PART 1

Mathematical Preliminaries, Definitions and Properties of Fractional Integrals and Derivatives

Chapter 1

Mathematical Preliminaries

1.1. Notation and definitions

Sets of natural, integer real and complex numbers are denoted, respectively, by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0, \infty)$.

Let Ω be an arbitrary subset of $\mathbb{R}$. We denote by $C_b(\Omega)$ the set of continuous functions on Ω such that

$$\|f\|_{C_b(\Omega)} = \sup_{x \in \Omega} |f(x)| < \infty.$$

It is well known that $C_b(\Omega)$ is a Banach space. If Ω is open, then we consider compact subsets of Ω, $K \subset\subset \Omega$, continuous functions f on Ω and the semi-norms

$$\|f\|_K = \sup_{x \in K} |f(x)| \, .$$

We can take a sequence of compact sets $K_1 \subset K_2 \subset, \ldots,$ so that $\cup_{i=1}^{\infty} K_i = \Omega$. Then, the sequence of semi-norms defines the Fréchet topology on $C\,(\Omega)$. This topology does not depend on a sequence $\{K_i\}_{i \in \mathbb{N}}$, with the given property. If K is compact, then $C\,(K)$ always denotes the set of continuous functions on K with the sup-norm over K.

Let Ω be open in $\mathbb{R}^n$. Then, we consider $C^k(\Omega) \subset C(\Omega)$: the space of functions having all the derivatives, up to order $k \in \mathbb{N}_0$, continuous. The Fréchet topology is defined by the semi-norms

$$\|f\|_{k,K} = \sup_{\substack{x\in K,\\ |i|\leq k}} |f^{(i)}(x)|, \quad K \subset\subset \Omega, \ |i| = i_1 + \ldots + i_n. \qquad [1.1]$$

The same topology is obtained if we again take, for compact sets in [1.1], a sequence of compact sets $K_1 \subset K_2 \subset \ldots$, so that $\cup_{i=1}^{\infty} K_i = \Omega$.

If K is compact, we use the notation $C^k(K)$ for the Banach space of functions with all derivatives continuous up to order k, with the norm [1.1]. In the case $k = 0$, we use the notation $C^0(\Omega) = C(\Omega)$ and $C^0(\mathbb{R}) = C(\mathbb{R})$. If $k = \infty$, then we call $C^\infty(\Omega)$ the space of smooth functions. It is the Fréchet space with the sequence of semi-norms $\|f\|_{p,K_p}$, $K_p \subset K_{p+1}$, $K_p \subset\subset \Omega$, $p \in \mathbb{N}$, $\cup_{p=1}^{\infty} K_p = \Omega$. Its subspace $C_0^\infty(\mathbb{R})$ consists of compactly supported smooth functions, that is of the smooth functions equal to zero outside the compact sets.

Analytic functions on $(a,b) \subset \mathbb{R}$ are smooth functions on (a,b), so that their Taylor series converges in any point a_0 of (a,b) on a suitable interval around a_0. A space $\mathcal{A}((a,b))$ of such functions is a Fréchet space under the convergence structure from $C^\infty((a,b))$.

$BV_{loc}(\mathbb{R}_+)$ denotes the space of functions f of locally bounded variations on $\mathbb{R}_+$. This means: for every interval $[a,b] \subset \mathbb{R}_+$, there exists a constant M such that $\sum_{i=0}^{n} |f(t_{i+1}) - f(t_i)| < M$ for every finite choice of points $t_0 = a, \ldots, t_n = b$.

$L^p((a,b)) = L^p([a,b]),\ p \geq 1$, is the space of measurable functions for which $\left(\int_a^b |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} < \infty$. We shorten the notation and use the symbol $L^p(a,b)$. In $L^p(a,b),\ p \geq 1$, the norm is defined as

$$\|f\|_p = \left(\int_{(a,b)} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

In $L^\infty(a,b)$, we have $\|f\|_\infty = \operatorname{vrai\,sup}_{x\in(a,b)} |f(x)|$.

More precisely, above we consider spaces of Lebesgue measurable functions on a Lebesgue measurable set $A \subset \mathbb{R}^n$ (above $A = (a,b)$) and identify them through the equivalence relation: $f \sim g$ over A if $f(x) = g(x)$, $x \in A \setminus N$ where N is of zero Lebesgue measure. This relation determines the classes $[f], \ldots$, and in the following, we put f for $[f]$. In this sense, when $f \sim g$, we say that these functions are equal almost everywhere on A ($f = g$ almost everywhere (a.e.) on A) and we just identify f and g. So, the notation above $\operatorname{vrai\,sup}_{x\in(a,b)} f(x)$, for a measurable function bounded almost everywhere on (a, b), means: supremum up to a set of points in (a, b) with the zero measure. In the sequel, we will consider equality almost everywhere, as well as the integration in the sense of Lebesgue. $L^1_{loc}(a, b) = L^1_{loc}((a, b))$, $a, b \in \mathbb{R}^n$, $a < b$, is the space of measurable functions f on $(a, b) \in \mathbb{R}^n$ such that for every compact set $K \subset (a, b)$, there holds $\int_K |f(x)|\mathrm{d}x < \infty$. It is clear that $L^1_{loc}([a, b)) \neq L^1_{loc}(a, b)$.

If p and q are real numbers such that $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ (for $p = 1$, $q = \infty$), and if $f \in L^p(a, b)$, $g \in L^q(a, b)$, then $fg \in L^1(a, b)$ and

$$\int_{(a,b)} |f(x)g(x)|\mathrm{d}x \leq \|f\|_p \|g\|_p. \qquad \text{[Hölder inequality]}$$

A real-valued function f defined on $[a, b] \subset \mathbb{R}$ is said to be absolutely continuous on $[a, b]$, if for given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x'_i) - f(x_i)| < \varepsilon$$

for every finite collection $\{(x'_i, x_i)\}_{i\in\mathbb{N}}$ of non-overlapping intervals with

$$\sum_{i=1}^{n} |x'_i - x_i| < \delta.$$

The space of absolutely continuous functions is denoted by $AC([a, b]) = AC^1([a, b])$. There holds $C([a, b]) \subset AC([a, b])$. Moreover, $f \in AC([a, b])$, if and only if there exists an integrable function g over $[a, b]$ such that

$$f(x) = c + \int_a^x g(t)\mathrm{d}t, \quad g = f' \text{ a.e. on } [a, b].$$

$AC^n([a, b])$, $n \in \mathbb{N}$, $n \geq 2$, is the space of functions f, which have continuous derivatives up to the order n 1 on $[a, b]$ and $f^{(n-1)} \in AC([a, b])$. Notation $AC^n_{loc}([0, \infty))$ means that the function $f \in AC^n([0, b])$, for every $b > 0$.

A function f on $[a,b]$ is Hölder continuous at $x_0 \in [a,b]$ if there exist $A > 0$ and $\lambda > 0$, such that

$$|f(x) - f(x_0)| \leq A\,|x - x_0|^{\lambda}$$

in a neighborhood of x_0. Hölder-type spaces on an interval $[a,b]$ are defined as subspaces of integrable functions on this interval with the following properties:

– $\mathcal{H}^{\lambda} \equiv \mathcal{H}^{\lambda}([a,b]) = \{f \mid |f(x_1) - f(x_2)| \leq A\,|x_1 - x_2|^{\lambda},\ x_1, x_2 \in [a,b]\},\ \lambda \in (0,1]$;

– $\mathcal{H} \equiv \mathcal{H}([a,b]) = \cup_{0<\lambda\leq 1}\mathcal{H}^{\lambda}([a,b])$;

– $\mathcal{H}^{*} \equiv \mathcal{H}^{*}([a,b]) = \Big\{f \mid f(x) = \frac{f^{*}(x)}{(x-a)^{1-\epsilon_1}(b-x)^{1-\epsilon_2}},\ x \in (a,b),\ \epsilon_1, \epsilon_2 > 0,\ f^{*} \in \mathcal{H}^{\lambda}([a,b]),\ \lambda \in (0,1]\Big\}$;

– $\mathcal{H}_0^{\lambda}(\epsilon_1, \epsilon_2) = \{f \in \mathcal{H}^{*} \mid f^{*}(0) = f^{*}(b) = 0\}$;

– $\mathcal{H}_{\alpha}^{*} = \cup_{\alpha<\lambda\leq 1,\ \epsilon_1,\epsilon_2>0}\mathcal{H}_0^{\lambda}(\epsilon_1, \epsilon_2)$;

– $h^{\lambda} \equiv h^{\lambda}([a,b]) = \Big\{f \mid \frac{f(x_1-x_2)}{|x_1-x_2|^{\lambda}} \to 0,\ x_2 \to x_1\Big\};\ h^{\lambda} \subset \mathcal{H}^{\lambda}$.

1.2. Laplace transform of a function

Let $f \in L^1_{loc}(\mathbb{R})$ and $f(t) = 0,\ t \in (-\infty, 0)$. The Laplace transform of f is defined by

$$\mathcal{L}\left[f(t)\right](s) = \tilde{f}(s) = \lim_{A\to\infty}\int_0^A f(t)\mathrm{e}^{-st}\mathrm{d}t, \qquad [1.2]$$

for those complex numbers s for which this limit exists. It is well known that the existence of the limit in [1.2] at $s = s_0$ implies the existence of this limit for any $s \in \mathbb{C}$ with the property $\mathrm{Re}\,s > \mathrm{Re}\,s_0$. We can consider the integral $\int_0^{\infty}|f(t)|\mathrm{e}^{-t\,\mathrm{Re}\,s}\mathrm{d}t$. If it is finite (we say an integral exists, or converges) for $s = s_1$, then f is called an absolutely convergent Laplace transformable function. In this case

$$\mathcal{L}\left[f(t)\right](s) = \int_0^{\infty} f(t)\mathrm{e}^{-st}\mathrm{d}t$$

is absolutely convergent for any $s \in \mathbb{C}$ such that $\mathrm{Re}\,s > \mathrm{Re}\,s_1$.

The number $a_e = \inf\{\mathrm{Re}\, s_0 \in \mathbb{R}\}$ representing the infimum of those $s_0 \in \mathbb{C}$ for which the Laplace transform is defined is called the abscissa of existence. The abscissa of absolute convergence a_a is defined in the same way. We have $a_a \geq a_e$.

It is clear that $\tilde{f}(s)$ exists (absolutely exists) for every $s \in \mathbb{C}$, $\mathrm{Re}\, s > a_e$ ($s \in \mathbb{C}$, $\mathrm{Re}\, s > a_a$). It is an analytic function in the half-plane $\mathrm{Re}\, s > a_e$, since, by partial integration, it can be represented as an absolutely convergent Laplace transform.

In the following, we consider the following class of Laplace transformable functions. Function $f \in L^1_{loc}([0,\infty))$ is called exponentially bounded if there exist constant $C = C_f > 0$, $r = r_f \in \mathbb{R}$ and $c = c_f \geq 0$ such that

$$|f(t)| \leq C\mathrm{e}^{rt}, \quad t > c. \tag{1.3}$$

We denote by $L^{exp}([0,\infty))$ the space of such functions. The growth order r is greater or equal than the abscissa of absolute convergence, $r \geq a_a$.

The Laplace transform is a linear operation on the space of exponentially bounded functions. If a function and its derivatives on $[0,\infty)$ up to order k are of exponential growth, then

$$\mathcal{L}\left[f^{(k)}(t)\right](s) = s^k \tilde{f}(s) - s^{k-1} f(0) - \ldots - f^{k-1}(0), \quad \mathrm{Re}\, s > c,$$

for suitable $c > 0$. Let us mention several useful properties of the Laplace transform, based on appropriate assumptions and on corresponding domains

$$\begin{aligned}
\mathcal{L}\left[f(t)\mathrm{e}^{at}\right](s) &= \tilde{f}(s-a), \quad \mathcal{L}\left[f(at)\right](s) = \frac{1}{a}\tilde{f}\left(\frac{s}{a}\right), \\
\mathcal{L}\left[t^n f(t)\right](s) &= (-1)^n \tilde{f}^{(n)}(s), \quad \mathcal{L}\left[f(t) * g(t)\right](s) = \tilde{f}(s)\tilde{g}(s),
\end{aligned}$$

where the convolution of two locally integrable functions on $[0,\infty)$ is defined by

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau, \quad t \geq 0.$$

The inverse Laplace transform is defined by

$$f(t) = \mathcal{L}^{-1}\left[\tilde{f}(s)\right](t) = \frac{1}{2\pi\mathrm{i}} \lim_{q\to\infty} \int_{p-\mathrm{i}q}^{p+\mathrm{i}q} \tilde{f}(s)\mathrm{e}^{st}\mathrm{d}s, \quad t \geq 0,$$

where $p > r$ (see [1.3]).

1.3. Spaces of distributions

The reader of this book has to have a knowledge of the theory of the generalized functions; here we call them distributions, as they are commonly known. This theory is a powerful tool used in mathematical theory and applications. Apart from books that discuss the basic theory, for example [SCH 51, VLA 73], there are a number of application-oriented textbooks such as [DUI 10].

We refer to [SCH 51, VLA 73] for the material of this section. By $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, the well-known Schwartz spaces are denoted. Norms in the space $\mathcal{D}_K(\mathbb{R}^n)$ of smooth functions supported by K are

$$p_{K,m}(\varphi) = \sup_{x\in K, |\alpha|\leq m} |\varphi^{(\alpha)}(x)|, \quad m \in \mathbb{N}_0,$$

while in $\mathcal{S}(\mathbb{R}^n)$ are

$$q_m(\varphi) = \sup_{x\in \mathbb{R}, |\alpha|\leq m} (1+|x|)^m |\varphi^{(\alpha)}(x)|, \quad m \in \mathbb{N}_0.$$

Then, $\mathcal{D}(\mathbb{R})$ is the inductive limit

$$\mathcal{D}(\mathbb{R}) = \operatorname*{ind\,lim}_{K_n \subset\subset \mathbb{R}} \mathcal{D}_{K_n},$$

where K_n, $n \in \mathbb{N}$, is an increasing sequence of compact sets so that $\cup_n K_n = \mathbb{R}$.

The corresponding duals, spaces of continuous linear functionals, $\mathcal{D}'(\mathbb{R}^n)$ and its subspace $\mathcal{S}'(\mathbb{R}^n)$, with the strong topologies, are the space of distributions and the space of tempered distributions. The space of compactly supported distributions is denoted by $\mathcal{E}'(\mathbb{R}^n)$. It is the dual space for the Fréchet space see section 1.1.

Operations of multiplication and differentiation in $\mathcal{D}'(\mathbb{R})$ are defined in a usual way

$$\langle af, \varphi\rangle = \langle f, a\varphi\rangle, \quad \langle f^{(k)}, \varphi\rangle = (-1)^k \langle f, \varphi^{(k)}\rangle, \quad a \in C^\infty(\mathbb{R}), \ \varphi \in \mathcal{D}(\mathbb{R}), \ k \in \mathbb{N}.$$

Note $a(x)\,\delta(x-x_0) = a(x_0)\,\delta(x-x_0)$, where δ is the Dirac distribution,

$$\langle \delta(x-x_0), \varphi(x)\rangle = \varphi(x_0), \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

We note that $\mathcal{D}' = \mathcal{D}'(\mathbb{R})$ contains regular elements defined by $f \in L_{loc}(\mathbb{R})$; they are denoted by f_{reg} and defined by

$$f_{reg} : \varphi \mapsto \langle f_{reg}, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)\mathrm{d}x, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

We can see that $\varphi_n \to 0$ in $\mathcal{D}$ implies $\langle f_{reg}, \varphi_n \rangle \to 0$, $n \to \infty$.

Polynomially bounded and locally integrable functions on $\mathbb{R}$ define, in the same way, regular tempered distributions. We will usually denote, by the same symbol f, a function and a corresponding distribution f_{reg}. Only if we want to explain in detail the relation between them do we use the symbol f_{reg}.

