

Chapter 1

Diffusion Phenomena and Models

The aim of this chapter is to obtain the differential diffusion equation from the macroscopic point of view starting from a microscopic point of view. The approach is heuristic and a rigorous analysis is found in the current literature as also suggested in the following sections. The equation is obtained with reference to the mass diffusion phenomena and also by analogy to heat conduction. Then the analysis is carried out with reference to this last physical aspect. The parabolic and elliptic equations are presented and the initial and boundary conditions are also given.

In doing so, we can see in the following chapters why stochastic finance uses the results of diffusion theory.

1.1. General presentation of diffusion process

In general, a diffusion phenomenon is a process in which some physical properties are transported at molecular or atomic level from one part of the space to another part. The process is the result of random migration of small particles inside the physical space. It determines the motion of matter as well as energy. From a general point of view, the diffusion concept or phenomenon is also related to the random movement of small particles, and a very simple example is given by an observer on a skyscraper watching a crowded square: people move in all directions randomly but uniformly. Another example is a red wine drop in a glass filled with water. After some time the water becomes uniformly light pink in color. This suggests that the wine overruns the water, the molecules of wine are everywhere and the wine is said to have diffused into the water. This mass transport is due to the molecular agitation with the result that zones with a high concentration

of wine determine a net molecular mass movement in all directions toward zones with lower wine concentration. In fact, an individual molecule of wine moves randomly and in a dilute solution each molecule of wine acts independently of the other molecules and undergoes collisions with the water molecules. The motion of a single molecule of wine can be depicted by the term of a “random walk” as shown in Figure 1.1. The picture of random molecular motions should adapt with the fact that a transfer of molecules from the region of higher concentration to the region of lower concentration is observed. If two thin zones are considered with equal volumes, one with a higher concentration and the other with a lower concentration, there is a dynamic exchange. A net transfer of molecules from the higher concentration to the lower concentration is obtained according to the second law of thermodynamics. Some other examples and descriptions are found in several books on this topic [BAK 08, CRA 75, CUS 09, GHE 88].

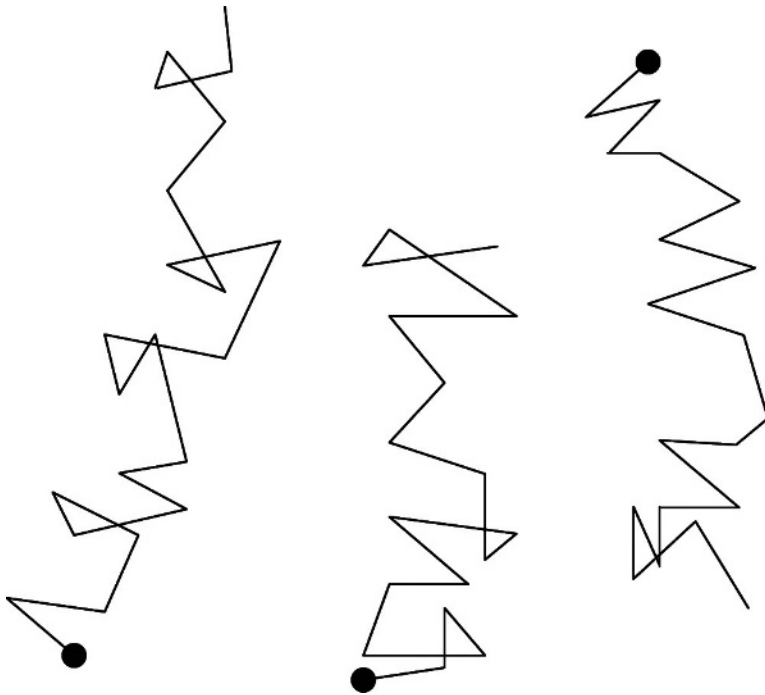


Figure 1.1. *Path of Brownian motion*

The molecular transfer determines a mass diffusion and, consequently, a diffusion of the other physical properties, such as the energy or more precisely an energy flux in conduction mode, is present. It needs to describe mathematically the molecular random transfer and to obtain a macroscopic description by means of a

continuous model [BAK 08, BER 93, GHE 88, MAZ 09, WEI 94]. In the following, the term particle will be substitute for molecule. To characterize diffusive spreading, it is convenient to consider points on a line with an arbitrary origin, as indicated in Figure 1.2.

