. The chosen laws of behavior, for example, are the ones that poorly or incompletely represent physical phenomena, the errors due to the choice of the finite element mesh, its finesse and the elements chosen. These uncertainties are in general difficult to assess.

To represent parametric uncertainty, different approaches, probabilistic theory [IBR 87] [SAR 11], fuzzy sets [CAM 00] and convex models [BEN 90] have been proposed in the literature, to which the reader should refer for more details concerning the main methods used in the treatment of uncertainties which are presented here, without going into details.

1.4.1. *Monte Carlo simulation*

For addressing uncertainty problems, whether experimental or numerical, the approach that was naturally used in the first place is based on the principle of sampling: Monte Carlo simulation, known and used intensively in several scientific fields, was introduced into structural mechanics by Schinozuka. The method is almost chosen as a reference and is both the simplest and the costliest.

The method is comprised of three stages. The first step is the most important; it consists of generating, by numerical simulation, a large set of realizations associated with the stochastic parameters of the physical problem. The next step is to solve the deterministic problem for each of the population elements, in order to obtain a population corresponding to the random response quantities of the system under study. In the last step, a large number of outputs is obtained, from which different static values of the interesting response variables can be computed.

The computing power required to perform all of the simulations severely restricts the applications of this method, which is why approximation techniques are preferred.

1.4.2. *Analytical method*

The different methods for analyzing uncertainty problems are in particular the perturbation method and the Neumann series decomposition. The principle of these methods is recalled in [IBR 87]. The use of Neumann series is based on finding the inverse of the operator of the problem by decomposing it into a Neumann series. This limits its application to certain types of differential equations, whereas the perturbation method does not have this limitation, since it corresponds to a (Taylor-type) decomposition of the output field. These methods are generally simple to implement, but only give good results for small perturbations and when the problems are linear.

1.4.3. *Stochastic finite element method*

Much of the current research deals with the problem of structures with uncertain parameters using stochastic finite element methods. These combine two techniques, namely “classical” finite element analysis, and statistical analysis. In general, we seek to determine the stochastic characteristics of random responses, based on knowledge of the hazards on: structural and geometric parameters, boundary conditions or loads in a system.

[SCH 01] outlined the various developments developed in this field, and proposed numerous references for each of the aspects of this problem. An explanation of the stochastic finite element principle described by [GHA 96] involves the use of a serial decomposition of stochastic processes, truncated to a certain order according to the desired precision. There are two levels of description of the stochastic aspect of the problem: the first consists of considering the characteristics of the structure as known stochastic fields. In this case, a Karhunen–Loeve decomposition [SOI 93] is generally used, which is analogous to the modal superposition used in structural calculation, since this decomposition has remarkable properties of orthogonality and convergence. The second level to be considered concerns the stochastic field formed by the solution at the nodes of the structure: in this case, the so-called polynomial chaos decomposition is used [ROG 91]. Therefore, stochastic solutions are projected on the basis of orthogonal polynomials whose variables are orthonormal Gaussians. The properties of this polynomial basis can be used to find either analytically or numerically the mean, standard deviation or distribution of the random solution. The main problem with this method lies in the choice of resolution algorithms and in storing the values. The possibility of implementing parallelism according to the characteristics of the matrices obtained are undoubtedly the essential aspects to be taken into account for the development of this method, which already gives satisfactory results, with a significant computation time.

A different approach to the stochastic formulation using the finite element method can be achieved by modifying the existing elements in order to allow the inclusion of defects; as such, [COM 01] present an axisymmetric element, based on a shell element. Nevertheless, it enables the analysis of structures including non-axisymmetric defects, such as a cylinder or cone radius, or a variable thickness on the circumference.

1.4.4. *Fuzzy logic method*

Fuzzy logic was founded by [ZAD 65]. Since its emergence, it has been the subject of several research projects. For [ZAD 94], fuzzy logic is fuzzy set theory (FST). FST is an extension of the theory of what is already known as “classical sets”. The membership function of a classical set A is defined by:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \qquad [1.34]$$

This means that an element x is either in A $\left(\mu_A(x) = 1\right)$ or not $\left(\mu_A(x) = 0\right)$.