The Fourier transform of a function $\varphi \in \mathcal{S}$ ($\mathcal{S} = \mathcal{S}(\mathbb{R})$) is defined by

$$\mathcal{F}[\varphi(x)](\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x)\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{d}x, \quad \xi \in \mathbb{R}.$$

The Fourier transform is an isomorphism on $\mathcal{S}$. If $f \in \mathcal{S}'$ ($\mathcal{S}' = \mathcal{S}'(\mathbb{R})$), then

$$\langle \mathcal{F}[f], \varphi \rangle = \langle f, \mathcal{F}[\varphi] \rangle, \quad \varphi \in \mathcal{S},$$

defines the Fourier transform of a tempered distribution. The Fourier transform is an isomorphism on $\mathcal{S}'$. The inverse Fourier transform of $\varphi \in \mathcal{S}$ is defined by

$$\varphi(x) = \mathcal{F}^{-1}[\hat{\varphi}(\xi)](x) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(\xi)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}\xi, \quad x \in \mathbb{R}.$$

If $\varphi, \psi \in \mathcal{S}$, their convolution is defined by

$$\varphi(x) * \psi(x) = \int_{\mathbb{R}} \varphi(x-\zeta)\,\psi(\zeta)\,\mathrm{d}\zeta, \quad x \in \mathbb{R}.$$

If $\operatorname{supp}\varphi, \operatorname{supp}\psi \in [0,\infty)$, which means that $\varphi = \psi = 0$ on $(-\infty, 0)$, then

$$\varphi(t) * \psi(t) = \int_0^t \varphi(t-\tau)\,\psi(\tau)\,\mathrm{d}\tau, \quad t \geq 0 \;\text{ and }\; \varphi(t) * \psi(t) = 0, \quad t < 0.$$

We know

$$\mathcal{F}\left[\varphi\left(x\right)*\psi\left(x\right)\right]\left(\xi\right)=\hat{\varphi}\left(\xi\right)*\hat{\psi}\left(\xi\right),\quad \varphi,\psi\in\mathcal{S}$$

and, as a consequence,

$$\mathcal{F}\left[f\left(x\right)*g\left(x\right)\right]\left(\xi\right)=\hat{f}\left(\xi\right)*\hat{g}\left(\xi\right),\quad f,g\in\mathcal{S}'.$$

Sobolev space $W^{k,p}(\mathbb{R})$, $p\in[1,\infty]$, $k\in\mathbb{N}_0$, is defined as the space of L^p-functions f with the property that all the distributional derivatives of f up to order k are elements of $L^p(\mathbb{R})$. It is a Banach space with the norm

$$\|f\|_{W^{k,p}}=\sum_{j=0}^{p}\left\|f^{(j)}\right\|_{L^p}.$$

Clearly, $W^{k,p}(\mathbb{R})\subset\mathcal{S}'(\mathbb{R})$.

$\mathcal{D}_{L^p},\ 1\le p<\infty$ is a space of smooth functions with all derivatives belonging to L^p. Note $\mathcal{D}_{L^p}\subset\mathcal{D}_{L^q}$ if $p\le q$. $\dot{\mathcal{B}}$ is a subspace of $\mathcal{D}_{L^\infty}=\mathcal{B}$, defined as follows: $\varphi\in\dot{\mathcal{B}}$ if and only if $|\varphi^{(\alpha)}(x)|\to 0$ as $|x|\to\infty$ for every $\alpha\in\mathbb{N}_0$.

$\mathcal{D}'_{L^p},\ 1<p\le\infty$ is the dual space of $\mathcal{D}_{L^q},\ 1\le q<\infty,\ \frac{1}{p}+\frac{1}{q}=1.$ $\mathcal{D}'_{L^1}$ is the dual of $\dot{\mathcal{B}}$ and $\mathcal{D}'_{L^\infty}$ is denoted by $\mathcal{B}'$ (see [SCH 51]).

$\mathcal{S}'_+$ denotes a subspace of tempered distributions consisting of distributions with supports in $[0,\infty)$. Note that $\mathcal{S}'_+$ is a convolution algebra.

The following structural theorem holds: $f\in\mathcal{S}'_+$ if and only if there exists a continuous function F on $\mathbb{R}$ such that $F(x)=0$, $x<0$, $|F(x)|\le C(1+|x|)^k$ for some $C>0$, $k>0$ and there exists $p\in\mathbb{N}_0$ such that

$$f(x)=F^{(p)}(x),\quad x\in\mathbb{R}, \qquad [1.4]$$

where the derivative is taken in the sense of distributions.

Let $f\in\mathcal{S}'_+$. Its Laplace transform is defined by

$$\mathcal{L}\left[f\left(t\right)\right](s)=\tilde{f}(s)=\langle f(t),\mathrm{e}^{-st}\rangle=s^p\mathcal{L}\left[F\left(t\right)\right](s),\quad \operatorname{Re}s>0, \qquad [1.5]$$

where we assume that f is of the form [1.4]. Clearly, $\tilde{f}(s)$ is a holomorphic function for $\operatorname{Re} s > 0$. We will often consider equations, with solutions u determining the tempered distributions, by the use of the Laplace transform. If we assume that u is of exponential growth, then we have $\tilde{u}(s)$, $\operatorname{Re} s > s_0$, for some $s_0 > 0$.

We consider the family $\{f_\alpha\}_{\alpha\in\mathbb{R}} \in \mathcal{S}'_+$ (see [VLA 84])

$$f_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} H(t), & t \in \mathbb{R},\ \alpha > 0, \\ \frac{\mathrm{d}^m}{\mathrm{d}t^m} f_{\alpha+m}(t), & \alpha \leq 0,\ \alpha + m > 0,\ m \in \mathbb{N}, \end{cases} \qquad [1.6]$$

where the mth derivative is understood in the distributional sense. Family $\{\check{f}_\alpha\}_{\alpha\in\mathbb{R}} \in \mathcal{S}'_-$ is defined by $\check{f}_\alpha(t) = f_\alpha(-t)$. The Heaviside function is defined as

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$$

Operators $f_\alpha *$ and $\check{f}_\alpha *$ are convolution operators

$$f_\alpha *, \check{f}_\alpha * : \mathcal{S}'_+ \to \mathcal{S}'_+ \ \text{ and } \ \check{f}_\alpha * : \mathcal{S}'_- \to \mathcal{S}'_-.$$

The semi-group property holds for f_α

$$f_\alpha * f_\beta = f_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{R}.$$

The Laplace transform of f_α is

$$\mathcal{L}[f_\alpha(t)] = \frac{1}{s^\alpha} \tilde{f}(s), \quad \operatorname{Re} s > 0.$$

EXAMPLE 1.1.– Let f be an absolutely continuous function on $[0, \infty)$ so that $f(0^+) = p \neq 0$. Assume that f and f' have the (classical) Laplace transform (denoted by $\mathcal{L}_c$) in the domain $\operatorname{Re} \lambda > \lambda_0 > 0$. Put f_{reg}, and $(f')_{reg}$, for the corresponding distributions. Note that $f_{reg} = fH$, with symbol $\mathcal{L}_d$ for [1.5]. We have

$$\begin{aligned} \mathcal{L}_d\left[(f')_{reg}(t)\right](s) &= \mathcal{L}_d[f'(t)H(t)](s) = \mathcal{L}_d[(f(t)H(t))'](s) \\ &\quad - \mathcal{L}_d[f(0)\delta(t)](s) \\ &= s\mathcal{L}_d[f_{reg}(t)](s) - f(0), \end{aligned}$$

where

$$\mathcal{L}_d\left[(f')_{reg}(t)\right](s) = \mathcal{L}_d\left[(f(t)\,H(t))'\right](s) = s\mathcal{L}_d\left[f_{reg}(t)\right](s), \quad \operatorname{Re} s > \lambda_0,$$

while in the classical case,

$$\mathcal{L}_c\left[f'(t)\right](s) = \mathcal{L}_c\left[f(t)\right](s) - f(0), \quad \operatorname{Re} s > \lambda_0.$$

EXAMPLE 1.2.– Let $u(x,t)$, $x \in \mathbb{R}^n$, $t > 0$, be a classical solution of the wave equation

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = f(x,t),\\
&u(x,0) = u_0(x), \quad \frac{\partial}{\partial t}u(x,0) = v_0(x),
\end{aligned}$$

that is the second derivative above are locally integrable functions on $[0,\infty) \times \mathbb{R}^n$, equal to zero for $t < 0$, u_0, v_0 are locally integrable functions on $\mathbb{R}^n$ and $f \in L^1_{loc}([0,\infty) \times \mathbb{R}^n)$ so that it has the classical Laplace transform with respect to t in the domain $\operatorname{Re} s > 0$.

Writing

$$u_{reg}(x,t) = u(x,t)\,H(t), \text{ and } f_{reg}(x,t) = f(x,t)\,H(t)$$

for the corresponding distributions, we rewrite the wave equation in the space of distributions as

$$\begin{aligned}
&H(t)\frac{\partial^2}{\partial t^2}u(x,t) - H(t)\frac{\partial^2}{\partial x^2}u(x,t) = H(t)f(x,t),\\
&\frac{\partial^2}{\partial t^2}\left(H(t)u(x,t)\right) - \delta(t)\frac{\partial}{\partial t}u(x,t) - \delta'(t)u(x,t) - \frac{\partial^2}{\partial x^2}\left(H(t)u(x,t)\right)\\
&\qquad = H(t)f(x,t),\\
&\frac{\partial^2}{\partial t^2}u_{reg}(x,t) - \delta(t)v_0(x) - \delta'(t)u_0(x) - \frac{\partial^2}{\partial x^2}u_{reg}(x,t) = f_{reg}(x,t),\\
&\qquad t > 0,\ x \in \mathbb{R},
\end{aligned}$$

where the last equation is written in the space of distributions. So with the application of the distributional Laplace transform with respect to t, for $\mathrm{Re}\, s > 0$, we have

$$s^2 \tilde{u}_{reg}(x,s) - v_0(x) - s u_0(x) - \frac{\partial^2}{\partial x^2}\tilde{u}_{reg}(x,s) = \tilde{f}_{reg}(x,s).$$

The space $\mathcal{K}(\mathbb{R})$ is the space of smooth functions φ with the property

$$\sup_{x\in\mathbb{R},\alpha\le m} \left|\varphi^{(\alpha)}(x)\right| \mathrm{e}^{m|x|} < \infty, \quad m \in \mathbb{N}_0. \qquad [1.7]$$

The space $\mathcal{K}'(\mathbb{R})$ is the dual of $\mathcal{K}(\mathbb{R})$ and elements of $\mathcal{K}'(\mathbb{R})$ are of the form $f = \sum_{\alpha=0}^{r} \Phi_\alpha^{(\alpha)}$, where Φ_α are continuous functions with the property $|\Phi_\alpha(t)| \le C e^{k_0|t|}$, $\alpha \le r,\ t \in \mathbb{R}$, for some $C > 0$, $r \in \mathbb{N}_0$ and some $k_0 \in \mathbb{N}_0$. $\mathcal{K}'_+(\mathbb{R}) = \mathcal{K}'_+$ is a subspace of $\mathcal{K}'(\mathbb{R})$ consisting of elements supported by $[0,\infty)$ (see [ABD 99, HAS 61]). Its elements are of the form

$$f(x) = (\Phi(x)\mathrm{e}^{kx})^{(p)}, \quad x \in \mathbb{R}, \qquad [1.8]$$

where Φ is a continuous bounded function such that $\Phi(t) = 0,\ t \le 0$. Note that $\mathcal{S}$ and $\mathcal{S}'_+$ are subspaces of $\mathcal{K}'(\mathbb{R})$ and $\mathcal{K}'_+$, respectively. The construction implies that elements of $\mathcal{K}'_+$ have the Laplace transform, that is if f is of the form [1.8], then its Laplace transform $\tilde{f}$ is an analytic function in the domain $\mathrm{Re}\, s > k$.

The Lizorkin space of test functions Φ is introduced so that Riesz integro-differentiation (and therefore symmetrized fractional derivative) is well defined (see [SAM 93]). Let

$$\Psi = \{\psi \mid \psi \in \mathcal{S}(\mathbb{R}),\ \psi^{(j)}(0) = 0,\ j = 0,1,2...\},$$

and consider the space Φ consisting of the Fourier transforms of functions in Ψ, i.e. $\Phi = \mathcal{F}[\Psi]$. Then, Φ consists of those functions $\varphi \in \mathcal{S}(\mathbb{R})$ that are orthogonal to polynomials

$$\int_{\mathbb{R}} x^k \varphi(x)\mathrm{d}x = 0, \quad k \in \mathbb{N}_0.$$

The space Ψ' and the space of Lizorkin generalized functions Φ' are dual spaces of Ψ and Φ, respectively. Recall, for $f \in \Phi'$, we have

$$\langle \mathcal{F}[f], \psi\rangle = \langle f, \mathcal{F}[\psi]\rangle, \quad \psi \in \Psi.$$

Let $f \in C^{\infty}(\mathbb{R} \setminus \{0\})$ be such that it has all the derivatives bounded by the polynomials in $\mathbb{R} \setminus \{0\}$. Then, product $f \cdot u$ is defined by

$$\langle f \cdot u, \psi \rangle = \langle u, f \cdot \psi \rangle, \quad \psi \in \Psi.$$

1.4. Fundamental solution

Let P be a linear partial integro-differential operator with constant coefficients. A fundamental solution of P, denoted by E, is a distributional solution to the equation $Pu = \delta$. Once the fundamental solution is determined, we find a solution to $Pu = f$ as $u = E * f$, if this convolution exits.

The Cauchy problem for the second-order linear partial integro-differential operator with constant coefficients P is given by

$$\begin{aligned} Pu(x,t) &= f(x,t), \quad x \in \mathbb{R}, \; t > 0, && [1.9] \\ u(x,0) &= u_0(x), \quad \frac{\partial}{\partial t} u(x,0) = v_0(x), && [1.10] \end{aligned}$$

where f is continuous for $t \geq 0$, $u_0 \in C^1(\mathbb{R})$ and $v_0 \in C(\mathbb{R})$. A classical solution $u(x,t)$ to the Cauchy problem [1.9], [1.10] is of class C^2 for $t > 0$ and of class C^1 for $t \geq 0$, satisfies equation [1.9] for $t > 0$, and initial conditions [1.10] when $t \to 0$. If functions u and f are continued by zero for $t < 0$, then the following equation is satisfied in $\mathcal{D}'(\mathbb{R}^2)$:

$$Pu(x,t) = f(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t). \qquad [1.11]$$

The explanation is given in example 1.2 in the case of the wave equation. The problem of finding generalized solutions (in $\mathcal{D}'(\mathbb{R}^2)$) of equation [1.11] that vanish for $t < 0$ will be called the generalized Cauchy problem for the operator P. If there is a fundamental solution E of the operator P and if $f \in \mathcal{D}'(\mathbb{R}^2)$ vanishes for $t < 0$, then there exists a unique solution to the corresponding generalized Cauchy problem and is given by

$$u(x,t) = E(x,t) * (f(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t)),$$

if the convolution $E * f$ exits. We refer to [DAU 00, TRE 75, VLA 84] for more details.

1.5. Some special functions

The Euler gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t, \quad \operatorname{Re} z > 0.$$

The gamma function can also be represented by

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad \operatorname{Re} z > 0.$$

It satisfies $\Gamma(z+1) = z\Gamma(z)$, $\operatorname{Re} z > 0$. By the analytic continuation, we have that $\Gamma(z)$, $z \neq -n$, $n \in \mathbb{N}_0$, is an analytic function. Gamma function has simple poles at $z = n$, $n \in \mathbb{N}_0$. Having $\Gamma(1) = 1$, we obtain $\Gamma(n+1) = n!, \quad n \in \mathbb{N}$. We refer to [POD 99] for the properties of the gamma function.

We refer to [GOR 97b, MAI 00] for the theory of Mittag-Leffler functions presented in this section. The one-parameter Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}, \ \alpha > 0. \qquad [1.12]$$

The one-parameter Mittag-Leffler function is an entire function of order $\rho = \frac{1}{\alpha}$ and type 1. In some special cases of α, the one-parameter Mittag-Leffler function becomes

$$E_2\left(z^2\right) = \cosh z, \quad E_2\left(-z^2\right) = \cos z, \quad z \in \mathbb{C},$$
$$E_{\frac{1}{2}}\left(\pm\sqrt{z}\right) = \mathrm{e}^z\left(1 + \operatorname{erf}\left(\pm\sqrt{z}\right)\right), \quad z \in \mathbb{C},$$

with $\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-u^2} \mathrm{d}u$ being the error function.