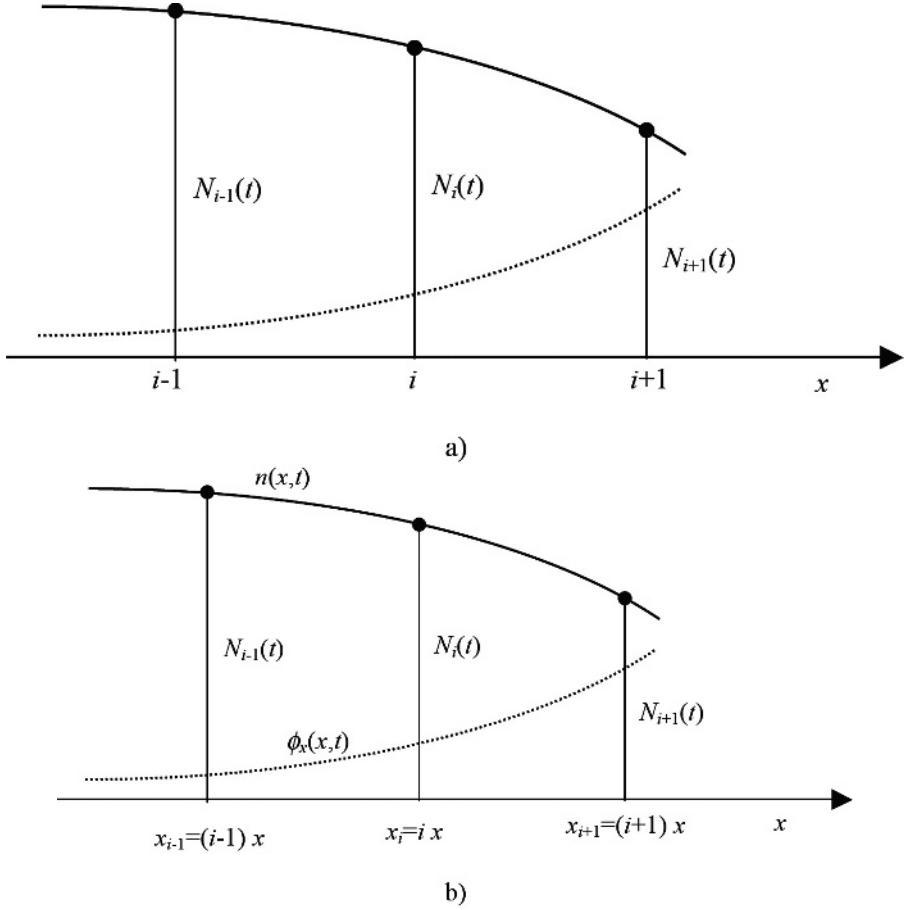


Figure 1.2. Particle exchange: a) particle number $N_i(t)$ at location (site) $i-1$, i and $i+1$ at time t ; b) the continuous distribution n and derived flux ϕ

The points or sites are named by $i = 0, \pm 1, \pm 2, \pm 3, \dots$. Then N_i is the number of particles at the site i and it is assumed that each particle can jump to adjacent sites with a frequency f which does not depend on the site i . On average, jumps to the right and to the left are equal in probability so that $(1/2)fN_i$ particles jump from site i to site $i+1$ per unit of time. The reverse transition, from $i+1$ to i , takes place at

the rate $(1/2) f N_i$ and the rate of exchange, i.e. the net number of particles, that jumps, per unit of time, from site i to site $i + 1$ is given by:

$$\Phi_{i+1/2} = \frac{1}{2} f (N_i - N_{i+1}) \quad [1.1]$$

and the net number of particles from site $i - 1$ to site i is given by:

$$\Phi_{i-1/2} = \frac{1}{2} f (N_{i-1} - N_i) \quad [1.2]$$

The difference between the two net numbers of particles determines the time rate of change and allows us to evaluate all possible transitions to and from the i -th site:

$$\frac{dN_i}{dt} = -(\Phi_{i+1/2} - \Phi_{i-1/2}) = \frac{1}{2} f (N_{i-1} - 2N_i + N_{i+1}) \quad [1.3]$$

Equation [1.3] expresses the rate of change in terms of a difference in the number of particles and of the nearest neighbor distribution around site i .

