# Chapter 1

# A One-Dimensional Beam Metamodel



We introduce a one-dimensional (1D) *metamodel* of a beam as a progenitor of specific models to be formulated later in the book. The metamodel establishes properties and rules that highlight the common logic structure of the particular models. It leads us to formulate equations in terms of abstract quantities (typically column-vectors and formal matrix differential operators), whose contents do not need to be specified at this stage. We first address internally unconstrained beams, i.e. models in which all the variables introduced in the kinematic description are not subject to additional limitations. Formulation of the balance equations via the virtual power principle (VPP) straightforwardly leads us to recognize kinematic and equilibrium operators as mutually adjoint. Then, we analyze internally constrained beams, in which one or more strains are prescribed to identically vanish along the beam, for which we illustrate two alternative approaches: (a) the *mixed formulation*, in which reactive stresses enter the set of the main unknowns; and (b) the *displacement formulation*, in which kinematic and dynamic equations are condensed in order to satisfy constraints and to filter reactive stresses, respectively. Then, prestressed beams are considered, for which the reference state differs from the natural state, since stresses there are different from zero for the existence of preloads. Both cases of internally unconstrained and constrained prestressed beams are analyzed, and the previous analysis is entirely repeated to account for prestress. In this context, attention is devoted to the linearized theory, widely used in technical applications, able to furnish critical loads (in buckling problems), eigenfrequencies (of strings and cables), as well as the response to small incremental loads. For each problem addressed, a brief sketch of the variational formulation is also given as an alternative approach.

## 1.1 Models and metamodel

In the modeling process of the mechanical behavior of a beam or cable, different phenomenological aspects can be taken into account, and/or the same aspect described at different sophistication levels. Thus, a beam can be 3D or 1D, with rigid or deformable cross-sections, with deformability permitted in the plane and/or out-of-plane of the section. Each of these assumptions leads to a specific *model*; thus, for example, we have the "Timoshenko beam", which is able to describe the relative rotation between the rigid cross-section and the centerline (the so-called *shearable beam*), as well as the "Euler–Bernoulli beam", in which the cross-section keeps its orthogonality to the centerline (the so-called *unshearable beam*), or the "Vlasov thin-walled beam", in which the cross-section is allowed to warp, but not to deform in its plane, or the "Brazier tubular beam" which does not warp, but ovalizes itself. As a further example, a cable can be considered as flexible, and therefore modeled as a (prestressed) Cauchy continuum (the "flexible cable"), or provided with flexural and torsional rigidity, and therefore modeled as a Cosserat continuum (the "stiff cable").

All these mathematical models, although different, and therefore leading to different equations, have common features, which refer to the logic underlying all of them. It is therefore convenient to introduce a *metamodel* (from the Greek "beyond the model"), which is independent of the specific aspects of the single model, but, in contrast, highlights the common structure of the models. A quite accepted definition of a metamodel is the following: "a precise definition of the constructs and rules needed for creating specific models". Accordingly, the metamodel is a system of inter-related "empty boxes"; once it is available, formulation of specific models consists of "filling in" these boxes.

As in all the problems of continuum mechanics, modeling requires analyzing three independent aspects: (a) geometrical (or kinematic), (b) dynamical, and (c) constitutive. Here, we will introduce these three aspects from an abstract point of view, in order to formulate a metamodel. However, to make the discussion clearer, we will often refer to (linear) models known to the reader, with the only purpose of exemplification. Although the metamodel would work for any spatial dimension, we will refer to a 1D problem because this will be the object of our successive studies.

## 1.2 Internally unconstrained beams

Let us consider a 1D deformable body, whose material points $P$ densely fill a curve in the space. We will say that the beam is *locally rigid*, when $P$ is capable of translations (non-polar continuum) and, possibly, also of rotations (polar continuum), i.e. it behaves as an evanescent rigid body. We will say that the beam is *locally non-rigid* when $P$ is also endowed with a "shape" susceptible to change in time. Standard models of cables and beams, possessing rigid cross-sections, fall into the first category; non-standard models, accounting for the change of shape of the cross-section, fall into the second category.

We will use the wording "position of the point $P$" in a generalized sense, including place, attitude and "shape" of the point. The collection of the positions is called a *configuration*. The configuration assumed by the body at $t = 0$ is called the *reference configuration*; the one assumed at time $t$ is called the *current configuration*. Let us consider a curve $\mathcal{S}$, of extremes $A, B$, on which the body lies at $t = 0$, and let $s \in [0, l]$ be a curvilinear abscissa taken on it; in such a way $s$ is a label for the material point $P$, in the sense that $\mathcal{Q}(s, t)$ represents the value assumed by the quantity $\mathcal{Q}$ at the position occupied by the material point $P$ at time $t$.

### 1.2.1 Kinematics

To describe the current configuration of the body, we follow the referential description of the continuum mechanics, by introducing suitable *generalized displacements* $\boldsymbol{w}(s,t) := (w_i\,(s,t))^T$, $i = 1, \ldots, N$, measured from the reference configuration. This is a set of kinematic descriptors (translations, rotations and distortion parameters) able to describe the change of position of the point $P$ in passing from the reference to the current configuration. The integer $N$ is also called the *number of degrees of freedom* of the point.

The change of configuration, except for special rigid transformations, entails a change of shape of the body, which we will call a *deformation*[1]. A measure of the *local* change of shape is called a *strain*; examples of strains are not only extension, shear, flexure and torsion of a bar, but also warping and ovalization of a pipe cross-section. These can be collected in the column-matrix of *generalized strains* $\boldsymbol{\varepsilon}(s,t) := (\varepsilon_j\,(s,t))^T$, with $j = 1, \ldots, M$. The number of generalized strains is closely related to the number of generalized displacements and the dimensions of the space in which the body is embedded. As a matter of fact, the change of shape of an infinitesimal element of length $ds$ depends on the displacements at its end, $\boldsymbol{w}(s,t)$ and $\boldsymbol{w}(s+ds,t)$;

1. We prefer to reserve the word deformation to *non-rigid* transformations, although it is used in the literature with a wider meaning.

the latter, in turn, can be expressed by Taylor series as $\boldsymbol{w}(s,t)+\boldsymbol{w}'(s,t)ds$[2], so that the configuration depends on $2N$ independent quantities, namely $(\boldsymbol{w}(s,t),\ \boldsymbol{w}'(s,t))$[3]. Since $R$ of them describe a rigid motion, the number of strains is $M := 2N - R$. Usually, $R = 6, 3, 1$ in the spatial, planar and linear cases, respectively; however, it can be lower, if some rotation remains undefined when $\boldsymbol{w}(s,t)$ and $\boldsymbol{w}'(s,t)$ are prescribed. For example, for a Timoshenko beam in the space, it is $N = 6$ (three translations and three rotation angles) and $R = 6$, so that $M = 6$ (one extension, two shear strains and three curvatures); in the plane, it is $N = 3$ (two translations and one rotation) and $R = 3$, and therefore $M = 3$ (one extension, one shear strain and one curvature). For a flexible cable in the plane, it is $N = 2$ (two translations) and $R = 3$, so that $M = 1$ (the extension); however, in the spatial case, it is $N = 3$ (three translations) but $R = 5$, since the displacements and their derivatives are unable to describe the rotation around the tangent to the element, so that it is still $M = 1$ (the extension).