The asymptotics of [1.12] are as follows:

$$E_\alpha(z) \approx \frac{1}{\alpha}\mathrm{e}^{\sqrt{z}} - \sum_{k=1}^\infty \frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad |z| \to \infty, \ |\arg z| < \frac{\alpha\pi}{2}, \ \alpha \in (0,2),$$

$$E_\alpha(z) \approx -\sum_{k=1}^\infty \frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad |z| \to \infty, \ \arg z \in \left(\frac{\alpha\pi}{2}, 2\pi - \frac{\alpha\pi}{2}\right), \ \alpha \in (0,2),$$

$$E_\alpha(z) \approx \frac{1}{\alpha}\sum_m \mathrm{e}^{\sqrt[\alpha]{z}\mathrm{e}^{\frac{2\pi i m}{\alpha}}} - \sum_{k=1}^\infty \frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad \begin{array}{c} |z| \to \infty, \ \arg z \in (-\pi, \pi), \\ \alpha \geq 2, \\ m \in \mathbb{N}, \ \arg(z + 2\pi m) \\ \in \left(-\frac{\alpha\pi}{2}, \frac{\alpha\pi}{2}\right). \end{array}$$

The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}\left(z\right)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)},\quad z\in\mathbb{C},\ \alpha>0,\ \beta\in\mathbb{C}.$$

It is an entire function of order $\rho=\frac{1}{\alpha}$ and type 1. In some special cases of α and β, it becomes

$$E_{1,2}\left(z\right)=\frac{\mathrm{e}^{z}-1}{z},\qquad E_{2,2}\left(z\right)=\frac{\sinh\sqrt{z}}{\sqrt{z}},\quad z\in\mathbb{C}.$$

We define one- and two-parameter Mittag-Leffler-type functions, respectively, by

$$e_{\alpha}\left(t,\lambda\right)=E_{\alpha}\left(-\lambda t^{\alpha}\right)\ \text{ and }\ e_{\alpha,\beta}\left(t,\lambda\right)=t^{\beta-1}E_{\alpha,\beta}\left(-\lambda t^{\alpha}\right),\quad t\geq0,\ \lambda\in\mathbb{C}.$$

In applications, we will often omit the parameter λ. According to [MAI 00], if $\alpha\in(0,1)$ and $\lambda>0$, we have $e_{\alpha}\in C^{\infty}((0,\infty))\cap C([0,\infty))$ and $\frac{\mathrm{d}}{\mathrm{d}t}e_{\alpha}\in C^{\infty}((0,\infty))\cap L_{loc}^{1}([0,\infty))$. Also, e_{α} is a completely monotonic function, i.e. $(-1)^{k}\frac{\mathrm{d}^{k}}{\mathrm{d}t^{k}}e_{\alpha}(t)>0$.

The Laplace transforms of e_{α} and $e_{\alpha,\beta}$ are

$$\mathcal{L}\left[e_{\alpha}\left(t,\lambda\right)\right]\left(s\right)=\frac{s^{\alpha-1}}{s^{\alpha}+\lambda},\qquad\mathcal{L}\left[e_{\alpha,\beta}\left(t,\lambda\right)\right]\left(s\right)=\frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda},\quad\operatorname{Re}s>\sqrt[\alpha]{|\lambda|},$$

respectively.

Functions e_{α} and $e_{\alpha,\beta}$ admit integral representations given by

$$\begin{aligned}e_{\alpha}\left(t,\lambda\right)&=\frac{1}{\pi}\int_{0}^{\infty}\frac{\lambda q^{\alpha-1}\sin\left(\alpha\pi\right)}{q^{2\alpha}+2\lambda q^{\alpha}\cos\left(\alpha\pi\right)+\lambda^{2}}\mathrm{e}^{-qt}\mathrm{d}q,\quad t\geq0,\ \alpha\in\left(0,1\right),\\&\qquad\lambda>0,\\e_{\alpha,\beta}\left(t,\lambda\right)&=\frac{1}{\pi}\int_{0}^{\infty}\frac{\lambda\sin\left(\left(\beta-\alpha\right)\pi\right)+q^{\alpha}\sin\left(\beta\pi\right)}{q^{2\alpha}+2\lambda q^{\alpha}\cos\left(\alpha\pi\right)+\lambda^{2}}q^{\alpha-\beta}\mathrm{e}^{-qt}\mathrm{d}q,\\&\qquad t\geq0,\ 0<\alpha\leq\beta<1,\ \lambda>0.\end{aligned}$$

Chapter 2

Basic Definitions and Properties of Fractional Integrals and Derivatives

2.1. Definitions of fractional integrals and derivatives

In this section, we review some basic properties of fractional integrals and derivatives, which we will need later in the analysis of concrete problems. This section contains results from various books and papers [ALM 12, ATA 14a, ATA 13a, ATA 07a, ATA 09b, ATA 09d, ATA 08b, BUT 00, CAN 87, CAP 67, CAP 71b, DIE 10, HER 11, KIL 04, KIL 06, KIR 94, NAH 03, ODI 07, POO 12a, POO 12b, POO 13, ROS 93, SAM 95, SAM 93, TAR 06, TRU 99, UCH 08, WES 03].

2.1.1. *Riemann–Liouville fractional integrals and derivatives*

There are many possible generalizations of the notion of a derivative of a function that would lead to the answer of the question: what is $\frac{\mathrm{d}^n}{\mathrm{d}x^n} y(x)$ when n is any real number? We start from the Cauchy formula for an n-fold primitive of a function f given as

$$ {}_a\mathrm{I}_t^n f(t) = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau) \, \mathrm{d}\tau, \quad t \in [a,b], \ n \in \mathbb{N}, \qquad [2.1] $$

where it is assumed that $f(t) = 0$, for $t < a$. Note that $(n-1)! = \Gamma(n)$, where Γ is the Euler gamma function (see section 1.5).

DEFINITION 2.1.– *The left Riemann–Liouville fractional integral of order* $\alpha \in \mathbb{C}$ *is formally given by*

$$ {}_a\mathrm{I}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \quad t \in [a,b], \ \mathrm{Re}\,\alpha > 0. \qquad [2.2] $$

In the special case of positive real α ($\alpha \in \mathbb{R}_+$) and $f \in L^1(a,b)$, the integral ${}_a\mathrm{I}_t^\alpha f$ exists for almost all $t \in [a,b]$. Also, ${}_a\mathrm{I}_t^\alpha f \in L^1(a,b)$ (see [DIE 10, p. 13]). For $\alpha = 0$, we define ${}_a\mathrm{I}_t^0 f = f$. This definition is motivated by the following reasoning. Suppose that $f \in C^1([a,b])$. Then, after integration by parts, from [2.2], we have

$$ {}_a\mathrm{I}_t^\alpha f(t) = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} f(a) + \frac{1}{\Gamma(\alpha+1)} \int_a^t (t-\tau)^\alpha f^{(1)}(\tau)\, \mathrm{d}\tau, $$

so that

$$ \lim_{\alpha \to 0} {}_a\mathrm{I}_t^\alpha f(t) = f(a) + \int_a^t f^{(1)}(\tau)\, \mathrm{d}\tau = f(t). $$

DEFINITION 2.2.– *The right Riemann–Liouville fractional integral of order* $\alpha \in \mathbb{C}$ *is formally given by*

$$ {}_t\mathrm{I}_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \quad t \in [a,b], \ \mathrm{Re}\,\alpha > 0. \qquad [2.3] $$

The existence is the same as in the case of the left Riemann–Liouville fractional integral given above.

In the special case when $f(t) = (t-a)^{\beta-1}$ and $g(t) = (b-t)^{\beta-1}$, $t \in [a,b]$, $\alpha, \beta \in \mathbb{C}$, we have

$$ \begin{aligned} {}_a\mathrm{I}_t^\alpha (t-a)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}, \quad \mathrm{Re}\,\alpha > 0, \ \mathrm{Re}\,\beta > 0, \\ {}_t\mathrm{I}_b^\alpha (b-t)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (b-t)^{\beta+\alpha-1}, \quad \mathrm{Re}\,\alpha > 0, \ \mathrm{Re}\,\beta > 0. \end{aligned} $$

Operators ${}_a\mathrm{I}_t^\alpha$ and ${}_t\mathrm{I}_b^\alpha$ with $\mathrm{Re}\,\alpha > 0$ are bounded operators from $L^p(a,b)$ into $L^p(a,b)$, $p \geq 1$. The following estimates hold:

$$ \|\mathrm{I}_t^\alpha f\|_{L^p(a,b)} \leq \frac{(b-a)^{\mathrm{Re}\,\alpha}}{|\Gamma(\alpha)|\,\mathrm{Re}\,\alpha} \|f\|_{L^p(a,b)}, \quad \|{}_t\mathrm{I}_b^\alpha f\|_{L^p(0,b)} \leq \frac{(b-a)^{\mathrm{Re}\,\alpha}}{|\Gamma(\alpha)|\,\mathrm{Re}\,\alpha} \|f\|_{L^p(a,b)}, \qquad [2.4] $$

see [SAM 93, p. 48]. If $\alpha \in (0,1)$ and $1 < p < \frac{1}{\alpha}$, then the operators ${}_0\mathrm{I}_t^{\alpha}$ and ${}_t\mathrm{I}_b^{\alpha}$ are bounded from $L^p(a,b)$ into $L^q(a,b)$ for $q = \frac{p}{1-\alpha p}$ (see [SAM 93, p. 66]).

Introducing the function

$$f_{\alpha}(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1}, & t > a, \\ 0, & t < a, \end{cases} \quad \operatorname{Re}\alpha > 0, \tag{2.5}$$

we conclude that the integral [2.2] may be written in the form of convolution as

$${}_a\mathrm{I}_t^{\alpha} y(t) = f_{\alpha}(t) * y(t) = \int_a^t f_{\alpha}(t-\tau)\, y(\tau)\, \mathrm{d}\tau. \tag{2.6}$$

REMARK 2.1.– Expression [2.6] may be used to define the generalized fractional integral with the different choice of f_{α}. For example, in [KIL 04], various generalizations of the fractional integral were presented, including the generalization that uses the two-parameter Mittag-Leffler function

$$K^{(\alpha)} f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} E_{\rho,\alpha}(\omega t^{\rho}) * f(t), \quad \omega \in \mathbb{R}.$$

The fractional integral of purely imaginary order is defined as

$${}_a\mathrm{I}_t^{\mathrm{i}\theta} y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left({}_a\mathrm{I}_t^{1+\mathrm{i}\theta} y(t)\right) = \frac{1}{\Gamma(1+\mathrm{i}\theta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\tau)^{\mathrm{i}\theta} y(\tau)\, \mathrm{d}\tau, \tag{2.7}$$

with $\theta \neq 0$.

The asymptotic behavior of the left Riemann–Liouville fractional integral may be characterized as follows.

PROPOSITION 2.1.– [UCH 08, p. 165] Suppose that $f \in L^1_{loc}([0,\infty))$ is an analytic function in $(0,\infty)$. Then

$${}_a\mathrm{I}_t^{\alpha} f(t) \sim {}_0\mathrm{I}_t^{\alpha} f(t) + \frac{a}{\pi}\Gamma(\alpha+1)\sin(\alpha\pi) f(0)\, t^{-\alpha-1} \sim {}_0\mathrm{I}_t^{\alpha} f(t), \quad \text{as } t \to \infty. \tag{2.8}$$

If ${}_a\mathrm{I}_t^{\alpha} f$ is used to model a hereditary process, then a physical meaning of [2.8] is that for large times, the importance of the initial state of the system is small.

DEFINITION 2.3.– *The left and right Riemann–Liouville fractional derivatives* ${}_a\mathrm{D}_t^\alpha f$ *and* ${}_t\mathrm{D}_b^\alpha f$ *of the order* $\alpha \in \mathbb{C}$, $\mathrm{Re}\,\alpha \geq 0$, $n-1 \leq \mathrm{Re}\,\alpha < n$, $n \in \mathbb{N}$, *with the appropriate assumptions on* f *(see below), are defined as*

$$\begin{aligned} {}_a\mathrm{D}_t^\alpha f(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}_a\mathrm{I}_t^{n-\alpha} f(t)\right) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \quad t \in (a,b), \\ {}_t\mathrm{D}_b^\alpha f(t) &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}_t\mathrm{I}_b^{n-\alpha} f(t)\right) = (-1)^n \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha-n+1}}\mathrm{d}\tau, \\ & \qquad t \in (a,b). \end{aligned} \qquad [2.9]$$

If $f \in AC^n([a,b])$ and $n-1 \leq \mathrm{Re}\,\alpha < n$, $n \in \mathbb{N}$, then ${}_a\mathrm{D}_t^\alpha f$ and ${}_t\mathrm{D}_b^\alpha f$ exist almost everywhere on $[a,b]$ and

$${}_a\mathrm{D}_t^\alpha f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \qquad [2.10]$$

$${}_t\mathrm{D}_b^\alpha f(t) = \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(b)}{\Gamma(1+k-\alpha)}(b-t)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha-n+1}}\mathrm{d}\tau, \qquad [2.11]$$

see [KIL 06, p. 73]. From the definitions, it follows that in the special case when $f(t) = (t-a)^{\beta-1}$, $t > a$, and $f(t) = (b-t)^{\beta-1}$, $t < b$, $\beta \in \mathbb{C}$, we have

$$\begin{aligned} {}_a\mathrm{D}_t^\alpha (t-a)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1} \quad \text{and} \\ {}_t\mathrm{D}_b^\alpha (b-t)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1}. \end{aligned} \qquad [2.12]$$

Again, from [2.12], for constant function $f = C$, we have

$${}_a\mathrm{D}_t^\alpha C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \quad \text{and} \quad {}_t\mathrm{D}_b^\alpha C = \frac{C}{\Gamma(1-\alpha)}(b-t)^{-\alpha}.$$

Also, ${}_a\mathrm{D}_t^\alpha f(t) = 0$ and ${}_t\mathrm{D}_b^\alpha g(t) = 0$, $n-1 \leq \mathrm{Re}\,\alpha < n$, if and only if, respectively,

$$f(t) = \sum_{k=1}^{n} c_k (t-a)^{\alpha-k} \text{ and } g(t) = \sum_{k=1}^{n} d_k (b-t)^{\alpha-k}, \qquad [2.13]$$

where c_k and d_k, $k = 1, \ldots, n$, are arbitrary constants. Thus, functions f and g in [2.13] play the role of constants for the left and right Riemann–Liouville fractional derivatives, respectively.

Let $\alpha = k + \gamma,\ k \in \mathbb{N}_0,\ \gamma \in [0,1)$. Then, ${}_0\mathrm{D}_t^\alpha$ and ${}_t\mathrm{D}_b^\alpha$ may be written as

$$\begin{aligned} {}_0\mathrm{D}_t^\alpha f(t) &= \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \int_0^t \frac{f(\tau)}{(t-\tau)^\gamma} \mathrm{d}\tau, \quad t > 0, \\ {}_t\mathrm{D}_b^\alpha f(t) &= (-1)^{k+1} \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \int_t^b \frac{f(\tau)}{(\tau-t)^\gamma} \mathrm{d}\tau, \quad t < b. \end{aligned}$$

Sometimes, in short, it is written ${}_a\mathrm{D}_t^\alpha f = f^{(\alpha)}$.

Let $\alpha \in [0,1)$. Then, for $t > a$ and $t < b$, we have

$$\begin{aligned} {}_a\mathrm{D}_t^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau \text{ and} \\ {}_t\mathrm{D}_b^\alpha f(t) &= -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b \frac{f(\tau)}{(\tau-t)^\alpha} \mathrm{d}\tau. \end{aligned} \qquad [2.14]$$

In the case when α is purely imaginary, i.e. $\alpha = \mathrm{i}\theta$, the left Riemann–Liouville fractional derivative is defined as

$${}_a\mathrm{D}_t^{\mathrm{i}\theta} f(t) = \frac{1}{\Gamma(1-\mathrm{i}\theta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{f(\tau)}{(t-\tau)^{\mathrm{i}\theta}} \mathrm{d}\tau, \quad t \geq a.$$

Consider the problem of determining $\lim_{\alpha\to1^-} {}_a\mathrm{D}_t^\alpha f$. Then, we have the following proposition.