The main interest of most engineering problems is not addressed in the molecular behavior of a substance but how the substance acts as a continuum medium. The following step is to find an analog of equation [1.3] in the continuum, treating the problem as a molecular motion along a single axis coordinate x . It is assumed for simplicity that the sites are equidistant and Δx is the jump or lattice distance. The i -th site has the coordinate x_i equal to $i\Delta x$. If the number of particles at each point along the x -axis, at time t , is known, it is possible to evaluate the net number of particles that will move across the unit area in the unit time from point x_i to point $x_i + \Delta x$, the flux of particles $\phi_x(x, t)$. A continuous distribution $n(x, t)$ that satisfies the relation is defined as:

$$n(x_i, t) = N_i(t) \quad [1.4]$$

at location $x = x_i$, as depicted in Figure 1.2(b). It is then obtained, per unit area and time:

$$\phi_x = -\frac{1}{2} \frac{(n(x_i + \Delta x) - n(x_i))}{A \Delta t} \quad [1.5]$$

with A the area normal to the x coordinate and Δt the considered time interval. It should be underlined that equation [1.5] is a general expression of the generic flux along the assigned direction x of the net exchange of particles. The flux could

be a net exchange of molecules or a current or heat. By simple manipulations it is obtained:

$$\phi_x = -\frac{\Delta x^2}{2\Delta t} \frac{1}{\Delta x} \frac{(n(x_i + \Delta x, t) - n(x_i, t))}{A\Delta x} = -\Gamma \frac{1}{\Delta x} (\eta(x_i + \Delta x, t) - \eta(x_i, t)) \quad [1.6]$$

where Γ and η depend on the particles exchanged, the mass or the current or the heat diffusion. Further, Γ is a characteristic coefficient linked to the material whereas a difference of η determines a flux along the considered direction. In the limit $\Delta x \rightarrow 0$, the partial variation of η is obtained along the x coordinate and a generic relation is obtained between the flux and the component of gradient along the x direction:

$$\phi_x(x, t) = -\Gamma \frac{\partial \eta(x, t)}{\partial x} \quad [1.7]$$

Equation [1.7] indicates that the flux along the assigned direction is related to the component of gradient along the assigned direction. Γ is the diffusivity or diffusion coefficient. Moreover, equation [1.2] in the continuum domain and for the unit area allows us to carry out the following relation:

$$\frac{\partial \eta(x, t)}{\partial t} = -\frac{\partial \phi_x(x, t)}{\partial x} \quad [1.8]$$

In equation [1.8], it is indicated that the rate of the η variation inside a spatial neighbor depends on the variation of ϕ_x along the x -axis. It is obtained by equations [1.7] and [1.8]:

$$\frac{\partial \eta(x, t)}{\partial t} = \frac{\partial}{\partial x} \left(\Gamma \frac{\partial \eta(x, t)}{\partial x} \right) \quad [1.9]$$

When the diffusion coefficient is independent of $\eta(x, t)$, equation [1.9] becomes:

$$\frac{\partial \eta(x, t)}{\partial t} = \Gamma \frac{\partial^2 \eta(x, t)}{\partial x^2} \quad [1.10]$$

Equations [1.9] and [1.10] are the diffusion equations in a one-dimensional space variable x . It allows us to evaluate the distribution of the $\eta(x, t)$ with respect to the time and the spatial variable x . Equations [1.9] and [1.10] can be extended

to a three-dimensional time-dependent problem and the diffusion equation takes the form:

$$\frac{\partial \eta(P, t)}{\partial t} = \underline{\nabla} \cdot (\underline{\Gamma} \underline{\nabla} \eta(P, t)) \quad [1.11]$$

where P is the point in the three-dimensional space and the flux component along the generic direction, x_k , is given by:

$$\phi_{x_k}(P, t) = -\Gamma \frac{\partial \eta(P, t)}{\partial x_k} \quad [1.12]$$

and in vector form:

$$\underline{\phi}(P, t) = -\underline{\Gamma} \underline{\nabla} \eta(P, t) \quad [1.13]$$

In equation [1.13], the flux is a vector and due to a gradient of a property η it represents a constitutive relation. Examples of constitutive relations are Fick's law for mass diffusion and Fourier's law for heat conduction. The first is a linear relationship between the mass flux and the concentration gradient [CRA 75, CUS 09, GHE 88, MIK 84]. The second determines a link between the heat diffusion and the temperature gradient [MIK 84, ÖZI 93, YEN 08, WAN 08].

The next step is to obtain the diffusion equation starting from a description in terms of continuum and to employ a global balance of an extensive quantity which is additive in the sense that its value for a set of subsystems is an algebraic sum of its values for each subsystem.

1.2. General balance equations

The balance of a generic extensive quantity or extensive property related to a physical, economic, financial or insurance phenomena should be given on an assigned system which is, from a thermodynamics point of view, an assigned portion of space or matter or, from a more general point of view, a set of points defined by independent variables. In financial as well as in economic problems, the independent variables are the financial or the economic ones, respectively.