This definition makes it possible to make the link with classical probabilistic analysis, for which the definition does not differ much, except that A is in this case a point in R and not an interval of R. This difference, which is not easy to interpret from a practical point of view, leads to one that shows the limitations of probabilistic analysis: if an event A has a probability $p(A)$, then the probability of the opposite event of A is known and is equal to $p(\bar{A}) = 1 - p(A)$. This is not true in the theory of possibilities: if we denote by $P(A)$ the possibility of an event A, the relation that links this possibility to the possibility of the complementary event $\bar{A}$ is written as $P(A) + P(\bar{A}) \geq 1$, because the relation $\max\{P(A), P(\bar{A})\} = 1$ is always verified. This reflects the fact that if two events are considered to be opposite, one of them is always completely possible, and that if an event is considered possible, its constraint may very well be so as well. All this is extensively detailed in [DUB 88], in which all of the mathematical bases of the theory can be found, as well as many computational elements allowing for a first application of fuzzy arithmetic.

1.4.5. *Bibliography study of reliability methods*

Most of the approaches studied in structural reliability can be generalized into two technical categories of simulation and analytics. These techniques are used as a tool to assist the design engineer in taking into account all possible uncertainties during the design and construction phase in order to calculate the reliability index or probability of failure corresponding to one or more failure scenarios [MOH 04].

1.4.5.1. *Simulation technique*

Simulation techniques are a means to accurately evaluate the probability of failure. These techniques, also known as Monte Carlo techniques, involve different simulation methods. These include: direct Monte Carlo, importance sampling [ENE 93], conditional simulations, directional simulations, adaptive sampling and the Pavé method. The Monte Carlo simulation technique was originally proposed in the early 1940s for testing technology systems using the low-cost simulation technique. The application of the Monte Carlo simulation is performed for the calculation of the probability of failure.

In the Monte Carlo simulation [AYY 97], the failure probability is given by:

$$P_f = \frac{N}{N_f} \qquad [1.35]$$

where N_f is the number of failure events obtained and N is the total number of simulated events. The statistical precision of the failure probability is measured by its coefficient of covariance and is

$$\mathrm{cov}\left(P_f\right) = \sqrt{\frac{\left(1-P_f\right)P_f}{N}} \Big/ P_f \qquad [1.36]$$

The equation indicates that a small failure probability implies a large number of simulation cycles to maintain an acceptable level of precision. Consequently, this results in increased computational costs. In addition, for a practical problem with several random variables, the Monte Carlo simulation becomes very expensive. To overcome this difficulty, several more effective alternative methods such as the method of importance sampling [AYY 97] have been developed. The fundamental principle for the method of importance sampling is to introduce random parameters with different distributions and that their averages are as close to the design point as that of the original distribution. As a result, simulation efficiency is increased since failure scenarios are obtained more frequently.

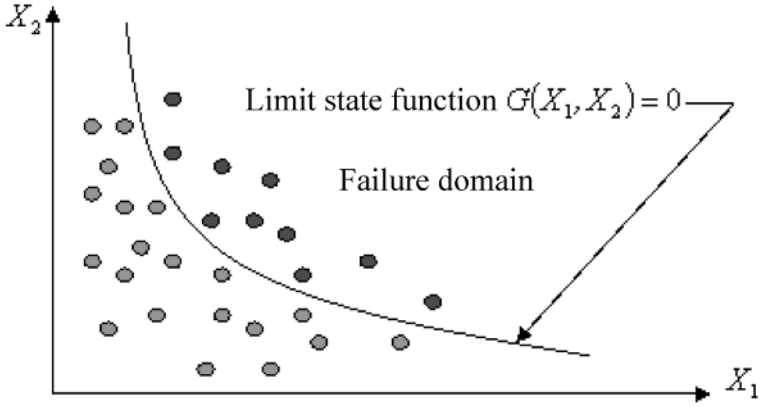

Figure 1.9. *Principle of the Monte Carlo method. For a color version of this figure, see www.iste.co.uk/elhami/multi-physics.zip*

[HAR 83] presented a general procedure based on the importance sampling technique for the calculation of the probability of failure; this procedure was applied to a fatigue problem. Therefrom, they showed the effectiveness of the importance sampling method compared to the Monte Carlo simulation.