PROPOSITION 2.2.– [NAH 03, p. 174] Suppose that $f \in C^1([0,T])$. Then, $\lim_{\alpha\to1^-} {}_a\mathrm{D}_t^\alpha f = f^{(1)}$.

We put $\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\cdot) = \mathrm{D}^n(\cdot)$. The index rule holds for the integer-order integrals and derivatives

$$\begin{aligned}
\left({}_a\mathrm{I}_t^n\, {}_a\mathrm{I}_t^m\right) f(t) &= \left({}_a\mathrm{I}_t^m\, {}_a\mathrm{I}_t^n\right) f(t) = {}_a\mathrm{I}_t^{n+m} f(t), \quad n, m \in \mathbb{N}_0, \\
\left(\mathrm{D}^n \mathrm{D}^m\right) f(t) &= \left(\mathrm{D}^m \mathrm{D}^n\right) f(t) = \mathrm{D}^{m+n} f(t), \quad n, m \in \mathbb{N}_0.
\end{aligned} \qquad [2.15]$$

The semi-group property $[2.15]_1$ holds for fractional integrals only.

PROPOSITION 2.3.– [DIE 10, p. 14] The fractional integral ${}_a\mathrm{I}_t^\alpha$ as a mapping from $L^1(a,b) \to L^1(a,b)$ forms a commutative semi-group with respect to orders of integrals. The identity operator ${}_a\mathrm{I}_t^0$ is the neutral element. Thus, if $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$

$$\begin{aligned}
\left({}_a\mathrm{I}_t^\alpha\, {}_a\mathrm{I}_t^\beta\right) f(t) &= \left({}_a\mathrm{I}_t^\beta\, {}_a\mathrm{I}_t^\alpha\right) f(t) = {}_a\mathrm{I}_t^{\alpha+\beta} f(t), \\
\left({}_t\mathrm{I}_b^\alpha\, {}_t\mathrm{I}_b^\beta\right) f(t) &= \left({}_t\mathrm{I}_b^\beta\, {}_t\mathrm{I}_b^\alpha\right) f(t) = {}_t\mathrm{I}_b^{\alpha+\beta} f(t),
\end{aligned}$$

holds for almost all $t \in [a,b]$ (almost everywhere (a.e.) in $[a,b]$) if $f \in L^p(a,b)$, $1 \leq p \leq \infty$.

Also, it can be shown that for $\operatorname{Re}\alpha > 0$, $f \in L^p(a,b)$, $1 \leq p \leq \infty$, the composition of fractional derivatives and fractional integrals holds, for almost all $t \in (a,b)$ (see [SAM 93, p. 44]),

$$\left({}_a\mathrm{D}_t^\alpha\, {}_a\mathrm{I}_t^\alpha\right) f(t) = f(t), \quad \text{and} \quad \left({}_t\mathrm{D}_b^\alpha\, {}_t\mathrm{I}_b^\alpha\right) f(t) = f(t),$$

showing that ${}_a\mathrm{D}_t^\alpha$, ${}_t\mathrm{D}_b^\alpha$ are the left inverses of ${}_a\mathrm{I}_t^\alpha$, ${}_t\mathrm{I}_b^\alpha$, respectively. However by applying ${}_a\mathrm{D}_t^\alpha$ and ${}_t\mathrm{D}_b^\alpha$ to the right of ${}_a\mathrm{I}_t^\alpha$ and ${}_t\mathrm{I}_b^\alpha$, we have different situation. To examine the resulting relations, we define the following spaces:

$$\begin{aligned}
{}_a\mathrm{I}_t^\alpha(L^p) &= \left\{f \mid f = {}_a\mathrm{I}_t^\alpha \varphi,\ \varphi \in L^p(a,b)\right\} \quad \text{and} \\
{}_t\mathrm{I}_b^\alpha(L^p) &= \left\{g \mid g = {}_t\mathrm{I}_b^\alpha \phi,\ \phi \in L^p(a,b)\right\}.
\end{aligned} \qquad [2.16]$$

PROPOSITION 2.4.– [KIL 06, p. 74] Let $\operatorname{Re}\alpha > 0$, $n-1 < \operatorname{Re}\alpha < n$. Then, the following holds:

i) If $f \in {}_a\mathrm{I}_t^\alpha(L^p)$, $1 \leq p \leq \infty$, then

$$\left({}_a\mathrm{I}_t^\alpha\, {}_a\mathrm{D}_t^\alpha\right) f(t) = f(t), \quad \text{a.e., in } [a,b]. \qquad [2.17]$$

ii) If $f \in L^1(a,b)\,,\; {}_a\mathrm{I}_t^{n-\alpha} f \in AC^n([a,b])\,,$ then

$$\left({}_a\mathrm{I}_t^{\alpha}\;{}_a\mathrm{D}_t^{\alpha}\right) f(t) = f(t) - \sum_{j=1}^{n} \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} \left[\frac{\mathrm{d}^{n-j}}{\mathrm{d}t^{n-j}}\left({}_a\mathrm{I}_t^{n-\alpha} f\right)\right]_{t=a} \qquad [2.18]$$

holds for almost all $t \in [a,b]\,.$

We state the results about the index rule for the fractional derivatives.

PROPOSITION 2.5.– [KIL 06, p. 75] Let $\alpha, \beta > 0$, $n-1 \le \alpha < n$, $m-1 \le \beta < m$ and $\alpha+\beta < n$. Let $f \in L^1(a,b)$ and ${}_a\mathrm{I}_t^{m-\alpha} f \in AC^m([a,b])\,.$ Then, the following index rule holds:

$$\left({}_a\mathrm{D}_t^{\alpha}\;{}_a\mathrm{D}_t^{\beta}\right) f(t) = {}_a\mathrm{D}_t^{\alpha+\beta} f(t) - \sum_{j=1}^{m} \frac{(t-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)} \left[{}_a\mathrm{D}_t^{\beta-j} f(t)\right]_{t=a}, \quad t \in [a,b]\,. \qquad [2.19]$$

There are special cases when the index rule holds (see [KIL 06, p. 74]).

The composition rule for the left Riemann–Liouville derivative and the right Riemann–Liouville integral takes a rather complicated form (see [NAH 03, p. 22]). Suppose that f is Hölder continuous in $[a,b]$ and $f \in L^1(a,b)\,.$ Then, for $\alpha \in (0,1)$, we have

$$\left({}_a\mathrm{D}_t^{\alpha}\;{}_t\mathrm{I}_b^{\alpha}\right) f(t) = f(t)\cos(\alpha\pi) + \mathrm{S}_{ab}^{\alpha} f(t)\cos(\alpha\pi), \quad \text{a.e. in } [a,b]\,,$$

where

$$\mathrm{S}_{ab}^{\alpha} f(t) = \frac{1}{\pi}\int_a^b \left|\frac{u-a}{t-a}\right|^{\alpha} \frac{f(u)}{u-t}\mathrm{d}u. \qquad [2.20]$$

The integral in [2.20] should be taken as a Cauchy principal value.

In the variational problems, important result is integration by parts formula. We state it as follows.

PROPOSITION 2.6.– [SKM, pp. 46 and 67]

i) Suppose $0 < \alpha < 1$, $f \in L^p(a,b)\,,\; g \in L^q(a,b)\,.$ Then

$$\int_a^b f(t)\left({}_a\mathrm{I}_t^{\alpha} g(t)\right)\mathrm{d}t = \int_a^b \left({}_t\mathrm{I}_b^{\alpha} f(t)\right) g(t)\,\mathrm{d}t, \qquad [2.21]$$

for $p \ge 1$, $q \ge 1$ and $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$.

ii) Suppose $0 < \operatorname{Re}\alpha < 1$, $f \in {}_t\mathrm{I}_b^\alpha (L^p)$ and $g \in {}_a\mathrm{I}_t^\alpha (L^q)$. Then

$$\int_a^b f(t) \left({}_a\mathrm{D}_t^\alpha g(t)\right) \mathrm{d}t = \int_a^b \left({}_t\mathrm{D}_b^\alpha f(t)\right) g(t) \,\mathrm{d}t, \qquad [2.22]$$

for $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$.

For a generalization of the integration by parts formula [2.22], see [11.81].

Fractional derivatives could be expressed in terms of integer-order derivatives through expansion formula.

PROPOSITION 2.7.– [SAM 93, p. 278] Suppose $\alpha \in \mathbb{R}_+$ and that f is an analytic function on (a, b) . Then

$${}_a\mathrm{D}_t^\alpha f(t) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{(t-a)^{n-\alpha}}{\Gamma(n+1-\alpha)} f^{(n)}(t), \quad t \in (a,b), \qquad [2.23]$$

where

$$\binom{\alpha}{n} = (-1)^{n-1} \frac{\alpha \Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)},$$

denotes the binomial coefficients.

The fractional derivatives can be expressed in terms of a function and its moments. The following expansion formula may be proved.

PROPOSITION 2.8.– [ATA 14a] Let $f \in C^1([0,T])$ and $0 < \alpha < 1$. Then

$${}_0\mathrm{D}_t^\alpha f(t) = \frac{f(t)}{t^\alpha}\mathcal{A}(N) - \sum_{p=1}^{N} \mathcal{C}_{p-1} \frac{V_{p-1}(f)(t)}{t^{p+\alpha}} + Q_{N+1}(f)(t), \quad t \in (0,T], \qquad [2.24]$$

where

$$\mathcal{A}(N) = \frac{1}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{p=1}^{N} \frac{\Gamma(p+\alpha)}{p!} = \frac{\sin(\alpha\pi)}{\alpha\pi} \frac{\Gamma(N+1+\alpha)}{\Gamma(N+1)}, \qquad [2.25]$$

$$\mathcal{C}_{p-1} = \frac{\Gamma(p+\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma(p)}, \qquad [2.26]$$

$$V_{p-1}(f)(t) = \int_0^t \tau^{p-1} f(\tau) \mathrm{d}\tau, \quad t \in [0,T], \ p \in \mathbb{N}, \qquad [2.27]$$

and the reminder term $Q_{N+1}(f)$ satisfies the estimate

$$|Q_{N+1}(f)(t)| \leq \frac{C \cdot M_t}{\Gamma(\alpha)\Gamma(1-\alpha)} \cdot \frac{t^{1-\alpha}}{N^{\alpha_1}}, \quad t \in [0,T]\,, \; 0 < \alpha_1 < 1-\alpha, \quad [2.28]$$

with $M_t = \max_{0 \leq \tau \leq t} |y^{(1)}(\tau)|$ and certain constant $C > 0$.

Thus

$$\lim_{N\to\infty} Q_{N+1}(f)(t) = 0 \text{ uniformly on } [0,T]$$

and the approximation formula for the left Riemann–Liouville fractional derivative becomes

$${}_0\mathrm{D}_t^{\alpha} f(t) \approx \frac{f(t)}{t^{\alpha}} \mathcal{A}(N) - \sum_{p=1}^{N} \mathcal{C}_{p-1} \frac{V_{p-1}(f)(t)}{t^{p+\alpha}}, \quad t \in (0,T]\,. \qquad [2.29]$$

From the expansion formula [2.24], approximation to the right fractional derivative could be derived.

PROPOSITION 2.9.– [ATA 14a] Let $g \in C^1([0,T])$ and $0 < \alpha < 1$. Then, the right Riemann–Liouville fractional derivative can be approximated by

$${}_t\mathrm{D}_T^{\alpha} g(t) \approx \frac{g(t)}{t^{\alpha}} \mathcal{A}(N) + \sum_{p=1}^{N} \mathcal{C}_{p-1} t^{p-1} W_{p-1}(g)(t), \quad t \in (0,T]\,, \qquad [2.30]$$

where $A(N,\alpha)$ and $C_{p-1}(\alpha)$ are defined by [2.25] and [2.26], respectively, and $W_{p-2}(g)$ is

$$W_{p-1}(g)(t) = \int_t^T \frac{g(\tau)}{\tau^{p+\alpha}}\, \mathrm{d}\tau.$$

REMARK 2.2.– Expansion formula [2.24] may be expressed in a different form (see [ATA 08b]) in which the first derivative of a function appears. Thus, for $t \in (0,T]$,

$${}_0\mathrm{D}_t^{\alpha} f(t) = \mathcal{A}(N) \frac{f(t)}{t^{\alpha}} + \mathcal{B}(N) t^{1-\alpha} f^{(1)}(t) - \sum_{p=1}^{N} \mathcal{C}_{p-1} \frac{V_{p-1}(f)(t)}{t^{p+\alpha}} + R_{N+1}(t)\,, [2.31]$$

where

$$\mathcal{A}(N) = \frac{1}{\Gamma(1-\alpha)} - \frac{1}{\Gamma(\alpha-1)\Gamma(2-\alpha)} \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{(p-1)!} = \frac{\sin(\alpha\pi)}{\alpha\pi} \frac{\Gamma(N+\alpha)}{\Gamma(N)},$$

$$\mathcal{B}(N) = \frac{1}{\Gamma(2-\alpha)} \left(1 + \frac{1}{\Gamma(\alpha-1)} \sum_{p=1}^{N} \frac{\Gamma(p-1+\alpha)}{p!}\right) = \frac{\Gamma(N+\alpha)}{\Gamma(\alpha)\Gamma(2-\alpha)\Gamma(N+1)},$$

$$\mathcal{C}_{p-1} = \frac{\Gamma(p+\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma(p)},$$

$$R_{N+1}(t) = \frac{t^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t y^{(2)}(\tau) \left(\sum_{p=N+1}^{\infty} \frac{\Gamma(p+\alpha)}{p!} \left(\frac{\tau}{t}\right)^p \mathrm{d}\tau\right), \quad t \in (0,T] \tag{2.32}$$

and

$$V_p^{(1)}(f)(t) = t^p f(t), \quad V_p(f)(0) = 0, \quad t \in [0,T], \ p \in \mathbb{N}.$$

There are several definitions of the fractional derivatives of variable order. In [ROS 93, SAM 95], the following definition was proposed. The left Riemann–Liouville fractional derivative of variable α for $0 \le \alpha(t) < 1$ is

$${}_0\mathrm{D}_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha(t)}} \mathrm{d}\tau, \quad t \in [0,T]. \tag{2.33}$$

In [ATA 13a], the following expansion formula for [2.33] is proved.