A logical form of a balance of an extensive quantity is applied on a system, V , as depicted in Figure 1.3, defined by means of a surface or a border, A , which contains the system in an assigned time interval Δt . It is given by:

$$\begin{aligned}
 & \left(\begin{array}{l} \text{Entering extensive} \\ \text{quantity in the} \\ \text{system during } \Delta t \end{array} \right) + \left(\begin{array}{l} \text{Production of extensive} \\ \text{quantity in the} \\ \text{system during } \Delta t \end{array} \right) = \\
 & \left(\begin{array}{l} \text{Outgoing extensive} \\ \text{quantity from the} \\ \text{system during } \Delta t \end{array} \right) + \left(\begin{array}{l} \text{Destruction of extensive} \\ \text{quantity in the system} \\ \text{during } \Delta t \end{array} \right) + \left(\begin{array}{l} \text{Variation of extensive} \\ \text{quantity in the system} \\ \text{during } \Delta t \end{array} \right) \quad [1.14]
 \end{aligned}$$

If the extensive quantity does not present any source term, positive (production) or negative (destruction), the equation [1.14] is a conservation equation of the extensive quantity or property and is given by:

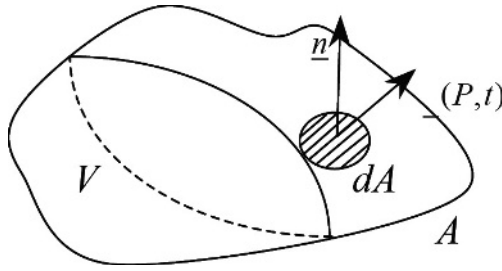


Figure 1.3. Assigned system to apply the balance of an extensive quantity

$$\left(\begin{array}{l} \text{Entering extensive} \\ \text{quantity in the} \\ \text{system during } \Delta t \end{array} \right) = \left(\begin{array}{l} \text{Outgoing extensive} \\ \text{quantity from the} \\ \text{system during } \Delta t \end{array} \right) + \left(\begin{array}{l} \text{Variation of extensive} \\ \text{quantity in the system} \\ \text{during } \Delta t \end{array} \right) \quad [1.15]$$

The various terms of this equation [1.14] are evaluated starting from the flux and the normal to the external surface and considering the variation per unit of time (\$\Delta t \rightarrow 0\$), i.e. rate of extensive quantity through the bounding surfaces, as:

$$\begin{aligned}
 & \left[\frac{\left(\begin{array}{l} \text{Entering extensive quantity} \\ \text{in the system during } \Delta t \end{array} \right) - \left(\begin{array}{l} \text{Outgoing extensive quantity} \\ \text{from the system during } \Delta t \end{array} \right)}{\Delta t} \right]_{\Delta t \rightarrow 0} = \\
 & \left(\begin{array}{l} \text{rate of extensive quantity} \\ \text{through the surface } A \end{array} \right) = - \int_A \underline{\phi} \cdot \underline{n} \, dA \quad [1.16]
 \end{aligned}$$

where \$\underline{\phi}\$ is the flux vector of the extensive quantity and \$\underline{n}\$ is the outward-drawn normal unit vector to the surface element \$dA\$. The minus sign assures that if the net

rate is entering, the scalar product is negative but the total quantity should be increased and the opposite is obtained if the net rate is exiting. Moreover, applying the divergence theorem to convert the surface integral to a volume integral, we obtain:

$$\int_A \underline{\phi}(P, t) \cdot \underline{n} \, dA = \int_V \nabla \cdot \underline{\phi}(P, t) \, dV \quad [1.17]$$

The rate of extensive quantity generation in the considered system is evaluated as:

$$\left[\frac{\left(\begin{array}{c} \text{Production of extensive} \\ \text{quantity in the} \\ \text{system during } \Delta t \end{array} \right) - \left(\begin{array}{c} \text{Destruction of extensive} \\ \text{quantity in the system} \\ \text{during } \Delta t \end{array} \right)}{\Delta t} \right]_{\Delta t \rightarrow 0} = \quad [1.18]$$

$$\left(\begin{array}{c} \text{Rate of generation of extensive} \\ \text{quantity in the system} \end{array} \right) = \int_V g \, dV$$

where g is the generation per unit volume and time. The rate of extensive quantity storage in the system is evaluated as:

$$\left(\begin{array}{c} \text{Variation of extensive} \\ \text{quantity in the system} \\ \text{during } \Delta t \end{array} \right) = \left(\begin{array}{c} \text{Extensive quantity} \\ \text{in the system at} \\ \text{time } t + \Delta t \end{array} \right) - \left(\begin{array}{c} \text{Extensive quantity} \\ \text{in the system at} \\ \text{time } t \end{array} \right) \quad [1.19]$$