The generalized strains depend on displacements and their first derivatives via the *strain–displacement relationships*. These are *nonlinear* differential equations of the type:

$$\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right) \tag{1.1}$$

where the arguments $s, t$ have been understood. Displacements and strains related by equation [1.1] are called *kinematically admissible*.

The kinematic description is completed by the geometric boundary conditions, which prescribe (part of) the displacements at the ends, where mechanical devices are applied, namely:

$$\boldsymbol{w}_H\left(t\right) = \breve{\boldsymbol{w}}_H\left(t\right), \qquad H = A, B \tag{1.2}$$

where the overmark denotes a known term.

However, not only strains, but even *strain-rates* are of interest. They are obtained by time-differentiating equation [1.1], thus obtaining[4]:

$$\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}\left(\boldsymbol{w}, \boldsymbol{w}'\right)\dot{\boldsymbol{w}} \tag{1.3}$$

where[5]:

$$\boldsymbol{D} := \frac{\partial \boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right)}{\partial \boldsymbol{w}} + \frac{\partial \boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right)}{\partial \boldsymbol{w}'}\frac{\partial}{\partial s} \tag{1.4}$$

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2. Here and further on a dash denotes $s$-differentiation.

3. There exist richer continua of *higher gradient* (see e.g. [DEL 09]), which call for higher derivatives of $\boldsymbol{w}(s,t)$, which, however, we will not consider in this book.

4. Here and further on a dot denotes $t$-differentiation.

5. The derivative of a vector with respect to a vector is a matrix; thus, e.g. $\frac{\partial \boldsymbol{\mathcal{E}}}{\partial \boldsymbol{w}} = \left[\frac{\partial \mathcal{E}_i}{\partial w_j}\right]$, where $i$ is the row index and $j$ the column index.

is a (formal) $M \times N$ matrix, containing the displacements and the space-differential operator $\partial_s := \partial/\partial s$[6]. $\boldsymbol{D}$ is a linear differential operator which transforms the generalized velocities in generalized strain-rates, called the *kinematic operator*. This depends, via $\boldsymbol{w}$ and $\boldsymbol{w}'$, on the configuration assumed by the body at time $t$.

If we consider an infinitesimal time interval $dt$, and we want to evaluate the strains that have been experienced by the beam in this interval, we have:

$$\delta\boldsymbol{\varepsilon} = \boldsymbol{D}\left(\boldsymbol{w}, \boldsymbol{w}'\right)\delta\boldsymbol{w} \qquad [1.5]$$

where $\delta\boldsymbol{\varepsilon} := \dot{\boldsymbol{\varepsilon}}dt$ are infinitesimal strains produced by infinitesimal displacements $\delta\boldsymbol{w} := \dot{\boldsymbol{w}}dt$, superimposed to a deformed state $\boldsymbol{w}$. If we take $\boldsymbol{w} \equiv \boldsymbol{0}$, i.e. if we consider infinitesimal displacements undergone by the beam starting from its undeformed state, then the *infinitesimal kinematic operator* $\boldsymbol{D}_0 := \boldsymbol{D}\left(\boldsymbol{0}, \boldsymbol{0}\right)$ must be considered, which is well-known in the linear theory.

Boundary conditions for equation [1.3] are $\dot{\boldsymbol{w}}_H = \boldsymbol{0}$ and for equation [1.5] are $\delta\boldsymbol{w}_H = \boldsymbol{0}$.

### 1.2.2 Dynamics

Dynamics concerns the study of the contact internal actions exchanged by adjacent points, when the body is loaded by external (active, dissipative or inertial) forces. The internal action is described by *generalized stresses* (forces and couples in locally rigid beams, but also more complex actions like the "bimoment", in locally non-rigid beams). The relationships linking generalized stresses and external forces are called *balance equations* (or equilibrium equations, when they refer to the static case). To define generalized stresses and to derive balance equations, we can follow two alternative philosophies, both popular in the literature: (a) the *power balance formulation*, based on the "virtual power principle", or (b) the *force balance formulation*, based on the "momentum principles" (or cardinal equations of motion, or, in the static case, equilibrium equations)[7]. Both the approaches are (not independent) postulates of the continuum mechanics, leading to the same results, so that we can choose which of them to use. However, if the choice is just a question of taste when dealing with locally rigid beams, the first approach is mandatory when locally non-rigid beams are addressed in the context of 1D models because the cardinal equations are not sufficient to describe the motion of a non-rigid body[8]. In

6. As an example, for a rod embedded in a 1D space, we have $\varepsilon = w_1'$, hence $\boldsymbol{D} = (\partial_s)$.

7. These two approaches are also known in literature as *integral (or weak) formulation*, and *differential (or strong) formulation*, respectively.

8. Of course, the force balance approach could be used for a 3D model, as for example is usually done in the de Saint-Venant Problem.

this book, we use both the approaches, as discussed later, guided by convenience reasons.

### The virtual power principle

We consider a beam loaded by generalized *external* forces (possibly including inertia and damping forces), acting in the domain, of linear density $\boldsymbol{p} := (p_i\,(s,t))^T$ (with $i = 1,\ldots,N$, i.e. a force component for each degree of freedom, d.o.f), as well as boundary external forces $\boldsymbol{P}_H := (P_{iH}\,(s,t))^T$, applied at $H = A, B$. These forces, except the trivial (but frequent) case of dead loads, depend on the configuration (e.g. if they are of follower type), for which $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{w}),\ \boldsymbol{P}_H = \boldsymbol{P}_H(\boldsymbol{w})$, although we will often understand the argument. We then consider the beam *frozen* in the current configuration and superimpose on it a *virtual motion* (i.e. a motion unrelated to the forces), made of a velocity field $\dot{\boldsymbol{w}}$ and a strain-rate field $\dot{\boldsymbol{\varepsilon}}$. The following quantities are introduced:

$$\begin{aligned}\mathcal{P}_{ext} &:= \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \\ \mathcal{P}_{int} &:= \int_{\mathcal{S}} \boldsymbol{\sigma}^T \dot{\boldsymbol{\varepsilon}} ds\end{aligned} \qquad [1.6]$$

called the "external virtual power" and the "internal virtual power" of the beam, respectively. The first of them is the usual definition of the power of a system of forces, except for the fact that forces and velocities are unrelated. In the second definition, $\boldsymbol{\sigma} := (\sigma_j\,(s,t))^T$, with $j = 1,\ldots,M$, are *generalized stresses*. According to this approach, no physical meaning is given to them, but, in analogy with external forces and velocities, they must be recognized as the dynamic action *dual* of the strain-rate (which, in contrast, do have a geometrical meaning).