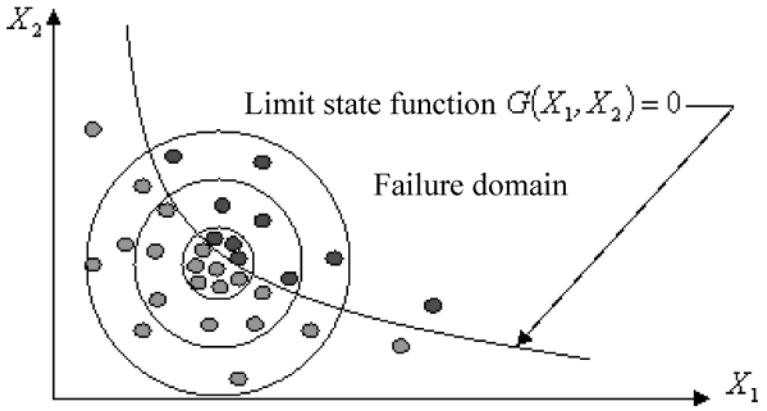

Figure 1.10. *Principle of importance sampling. For a color version of this figure, see www.iste.co.uk/elhami/multi-physics.zip*

1.4.5.2. *Analytical method*

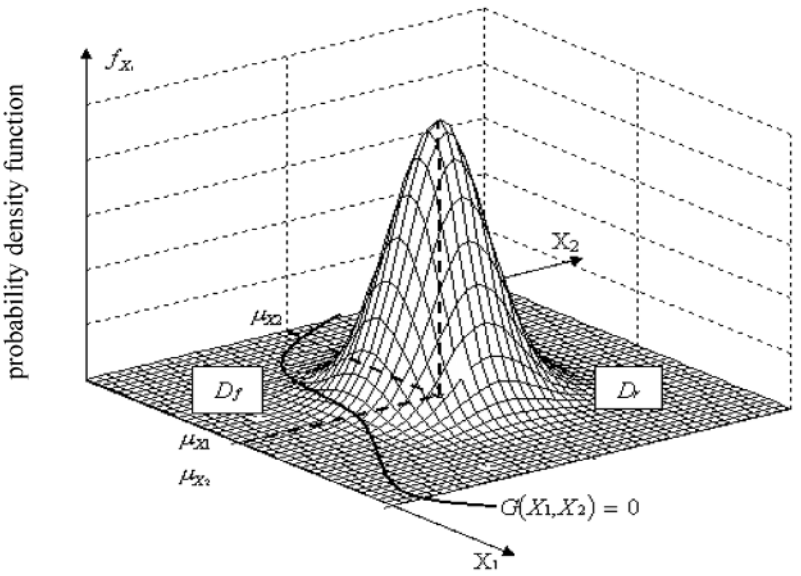

Figure 1.11. *Representation in the physical space of the probability density function for two variables*

Analytical techniques commonly used for structural reliability analysis include first-order and second-order reliability methods (FORM: First-Order Reliability Method; SORM: Second-Order Reliability Method). The fundamental concept behind these techniques consists of transforming random variables into statistically independent reduced Gaussian variables in a normalized space. Within this space, the FORM or SORM techniques are applied to approximate the limit state described by the failure criterion. The probability of failure is evaluated by a reliability index β such that $p_f \approx \Phi^{-1}(\beta)$, where Φ is the cumulative distribution function of the reduced centered normal distribution.