$$\left[\frac{\left(\begin{array}{c} \text{Extensive quantity in the} \\ \text{system at time } t + \Delta t \end{array} \right) - \left(\begin{array}{c} \text{Extensive quantity in the} \\ \text{system at time } t \end{array} \right)}{\Delta t} \right]_{\Delta t \rightarrow 0} = \quad [1.20]$$

$$\left(\begin{array}{c} \text{Rate of extensive} \\ \text{quantity storage in } V \end{array} \right) = \frac{D}{Dt} \int_V \chi \, dV$$

where χ is the specific value of the extensive quantity per unit volume and the derivative D/Dt is the substantial derivative or total derivative of the substance of the extensive quantity contained in the volume V . Applying the Reynolds transport theorem [ARP 84] and considering a velocity field \underline{v} , we obtain:

$$\frac{D}{Dt} \int_V \chi \, dV = \int_V \frac{\partial \chi(P, t)}{\partial t} \, dV + \int_A \chi \underline{v} \cdot \underline{n} \, dA \quad [1.21]$$

The second term in the left-hand side of equation [1.21] is the convective flux through the surface A which enters and exits together with the macroscopic motion of the substance. The balance equation in logical form, equation [1.14], can be written in its integral form as:

$$\frac{D}{Dt} \int_V \chi dV + \int_V \underline{\nabla} \cdot \underline{\phi}(P, t) dV = \int_V g dV \quad [1.22]$$

Considering equations [1.18] and [1.21] and the Gauss divergence theorem [1.17], we obtain:

$$\int_V \frac{\partial \chi(P, t)}{\partial t} dV + \int_V \underline{\nabla} \cdot [\chi(P, t) \underline{v}(P, t)] dV + \int_V \underline{\nabla} \cdot \underline{\phi}(P, t) dV = \int_V g(P, t) dV \quad [1.23]$$

then

$$\int_V \left\{ \frac{\partial \chi(P, t)}{\partial t} + \underline{\nabla} \cdot [\chi(P, t) \underline{v}(P, t)] + \underline{\nabla} \cdot \underline{\phi}(P, t) - g(P, t) \right\} dV = 0 \quad [1.24]$$

Equation [1.24] is derived on an arbitrary volume V and is valid for each V , and, consequently, the local balance equation is given by:

$$\frac{\partial \chi(P, t)}{\partial t} + \underline{\nabla} \cdot [\chi(P, t) \underline{v}(P, t)] = -\underline{\nabla} \cdot \underline{\phi}(P, t) + g(P, t) \quad [1.25]$$

The two terms $\chi(P, t) \underline{v}(P, t)$ and $\underline{\phi}(P, t)$ are the convective flux or flow and the diffusive flux or flux of the extensive property χ . They are associated with the macroscopic mass flow rate and the microscopic diffusion of the considered property, respectively.

If the volume V is fixed in the space (a solid or a fluid in rest), we obtain [MIK 84, ÖZI 93]:

$$\frac{D}{Dt} \int_V \chi dV = \int_V \frac{\partial \chi(P, t)}{\partial t} dV \quad [1.26]$$

In this case, through the surface A there is only the molecular flux or diffusion which enters and exits together with the molecular motion of the substance. The

substitution of equations [1.16], [1.18] and [1.26] into equation [1.14] provides the following relation:

$$\int_V \left[-\underline{\nabla} \cdot \underline{\phi}(P, t) + g(P, t) - \frac{\partial \chi(P, t)}{\partial t} \right] dV = 0 \quad [1.27]$$

Equation [1.27] is evaluated on an arbitrary volume V and is valid for each V , which is:

$$-\underline{\nabla} \cdot \underline{\phi}(P, t) + g(P, t) - \frac{\partial \chi(P, t)}{\partial t} = 0 \quad [1.28]$$

and, consequently, the following is obtained:

$$-\underline{\nabla} \cdot \underline{\phi}(P, t) + g(P, t) = \frac{\partial \chi(P, t)}{\partial t} \quad [1.29]$$

Considering equation [1.13], we obtain:

$$\underline{\nabla} \cdot (\Gamma \underline{\nabla} \eta(P, t)) + g(P, t) = \frac{\partial \chi(P, t)}{\partial t} \quad [1.30]$$