The VPP establishes that, *in any kinematically admissible virtual motion* $(\dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}})$, *the external virtual power spent by the generalized forces* $\boldsymbol{p}, \boldsymbol{P}_H$ *on the velocity field* $\dot{\boldsymbol{w}}$*, equates the internal virtual power spent by the stresses* $\boldsymbol{\sigma}$ *on the strain-rate field* $\dot{\boldsymbol{\varepsilon}}$, i.e.:

$$\int_{\mathcal{S}} \boldsymbol{\sigma}^T \dot{\boldsymbol{\varepsilon}} ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \qquad \forall\, (\dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}})\,|\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}\dot{\boldsymbol{w}} \qquad [1.7]$$

The VPP furnishes the balance equations via the following procedure. By using equation [1.3], and integrating by parts to "free" the velocities by the derivatives, the internal power reads:

$$\int_{\mathcal{S}} \boldsymbol{\sigma}^T \boldsymbol{D}\dot{\boldsymbol{w}} ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{D}^\star \boldsymbol{\sigma} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{\mathcal{D}}_H^\star \boldsymbol{\sigma} \qquad [1.8]$$

where the following operators have been introduced, accounting for equation [1.4][9]

$$\begin{aligned} \boldsymbol{D}^\star &:= \left( \frac{\partial \boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}')}{\partial \boldsymbol{w}} - \frac{\partial}{\partial s}\left( \frac{\partial \boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}')}{\partial \boldsymbol{w}'} \right) - \frac{\partial \boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}')}{\partial \boldsymbol{w}'} \frac{\partial}{\partial s} \right)^T \\ \boldsymbol{\mathcal{D}}_H^\star &:= \mp \left( \frac{\partial \boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}')}{\partial \boldsymbol{w}'} \right)_H^T \end{aligned} \qquad [1.9]$$

The VPP, therefore, reads:

$$\int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \left( \boldsymbol{D}^\star \boldsymbol{\sigma} - \boldsymbol{p} \right) ds - \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \left( \boldsymbol{\mathcal{D}}_H^\star \boldsymbol{\sigma} - \boldsymbol{P}_H \right) = 0 \qquad \forall \dot{\boldsymbol{w}} \qquad [1.10]$$

and, since $\dot{\boldsymbol{w}}$ is arbitrary, it leads to the following field equations:

$$\boldsymbol{D}^\star (\boldsymbol{w}, \boldsymbol{w}') \, \boldsymbol{\sigma} = \boldsymbol{p} \qquad [1.11]$$

and to the boundary conditions:

$$\left[ \dot{\boldsymbol{w}}^T \left( \boldsymbol{\mathcal{D}}^\star (\boldsymbol{w}, \boldsymbol{w}') \, \boldsymbol{\sigma} - \boldsymbol{P} \right) \right]_H = \boldsymbol{0} \qquad H = A, B \qquad [1.12]$$

Equation [1.11] is the balance (or equilibrium) equation sought. Equation [1.12] supplies the relevant boundary conditions, called mechanical (or natural) conditions. They supplement the geometric (or essential) boundary conditions [1.2] in the following senses: (a) if $H$ is fully constrained, then $\dot{\boldsymbol{w}}_H = \boldsymbol{0}$, and therefore no mechanical conditions hold there; (b) if $H$ is fully free, then $\dot{\boldsymbol{w}}_H \neq \boldsymbol{0}$ and it is arbitrary, and therefore $\boldsymbol{\mathcal{D}}_H^\star \boldsymbol{\sigma} = \boldsymbol{P}_H$ must hold there. Similar properties hold for partially restrained ends. In conclusion, if a displacement component is prescribed, no mechanical condition must be added; if a displacement component is free, a scalar mechanical condition must be enforced. Therefore, geometric and mechanical conditions are alternative.

The operator $\boldsymbol{D}^\star (\boldsymbol{w}, \boldsymbol{w}')$, which appears in the balance equations, is a formal $N \times M$ matrix, depending on the operator $\partial_s$[10]. It is a linear differential operator that

9. For example, $\int_{\mathcal{S}} \boldsymbol{\sigma}^T \left( \boldsymbol{A} \frac{d}{ds} \right) \dot{\boldsymbol{w}} ds = - \int_{\mathcal{S}} \frac{d}{ds} \left( \boldsymbol{\sigma}^T \boldsymbol{A} \right) \dot{\boldsymbol{w}} ds + \left[ \boldsymbol{\sigma}^T \boldsymbol{A} \dot{\boldsymbol{w}} \right]_A^B = - \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \frac{d}{ds} \left( \boldsymbol{A}^T \boldsymbol{\sigma} \right) ds + \left[ \dot{\boldsymbol{w}}^T \boldsymbol{A}^T \boldsymbol{\sigma} \right]_A^B$.

10. As an example, for a rod embedded in a 1D space, the equilibrium equation reads $N' + p_1 = 0$, hence $\boldsymbol{D}^\star (\cdot) = (-\partial_s)$.

transforms the generalized stresses into generalized forces, and it is called the *equilibrium operator*. Note that it depends on the state $\boldsymbol{w}$ because it describes the equilibrium of the beam in the current configuration, thus encompassing the nonlinear nature of the problem. When, in contrast, the effects of deformation are ignored (i.e. the current configuration is confused with the reference configuration), then the equilibrium operator reduces to $\boldsymbol{D}_0^\star := \boldsymbol{D}^\star(\mathbf{0}, \mathbf{0})$, which is the well-known *linear equilibrium operator* of the linear theory. The operators $\boldsymbol{\mathcal{D}}_H^\star$ are (algebraic) *boundary equilibrium operators*.

REMARK 1.1. The VPP expression [1.8] is also called, in a wider context, the *extended Green identity*. It states that *the kinematic operator and the equilibrium operator, as well as the associated boundary conditions, are mutually adjoint*. Such an occurrence is called *duality property*. It is well-known in the linear field (where it concerns the adjointness property between $\boldsymbol{D}_0$ and $\boldsymbol{D}_0^\star$), but it also holds in the nonlinear field, when use is made of the kinematic and equilibrium operators relevant to the current configuration.

REMARK 1.2. The VPP could be reformulated as a *virtual work* principle (VWP). Indeed, it is sufficient to multiply both members of equation [1.7] by an infinitesimal time interval $dt$, and to refer to infinitesimal displacements $\delta\boldsymbol{w} := \dot{\boldsymbol{w}}dt$ and infinitesimal strains $\delta\boldsymbol{\varepsilon} := \dot{\boldsymbol{\varepsilon}}dt$, namely:

$$\int_{\mathcal{S}} \boldsymbol{\sigma}^T \delta\boldsymbol{\varepsilon} ds = \int_{\mathcal{S}} \delta\boldsymbol{w}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \delta\boldsymbol{w}_H^T \boldsymbol{P}_H \qquad \forall\left(\delta\boldsymbol{w}, \delta\boldsymbol{\varepsilon}\right) | \delta\boldsymbol{\varepsilon} = \boldsymbol{D}\delta\boldsymbol{w} \quad [1.13]$$

The formulation in terms of velocities is preferred in formal treatments because it does not call for resorting to the concept of “infinitesimal” displacements and strains, which understands a series expansion.