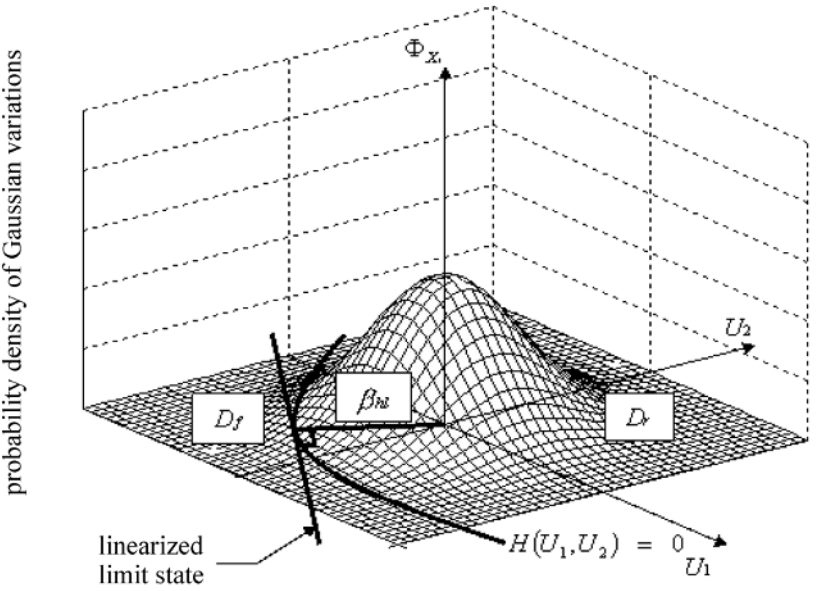

Figure 1.12. *Representation in the reduced space of the probability density function for two variables*

To evaluate the reliability index for the FORM method, [HAS 74] proposed an iterative algorithm in which the limit-state surface is approximated by a hyperplane tangent to the design point that corresponds to the most probable failure point (MMP: most probable failure point).

The reliability index is defined as the shortest distance between the origin of the axis system and the limit-state surface. [RAC 78] include information about the distribution. The [HAS 74] algorithms with that of [RAC 78] are called the HL-RF algorithm.

The FORM-based approach for calculating system reliability is due to [HOH 83]. They first reduced the system to a series of parallel systems, and then they gave a first-order solution to the multi-normal integral for the simple series or parallel systems. Finally, they determined the probability of failure.

[GOL 83] simplified the evaluation of system reliability by replacing the subsets with equivalent components within the FORM context.

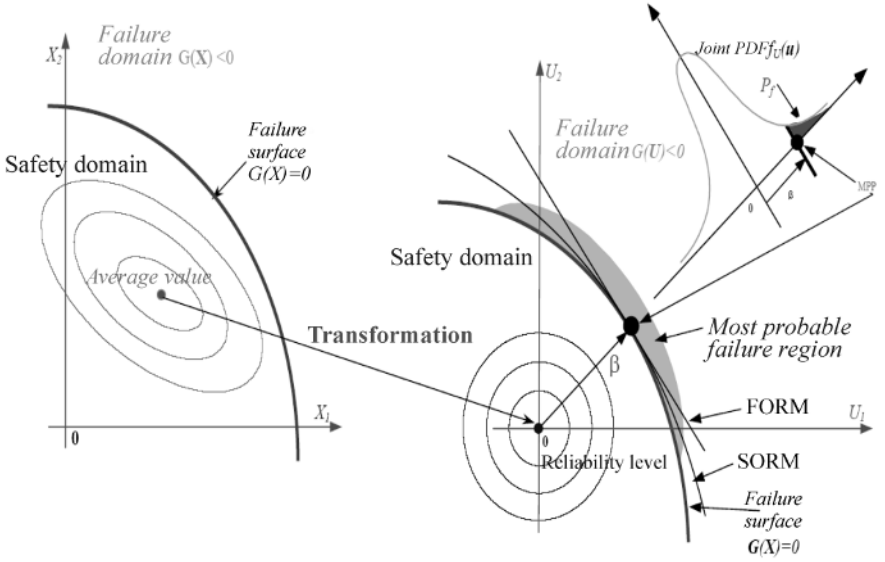

Figure 1.13. *Transformation of the probabilistic space into a normal space. For a color version of this figure, see www.iste.co.uk/elhami/multi-physics.zip*

[ENE 93] described a FORM-based method for predicting the system reliability index by modeling the serial system as a parallel system.