1.3. Heat conduction equation

Equation [1.30] is a general form of a diffusion equation in a differential form and it is a partial differential equation (PDE). In a physical space, it is a three-dimensional PDE. Equation [1.30] describes the physical diffusions phenomena such as heat conduction and mass diffusion. In the following, equation [1.30] is considered with reference to heat conduction. In this case, the flux vector, $\underline{\phi}$, is the heat flux vector, \underline{q} , which represents the heat flow per unit time, per unit area of an isothermal surface in the direction of decreasing temperature, T , and the physical dimensions of the heat flux are expressed in W/m^2 and the temperature in K or $^{\circ}\text{C}$. Equation [1.13] is written as [MIK 84, ÖZI 93, YEN 08, WAN 08]:

$$\underline{q}(P, t) = -k \underline{\nabla} T(P, t) \quad [1.31]$$

where k is the thermal conductivity of the considered material and it is expressed in $\text{W}/(\text{m K})$ or $\text{W}/(\text{m}^{\circ}\text{C})$. Equation [1.30] is referred to the internal energy of the considered system and χ is equal to ρu , the internal energy per unit volume, expressed in J/m^3 whereas ρ is the density, kg/m^3 , and u is the internal energy per

unit mass, J/kg. Further, $u = cT$ with c the specific heat of the considered material, J/(kg K) or J/(kg °C). g in this case represents the internal energy production per unit volume and time, J/m³. It is related, for example, to the chemical reaction or nuclear reaction that determines an energy conversion from chemical and nuclear energy to internal energy. The heat conduction equation can be written as [ÖZI 93, WAN 08]:

$$\underline{\nabla} \cdot (k \underline{\nabla} T(P, t)) + g(P, t) = \rho c \frac{\partial T(P, t)}{\partial t} \quad [1.32]$$

Equation [1.32] is the general form of the heat conduction equation for an isotropic medium. When the thermal conductivity is assumed to be independent of position and temperature, equation [1.32] simplifies to:

$$\nabla^2 T(P, t) + \frac{1}{k} g(P, t) = \frac{1}{\alpha} \frac{\partial T(P, t)}{\partial t} \quad [1.33]$$

where $\alpha = (k/\rho c)$ is the thermal diffusivity, m²/s. For constant thermal conductivity and no heat generation, the Fourier equation is obtained:

$$\nabla^2 T(P, t) = \frac{1}{\alpha} \frac{\partial T(P, t)}{\partial t} \quad [1.34]$$

Equations [1.32] and [1.34] in a physical domain, the Euclidian geometrical space, can be expressed in different geometrical coordinate systems, such as a Cartesian or a rectangular coordinate system, a cylindrical coordinate system or a spherical coordinate system.

For a one-dimensional geometrical heat conduction problem, along the x -axis, equation [1.32] is similar to equation [1.9] and becomes:

$$\rho c \frac{\partial T(x, t)}{\partial t} = \frac{\partial}{\partial x} \left(k(T) \frac{\partial T(x, t)}{\partial x} \right) + g(x, t) \quad [1.35]$$

and for k which is independent of temperature or space and time, equation [1.35] simplifies to:

$$\frac{\partial^2 T(x, t)}{\partial x^2} + \frac{1}{k} g(x, t) = \frac{1}{\alpha} \frac{\partial T(x, t)}{\partial t} \quad [1.36]$$

If the term $g(x, t)$ is equal to 0, a one-dimensional Fourier equation is obtained:

$$\frac{\partial^2 T(x, t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T(x, t)}{\partial t} \quad [1.37]$$

Equation [1.36] becomes the Poisson equation if the problem is independent of time, a steady state condition, and is given by:

$$\frac{\partial^2 T(x, t)}{\partial x^2} = -\frac{1}{k} g(x, t) \quad [1.38]$$

and it is the Laplace equation for $g(x, t) = 0$:

$$\frac{d^2 T(x, t)}{dx^2} = 0 \quad [1.39]$$

The Laplace equation in a three-dimensional problem is given by:

$$\nabla^2 T(P, t) = 0 \quad [1.40]$$

and in rectangular coordinates, $P = (x, y, z)$, it assumes the following form:

$$\frac{\partial^2 T(x, y, z, t)}{\partial x^2} + \frac{\partial^2 T(x, y, z, t)}{\partial y^2} + \frac{\partial^2 T(x, y, z, t)}{\partial z^2} = 0 \quad [1.41]$$

1.4. Initial and boundary conditions

The differential equation, both the ordinary differential equation (ODE) or PDE, will have numerous solutions unless a set of conditions, initial and on the boundary, are prescribed. The initial condition assigns the temperature distribution inside the physical domain, the system V , at the origin of the time coordinate, i.e. when the observation of phenomena starts. In general, it is assumed that the initial time is at $t = 0$. The boundary conditions specify the different interactions between the system V and its ambient or environment through the border surface S_V , i.e. the constraints of the ambient on the assigned system. The boundary conditions are specified in terms of temperature or heat transfer on the external surface.