PROPOSITION 2.10.– [ATA 13a] Suppose that $f \in C^2([0,T])$, $\alpha \in C^1([0,T])$. Then, the fractional derivative of the order α, defined by [2.33], may be written as

$$\begin{aligned} {}_0\mathrm{D}_t^{\alpha(t)} f(t) &= A_1\left(f(t), f^{(1)}(t), \alpha(t), t\right) \\ &\quad -\alpha^{(1)}(t) A_2\left(f(t), f^{(1)}(t), \alpha(t), t\right) + R_1^N(t) \\ &\quad + R_2^{N,M}(t), \quad t \in (0,T], \end{aligned}$$

where

$$A_1\left(f(t), f^{(1)}(t), \alpha(t), t\right)$$
$$= \frac{f(t)}{t^{\alpha(t)}}\left(\frac{1}{\Gamma(1-\alpha(t))} - \frac{1}{\Gamma(\alpha(t)-1)\Gamma(2-\alpha(t))}\sum_{p=2}^{N}\frac{\Gamma(p-1+\alpha(t))}{(p-1)!}\right)$$
$$+ \frac{f^{(1)}(t)t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))}\left(1+\frac{1}{\Gamma(\alpha(t)-1)}\sum_{p=1}^{N}\frac{\Gamma(p-1+\alpha(t))}{p!}\right)$$
$$- \frac{1}{\Gamma(\alpha(t)-1)\Gamma(2-\alpha(t))}\sum_{p=0}^{N}\frac{\Gamma(p+1+\alpha(t))}{p!}\frac{V_p(f)(t)}{t^{p+1+\alpha(t)}}, \quad t\in(0,T],$$

and ($t \in [0,T]$)

$$A_2\left(f(t), f^{(1)}(t), \alpha(t), t\right)$$
$$= \frac{1}{\Gamma(1-\alpha(t))}\left(\frac{f(t)\,t^{1-\alpha(t)}}{1-\alpha(t)}\left(\ln t - \frac{1}{1-\alpha(t)}\right)\right.$$
$$- \frac{f^{(1)}(t)t^{2-\alpha(t)}}{2-\alpha(t)}\left(\ln t - \frac{1}{2-\alpha(t)}\right)$$
$$+ \frac{t^{1-\alpha(t)}\ln t}{\Gamma(\alpha(t))}\sum_{k=0}^{N}\frac{\Gamma(k+\alpha(t))}{k!}\left(\frac{tf^{(1)}(t)}{(k+1)(k+2)} - \frac{f(t)}{k+1} + \frac{V_k(f)(t)}{t^{k+1}}\right)$$
$$+ \frac{t^{1-\alpha(t)}}{\Gamma(\alpha(t))}\sum_{p=1}^{M}\frac{1}{p}\sum_{k=0}^{N}\frac{\Gamma(k+\alpha(t))}{k!}\left(\frac{tf^{(1)}(t)}{(k+p+1)(k+p+2)} - \frac{f(t)}{k+p+1}\right.$$
$$\left.\left.+\frac{V_{k+p}(f)(t)}{t^{k+p+1}}\right)\right),$$

with $V_p(f)(t) = \int_0^t \tau^p f(\tau)\,\mathrm{d}\tau$ being moments of the function f and satisfying

$$V_p^{(1)}(f)(t) = t^p f(t), \quad V_p(0) = 0, \quad t\in[0,T], \; p = 0,1,\ldots \,. \qquad [2.34]$$

Also, there exists $N_\varepsilon \in \mathbb{N}$ such that for any $\varepsilon > 0$, and for $N, M > N_\varepsilon$, it holds that

$$\left|R_1^N(t) + R_2^{N,M}(t)\right| < \varepsilon.$$

Thus, the approximation formula for ${}_0\mathrm{D}_t^{\alpha(t)} f(t)$, $t \in (0, T]$, becomes

$$
\begin{aligned}
{}_0\mathrm{D}_t^{\alpha(t)} f(t) &\approx \widehat{{}_0\mathrm{D}_t^{\alpha(t)}} f(t) \\
&= \frac{f(t)}{t^{\alpha(t)}} \left(\frac{1}{\Gamma(1-\alpha(t))} - \frac{1}{\Gamma(\alpha(t)-1)\Gamma(2-\alpha(t))} \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha(t))}{(p-1)!} \right) \\
&+ \frac{f^{(1)}(t) t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))} \left(1 + \frac{1}{\Gamma(\alpha(t)-1)} \sum_{p=1}^{N} \frac{\Gamma(p-1+\alpha(t))}{p!} \right) \\
&- \frac{1}{\Gamma(\alpha(t)-1)\Gamma(2-\alpha(t))} \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha(t))}{(p-2)!} \frac{V_{p-2}(f)(t)}{t^{p-1+\alpha(t)}} \\
&- \frac{\alpha^{(1)}(t)}{\Gamma(1-\alpha(t))} \left(f(t) \left(\frac{1}{1-\alpha(t)} t^{1-\alpha(t)} \ln t - \frac{t^{1-\alpha(t)}}{(1-\alpha(t))^2} \right) \right. \\
&- \frac{f^{(1)}(t) t^{2-\alpha(t)}}{2-\alpha(t)} \left(\ln t - \frac{1}{2-\alpha(t)} \right) \\
&+ \frac{t^{1-\alpha(t)} \ln t}{\Gamma(\alpha(t))} \sum_{k=0}^{N} \frac{\Gamma(k+\alpha(t))}{k!} \left(\frac{t f^{(1)}(t)}{(k+1)(k+2)} - \frac{f(t)}{k+1} + \frac{V_k(f)(t)}{t^{k+1}} \right) \\
&+ \frac{t^{1-\alpha(t)}}{\Gamma(\alpha(t))} \sum_{p=1}^{M} \frac{1}{p} \sum_{k=0}^{N} \frac{\Gamma(k+\alpha(t))}{k!} \left(\frac{t f^{(1)}(t)}{(k+p+1)(k+p+2)} - \frac{f(t)}{k+p+1} \right. \\
&+ \left. \left. \frac{V_{k+p}(f)(t)}{t^{k+p+1}} \right) \right). \qquad [2.35]
\end{aligned}
$$

Note that for the case $\alpha = const.$, expressions [2.29] and [2.35] coincide.

REMARK 2.3.– The procedure of expressing fractional derivatives in terms of function, its first derivative and moments of function is extended in different directions in a series of papers [POO 12a, POO 12b, POO 13].

2.1.1.1. *Laplace transform of Riemann–Liouville fractional integrals and derivatives*

Suppose that f is exponentially bounded (see section 1.2), that is $f \in L^1(0,\infty)$, $|f(t)| \leq A\mathrm{e}^{s_0 t}$, $t > 0$, where $A > 0$, $s_0 > 0$. Then

$$\mathcal{L}\left[{}_0\mathrm{I}_t^{\alpha} f(t)\right](s) = \frac{1}{s^{\alpha}} \tilde{f}(s), \quad \mathrm{Re}\, s > s_0, \qquad [2.36]$$

see [KIL 06, p. 84]. Expression [2.36] follows from the well-known property of the Laplace transform of convolution and $\mathcal{L}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right](s) = \frac{1}{s^{\alpha}}$ (see section 1.3).

For the fractional derivatives, we have the following result.

PROPOSITION 2.11.– [KIL 06, p. 84] Let $n-1 < \operatorname{Re}\alpha < n$, $f \in AC^n_{loc}([0,\infty))$ and f be of exponential growth. Suppose that there exist finite limits

$$\lim_{t\to 0}\left(\mathrm{D}^k\, {}_0\mathrm{I}_t^{n-\alpha} f(t)\right) \quad \text{and} \quad \lim_{t\to\infty}\left(\mathrm{D}^k\, {}_0\mathrm{I}_t^{n-\alpha} f(t)\right) = 0, \quad k = 0, 1, \ldots, n-1.$$

Then

$$\mathcal{L}\left[{}_0\mathrm{D}_t^{\alpha} f(t)\right](s) = s^{\alpha}\tilde{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1}\left[\mathrm{D}^k\, {}_0\mathrm{I}_t^{n-\alpha} f(t)\right]_{t=0}, \quad \operatorname{Re} s > s_0. \qquad [2.37]$$

For $0 < \alpha < 1$

$$\mathcal{L}\left[{}_0\mathrm{D}_t^{\alpha} f(t)\right](s) = s^{\alpha}\tilde{f}(s) - \left[{}_0\mathrm{I}_t^{1-\alpha} f(t)\right]_{t=0} = s^{\alpha}\tilde{f}(s), \quad \operatorname{Re} s > s_0. \qquad [2.38]$$

Relation [2.38] could be used for the (heuristic) definition of the fractional derivative.

The Leibnitz rule for fractional derivatives does not hold in its usual form. It could be shown that for analytic functions, we have the following.

PROPOSITION 2.12.– [SAM 93, p. 280] Suppose that f and g are analytic for $t > 0$ and $\alpha > 0$. Then

$${}_a\mathrm{D}_t^{\alpha}\left(f(t)\,g(t)\right) = \sum_{k=0}^{\infty}\binom{\alpha}{k}\left(\mathrm{D}^k g(t)\right)\left({}_a\mathrm{D}_t^{\alpha-k} f(t)\right), \quad t > a. \qquad [2.39]$$

Note that in [2.39], on the right-hand side we have integer-order derivatives of g and fractional-order derivatives of f. There is an apparent lack of symmetry in the derivatives of the two functions. The left-hand side of [2.39] does not depend on the order of the functions f and g, while on the right-hand side there are only integer derivatives of g and non-integer derivatives (integrals) of f. It could be shown that the two functions f and g can be interchanged without changing the value of the fractional derivative of their product.

2.1.2. *Riemann–Liouville fractional integrals and derivatives on the real half-axis*

The Riemann–Liouville fractional integrals and derivatives defined on a finite interval $[a,b]$ can be naturally extended to a half-line $\mathbb{R}_+$ as

$$\begin{aligned}
\mathrm{I}_+^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad t > 0,\ \operatorname{Re}\alpha > 0, \\
\mathrm{I}_-^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^{\infty} (\tau-t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad t > 0,\ \operatorname{Re}\alpha > 0, \qquad [2.40]
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{D}_{+}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \quad t>0,\ n-1 \leq \operatorname{Re}\alpha < n, \\
\mathrm{D}_{-}^{\alpha} f(t) &= (-1)^n \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_t^{\infty} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \quad t>0, \\
&\qquad n-1 \leq \operatorname{Re}\alpha < n.
\end{aligned}$$

Operator $[2.40]_2$ is sometimes called the Weyl integral.

Fourier transform of the Riemann–Liouville fractional integrals $\mathrm{I}_{+}^{\alpha} f$ and $\mathrm{I}_{-}^{\alpha} f$ are given as

$$\mathcal{F}\left[\mathrm{I}_{+}^{\alpha} f(t)\right](\omega) = \frac{1}{(\mathrm{i}\omega)^{\alpha}} \hat{f}(\omega) \ \text{ and } \ \mathcal{F}\left[\mathrm{I}_{-}^{\alpha} f(t)\right](\omega) = \frac{1}{(-\mathrm{i}\omega)^{\alpha}} \hat{f}(\omega), \quad \omega \in \mathbb{R}, \quad [2.41]$$

for $0 < \operatorname{Re}\alpha < 1$ and for $f \in L^1(\mathbb{R})$. Equation [2.41] cannot be extended directly to the case $\operatorname{Re}\alpha \geq 1$. For the Riemann–Liouville derivatives, we have

$$\mathcal{F}\left[\mathrm{D}_{+}^{\alpha} f(t)\right](\omega) = (\mathrm{i}\omega)^{\alpha} \hat{f}(\omega) \text{ and } \mathcal{F}\left[\mathrm{D}_{-}^{\alpha} f(t)\right](\omega) = (-\mathrm{i}\omega)^{\alpha} \hat{f}(\omega), \quad \omega \in \mathbb{R}. \quad [2.42]$$

In [2.41] and [2.42], we have $(\pm\mathrm{i}\omega)^{\alpha} = |\omega|^{\alpha} e^{\mp \frac{\alpha\pi}{2} \operatorname{sgn}\omega}$. In the case when $f \in \mathcal{S}'$, we have that [2.41] and [2.42] remain the same (see [VLA 73, p. 110]).

2.1.3. *Caputo fractional derivatives*

We present the definition of fractional derivative from Caputo [CAP 67] and Caputo and Mainardi [CAP 71b]. The left Caputo fractional derivative of a function of order α, denoted by ${}_{a}^{C}\mathrm{D}_{t}^{\alpha} f$, is

$${}_{a}^{C}\mathrm{D}_{t}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau, & n-1 \leq \alpha < n, \\ \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), & \alpha = n, \end{cases} \quad t \in [a,b]. \quad [2.43]$$

Similarly, the right Caputo derivative is defined as

$${}_{t}^{C}\mathrm{D}_{b}^{\alpha} f(t) = \begin{cases} (-1)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha+1-n}} \mathrm{d}\tau, & n-1 \leq \alpha < n, \\ (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), & \alpha = n, \end{cases} \quad t \in [a,b]. \quad [2.44]$$

It is easy to see that

$$ {}_a^C\mathrm{D}_t^{\alpha} f(t) = {}_a\mathrm{I}_t^{n-\alpha}\left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right) \quad \text{and} \quad {}_t^C\mathrm{D}_b^{\alpha} f(t) = (-1)^n {}_t\mathrm{I}_b^{n-\alpha}\left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right), $$

where ${}_a\mathrm{I}_t^{n-\alpha}$ and ${}_t\mathrm{I}_b^{n-\alpha}$ are the Riemann–Liouville fractional integrals [2.2] and [2.3], respectively.

Observe that [2.43] for $a = 0$ can be written as

$$ {}_0^C\mathrm{D}_t^{\alpha} f(t) = \frac{t^{n-1-\alpha}}{\Gamma(n-\alpha)} * \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), \quad t > 0, \ n-1 \le \operatorname{Re}\alpha < n. \qquad [2.45] $$

Note that the Caputo derivative of a constant function is zero

$$ {}_a^C\mathrm{D}_t^{\alpha} C = 0 \ \text{ and } \ {}_t^C\mathrm{D}_b^{\alpha} C = 0. \qquad [2.46] $$

For $n-1 \le \alpha < n$, the Caputo derivatives ${}_a^C\mathrm{D}_t^{\alpha}$ and ${}_t^C\mathrm{D}_b^{\alpha}$ are operators mapping $C^n([a,b])$ into

$$ \begin{aligned} C_a([a,b]) &= \{ f \mid f \in C([a,b]), \ f(a) = 0 \}, \quad \|f\|_{C_a} = \|f\|_C, \\ C_b([a,b]) &= \{ f \mid f \in C([a,b]), \ f(b) = 0 \}, \quad \|f\|_{C_b} = \|f\|_C, \end{aligned} $$

respectively.

PROPOSITION 2.13.– [KIL 06, p. 94] Let $n - 1 \le \operatorname{Re}\alpha < n$, $\alpha \ne \mathbb{N}$. Then, the Caputo derivatives ${}_a^C\mathrm{D}_t^{\alpha}$ and ${}_t^C\mathrm{D}_b^{\alpha}$ are bounded operators from $C^n([a,b])$ into $C_a([a,b])$ and $C_b([a,b])$, respectively, and the following estimates hold

$$ \begin{aligned} \left\| {}_a^C\mathrm{D}_t^{\alpha} f \right\|_{C_a} &\le \frac{(b-a)^{n-\operatorname{Re}\alpha}}{|\Gamma(n-\alpha)|(n-\operatorname{Re}\alpha+1)} \|f\|_{C^n}, \\ \left\| {}_t^C\mathrm{D}_b^{\alpha} f \right\|_{C_b} &\le \frac{(b-a)^{n-\operatorname{Re}\alpha}}{|\Gamma(n-\alpha)|(n-\operatorname{Re}\alpha+1)} \|f\|_{C^n}. \end{aligned} $$

In general, the Caputo and the Riemann–Liouville fractional derivatives do not coincide. The connections between them are given as

$$\begin{aligned}
{}_a^C\mathrm{D}_t^{\alpha} f(t) &= {}_a\mathrm{D}_t^{\alpha}\left(f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^k}{k!}f^{(k)}(a)\right), \quad t\in[a,b], \\
{}_t^C\mathrm{D}_b^{\alpha} f(t) &= {}_t\mathrm{D}_b^{\alpha}\left(f(t) - \sum_{k=0}^{n-1}\frac{(b-t)^k}{k!}f^{(k)}(b)\right), \quad t\in[a,b].
\end{aligned}$$

In particular, if $0 < \operatorname{Re}\alpha < 1$, for $t \in [a, b]$, we have

$$ {}_a^C\mathrm{D}_t^{\alpha} f(t) = {}_a\mathrm{D}_t^{\alpha}\left(f(t) - f(a)\right) \quad \text{and} \quad {}_t^C\mathrm{D}_b^{\alpha} f(t) = {}_t\mathrm{D}_b^{\alpha}\left(f(t) - f(b)\right), $$

or ($t \in [a, b]$)

$$\begin{aligned}
{}_a^C\mathrm{D}_t^{\alpha} f(t) &= {}_a\mathrm{D}_t^{\alpha} f(t) - \frac{f(a)}{\Gamma(1-\alpha)(t-a)^{\alpha}} \quad \text{and} \\
{}_t^C\mathrm{D}_b^{\alpha} f(t) &= {}_t\mathrm{D}_b^{\alpha} f(t) - \frac{f(b)}{\Gamma(1-\alpha)(b-t)^{\alpha}}.
\end{aligned} \qquad [2.47]$$

Thus, the Caputo fractional derivatives are regularized Riemann–Liouville fractional derivatives.