### The force balance formulation for locally rigid beams

When the local structure of the 1D beam is rigid, the force balance formulation is viable. With respect to the power balance formulation, it has the advantage to endow the stresses of a physical meaning.

We consider the internal action that two parts of the beam mutually exchange at the abscissa $s$ and time $t$, and denote by $\boldsymbol{f} = \left(f_i\left(s,t\right)\right)^T$, $i = 1, \ldots, N$ the generalized forces (i.e. forces and couples) acting on one of the two parts, conventionally assumed as positive. Note that the internal force components are in the same number of the degrees of freedom (translations and rotations) of the “rigid” point $P$. We then consider an infinitesimal element of length $ds$, loaded by external forces per unit length $\boldsymbol{p} := \left(p_i\left(s,t\right)\right)^T$, at whose ends, internal forces $\boldsymbol{f}(s,t)$ and $\boldsymbol{f}(s+ds,t) = \boldsymbol{f}(s,t) + \boldsymbol{f}'(s,t)ds$ act. Therefore, the contact action that the element exchanges with the adjacent ones depends on $2N$ independent scalar quantities, namely $\left(\boldsymbol{f}(s,t),\ \boldsymbol{f}'(s,t)\right)$.

We define *stresses* the independent internal forces able to describe the more general self-equilibrated state of the element (i.e. when $\boldsymbol{p} = \mathbf{0}$). We *postulate* that equilibrium of this elementary body is governed by the *same cardinal equations* of rigid-body mechanics. Since the scalar equilibrium equations are in number of $R$, i.e. one for each independent rigid motion of the element, we conclude that the self-equilibrated states are $M := 2N - R$, described by $M$ independent stresses, i.e. stresses are in the same number of strains. We collect all the stresses in a column-matrix $\boldsymbol{\sigma} := \left(\sigma_j\left(s,t\right)\right)^T$, with $j = 1, \ldots, M$. For example, for a spatial cable, we have $2N = 6$ internal end-forces, which have to satisfy $R = 5$ independent equilibrium equations (since the sixth one, relevant to the moment with respect to the tangent axis, is trivially satisfied); hence, $M = 1$ stresses exist (namely the axial force).

When the external forces are non-zero, then the cardinal equations must express the balance of external forces and stresses acting on the element. However, just $N$ of them are significant, the remaining $R - N$ being satisfied by the stresses alone (e.g. in the spatial cable, all the moment equations are identically satisfied, so that only the translational equilibrium has to be satisfied). In conclusion, the $M$ generalized stresses must satisfy $N$ (differential) field balance equations; since these are linear, they assume the form [1.11].

When the element is taken at the boundary of the beam, and this is free of constraints, then the stresses are there prescribed; a suitable linear combination of the stresses must equate the external forces at the end, as stated in equation [1.12].

### 1.2.3 The hyperelastic law

To complete the model, we need to introduce a *constitutive law*, able to link generalized stresses $\boldsymbol{\sigma}$ to generalized strains $\varepsilon$, thus realizing the "bridge" between kinematics and dynamics. This topic is quite difficult, when tackled in a general context. Indeed, a general law linking stresses and strains should account for the *deformation history* of the material, this requiring a quite complex mathematical apparatus (i.e. the use either of functionals or of incremental forms in terms of strain-rates and stress-rates). This, however, is a peculiarity of plasticity; if we are, instead, interested in comparatively small strains, then we can ignore the past events, and refer just to the current state, by writing $\boldsymbol{\sigma}(s,t) = \mathcal{F}(\boldsymbol{\varepsilon}(s,t),\, t)$. The explicit dependence on time is a peculiarity of viscosity, whose brief treatment will be postponed. If we admit that the constitutive law does not depend explicitly on time, we simply write $\boldsymbol{\sigma}(s,t) = \mathcal{F}(\boldsymbol{\varepsilon}(s,t))$ or, by omitting the arguments, $\boldsymbol{\sigma} = \mathcal{F}(\boldsymbol{\varepsilon})$. This law characterizes elasticity.

If a beam is elastic, then stresses at the abscissa $s$ and time $t$ only depend on strains existing at the same place at the same instant. However, this concept of elasticity (said to be of Cauchy) does not match the idealization of the perception everybody has in real life, i.e. an elastic body requires some energy to be deformed, but it entirely returns when the deformation is removed, *regardless of the way the unloading is performed*. In other words, this more refined idea of elasticity (said to be of Green, or *hyperelasticity*) expresses the *conservation of energy*, i.e. the absence of dissipation in any cyclic process the body can undergo (or, equivalently, the independence of the energy spent on any paths followed to connect two states). Since just Green-elastic bodies are of interest, very often the adjective "elastic" is used as "hyperelastic", and we will comply with this tradition.

### The elastic potential

The work spent by the external forces to deform the beam in an interval of time $dt$ is equal to $\mathcal{P}_{ext}dt$, with $\mathcal{P}_{ext}$ the *deformation* external power, formally still given by equation [1.6a], but with velocities now denoting quantities related to the real (not virtual) process. Since $\mathcal{P}_{ext} = \mathcal{P}_{int}$ for the VPP [1.7], then we can define a *deformation work for unit length* as $\frac{d}{ds}\left(\mathcal{P}_{int}dt\right) = \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}dt = \boldsymbol{\sigma}^T\delta\boldsymbol{\varepsilon}$. To evaluate the work needed to lead a unitary element of beam from the state $\boldsymbol{\varepsilon}_1$ to the state $\boldsymbol{\varepsilon}_2$, we have to integrate the linear differential form $\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon}\right)^T\delta\boldsymbol{\varepsilon}$ along a line connecting the two states in the space of the strains. The result, in general, depends on the path chosen for integration (i.e. on the sequence of the deformation imposed), unless $\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon}\right)^T\delta\boldsymbol{\varepsilon}$ is an *exact differential*, i.e. it is the differential of a scalar function $\phi(\boldsymbol{\varepsilon})$. By requiring $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})^T\delta\boldsymbol{\varepsilon} = \delta\phi(\boldsymbol{\varepsilon}) \equiv \left(\delta\phi/\delta\boldsymbol{\varepsilon}\right)^T\delta\boldsymbol{\varepsilon}$, it follows that[11]:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \frac{\delta\phi(\boldsymbol{\varepsilon})}{\delta\boldsymbol{\varepsilon}} \qquad [1.14]$$

Equation [1.14] is the hyperelastic law sought. The law *postulates* the existence of a function $\phi(\boldsymbol{\varepsilon})$, called the *density of elastic potential energy* or, simply, the *elastic potential*.