The initial condition is written, with reference to Figure 1.4, as the temperature distribution in the assigned domain V at time equal to 0:

$$T(P, 0) = f(P, 0) \quad \text{for } \forall P \in V \quad [1.42]$$

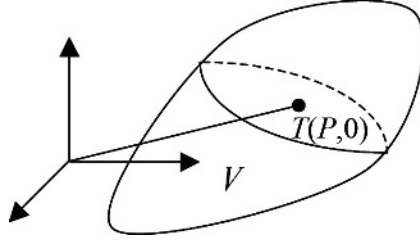


Figure 1.4. Temperature distribution in the domain V at time $t = 0$

In general, three different types of boundary conditions can be considered in heat conduction problems [DUF 06, MIK 84, ÖZI 93, YEN 08, WAN 08]:

1) Boundary condition of the first kind or a Dirichlet condition when the temperature distribution is assigned to the boundary surface, as indicated in Figure 1.5:

$$T(P, t) = f(P, t) \quad \text{for } \forall P \in S_V \text{ and } \forall t > 0 \quad [1.43]$$

where the surface temperature $f(P, t)$ is a function of position and time. The particular case

$$T(P, t) = 0 \quad \text{for } \forall P \in S_V \text{ and } \forall t > 0 \quad [1.44]$$

is the homogeneous boundary condition of the first kind or homogeneous Dirichlet condition.

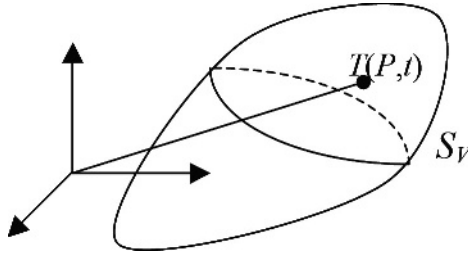


Figure 1.5. Assigned temperature distribution at the boundary surface

2) Boundary condition of the second kind or a Neumann condition when the heat flux is assigned to the boundary surface, as depicted in Figure 1.6:

$$k \frac{\partial T(P, t)}{\partial n} = f(P, t) \quad \text{for } \forall P \in S_V \text{ and } \forall t > 0 \quad [1.45]$$

where the derivative is along the outward drawn normal to the surface and $f(P, t)$ is the assigned heat flux. The particular case:

$$\frac{\partial T(P, t)}{\partial n} = 0 \quad \text{for } \forall P \in S_V \text{ and } \forall t > 0 \quad [1.46]$$

3) Boundary condition of the third kind or a Robin condition when the surface heat transfer is assigned to the boundary surface, as shown in Figure 1.7:

$$k \frac{\partial T(P, t)}{\partial n} + hT(P, t) = f(P, t) \quad \text{for } \forall P \in S_V \text{ and } \forall t > 0 \quad [1.47]$$

where the derivative is again along the outward drawn normal to the surface and $f(P, t)$ is an assigned function which represents a heat flux at boundary surface. The particular case:

$$k \frac{\partial T(P, t)}{\partial n} + hT(P, t) = 0 \quad \text{for } \forall P \in S_V \text{ and } \forall t > 0 \quad [1.48]$$

is the homogeneous boundary condition of the third kind or homogeneous Robin condition.

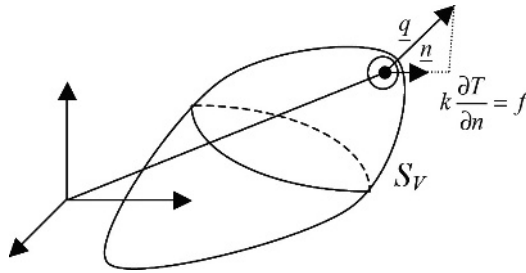


Figure 1.6. Assigned heat flux distribution at the boundary surface

If two different materials having two different thermal conductivity are in contact, such as shown in Figure 1.8 and there is no relative motion and the contact is perfect between the materials, then the heat flux and the temperature at two surfaces at the interface are equal and are defined as:

$$k_1 \frac{\partial T_1(P, t)}{\partial n} = k_2 \frac{\partial T_2(P, t)}{\partial n} \quad \text{for } \forall P \in S \text{ and } \forall t > 0 \quad [1.49a]$$

$$T_1(P, t) = T_2(P, t) \quad \text{for } \forall P \in S \text{ and } \forall t > 0 \quad [1.49b]$$