The expansion formula [2.24] for the Caputo derivative ${}_0^C\mathrm{D}_t^{\alpha} f$, $0 < \alpha < 1$, with [2.47], becomes ($t \in (0, T]$)

$$\begin{aligned}
{}_0\mathrm{D}_t^{\alpha} f(t) &= \frac{f(t)}{t^{\alpha}}\mathcal{A}(N,\alpha) - \frac{f(0)}{t^{\alpha}\Gamma(1-\alpha)} - \sum_{p=1}^{N}\mathcal{C}_{p-1}(\alpha)\frac{V_{p-1}(f)(t)}{t^{p+\alpha}} \\
&\quad + Q_{N+1}(f)(t),
\end{aligned} \qquad [2.48]$$

where $\mathcal{A}$, $\mathcal{C}_{p-1}$ and V_p are given by [2.25], [2.26] and [2.27], respectively.

The case when α is close to 1 is called the case of low-level fractionality. For this case, we refer to [HER 11, TAR 06]. We have, for $t \in [a, b]$,

$$ {}_a^C\mathrm{D}_t^{1-\varepsilon} f(t) = f^{(1)}(t) + \varepsilon\left(f^{(1)}(0)\ln t + \int_0^t f^{(2)}(\xi)\ln(t-\xi)\,\mathrm{d}\xi\right) + o\left(\varepsilon^2\right), \quad \varepsilon \to 0^+, \qquad [2.49]$$

where o is the Landau symbol "small o". Recall that $a(x) = o(|x|^{\alpha})$ means $\frac{a(x)}{|x|^{\alpha}} \to 0$ as $|x| \to 0$, or $|x| \to \infty$.

We have the following integration by parts formula for the Caputo derivatives.

PROPOSITION 2.14.– [ALM 12, p. 112] Let $n-1 \leq \alpha < n$ and let $f, g \in C^{n}([a,b])$. Then

$$\begin{aligned}\int_a^b g(t)\left({}_t^C\mathrm{D}_b^{\alpha} f(t)\right)\mathrm{d}t &= \int_a^b \left({}_a^C\mathrm{D}_t^{\alpha} g(t)\right) f(t)\,\mathrm{d}t \\ &+\sum_{j=0}^{n-1}(-1)^{n+j}\left[\left({}_a^C\mathrm{D}_t^{\alpha+j-n} g(t)\right)\left({}_a^C\mathrm{D}_t^{n-1-j} f(t)\right)\right]_{t=a}^{t=b}. \qquad [2.50]\end{aligned}$$

The Laplace transform of the left Caputo derivative is given as follows.

PROPOSITION 2.15.– [KIL 06, p. 98] Suppose that $n-1 < \alpha \leq n$ and let f be such that $f \in C^{n}(\mathbb{R}_{+}), |f(t)|, \left|f^{(1)}(t)\right|, \ldots, \left|f^{(n)}(t)\right| \leq Be^{s_0 t}$, $B, s_0 > 0$, $t > 0$. Suppose that $\lim_{t\to\infty} f^{(k)}(t) = 0$, for $k = 0, 1, \ldots, n-1$. Then

$$\mathcal{L}\left[{}_0^C\mathrm{D}_t^{\alpha} f(t)\right](s) = s^{\alpha}\tilde{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad \operatorname{Re} s > s_0. \qquad [2.51]$$

For $0 < \alpha < 1$, expression [2.51] becomes

$$\mathcal{L}\left[{}_0^C\mathrm{D}_t^{\alpha} f(t)\right](s) = s^{\alpha}\tilde{f}(s) - s^{\alpha-1} f(0), \quad \operatorname{Re} s > s_0. \qquad [2.52]$$

2.1.4. *Riesz potentials and Riesz derivatives*

Let $0 < \alpha < 1$. Consider the following integrals:

$$\mathrm{R}^{\alpha} f(x) = \frac{1}{2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}}\int_{-\infty}^{\infty}\frac{1}{|x-\zeta|^{1-\alpha}} f(\zeta)\,\mathrm{d}\zeta, \quad x \in \mathbb{R}, \qquad [2.53]$$

$$\mathrm{H}^{\alpha} f(x) = \frac{1}{2\Gamma(\alpha)\sin\frac{\alpha\pi}{2}}\int_{-\infty}^{\infty}\frac{\operatorname{sgn}(x-\zeta)}{|x-\zeta|^{1-\alpha}} f(\zeta)\,\mathrm{d}\zeta, \quad x \in \mathbb{R}. \qquad [2.54]$$

Then, $\mathrm{R}^{\alpha} f$ is called the Riesz potential of f of the order α on $\mathbb{R}$, while $\mathrm{H}^{\alpha} f$ is the conjugate Riesz potential of the order α on $\mathbb{R}$. The following proposition holds for [2.53] and [2.54]. Note that equality is almost everywhere (see section 1.1).

PROPOSITION 2.16.– [UCH 08, p. 200] The Riemann–Liouville fractional integral and the Riesz potential, for $x \in \mathbb{R}$, are connected as follows:

$$\begin{aligned} {}_{-\infty}\mathrm{I}_x^{\alpha} f(x) &= \cos\frac{\alpha\pi}{2}\,\mathrm{R}^{\alpha} f(x) + \sin\frac{\alpha\pi}{2}\,\mathrm{H}^{\alpha} f(x), \\ {}_x\mathrm{I}_{\infty}^{\alpha} f(x) &= \cos\frac{\alpha\pi}{2}\,\mathrm{R}^{\alpha} f(x) - \sin\frac{\alpha\pi}{2}\,\mathrm{H}^{\alpha} f(x), \\ \mathrm{R}^{\alpha} f(x) &= \frac{1}{2\cos\frac{\alpha\pi}{2}}\left({}_{-\infty}\mathrm{I}_x^{\alpha} f(x) + {}_x\mathrm{I}_{\infty}^{\alpha} f(x)\right), \\ \mathrm{H}^{\alpha} f(x) &= \frac{1}{2\sin\frac{\alpha\pi}{2}}\left({}_{-\infty}\mathrm{I}_x^{\alpha} f(x) - {}_x\mathrm{I}_{\infty}^{\alpha} f(x)\right). \end{aligned} \quad [2.55]$$

Also, if $\alpha, \beta > 0$, $\alpha + \beta < 1$, then

$$\mathrm{R}^{\alpha}\,\mathrm{R}^{\beta} f(x) = \mathrm{R}^{\alpha+\beta} f(x) \quad \text{and} \quad \mathrm{H}^{\alpha}\,\mathrm{H}^{\beta} f(x) = -\mathrm{R}^{\alpha+\beta} f(x), \quad x \in \mathbb{R}. \quad [2.56]$$

Integrals [2.53] and [2.54] exist with the appropriate assumptions on f. For example, if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\mathrm{R}^{\alpha} f$ and $\mathrm{H}^{\alpha} f$ exist almost everywhere and belong to $L^1_{loc}(\mathbb{R})$, as stated in [BUT 00, p. 46]. More generally, the following result holds.

PROPOSITION 2.17.– [KIL 06, p. 129] Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Then, R^{α} is a bounded operator from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$ if and only if

$$0 < \alpha < 1, \quad 1 < p < \frac{1}{\alpha}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{\alpha}.$$

The Fourier transforms of R^{α} and H^{α} are

$$\begin{aligned} \mathcal{F}\left[\mathrm{R}^{\alpha} f(x)\right](\omega) &= \frac{1}{|\omega|^{\alpha}}\hat{f}(\omega) \quad \text{and} \\ \mathcal{F}\left[\mathrm{H}^{\alpha} f(x)\right](\omega) &= -\mathrm{i}\frac{\operatorname{sgn}(\omega)}{|\omega|^{\alpha}}\hat{f}(\omega), \quad \omega \in \mathbb{R}\setminus\{0\}, \end{aligned} \quad [2.57]$$

see [BUT 00].

There is another important property of R^{α}. Namely, it maps the Lizorkin space of test functions Φ into itself, i.e. $\mathrm{R}^{\alpha}(\Phi) = \Phi$ (see [SAM 93, p. 493]).

DEFINITION 2.4.– *[BUT 00, UCH 08] The Riesz fractional derivative of the order α is defined as*

$$
{}^{R}\mathrm{D}^{\alpha} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{H}^{1-\alpha} f(x) = \frac{1}{2\Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-\zeta)}{|x-\zeta|^{\alpha}} f(\zeta)\,\mathrm{d}\zeta, \quad x \in \mathbb{R}, \qquad [2.58]
$$

while the conjugate Riesz derivative of the order α is defined as

$$
{}^{RC}\mathrm{D}^{\alpha} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{R}^{1-\alpha} f(x) = \frac{1}{2\Gamma(1-\alpha)\sin\frac{\alpha\pi}{2}} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} \frac{1}{|x-\zeta|^{\alpha}} f(\zeta)\,\mathrm{d}\zeta, \quad x \in \mathbb{R}. \qquad [2.59]
$$

It is easy to see that the Fourier transforms of ${}^{R}\mathrm{D}^{\alpha} f$ and ${}^{RC}\mathrm{D}^{\alpha} f$, for $0 < \alpha < 1$, are

$$
\begin{aligned}
\mathcal{F}\left[{}^{R}\mathrm{D}^{\alpha} f(x)\right](\omega) &= |\omega|^{\alpha} \hat{f}(\omega) \text{ and} \\
\mathcal{F}\left[{}^{RC}\mathrm{D}^{\alpha} f(x)\right](\omega) &= \mathrm{i}\,|\omega|^{\alpha} \operatorname{sgn}(\omega)\, \hat{f}(\omega), \quad \omega \in \mathbb{R},
\end{aligned} \qquad [2.60]
$$

see [BUT 00]. Therefore, ${}^{R}\mathrm{D}^{\alpha}$ inverts R^{α}, while ${}^{RC}\mathrm{D}^{\alpha}$ inverts H^{α}, if the functions involved satisfy certain additional conditions. Namely, ${}^{R}\mathrm{D}^{\alpha}\,\mathrm{R}^{\alpha} f = f$ holds for functions f belonging to the Lizorkin space Φ (see [SAM 93, p. 150]).

In various applications, there is a need to study the following potential:

$$
{}_{a}^{R}\mathrm{I}_{b}^{\alpha} f(x) = \int_{a}^{b} \frac{1}{|x-\zeta|^{1-\alpha}} f(\zeta)\,\mathrm{d}\zeta, \quad x \in (a,b), \qquad [2.61]
$$

where $0 < \alpha < 1$. The Carleman equation involves the potential of the type [2.61]. The inverse of ${}_{a}^{R}\mathrm{I}_{b}^{\alpha}$, i.e. the solution to the equation

$$
{}_{a}^{R}\mathrm{I}_{b}^{\alpha} f(x) = \int_{a}^{b} \frac{1}{|x-\zeta|^{1-\alpha}} f(\zeta)\,\mathrm{d}\zeta = g(x), \quad x \in (a,b), \qquad [2.62]
$$

for a specified class of functions f and g is given in [SAM 93, p. 627]. Let $g(x) = (x-a)^n$, $n = 0, 1, 2, \ldots$. Then, the solution to [2.62], for $x \in (a, b)$, becomes

$$\begin{aligned} f(x) &= \frac{n!}{\pi}(b-a)^n \sin\frac{\alpha\pi}{2} \frac{\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)} \frac{1}{((x-a)(b-x))^{\frac{\alpha}{2}}} \\ &\quad \times \sum_{k=0}^{n} \binom{n-\frac{\alpha}{2}}{n-k} \left(\frac{b-x}{x-a}\right)^k . \end{aligned} \quad [2.63]$$

If $g = 0$, then the solution to [2.62] is $f = 0$ (see [SAM 93, eq. (30.84)], or [NAH 03, p. 47]).

2.1.5. *Symmetrized Caputo derivative*

Let $0 \leq \beta < 1$, $-\infty \leq a < b \leq \infty$. The symmetrized Caputo fractional derivative of an absolutely continuous function f is defined as

$$ {}_a^C\mathcal{E}_b^\beta f(x) = \frac{1}{2}\left({}_a^C\mathrm{D}_x^\beta - {}_x^C\mathrm{D}_b^\beta\right) f(x) = \frac{1}{2}\frac{1}{\Gamma(1-\beta)} \int_a^b \frac{f^{(1)}(\theta)}{|x-\theta|^\beta} \mathrm{d}\theta, \quad x \in [a, b]. \quad [2.64]$$

For $a = -\infty$ and $b = \infty$, we write $\mathcal{E}_x^\beta$ instead of ${}_a^C\mathcal{E}_b^\beta$ and then

$$\mathcal{E}_x^\beta f(x) = \frac{1}{2}\frac{1}{\Gamma(1-\beta)}|x|^{-\beta} * f^{(1)}(x), \quad x \in \mathbb{R}. \quad [2.65]$$

Note that $\mathcal{E}_x^0 f(x) = 0$ and $\mathcal{E}_x^\beta f(x) \to f^{(1)}(x)$, as $\beta \to 1$. This yields that the symmetrized fractional derivative generalizes the first derivative of a function, but it does not generalize the derivative of zero order: the zeroth-order symmetrized fractional derivative of a function is not a function itself, but zero.

For $f = const.$, we have that ${}_a^C\mathcal{E}_b^\beta f = 0$, and conversely, the fact that $f = const.$ is the unique solution to equation ${}_a^C\mathcal{E}_b^\beta f = 0$ is shown in [ATA 09f].

In studying [2.64], we need some properties of the function $|x|^{-\beta}$. It is an element of the Lizorkin space Ψ' for all $\beta \in \mathbb{R}$ and for $\beta \in [0, 1)$ it holds that

$$\begin{aligned} \mathcal{F}\left[|x|^{-\beta}\right](\xi) &= 2\Gamma(1-\beta)\sin\frac{\beta\pi}{2}\frac{1}{|\xi|^{1-\beta}}, \quad \xi \in \mathbb{R}, \\ \mathcal{F}\left[\mathcal{E}_x^\beta f(x)\right](\xi) &= \mathrm{i}\frac{\xi}{|\xi|^{1-\beta}}\sin\frac{\beta\pi}{2}\hat{f}(\xi), \quad \xi \in \mathbb{R}. \end{aligned}$$

2.1.6. *Other types of fractional derivatives*

In this section, we present other types of fractional integrals and derivatives. For extensive review of definitions, see [KIR 94].

2.1.6.1. *Canavati fractional derivative*

There is another definition of fractional derivatives that is useful in deriving inequalities. This is the Canavati fractional derivative. It is "between" the Riemann–Liouville derivative and the Caputo derivative. Let $n-1<\alpha<n$. Then, the Canavati derivative of order α is defined as

$$ {}^{Can}_{\;\;a}\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{f^{(n-1)}(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \qquad [2.66] $$

see [CAN 87]. The Canavati derivative [2.66] is used in [ANA 09] for $f \in C^{\alpha}([a,b])$ where

$$ C^{\alpha}([a,b]) = \left\{ f \in C^{n-1}([a,b]) \mid {}_a\mathrm{I}_t^{n-1} f^{(n-1)} \in C^1([a,b]) \right\}. $$

2.1.6.2. *Marchaud fractional derivatives*

The left Marchaud fractional derivative of the order $0<\alpha<1$ for $f \in \mathcal{H}^{\lambda}([a,b])$, $\lambda>\alpha$ (see section 1.1) is defined as

$$ {}^{M}_{a}\mathrm{D}_t^{\alpha} f(t) = \frac{f(t)}{\Gamma(1-\alpha)(t-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^t \frac{f(t)-f(\tau)}{(t-\tau)^{1+\alpha}} \mathrm{d}\tau, \; t \in [a,b]. \qquad [2.67] $$

The right Marchaud fractional derivative is defined as

$$ {}^{M}_{t}\mathrm{D}_b^{\alpha} f(t) = \frac{f(t)}{\Gamma(1-\alpha)(b-t)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_t^b \frac{f(t)-f(\tau)}{(\tau-t)^{1+\alpha}} \mathrm{d}\tau, \; t \in [a,b]. \qquad [2.68] $$

The integrals in [2.67] and [2.68] are assumed to be convergent. To make it precise, let

$$ \psi_{\varepsilon}(t) = \int_a^{t-\varepsilon} \frac{f(t)-f(\tau)}{(t-\tau)^{1+\alpha}} \mathrm{d}\tau, \quad \varepsilon>0. \qquad [2.69] $$

Then

$$ {}^{M}_{t}\mathrm{D}_t^{\alpha} f(t) = \frac{f(t)}{\Gamma(1-\alpha)(t-a)^{\alpha}} + \lim_{\varepsilon\to 0} \psi_{\varepsilon}(t). \qquad [2.70] $$

If $f \in L^p(a,b)$, then the limit in [2.70] is considered in the norm of $L^p(a,b)$.