### The linear law

To write the elastic law, we have to assume a suitable form for the elastic potential, and then to differentiate it. If, for example, we adopt a polynomial of degree $n$, we obtain a stress–strain polynomial law of degree $n-1$. Of course, the simplest choice

11. This symbolism means that the $i$th component of the column-matrix $\boldsymbol{\sigma}$ is equal to the derivative of the scalar $\phi$ with respect to the $i$th component of the column-matrix $\boldsymbol{\varepsilon}$, i.e. $\sigma_i = \delta\phi/\delta\varepsilon_i$.

is to take a quadratic potential, from which a linear law follows. We start by assuming a *homogeneous* quadratic polynomial, as:

$$\phi(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}^T \boldsymbol{E}\boldsymbol{\varepsilon} \quad [1.15]$$

where $\boldsymbol{E}$ is a square matrix of constants, called the *elastic matrix*. It possesses two properties: (a) $\boldsymbol{E} = \boldsymbol{E}^T$ is symmetric because it is the matrix of a quadratic form (and therefore its non-symmetric part is unessential)[12]; and (b) $\boldsymbol{E}$ is *positive definite*, this assuring that a positive work must be spent on the body in order to deform it, i.e.: $\phi(\boldsymbol{\varepsilon}) > 0 \; \forall \boldsymbol{\varepsilon} \neq \mathbf{0}$. By applying equation [1.14], the *Hooke law* follows:

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon} \quad [1.16]$$

It establishes direct proportionality between stresses and strains; moreover, it states that stresses vanish when strains vanish. Since we decided to measure the strains starting from the reference configuration, the homogeneous form of the elastic law applies when the reference configuration is stress-free, also known as *unprestressed*. Such a state is called the *natural state* of the body, whose existence is postulated. We will return on this topic in the next section, when we will account for prestresses.

### 1.2.4 The Fundamental Problem

The *Fundamental Problem of Elasticity* (or elastic problem), relevant to a 1D beam, is stated as follows. A beam is given under assigned loads $\boldsymbol{p}(s,t)$ acting in the domain, and displacements $\breve{\boldsymbol{w}}_H(t)$ *or* forces $\boldsymbol{P}_H(t)$ prescribed/applied at the ends $H = A, B$. We want to determine the generalized displacements $\boldsymbol{w}(s,t)$, the strains $\boldsymbol{\varepsilon}(s,t)$ and the stresses $\boldsymbol{\sigma}(s,t)$. The problem is governed by the field equations [1.1], [1.11] and [1.16] and boundary conditions [1.2] and [1.12]. Overall, there are $N + 2M$ unknowns, appearing in as many field equations.

The equations can be combined according to the *displacement method*, which consists of expressing the balance equations and the boundary conditions in terms of displacements only, by using, in the order, the elastic law and the strain–displacement relationships. The stress–displacement relationships, therefore, read:

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right) \quad [1.17]$$

and, consequently, the balance equations and the mechanical boundary conditions transform into:

$$\begin{aligned} &\boldsymbol{D}^\star\left(\boldsymbol{w}, \boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right) = \boldsymbol{p} \\ &\boldsymbol{\mathcal{D}}_H^\star\left(\boldsymbol{w}, \boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}_H\left(\boldsymbol{w}, \boldsymbol{w}'\right) = \boldsymbol{P}_H \end{aligned} \quad [1.18]$$

12. We can also say that $E_{ij} = \partial^2\phi/\partial\varepsilon_i\partial\varepsilon_j = \partial^2\phi/\partial\varepsilon_j\partial\varepsilon_i = E_{ji}$.

to be joined to the geometric boundary conditions [1.2]. These equations constitute a nonlinear boundary value problem for the principal unknowns $\boldsymbol{w}$. Since the equations are nonlinear, the uniqueness of the solution is not ensured.

### The linear theory

If all terms in equations [1.18] are expanded around the reference configuration, and only the leading term is taken in each expansion, we have[13]:

$$\begin{aligned}
&\boldsymbol{D}^{\star}\left(\boldsymbol{w}, \boldsymbol{w}'\right) = \boldsymbol{D}_0^{\star} + \text{h.o.t.} \\
&\boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right) = \boldsymbol{D}_0 \boldsymbol{w} + \text{h.o.t.} \\
&\boldsymbol{p}\left(\boldsymbol{w}\right) = \boldsymbol{p}_0 + \text{h.o.t.} \\
&\boldsymbol{\mathcal{D}}_H^{\star}\left(\boldsymbol{w}, \boldsymbol{w}'\right) = \boldsymbol{\mathcal{D}}_{0H}^{\star} + \text{h.o.t.} \\
&\boldsymbol{P}_H\left(\boldsymbol{w}\right) = \boldsymbol{P}_{0H} + \text{h.o.t.}
\end{aligned} \qquad [1.19]$$

where use has been made of equation [1.4], and where the index 0 denotes evaluation at $\boldsymbol{w} = \mathbf{0}$. As a result, equations [1.18] become:

$$\begin{aligned}
&\boldsymbol{L}\boldsymbol{w} = \boldsymbol{p}_0 \\
&\boldsymbol{\mathcal{L}}_H \boldsymbol{w} = \boldsymbol{P}_{0H}
\end{aligned} \qquad [1.20]$$

where:

$$\boldsymbol{L} := \boldsymbol{D}_0^{\star} \boldsymbol{E} \boldsymbol{D}_0, \qquad \boldsymbol{\mathcal{L}}_H : \boldsymbol{\mathcal{D}}_{0H}^{\star} \boldsymbol{E} \boldsymbol{D}_{0H} \qquad [1.21]$$

are the familiar *(tangent) stiffness operators* (in the domain and at the boundary) of the linear theory. Note that $\boldsymbol{L}$ is self-adjoint, for the duality property and the symmetry of $\boldsymbol{E} = \boldsymbol{E}^T$[14].

REMARK 1.3. A consistent first-order expansion would also require the accounting of the first derivative of the loads, but this effect is ignored in the linear theory because equilibrium is referred to the reference configuration. Therefore, any information about dependence of the loads on displacements is lost.

## 1.3 Internally constrained beams

It is well-known, from Lagrangian mechanics, that internal constraints reduce the number of d.o.f. of the system. Thus, a collection of $N$ particles, free in the space,

13. Here, and further on, h.o.t. denotes "higher order terms".

14. Namely, $\boldsymbol{L}^{\star} = \left(\boldsymbol{D}_0^{\star} \boldsymbol{E} \boldsymbol{D}_0\right)^{\star} = \boldsymbol{D}_0^{\star} \boldsymbol{E}^T \boldsymbol{D}_0 = \boldsymbol{L}$.

possesses $3N$ d.o.f., but these reduce to 6 if the mutual distances among the particles are constrained to remain unaltered in any motions, i.e. if the system behaves as a rigid body. In addition, it is also well-known that introducing internal constraints, while simplifying kinematics, makes the study of dynamics more difficult because part of the forces are unrelated to displacements. Accordingly, we say that the internal forces are *active*, when they depend on kinematic quantities via a constitutive law, and *reactive*, when they are independent of them. Thus, by using again the example of a collection of particles and assuming that they attract each other, the internal forces depend on the mutual distances if the particles are unconstrained, but they assume any magnitude if the particles are rigidly connected.

The degenerateness of the constitutive law can also be understood if we consider a linear spring, whose stiffness quasi-statically increases to infinite. Until the stiffness is finite, there exists proportionality between the force and the elongation (i.e. the response curve is a straight line, whose slope is the stiffness); however, when the stiffness becomes infinitely large, any force can be obtained because the product of infinite (the stiffness) by zero (the elongation) is undetermined (the response curve is vertical, i.e. it is the graph of a degenerate, not single-valued, function). As a matter of fact, in the limit process, the spring becomes a rigid truss, able to supply any force aligned with its axis.