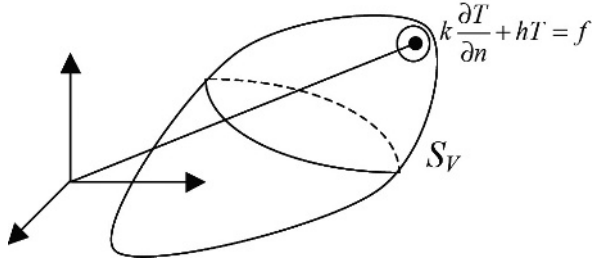


Figure 1.7. Surface heat transfer assigned at the boundary surface

There are other types of boundary conditions, such as radiative heat transfer, change of phase or the moving interface of two media, the interface of two solids in relative motion or a nonlinear convective heat transfer. These can be written following the procedure given to formulate the above boundary conditions.

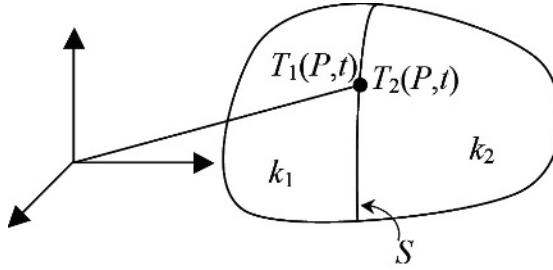


Figure 1.8. Two different materials in contact along the surface S

EXAMPLE 1.1.—

Consider a plane, depicted in Figure 1.9, with an assigned thickness, L , subject to surface heat transfer on the two external surfaces by means of the heat transfer surface coefficients, h_1 and h_2 . The two external temperatures are $T_{\infty 1}$ and $T_{\infty 2}$. The initial distribution of temperature inside the plate is uniform and equal to T_i . Write the heat conduction equation, the initial and boundary conditions in the hypothesis that the solid is homogeneous and isotropic and the thermal conductivity is k .

The problem is one-dimensional and a rectangular coordinate system is considered. Temperature distribution is a function of spatial coordinate x and time t , $T = T(x, t)$. The heat conduction equation is given by equation [1.37] as:

$$\frac{\partial^2 T(x, t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T(x, t)}{\partial t} \quad 0 \leq x \leq L, \quad t > 0 \quad [1.50a]$$

the initial and boundary conditions are obtained:

$$T(x, 0) = T_i \quad \text{for } 0 \leq x \leq L \quad [1.50b]$$

$$-k \frac{\partial T(0, t)}{\partial x} + h_1 T(0, t) = h_1 T_{\infty 1} \quad \text{for } t > 0 \quad [1.50c]$$

$$k \frac{\partial T(L, t)}{\partial x} + h_2 T(L, t) = h_2 T_{\infty 2} \quad \text{for } t > 0 \quad [1.50d]$$

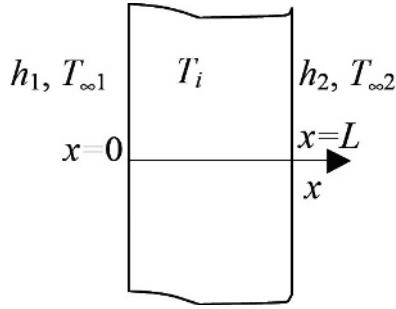


Figure 1.9. *Geometric configuration and temperature values at external surface and initial temperature distribution of example 1.1*

The conductive heat transfer problem [1.50], heat conduction equation [1.50a] together with initial [1.50b] and boundary [1.50c] and [1.50d] are a well-posed problem and should have one solution that satisfies equation [1.50a] and the conditions [1.50b]–[1.50d]. If the h_1 and h_2 values are very high with respect to the k value ($h_i \gg k$) equations [1.50c] and [1.50d] become:

$$T(0, t) = T_{\infty 1} \quad \text{for } t > 0 \quad [1.50e]$$

$$T(L, t) = T_{\infty 2} \quad \text{for } t > 0 \quad [1.50f]$$

and two boundary conditions of the first kind are assigned, i.e. the temperature values on the two external surfaces of the plate are known. Moreover, if the h_1 is equal to 0, we obtain:

$$\frac{\partial T(0, t)}{\partial x} = 0 \quad \text{for } t > 0 \quad [1.50g]$$

and an assigned heat flux value is given on the surface at $x = 0$. In this case, the surface does not allow the heat transfer between the ambient and the plate (the assigned system) and it is an adiabatic surface.