For functions belonging to $C^1([a,b])$, the Marchaud derivatives coincide with the corresponding Riemann–Liouville derivatives.

2.1.6.3. *Grünwald–Letnikov fractional derivatives*

The left Grünwald–Letnikov fractional derivative of the order α is, according to [KIL 06, p. 122], formally defined as

$$ {}^{G-L}_{\ a}\mathrm{D}_t^{\alpha} f(t) = \lim_{h\to 0}\left(\frac{1}{h^{\alpha}}\sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^j\binom{\alpha}{j} f(t-jh)\right),\quad t>a,\ \alpha>0. \qquad [2.71]$$

Similarly, the right Grünwald–Letnikov fractional derivative of the order α is defined as

$$ {}^{G-L}_{\ t}\mathrm{D}_b^{\alpha} f(t) = \lim_{h\to 0}\left(\frac{1}{h^{\alpha}}\sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^j\binom{\alpha}{j} f(t+jh)\right),\quad t>a,\ \alpha>0. \qquad [2.72]$$

There is a connection between the Marchaud and the Grünwald–Letnikov fractional derivatives.

PROPOSITION 2.18.– [SAM 93, p. 386] Let $f \in L^p(a,b)$, $1 \le p < \infty$. Then, limit [2.71] exists in the sense of $L^p(a,b)$ convergence, if and only if there exists the Marchaud fractional derivative in sense [2.70]. Both limits, if they exist, are equal almost everywhere.

2.2. Some additional properties of fractional derivatives

2.2.1. *Fermat theorem for fractional derivative*

Let $0<\alpha<1$. As a motivation, following [SAM 93, p. 111], we start from

$$\begin{aligned} {}_0\mathrm{D}_t^{\alpha} y(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{y(t-\tau)}{\tau^{\alpha}}\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{y(0)}{t^{\alpha}} + \frac{1}{\Gamma(1-\alpha)}\int_0^t y^{(1)}(t-\tau)\left(\alpha\int_{\tau}^{t}\xi^{-1-\alpha}\mathrm{d}\xi + \frac{1}{t^{\alpha}}\right)\mathrm{d}\tau \\ &= \frac{y(t)}{\Gamma(1-\alpha)\,t^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{y(t)-y(t-\tau)}{\tau^{1+\alpha}}\mathrm{d}\tau,\quad t>0. \end{aligned} \qquad [2.73]$$

Let $0 < \alpha < 1$. Similarly, for the Caputo derivative of an integrable function, we have

$$\begin{aligned} {}_0^C\mathrm{D}_t^\alpha y(t) &= {}_0\mathrm{D}_t^\alpha y(t) - \frac{y(0)}{\Gamma(1-\alpha)\, t^\alpha} \\ &= \frac{y(t) - y(0)}{\Gamma(1-\alpha)\, t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{y(t) - y(t-\tau)}{\tau^{1+\alpha}} \mathrm{d}\tau, \quad t > 0. \end{aligned} \qquad [2.74]$$

Suppose that y is an increasing positive function with the maximum at $t^* \in (0, t)$. Then

$$y(t^*) - y(t) \geq 0, \quad t \in [0, t^*].$$

From [2.73] and [2.74], we conclude that

$$\left[{}_0\mathrm{D}_t^\alpha y(t)\right]_{t=t^*} \geq \frac{y(t^*)}{\Gamma(1-\alpha)\,(t^*)^\alpha} > 0 \text{ and } \left[{}_0^C\mathrm{D}_t^\alpha y(t)\right]_{t=t^*} \geq \frac{y(t^*) - y(0)}{\Gamma(1-\alpha)\,(t^*)^\alpha} > 0. \qquad [2.75]$$

The above results may be used to prove the following proposition.

PROPOSITION 2.19.– [NAH 03, p. 104] Suppose that y is an integrable function on $[A, B]$. Suppose further that there exists $\delta > 0$ such that $y \in \mathcal{H}^\lambda([t^* - \delta, t^*])$, $\lambda > \alpha$, and that y attains a maximum at a point $t^* \in [A, B]$. Then, for any $\alpha \in [0, 1]$ and $a \in [t^* - \delta, t^*]$, $a \neq t^*$, we have

$$\left[{}_a\mathrm{D}_t^\alpha y(t)\right]_{t=t^*} \leq \frac{y(t^*)}{\Gamma(1-\alpha)\,(t^* - a)^\alpha} \text{ and } \left[{}_a^C\mathrm{D}_t^\alpha y(t)\right]_{t=t^*} \leq \frac{y(x^*) - y(a)}{\Gamma(1-\alpha)\,(t^* - a)^\alpha}. \qquad [2.76]$$

Thus, at the point of maximum, fractional derivatives either satisfy [2.76] or do not exist.

It could be easily shown that for a minimum of a function, the inequalities in [2.76] become.

2.2.2. *Taylor theorem for fractional derivatives*

The mean value theorem for the Riemann–Liouville fractional derivative reads as follows.

PROPOSITION 2.20.– [TRU 99] Let $\alpha \in [0,1]$ and suppose that $f \in C([a,b])$, such that ${}_a\mathrm{D}_t^{\alpha} f \in C([a,b])$. Then

$$f(t) = (t-a)^{\alpha-1}\left[(t-a)^{1-\alpha} f(t)\right]_{t=a^+} + [{}_a\mathrm{D}_t^{\alpha} f(t)]_{t=\xi}\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}, t \in (a,b], \quad [2.77]$$

with $a \leq \xi \leq b$.

The generalization of the Taylor formula for the Riemann–Liouville fractional derivative has several different forms. To state the formula, we need the following definition.

DEFINITION 2.5.– *A function* $f : [a,b] \to \mathbb{R}$ *is said to be* α*-continuous,* $0 \leq \alpha \leq 1$, *at* t_0 *if there exists* $\lambda \in [0, 1-\alpha)$ *such that* $g(t) = |t-t_0|^{\lambda} f(t)$ *is continuous at* t_0.

Function f is α-continuous in $[a,b]$ if it is α-continuous for every $t \in [a,b]$. Let $C_\alpha = \{f \mid [a,b] \to \mathbb{R},\ f \text{ is } \alpha\text{-continuous}\}$. Note that $C_1([a,b]) = C([a,b])$. Let ${}_a\mathrm{I}_b^{\alpha}(a,b) = \{f \mid [a,b] \to \mathbb{R},\ {}_a\mathrm{I}_t^{n} f(t) \text{ exists and it is finite for all } t \in [a,b]\}$. A function f is singular of order α at $t = t^*$ if $\lim_{t\to t^*} \frac{f(t)}{(t-t^*)^{\alpha-1}} = k < \infty$ and $k \neq 0$. Finally, we use ${}_a\mathrm{D}_t^{j\alpha}$ to denote the application of ${}_a\mathrm{D}_t^{\alpha}$ j-times, i.e. ${}_a\mathrm{D}_t^{j\alpha} = \underbrace{{}_a\mathrm{D}_t^{\alpha} \cdots {}_a\mathrm{D}_t^{\alpha}}_{j\text{-times}}$.

PROPOSITION 2.21.– [TRU 99] Let $\alpha \in [0,1]$, $n \in \mathbb{N}$. Let f be a continuous function in $(a,b]$ satisfying the following conditions:

i) ${}_a\mathrm{D}_t^{j\alpha} f \in C([a,b])$ and ${}_a\mathrm{D}_t^{j\alpha} f \in {}_a\mathrm{I}_b^{\alpha}(a,b)$, for all $j = 1, \ldots, n$;

ii) ${}_a\mathrm{D}_t^{(n+1)\alpha} f$ is continuous in $[a,b]$;

iii) if $\alpha < \frac{1}{2}$, then for each $j \in \mathbb{N}$, $1 \leq j \leq n$, such that $(j+1)\alpha < 1$, ${}_a\mathrm{D}_t^{(j+1)\alpha} f$ is γ-continuous at $t = a$ for some γ, $1-(j+1)\alpha \leq \gamma \leq 1$, or it is singular of order α at $t = a$.

Then, for $t \in (a,b]$,

$$f(t) = \sum_{j=0}^{n} \frac{c_j}{\Gamma((j+1)\alpha)}(t-a)^{(j+1)\alpha-1} + \frac{\left[{}_a\mathrm{D}_t^{(n+1)\alpha} f(t)\right]_{t=\xi}}{\Gamma((n+1)\alpha+1)}(t-a)^{(n+1)\alpha}, \quad a \leq \xi \leq b, \quad [2.78]$$

where

$$c_j = \Gamma(\alpha)\left[(t-a)_a^{1-\alpha}\, \mathrm{D}_t^{j\alpha} f(t)\right]_{t=a^+}, \quad j = 0, 1, \ldots, n. \quad [2.79]$$

The Taylor formula for the Caputo derivative is given in the following proposition.

PROPOSITION 2.22.– [ODI 07, p. 289] Suppose that ${}_{a}^{C}\mathrm{D}_{t}^{j\alpha} f \in C((a,b])$ for $j = 0, 1, \ldots, n+1$, where $0 < \alpha \leq 1$. Then

$$f(t) = \sum_{j=0}^{n} \frac{(t-a)^{j\alpha}}{\Gamma(j\alpha+1)} \left[{}_{a}^{C}\mathrm{D}_{t}^{j\alpha} f(t)\right]_{t=a^{+}} + \frac{\left[{}_{a}^{C}\mathrm{D}_{t}^{(n+1)\alpha} f(t)\right]_{t=\xi}}{\Gamma((n+1)\alpha+1)} (t-a)^{(n+1)\alpha}, \quad t \in (a,b], \qquad [2.80]$$

with $a \leq \xi \leq b$.

REMARK 2.4.– [ODI 07, p. 288] In the special case for the Caputo derivative, the corresponding result is stated as follows. Suppose that $f \in C([a,b])$ and ${}_{a}^{C}\mathrm{D}_{t}^{\alpha} f \in C((a,b])$, for $0 < \alpha \leq 1$. Then

$$f(t) = f(a) + \frac{\left[{}_{a}^{C}\mathrm{D}_{t}^{\alpha} f(t)\right]_{t=\xi}}{\Gamma(\alpha+1)} (t-a)^{\alpha}, \quad t \in (a,b], \qquad [2.81]$$

where $a \leq \xi \leq b$.

2.3. Fractional derivatives in distributional setting

Throughout this section, we will assume that functions that appear determine the tempered distributions.

2.3.1. *Definition of the fractional integral and derivative*

We introduce the following definition.

DEFINITION 2.6.– *The convolution operator $f_\alpha *$ in $\mathcal{S}'_+$ ($\check{f}_\alpha *$ in $\mathcal{S}'_-$) is the operator of fractional integration for $\alpha > 0$ and the operator of the left (right) fractional differentiation for $\alpha < 0$*

$$\begin{aligned}
{}_{D}\mathrm{I}_{t}^{\alpha} u &= f_\alpha * u, \quad \alpha > 0, \\
{}_{D}\mathrm{D}_{t}^{\alpha} u &= f_{-\alpha} * u = \frac{\mathrm{d}^m}{\mathrm{d}t^m} f_{m-\alpha} * u = f_{m-\alpha} * \frac{\mathrm{d}^m}{\mathrm{d}t^m} u = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[f_{m-\alpha} * u\right], \quad \alpha > 0, \\
&\qquad m \in \mathbb{N}, \\
{}_{D}\check{\mathrm{D}}_{t}^{\alpha} u &= \check{f}_{-\alpha} * u = (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}t^m} f_{m-\alpha} * u = (-1)^m f_{m-\alpha} * \frac{\mathrm{d}^m}{\mathrm{d}t^m} u, \quad \alpha > 0, \ m \in \mathbb{N},
\end{aligned} \qquad [2.82]$$

where f_α is given by [1.6] and $\check{f}_\alpha(t) = f_\alpha(-t)$.

Operator ${}_D\mathrm{I}_t^\alpha$ coincides with the operator of fractional derivation ${}_D\mathrm{D}_t^\alpha$ for $-\alpha \in \mathbb{N}$ and it is the operator of fractional integration for $\alpha \in \mathbb{N}$.

We have that the Laplace transforms of the distributional fractional integral and derivative are

$$\mathcal{L}\left[{}_D\mathrm{I}_t^\alpha u(t)\right](s) = \frac{1}{s^\alpha}\tilde{u}(s) \text{ and } \mathcal{L}\left[{}_D\mathrm{D}_t^\alpha u(t)\right](s) = s^\alpha \tilde{u}(s), \quad \operatorname{Re} s > 0.$$

We derive, in the following proposition, a connection between the Caputo fractional derivative of a function u belonging to $AC_{loc}^m([0,\infty))$ and the distributional fractional derivative of a distribution u_{reg} belonging to $\mathcal{S}'_+$. Also, we derive the connection between the corresponding Laplace transforms. Recall that notation u_{reg} means that we consider u as a distribution, i.e. regular distribution u_{reg} is determined by u.

PROPOSITION 2.23.– [ATA 09d] Let $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, $u \in AC_{loc}^m([0,\infty))$ and put

$$u_{reg}(t) = u(t)\, H(t), \quad t \in \mathbb{R}. \qquad [2.83]$$

i) Then

$${}_D\mathrm{D}_t^\alpha u_{reg}(t) = {}_0^C\mathrm{D}_t^\alpha u(t) + \sum_{j=0}^{m-1} \frac{\mathrm{d}^{m-1-j}}{\mathrm{d}t^{m-1-j}} f_{m-\alpha}(t) \frac{\mathrm{d}^j}{\mathrm{d}t^j} u(0), \quad t > 0, \qquad [2.84]$$

where ${}_0^C\mathrm{D}_t^\alpha$ is defined by [2.45].

ii) Also

$$\mathcal{L}\left[{}_0^C\mathrm{D}_t^\alpha u(t)\right](s) = \mathcal{L}\left[{}_D\mathrm{D}_t^\alpha u_{reg}(t)\right](s) - \sum_{j=0}^{m-1} s^{\alpha-1-j} \frac{\mathrm{d}^j}{\mathrm{d}t^j} u(0), \quad s \in \mathbb{C}_+. \qquad [2.85]$$

Using the notation

$$\mathcal{L}\left[{}_D\mathrm{D}_t^\alpha u_{reg}(t)\right](s) = s^\alpha \mathcal{L}\left[u_{reg}(t)\right](s) = s^\alpha \tilde{u}(s), \quad s \in \mathbb{C}_+,$$

[2.85] can be written as

$$\mathcal{L}\left[{}_0^C\mathrm{D}_t^\alpha u(t)\right](s) = s^\alpha \tilde{u}(s) - \sum_{j=0}^{m-1} s^{\alpha-1-j} \frac{\mathrm{d}^j}{\mathrm{d}t^j} u(0).$$

2.3.2. *Dependence of fractional derivative on order*

Recall that ${}_0\mathrm{D}_t^\alpha u = {}_D\mathrm{D}_t^\alpha u$ if $u \in AC^m\left([a,b]\right), m-1 \leq \alpha < m, m \in \mathbb{N}_0$.

We examine the mapping

$$\alpha \mapsto {}_D\mathrm{D}_t^\alpha u, \quad \alpha \in (-\infty, \infty),$$

for given $u \in L^1_{loc}(\mathbb{R})$, such that $u(t) = 0$ for $t < 0$, i.e. $u \in L^1_{loc}([0,\infty))$.