These ideas can be translated into the mechanics of a deformable body, in particular beams. We are encouraged to formulate internally constrained models, in which the configuration variables are not free, but are required to satisfy one or more geometrical constraints. Although, in principle, any conditions could be introduced, the most meaningful of them consist of *vanishing one or more strains*, identically along the beam. Thus, a beam is inextensible (or unshearable) if the elongation (or the shear-strain) is prevented. Later in the book (Chapter 4), we will discuss the conditions under which constrained models are applicable to real cases, and also consider more complex linear and nonlinear constraint conditions (Chapter 8).

Because of the internal constraints, the dual stresses (i.e. those spending power on the constrained strains) become reactive, so that they cannot be expressed by an elastic law. This drawback calls either for a mixed displacement–stress formulation, or for a special treatment to eliminate reactive stresses, which we are going to illustrate in the next sections.

### 1.3.1 The mixed formulation for the internally constrained beam kinematics and constraints

The deformation of a beam is described by $M$ independent quantities $\boldsymbol{\varepsilon}\left(s,t\right)$. We assume that $M_c < M$ of them are identically zero, i.e. $\boldsymbol{\varepsilon}_c\left(s,t\right) = \mathbf{0}\ \forall\left(s,t\right)$, and we call them the *constrained strains*. The remaining $M_u := M - M_c$ strains, $\boldsymbol{\varepsilon}_u \neq \mathbf{0}$, are referred to as the *unconstrained* (or admissible) *strains*. As a result, $\boldsymbol{\varepsilon} := (\boldsymbol{\varepsilon}_u, \boldsymbol{\varepsilon}_c)^T$, and equations [1.1] read:

$$\begin{pmatrix} \boldsymbol{\varepsilon}_u \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{E}}_u(\boldsymbol{w}, \boldsymbol{w}') \\ \boldsymbol{\mathcal{E}}_c(\boldsymbol{w}, \boldsymbol{w}') \end{pmatrix} \tag{1.22}$$

with the boundary conditions $\boldsymbol{w}_H = \breve{\boldsymbol{w}}_H$, where $H = A, B$. We will refer to the upper part of equations [1.22] as the strain–displacement relationships, and to the lower part as *a set of constraints*, limiting the arbitrariness of the displacements and their derivatives.

By time-differentiating equations [1.22], we obtain the strain-rate-velocity relationships:

$$\begin{pmatrix} \dot{\boldsymbol{\varepsilon}}_u \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{D}_u(\boldsymbol{w}, \boldsymbol{w}') \\ \boldsymbol{D}_c(\boldsymbol{w}, \boldsymbol{w}') \end{pmatrix} \dot{\boldsymbol{w}} \tag{1.23}$$

where $\boldsymbol{D}_u$, $\boldsymbol{D}_c$ are partitions of the kinematic operator appearing in equation [1.3].

**Dynamics**

To derive the balance equations for the constrained problem, we will apply the VPP principle [1.7]. To express the virtual internal power, it is convenient to partition the stress $\boldsymbol{\sigma}(s,t)$ in two subsets, namely $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_c)^T$, where $\boldsymbol{\sigma}_u$ is an $M_u$-vector collecting the stresses dual of the unconstrained strains, and $\boldsymbol{\sigma}_c$ is an $M_c$-vector listing the stresses dual of the constrained strains. Thus, the VPP reads:

$$\begin{gathered} \int_{\mathcal{S}} (\boldsymbol{\sigma}_u^T \dot{\boldsymbol{\varepsilon}}_u + \boldsymbol{\sigma}_c^T \dot{\boldsymbol{\varepsilon}}_c) ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \\ \forall (\dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}_u, \dot{\boldsymbol{\varepsilon}}_c) \,|\, (\dot{\boldsymbol{\varepsilon}}_u = \boldsymbol{D}_u \dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}_c = \mathbf{0} = \boldsymbol{D}_c \dot{\boldsymbol{w}}) \end{gathered} \tag{1.24}$$

where equations [1.23] have been accounted for. By using them in the internal power expression, we have:

$$\int_{\mathcal{S}} \boldsymbol{\sigma}_u^T \boldsymbol{D}_u \dot{\boldsymbol{w}} ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \quad \forall \dot{\boldsymbol{w}} | \boldsymbol{D}_c \dot{\boldsymbol{w}} = \mathbf{0}, \tag{1.25}$$

which is still a *constrained* problem, since $\dot{\boldsymbol{w}}$ cannot be taken arbitrarily, but it must satisfy an auxiliary condition. By following the well-known *Lagrange multipliers* technique, we add to the previous equation the integral of $\boldsymbol{\lambda}^T \boldsymbol{D}_c \dot{\boldsymbol{w}} = 0$, where

$\boldsymbol{\lambda} = \boldsymbol{\lambda}(s,t)$ are unknown Lagrangian multipliers, and rewrite equation [1.25] as an *unconstrained* problem[15]:

$$\int_{\mathcal{S}} \boldsymbol{\sigma}_u^T \boldsymbol{D}_u \dot{\boldsymbol{w}} ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H - \int_{\mathcal{S}} \boldsymbol{\lambda}^T \boldsymbol{D}_c \dot{\boldsymbol{w}} ds \quad \forall \dot{\boldsymbol{w}} \qquad [1.26]$$

But, if we rename $\boldsymbol{\lambda}$ as $\boldsymbol{\sigma}_c$, i.e., if we attribute to the constrained stresses the meaning of Lagrangian multipliers, this latter is equivalent to the original principle [1.24] with no constraints, i.e.:

$$\begin{gathered}\int_{\mathcal{S}} (\boldsymbol{\sigma}_u^T \dot{\boldsymbol{\varepsilon}}_u + \boldsymbol{\sigma}_c^T \dot{\boldsymbol{\varepsilon}}_c) ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \\ \forall \left(\dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}_u, \dot{\boldsymbol{\varepsilon}}_c\right) \mid \left(\dot{\boldsymbol{\varepsilon}}_u = \boldsymbol{D}_u \dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}_c = \boldsymbol{D}_c \dot{\boldsymbol{w}}\right)\end{gathered} \qquad [1.27]$$

Therefore, after integration by parts, results identical to those supplied by the principle in equation [1.7] are recovered, but in split form; namely, the *split balance equations*:

$$\begin{pmatrix} \boldsymbol{D}_u^\star & \boldsymbol{D}_c^\star \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_u \\ \boldsymbol{\sigma}_c \end{pmatrix} = \boldsymbol{p} \qquad [1.28]$$

and the *split boundary conditions*:

$$\left[ \dot{\boldsymbol{w}}^T \left( \begin{pmatrix} \boldsymbol{\mathcal{D}}_u^\star & \boldsymbol{\mathcal{D}}_c^\star \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_u \\ \boldsymbol{\sigma}_c \end{pmatrix} - \boldsymbol{P} \right) \right]_H = \boldsymbol{0} \qquad [1.29]$$

where $\boldsymbol{D}_u^\star$ and $\boldsymbol{D}_c^\star$ are the adjoint operators of $\boldsymbol{D}_u$ and $\boldsymbol{D}_c$.