FORM is usually accurate for linear limit-state functions and not for strongly nonlinear limit states. To address this problem of limit-state nonlinearity, SORM was proposed to improve the evaluation of reliability by using a quadratic approximation of the limit-state surface.

Such an approximation was initially verified by [FIE 79]. Breitung [BRE 84] proposed an asymptotic method for predicting probabilities of failure for a large β by applying the quadratic approximation to MMP. He showed that the probability of failure can be expressed as:

$$P_f \approx \Phi(-\beta)\prod_{i=1}^{n-1}\left(1-\beta\frac{\varphi(-\beta)}{\Phi(-\beta)}k_i\right)^{1/2},\ \beta\to\infty \qquad [1.37]$$

where k_i are the main curvatures of failure going through the MMP.

[TVE 90] offered an exact failure probability for a second-order approximation to MMP:

$$P_f = \frac{1}{2}-\frac{1}{\pi}\int_0^{\infty}\sin\left(\beta t+\frac{1}{2}\sum_{i=1}^{n-1}\tan^{-1}(-k_i t)\right)\frac{\exp\left(-\frac{1}{2}t^2\right)}{t\left[\prod_{i=1}^{n-1}\left(1+k_i^2 t^2\right)\right]^{1/4}}dt \qquad [1.38]$$

[DER 87] presented a paraboloid second-order approximation method around the most probable point. In another article, Der Kiureghian [DER 91] developed an efficient algorithm to determine the main curvature in an iterative way for the second-order reliability analysis. [KOY 94] proposed an alternative form to the second-order approach to reliability evaluation. It should also be mentioned that their algorithm can give accurate results for small or even negative values of β. [ZHA 98a] described a SORM process of which they first give a simple approximation; they then present a second-order reliability index to estimate the failure probability and finally evaluate the failure probability by a fast inverse Fourier transformation.

An improvement has been brought to Breitung's asymptotic formula by [HOH 87].

Although generally simulations or analytical techniques work very well for most problems, they could face computational problems when the limit-state function is not an explicit function of random variables but must be determined through complex structural analysis.

In this case, the response surface methodology (RSM) was used to obtain an analytical approximation of the implicit limit-state function, which is used by FORM/SORM or other approaches to estimate structural reliability. RSM is approved as a good approach for complex structural systems and particularly for those with complex limit states [SU 02].

In addition to the approaches mentioned, the artificial intelligence technique has emerged as a potential tool for the analysis of structural reliability. [SHA 97] proposed an approximation of the limit state using neural networks and they have also developed an active algorithm that allows the search for non-feasible regions.

1.4.6. *Bibliography study of reliability optimization methods*

[FRA 85] presented a sensitivity analysis technique that was later applied to design optimization in which the weight was taken as the objective function while the target reliability index was taken as constraint. [FRA 94] developed a vector optimization approach for structural design problems requiring multiple limit states and they are considered simultaneously. They also suggested a three-step reliability-based vector optimization strategy in the solution to the optimization problem.

[YAN 91] ran a reliability-based optimization system for an aircraft wing subjected to certain loads. FORM was used to predict the reliability index of various components, while Ditlevsen adopted a technique to obtain the reliability index of the system. A two-level optimization problem, in which weight was taken as an objective function and reliability index as a constraint, was presented by [YAN 90] for composite structures.