PROPOSITION 2.24.– [ATA 07a] Let $u \in L^1_{loc}(\mathbb{R})$, $u(t) = 0$, $t < 0$, so that it determines a tempered distribution. Then, $\alpha \mapsto {}_D\mathrm{D}_t^\alpha u$ is a smooth mapping from $(-\infty,\infty)$ to $\mathcal{S}'_+$. Also, for every $\alpha \in \mathbb{R}$ with $k > \alpha$,

$$\frac{\partial}{\partial\alpha}{}_D\mathrm{D}_t^\alpha u(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(f_k * u\right)(\alpha,t), \quad t \in \mathbb{R}, \tag{2.86}$$

where the derivatives are understood in the sense of the tempered distributions,

$$f_k(\alpha,t) = \frac{t^{k-\alpha-1}}{\Gamma(k-\alpha)}\left[\psi(k-\alpha) - \ln(t)\right], \quad \alpha \in (-\infty,k), \ t > 0, \tag{2.87}$$

$f_k(\alpha,t) = 0$, $t \leq 0$, and $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x)$, $x > 0$, is the Euler function.

Note that a locally integrable function f determines a tempered distribution in $\mathcal{S}'_+$ if it is polynomially bounded on $\mathbb{R}$ as $t \to \infty$. Thus, in sections 2.3.2 and 2.3.3 we will assume that f and its derivatives are polynomially bounded on $\mathbb{R}$ as $t \to \infty$.

PROPOSITION 2.25.– [ATA 07a] Let $u \in L^1_{loc}(\mathbb{R})$, $u(t) = 0$, $t < 0$, so that it determines a tempered distribution. Then

$$\left[\frac{\partial}{\partial\alpha}{}_D\mathrm{D}_t^\alpha u(t)\right]_{\alpha=0} = -\left(c + \ln t\right)u(t) + \int_0^t \frac{u(t) - u(t-\tau)}{\tau}\mathrm{d}\tau, \quad t > 0,$$

where $c = 0.5772$ is Euler's constant.

Note that for an analytic function, it holds that

$$\left[\frac{\partial}{\partial\alpha}{}_D\mathrm{D}_t^\alpha u(t)\right]_{\alpha=0} = -\left(c + \ln t\right)u(t) + \sum_{n=0}^{\infty}\frac{(-1)^k u^{(k)}(t)}{k\Gamma(k+1)}, \quad t > 0,$$

given in [WES 03, p. 112].

REMARK 2.5.– [ATA 07a] Proposition 2.25 allows the following representation of fractional derivative ${}_D\mathrm{D}_t^\alpha u$ for α small enough:

$$\begin{aligned}{}_D\mathrm{D}_t^\alpha u(t) &= u(t) + \alpha \left[\frac{\partial}{\partial\alpha}{}_D\mathrm{D}_t^\alpha u(t)\right]_{\alpha=0} + o(\alpha) \\ &= u(t) + \alpha\left[-(c+\ln t)\,u(t) + \int_0^t \frac{u(t)-u(t-\tau)}{\tau}\mathrm{d}\tau\right] + o(\alpha),\; t>0.\end{aligned} \quad [2.88]$$

A relation similar to [2.88] was used in [DIE 02a] for the study of fractional differential equations through the change of the order of integration.

PROPOSITION 2.26.– [ATA 07a] Let $\alpha \in \mathbb{R}$, $u \in L^1_{loc}(\mathbb{R})$, $u(t) = 0$, $t < 0$, so that it determines a tempered distribution. Then

$$\mathcal{L}\left[\frac{\partial}{\partial\alpha}{}_D\mathrm{D}_t^\alpha u(t)\right](s) = s^\alpha \tilde{u}(s)\ln s, \quad \operatorname{Re} s > 0. \quad [2.89]$$

2.3.3. *Distributed-order fractional derivative*

Let u be an element of $\mathcal{S}'_+$. Then, it is proved in [ATA 09b] that the mappings

$$\alpha \mapsto {}_D\mathrm{D}_t^\alpha u : \mathbb{R} \mapsto \mathcal{S}'_+ \;\text{ and }\; \alpha \mapsto \langle {}_D\mathrm{D}_t^\alpha u(t), \varphi(t)\rangle : \mathbb{R} \mapsto \mathbb{R} \quad [2.90]$$

are smooth (see proposition 2.24). We define the distributed-order fractional derivative by the use of the distributional fractional derivative.

DEFINITION 2.7.– *[ATA 09b] Let* $\phi \in \mathcal{E}'$, $\operatorname{supp}\phi \subset [0,2]$ *and* $u \in \mathcal{S}'_+$. *Then, the distributed-order fractional derivative of* u

$${}_D\mathrm{D}_\phi u(\cdot) = \int_{\operatorname{supp}\phi} \phi(\alpha)\,{}_D\mathrm{D}_t^\alpha u(\cdot)\,\mathrm{d}\alpha \quad [2.91]$$

is defined as element of $\mathcal{S}'_+$ *by*

$$\left\langle \int_{\operatorname{supp}\phi} \phi(\alpha)\,{}_D\mathrm{D}_t^\alpha u(t)\,\mathrm{d}\alpha, \varphi(t)\right\rangle = \langle \phi(\alpha), \langle {}_D\mathrm{D}_t^\alpha u(t), \varphi(t)\rangle\rangle, \quad \varphi \in \mathcal{S}. \quad [2.92]$$

Let $u \in AC^m_{loc}([0,\infty))$, $\alpha \in [0,m]$. Recall that the Caputo fractional derivative ${}_0^C\mathrm{D}_t^\alpha$ is defined on intervals $\alpha \in (j-1, j]$, $j \in \{1,\ldots,m\}$ and

$$\lim_{\alpha\to(j-1)+0} {}_0^C\mathrm{D}_t^\alpha u(t) = \frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}}u(t) - \frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}}u(0), \quad t > 0.$$

Moreover, $\lim_{\alpha \to j-0} {}_0^C\mathrm{D}_t^\alpha u(t) = \frac{\mathrm{d}^j}{\mathrm{d}t^j} u(t)$, $t > 0$. Thus, $[0,m] \ni \alpha \mapsto {}_0^C\mathrm{D}_t^\alpha u$ is continuous in intervals $\alpha \in (j-1, j)$, $j \in \{1, \dots, m\}$, left continuous at j, $j \in \{1, \dots, m\}$ and it has jumps that appear in the limit from the right at points $j-1$, $j \in \{1, \dots, m\}$. For fixed $\alpha \in [0,m]$, the function $[0,\infty) \ni t \mapsto {}_0^C\mathrm{D}_t^\alpha u(t)$ is locally integrable on $[0,\infty)$.

For the sake of the next proposition, we introduce the following definition. Recall that we have assumed that function u, equal to zero on $(-\infty, 0)$, has classical derivatives that are polynomially bounded as $t \to \infty$.

DEFINITION 2.8.– *[ATA 09d] Let $u \in AC_{loc}^2([0,\infty))$.*

i) Let $\alpha \mapsto \phi(\alpha)$ be continuous in $[0,2]$. Then, we define the distributed-order fractional derivative as

$$\mathrm{D}_\phi u(t) = \int_0^2 \phi(\alpha)\, {}_0^C\mathrm{D}_t^\alpha u(t)\, \mathrm{d}\alpha, \quad t > 0.$$

ii) Let $\alpha = \{\alpha_j\}_{j \in \{0,\dots,k\}}$, $\alpha_j \in [0,2]$, $j \in \{0, \dots, k\}$. Then, we define the distributed-order fractional derivative as

$$\mathrm{D}_\phi u(t) = \sum_{j=0}^{k} a_j\, {}_0^C\mathrm{D}_t^{\alpha_j} u(t), \quad t > 0.$$

iii) If ϕ is a continuous function in $[\mu, \eta] \subset [0,2]$ and $\phi(\alpha) = 0$, $\alpha \in [0,2] \setminus [\mu, \eta]$, then we define the distributed-order fractional derivative as

$$\mathrm{D}_\phi u(\cdot, t) = \int_\mu^\eta \phi(\alpha)\, {}_0^C\mathrm{D}_t^\alpha u(\cdot, t)\, \mathrm{d}\alpha. \qquad [2.93]$$

Let us derive connections between the distributed-order fractional derivative of $u \in AC_{loc}^2([0,\infty))$ and the corresponding distribution $u_{reg} \in \mathcal{S}'_+$ (in the sense of [2.83]) in cases that are analyzed above.

PROPOSITION 2.27.– [ATA 09d]

i) If ϕ belongs to $C([0,2])$ and $u \in AC_{loc}^2([0,\infty))$, then

$$\begin{aligned} {}_D\mathrm{D}_\phi u_{reg}(t) &= \mathrm{D}_\phi u(t) + u(0) \int_0^2 \phi(\alpha)\, f_{1-\alpha}(t)\, \mathrm{d}\alpha \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t} u(0) \int_1^2 \phi(\alpha)\, f_{2-\alpha}(t)\, \mathrm{d}\alpha. \end{aligned} \qquad [2.94]$$

ii) Let $\phi = \sum_{j=0}^{k} a_j \delta(\cdot - \alpha_j)$, $\operatorname{supp}\phi \subset [0,2]$, $a_j \in \mathbb{R}_+$, $0 \leq \alpha_k \leq \alpha_j \leq \alpha_0 \leq 2$, $\phi \in \mathcal{E}'$. Let $l \leq k$ be chosen so that $\alpha_l > 1$ and $\alpha_{l+1} \leq 1$. Then

$$\begin{aligned} {}_D\mathrm{D}_\phi u_{reg}(t) &= \mathrm{D}_\phi u(t) + u(0) \left(\sum_{j=l+1}^{k} a_j f_{1-\alpha_j}(t) \right. \\ &\quad \left. + \sum_{j=0}^{l} a_j \frac{\mathrm{d}}{\mathrm{d}t} f_{2-\alpha_j}(t) \right) + \frac{\mathrm{d}}{\mathrm{d}t} u(0) \sum_{j=0}^{l} a_j f_{2-\alpha_j}(t). \end{aligned} \quad [2.95]$$

iii) Both cases will be summarized by the use of [2.92] as

$$\begin{aligned} {}_D\mathrm{D}_\phi u_{reg}(t) &= \mathrm{D}_\phi u(t) + u(0) \left(\int_{\alpha\in[0,1]} \phi(\alpha) f_{1-\alpha}(t)\, \mathrm{d}\alpha \right. \\ &\quad \left. + \int_{\alpha\in(1,2]} \phi(\alpha) \frac{\mathrm{d}}{\mathrm{d}t} f_{2-\alpha}(t)\, \mathrm{d}\alpha \right) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t} u(0) \int_{\alpha\in(1,2]} \phi(\alpha) f_{2-\alpha}(t)\, \mathrm{d}\alpha. \end{aligned} \quad [2.96]$$

REMARK 2.6.– We use the order of points $0 \leq \alpha_k \leq \alpha_j \leq \alpha_0 \leq 2$ because we will consider two cases separately. The first case is when $\alpha_j \leq 1$, $j \in \{0, \ldots, k\}$. The second case is when some of the α_j are in $(1, 2]$. So, this notation is helpful from this point of view.

REMARK 2.7.– If $\operatorname{supp}\phi \subset [0,1]$ and u belongs to $AC^1_{loc}([0,\infty))$, then [2.96] reduces to

$${}_D\mathrm{D}_\phi u_{reg}(t) = \mathrm{D}_\phi u(t) + u(0) \int_{\alpha\in[0,1]} \phi(\alpha) f_{1-\alpha}(t)\, \mathrm{d}\alpha. \quad [2.97]$$

In the next proposition, we apply the Laplace transform to ${}_D\mathrm{D}_\phi u$, $u \in \mathcal{S}'_+$.

PROPOSITION 2.28.– [ATA 09b] Let $\phi \in \mathcal{E}'$, $\operatorname{supp}\phi \subset [0,2]$ and $u \in \mathcal{S}'_+$. Then:

i) $u \mapsto {}_D\mathrm{D}_\phi u$ is linear and continuous mapping from $\mathcal{S}'_+$ to $\mathcal{S}'_+$.

ii)

$$\mathcal{L}[{}_D\mathrm{D}_\phi u](s) = \langle \phi(\alpha), s^\alpha \tilde{u}(s) \rangle, \quad s \in \mathbb{C}_+. \quad [2.98]$$

iii) Let $\phi \in C([\mu,\eta])$, $[\mu,\eta] \subset [0,2]$ and $\phi(\alpha) = 0$, $\alpha \in [0,2] \diagdown [\mu,\eta]$. Then

$$\mathcal{L}[{}_D\mathrm{D}_\phi u](s) = \tilde{u}(s)\int_\mu^\eta \phi(\alpha)\, s^\alpha \mathrm{d}\alpha, \quad s \in \mathbb{C}_+. \qquad [2.99]$$

Further, we use proposition 2.28 in order to derive the Laplace transform of a function $u \in AC^2_{loc}([0,\infty))$, using the connection between its distributed-order fractional derivative and distributed-order fractional derivative of the corresponding distribution $u_{reg} \in \mathcal{S}'_+$ (in the sense of [2.83]). Again, we have two cases.

PROPOSITION 2.29.– [ATA 09d]

i) Let $\phi \in C([0,2])$ and $u \in AC^2_{loc}([0,\infty))$, then $(s \in \mathbb{C}_+)$

$$\begin{aligned}\mathcal{L}[\mathrm{D}_\phi u(t)](s) &= \tilde{u}(s)\int_0^2 \phi(\alpha)\, s^\alpha \mathrm{d}\alpha - u(0)\frac{1}{s}\int_0^2 \phi(\alpha)\, s^\alpha \mathrm{d}\alpha - \frac{\mathrm{d}}{\mathrm{d}t}u(0)\frac{1}{s^2} \\ &\quad \times \int_1^2 \phi(\alpha)\, s^\alpha \mathrm{d}\alpha. \qquad [2.100]\end{aligned}$$

ii) Let $\phi = \sum_{j=0}^{k} a_j \delta(\cdot - \alpha_j)$, $\operatorname{supp}\phi \subset [0,2]$, $a_j \in \mathbb{R}_+$, $0 \le \alpha_k \le \alpha_j \le \alpha_0 \le 2$, $\alpha_l > 1$ and $\alpha_{l+1} \le 1$. Let $u \in AC^2_{loc}([0,\infty))$, then $(s \in \mathbb{C}_+)$

$$\mathcal{L}[\mathrm{D}_\phi u(t)](s) = \tilde{u}(s)\sum_{j=0}^{k} a_j s^{\alpha_j} - u(0)\frac{1}{s}\sum_{j=0}^{k} a_j s^{\alpha_j} - \frac{\mathrm{d}}{\mathrm{d}t}u(0)\frac{1}{s^2}\sum_{j=0}^{l} a_j s^{\alpha_j}. \qquad [2.101]$$

iii) Both [2.100] and [2.101] are summarized by

$$\begin{aligned}\mathcal{L}[\mathrm{D}_\phi u(t)](s) &= \tilde{u}(s)\int_{\alpha\in[0,2]} \phi(\alpha)\, s^\alpha \mathrm{d}\alpha - u(0)\frac{1}{s}\int_{\alpha\in[0,2]} \phi(\alpha)\, s^\alpha \mathrm{d}\alpha \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}t}u(0)\frac{1}{s^2}\int_{\alpha\in(1,2]} \phi(\alpha)\, s^\alpha \mathrm{d}\alpha, \quad s \in \mathbb{C}_+. \qquad [2.102]\end{aligned}$$

REMARK 2.8.– If $\operatorname{supp}\phi \subset [0,1]$ and u belongs to $AC^1_{loc}([0,\infty))$, then equation [2.102] reduces to

$$\mathcal{L}[\mathrm{D}_\phi u(t)](s) = \tilde{u}(s)\int_{\alpha\in[0,1]} \phi(\alpha)\, s^\alpha \mathrm{d}\alpha - u(0)\frac{1}{s}\int_{\alpha\in[0,1]} \phi(\alpha)\, s^\alpha \mathrm{d}\alpha, \quad s \in \mathbb{C}_+. \qquad [2.103]$$