## Constitutive law

The left-hand member of equation [1.27] states that the internal virtual power $\mathcal{P}_{int}$ of a constrained system is made of two contributions: an *active* virtual power $\mathcal{P}_{act} := \int_{\mathcal{S}} \boldsymbol{\sigma}_u^T \dot{\boldsymbol{\varepsilon}}_u ds$ and a *reactive* virtual power $\mathcal{P}_{react} := \int_{\mathcal{S}} \boldsymbol{\sigma}_c^T \dot{\boldsymbol{\varepsilon}}_c ds$. Accordingly, $\boldsymbol{\sigma}_u$ is called the *active stress*, and $\boldsymbol{\sigma}_c$ the *reactive stress*. Note that, as suggested by the Lagrange multiplier technique, *the reactive stress spends zero virtual power in any admissible motion* (as reactive forces do in rigid-body mechanics), i.e. it satisfies the "perfect constraint postulate".

15. It is well-known, from the Variational Calculus, that the *constrained* problem $\delta I\left[u\left(s\right)\right] := \int_a^b \delta\mathcal{L}\left(u\left(s\right), u'\left(s\right)\right) ds = 0,\ \forall \delta f_i\left(u\left(s\right), u'\left(s\right)\right) = 0,\ i = 1, \ldots, n$ is equivalent to the *unconstrained* problem $\delta\tilde{I}\left[u\left(s\right), \lambda_i\left(s\right)\right] := \int_a^b \delta\mathcal{L} ds + \sum_{i=1}^n \int_a^b \lambda_i \delta f_i = 0$, where $\lambda_i\left(s\right)$ are Lagrange multipliers.

We can still formulate a hyperelastic law for the active stresses, by requiring that the deformation work for unit length of the beam, i.e. the work $\frac{d}{ds}\left(\mathcal{P}_{act}dt\right)$ spent by the active stresses in time interval $dt$, equates the differential $d\phi$ of the elastic potential $\phi = \phi\left(\boldsymbol{\varepsilon}_u\right)$; from this, $\boldsymbol{\sigma}_u = \partial\phi/\partial\boldsymbol{\varepsilon}_u$ follows. If the potential is assumed quadratic, i.e. $\phi = 1/2\boldsymbol{\varepsilon}_u^T E_{uu}\boldsymbol{\varepsilon}_u$, the linear law follows:

$$\boldsymbol{\sigma}_u = \boldsymbol{E}_{uu}\boldsymbol{\varepsilon}_u \qquad [1.30]$$

## General linear constraints

The constraints $\boldsymbol{\varepsilon}_c = 0$, so far considered, are probably so simple that they hide some interesting aspects of the problem. Therefore, we find useful a digression concerning more general kinematic constraints, of the kind:

$$\boldsymbol{B}\boldsymbol{\varepsilon} = \boldsymbol{0} \qquad [1.31]$$

where $\boldsymbol{B}$ is an $M_c \times M$ constant matrix. Of course, if $\boldsymbol{B} = [\boldsymbol{0},\ \boldsymbol{I}]$, the previous case is recovered. Later on in the book (Chapter 8), constraints like this will be addressed.

The VPP, with the constraint [1.31], reads:

$$\int_{\mathcal{S}} (\boldsymbol{\sigma}_a^T\dot{\boldsymbol{\varepsilon}} + \boldsymbol{\lambda}^T\boldsymbol{B}\dot{\boldsymbol{\varepsilon}})ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T\boldsymbol{p}ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T\boldsymbol{P}_H, \qquad \forall\left(\dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}\right)|\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}\dot{\boldsymbol{w}} \qquad [1.32]$$

where we denoted by $\boldsymbol{\sigma}_a$ the active stresses and by $\boldsymbol{\lambda}$ the Lagrangian multipliers. Note that, differently from the particular case examined previously, we did *not* introduce the constraint in the active part of the internal power. From the VPP, the balance equations are derived:

$$\boldsymbol{D}^\star\left(\boldsymbol{\sigma}_a + \boldsymbol{B}^T\boldsymbol{\lambda}\right) = \boldsymbol{0} \qquad [1.33]$$

together with the boundary conditions:

$$\left[\dot{\boldsymbol{w}}^T\left(\boldsymbol{\mathcal{D}}^\star\boldsymbol{\sigma} - \boldsymbol{P}\right)\right]_H = \boldsymbol{0} \qquad [1.34]$$

The internal virtual power states that the stress is the sum of an active and a passive quota, namely $\boldsymbol{\sigma} = \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_r$, with $\boldsymbol{\sigma}_r := \boldsymbol{B}^T\boldsymbol{\lambda}$. By assuming for the active stresses a linear elastic law, $\boldsymbol{\sigma}_a = \boldsymbol{E}\boldsymbol{\varepsilon}$, and taking into account the reactive part, the *elasto-reactive constitutive law* follows for the total stresses:

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon} + \boldsymbol{B}^T\boldsymbol{\lambda} \qquad [1.35]$$

This shows that, in general, each component of $\boldsymbol{\sigma}$ is partially active and partially reactive.

In the simplest constraint case, $\boldsymbol{\varepsilon}_c = \boldsymbol{0}$, for which $\boldsymbol{B} = [\boldsymbol{0},\ \boldsymbol{I}]$, the constitutive law [1.35] reads:

$$\begin{pmatrix}\boldsymbol{\sigma}_u\\ \boldsymbol{\sigma}_c\end{pmatrix} = \begin{pmatrix}\boldsymbol{E}_{uu} & \boldsymbol{E}_{uc}\\ \boldsymbol{E}_{cu} & \boldsymbol{E}_{cc}\end{pmatrix}\begin{pmatrix}\boldsymbol{\varepsilon}_u\\ \boldsymbol{0}\end{pmatrix} + \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{\lambda}\end{pmatrix} = \begin{pmatrix}\boldsymbol{E}_{uu}\boldsymbol{\varepsilon}_u\\ \boldsymbol{E}_{cu}\boldsymbol{\varepsilon}_u + \boldsymbol{\lambda}\end{pmatrix} \qquad [1.36]$$

so that $\boldsymbol{\sigma}_u$ is purely active, while $\boldsymbol{\sigma}_c$ is elasto-reactive. Since in equation [1.27] we zeroed $\dot{\boldsymbol{\varepsilon}}_c$ in the active internal power, the Lagrangian multiplier used in that equation accounts for both the active and reactive components of $\boldsymbol{\sigma}_c$. In the special (but frequent) case in which the elastic matrix $\boldsymbol{E}$ is diagonal, the constrained strains are *purely reactive*.