[ENE 93] suggested four different procedures to solve the problem of reliability optimization for serial and parallel systems. The first two approaches are based on sensitivity analysis and the last is based on sequential methods. In another article by the same authors [ENE 94], several aspects related to reliability optimization in structural engineering were discussed. A variety of reliability optimization problems were formulated; the FORM method was used to estimate the level of reliability of the system, and a two-level strategy was suggested to solve the reliability optimization problem. The choice of first-rate optimization algorithms at the same time as a sensitivity analysis increases the efficiency of the solution to the reliability-based optimization problem. They also proceeded with their discussion by examining several practical outcomes of reliability optimization including the use of finite element analysis. They also concluded the discussion with a description of a strategy for correcting and improving the model as well as evaluating the optimal outcome.

[ROY 01] presented a decoupling approach by which the optimization problem can be reformulated into a deterministic and semi-infinite problem (characterized by a finite number of design variables and an infinite number of constraints). This approach was then applied to the reliability optimization of serial structure systems with two optimization formulations. In the first formulation, cost is minimized under

the constraint of reliability and mechanical stresses, whereas in the other, reliability is taken as an objective function under mechanical stress. The advantage of this approach lies in its flexibility as well as in the fact that any optimization algorithm and any reliability method can be independently adopted for the reliability-based optimization solution since computation, optimization and reliability are entirely decoupled.

Analytical techniques, mainly FORM/SORM, will always be a good choice for a reliability assessment in the reliability optimization problem, when the analytical model of the limit state function equivalent approximation is available. A one-level optimization strategy proposed by [KUS 97] based on FORM and using optimality criteria was used to solve both formulations of the reliability optimization problem. The first problem consists of minimizing the cost under reliability constraints. The second optimizes reliability under cost constraints. The algorithm of the approach was illustrated with three examples, and the optimization results were compared to those obtained by other methods available in the literature in terms of cost and stability. The disadvantage of this algorithm is that it is limited to a single limit state, so it is not applicable to reliability optimization problems.

[FEN 86] performed a structural optimization treating the probability of failure of the entire system as a constraint. An optimality criterion was used to solve the optimization problem.

For reliability optimization, two approaches, including the conventional reliability index and the target performance approach, were used to determine and satisfy the constraints in structural reliability. [LEE 02] conducted a comparative study, and they found that the target performance approach was computationally more efficient and robust than the conventional reliability index approach for the examples considered.

[WAN 96] established a set of methodologies that have applications in optimization problems based on system reliability. In order to accurately assess the failure probability for nonlinear limit-state functions, they proposed the HORM method (HORM: Higher-Order Reliability Method) to predict the probability of failure based on the reliability index obtained from an efficient safety index algorithm. The upper limit was then used to calculate the reliability of the system for which the commonalities are localized using a faster algorithm and using a high-order approximation. The approximation of two nonlinear points was developed to execute the reliability optimization system. The advantage of this method was demonstrated by a numerical test example.

[PAP 02] presented a methodology for reliability optimization where MCM with importance sampling was adopted to assess reliability while neural networks were used to perform the optimization.

The application of RSM by calculating reliability for reliability optimization is described in the article by [GAS 97], and the one by [SU 02].

An effective method was developed by [KHA 01, 02, 03] by merging the two physical and normal spaces. This approach is called the hybrid method. The hybrid approach consists of integrating the two sub-problems into a single problem that is simultaneously solved in the spaces of deterministic and random variables. Under this scheme, we integrate the reliability problem into the formulation of the optimization problem, in order to arrive at a single objective function and, thus, to reduce the overall computational cost. This last method was the objective of a detailed study during this thesis in order to develop methods adapted to the dynamic problem; among the most recent works in this field, we proposed a new method for the detection of critical areas in the vicinity of resonant frequencies as well as a new method called improved hybrid which allowed us to improve the results of the conventional hybrid method [MOH 05a]. These results will be explained in detail in the following chapters.

1.5. Conclusion

In this chapter, a literature review was presented in all of the areas that will be covered; in particular, reliability-based structural optimization. In the first section, the different analysis methods for structural dynamics were presented. In the second section, the notion of uncertainty was taken into account, different stochastic finite element methods were presented, as well as a historical study of reliability and optimization analysis.