**The constrained Fundamental Problem: the mixed formulation**

By summarizing, the Fundamental Problem for the internally constrained beam is governed by: $M_u$ strain–displacements relationships, with $M_c$ constraints equations appended (equation [1.22]); $N$ balance equation [1.28]; $M_u$ purely elastic constitutive equation [1.30]; overall $2M_u + M_c + N$ equations. The unknowns involved are: $M_u$ unconstrained strains $\boldsymbol{\varepsilon}_u$, $M_u + M_c$ stresses $(\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_c)$, $N$ displacements $\boldsymbol{w}$, i.e. $2M_u + M_c + N$ unknowns. If we compare these numbers with that of the unconstrained problem, we note that $M_c$ constrained strains $\boldsymbol{\varepsilon}_c$ disappeared, and also $M_c$ constitutive elastic laws were canceled, this resulting in a contraction of the dimensions of the problem.

The fundamental equations cannot be combined according to the displacement method because the reactive stresses are independent of kinematic quantities. Therefore, a *mixed formulation* must be adopted, in terms of *both* displacements and reactions. Hence, the balance equations and constraints read (compare them with equations [1.18]):

$$\begin{aligned} &\boldsymbol{D}_u^\star \boldsymbol{E}_{uu} \boldsymbol{\mathcal{E}}_u + \boldsymbol{D}_c^\star \boldsymbol{\sigma}_c = \boldsymbol{p} \\ &\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w}, \boldsymbol{w}'\right) = \boldsymbol{0} \end{aligned} \qquad [1.37]$$

together with the boundary conditions:

$$\begin{aligned} &\boldsymbol{\mathcal{D}}_{uH}^\star \boldsymbol{E}_{uu} \boldsymbol{\mathcal{E}}_{uH} + \boldsymbol{\mathcal{D}}_{cH}^\star \boldsymbol{\sigma}_c = \boldsymbol{P}_H \\ &\boldsymbol{w}_H = \breve{\boldsymbol{w}}_H \end{aligned} \qquad [1.38]$$

of mechanical and geometric types, respectively (to be enforced alternatively). Equations [1.37] and [1.38] constitute a mixed boundary value problem, coupled in $N$ displacements and $M_c$ reactive stresses.

**The linear theory**

If equations [1.37] and [1.38a] are linearized around the trivial configuration, we have (compare them with equations [1.20] and [1.21]):

$$\begin{pmatrix} \boldsymbol{L}_u & \boldsymbol{D}_{0c}^\star \\ \boldsymbol{D}_{0c} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{\sigma}_c \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_0 \\ \boldsymbol{0} \end{pmatrix} \qquad [1.39]$$

with the mechanical boundary conditions:

$$\boldsymbol{\mathcal{L}}_{uH} \boldsymbol{w} + \boldsymbol{\mathcal{D}}_{0cH}^\star \boldsymbol{\sigma}_c = \boldsymbol{P}_{0H} \qquad [1.40]$$

where $\boldsymbol{L}_u, \boldsymbol{\mathcal{L}}_{uH}$ are *condensed* linear stiffness operators:

$$\boldsymbol{L}_u := \boldsymbol{D}_{0u}^\star \boldsymbol{E}_{uu} \boldsymbol{D}_{0u}, \qquad \boldsymbol{\mathcal{L}}_{uH} := \left[\boldsymbol{\mathcal{D}}_{0u}^\star \boldsymbol{E}_{uu} \boldsymbol{D}_{0u}\right]_H \qquad [1.41]$$

Note that $\boldsymbol{L}_u$ is self-adjoint.

### 1.3.2 The displacement method for the internally constrained beam

The mixed formulation leads to balance equations that contain the reactive stresses. In order to formulate a problem purely in terms of displacements, as for the unconstrained beam, we have to eliminate the reactions. This goal could be reached, in principle, by performing linear algebraic/differential combinations among the original balance equation [1.28], by exploiting the fact that they are linear in the stresses. Thus, by using $M_c < N$ balance equations, we could eliminate as many reactive stresses, so obtaining $N_m := N - M_c$ equations in the active stresses only. The operation is called *condensation of the reactive stresses*. This circumstance is analogous to that of Lagrangian mechanics of constrained bodies, where one looks for the "Lagrange equations of motion", i.e. equations free of reactive forces.

As for rigid bodies, however, the condensation of the stresses via linear combination is neither simple nor convenient, but a variational (or integral, i.e. based on the VPP) approach is advised. This is based on a preliminary study of kinematics, in which $N_m = N - M_c$ displacements must be chosen as "master (or free) variables", and the remaining $M_c$ "slave variables" related to them, in such a way to identically satisfy the $M_c$ constraints. This operation represents a *condensation of the displacements*, dual to that of stresses, which balances the problem ($M_c$ balance equations disappear, and $M_c$ unknown slave displacements are eliminated). The master variables play a role identical to that of the Lagrangian parameters in rigid-body mechanics, i.e. they describe the most general configuration that is admissible with the constrains.

The true difficulty of the problem, however, consists of solving the constraint equations. Since they are *nonlinear* equations, they can rarely be tackled in the exact form, but, in contrast, a perturbation procedure must be applied, by resorting to series expansions. There is, however, another problem that makes the elimination of the variables difficult, due to the fact that the constraints are differential (and not algebraic!) equations[16]. In lucky cases in which a not-differentiated variable $w_i$ appears in one equation, we can solve this equation (maybe, by means of a perturbation method) with respect to this variable, by using *algebraic* operations only, thus obtaining $w_i = f\left(w_j, w_j'\right)$ with $j \neq i$. If, in contrast, only first derivatives appear in the constraint equation, we could find $w_i' = f\left(w_j'\right)$ still using algebra. However, if $w_i$ is needed, for example to evaluate inertia forces proportional to $\ddot{w}_i$, we should integrate, thus obtaining $w_i = \int f\left(w_j'\right) ds$, and also using geometric boundary conditions. In these circumstances, elimination of the variables could be inconvenient, and the mixed formulation would be preferable. Hybrid procedures, in

16. A similar circumstance occurs in Lagrangian mechanics, when *non-holonomic constraints* exist, which involve time-differentiated displacements.

which only a sub-set of the variables is eliminated, are also possible, and relevant examples will be illustrated further on in the book (Chapter 4).

**Condensation of displacements: master and slave variables**

Let us assume that *all* the $M_c$ constraint equations $\boldsymbol{\mathcal{E}}_c(\boldsymbol{w}, \boldsymbol{w}') = \mathbf{0}$ can be solved with respect to $M_c$ *slave variables* $\boldsymbol{w}_s$, i.e. $\boldsymbol{w}_s = \boldsymbol{\mathcal{W}}_s(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots)$, where $\boldsymbol{w}_m$ are the remaining $N_m = N - M_c$ *master variables*; therefore, $\boldsymbol{w} := (\boldsymbol{w}_m, \boldsymbol{w}_s)^T = (\boldsymbol{w}_m, \boldsymbol{\mathcal{W}}_s(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots))^T$ or, in short[17]:

$$\boldsymbol{w} = \boldsymbol{\mathcal{W}}(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots) \qquad [1.42]$$

The generic configuration of the internally constrained beam is thus described by master configuration variables only. We will call this (nonlinear) relation the *constraint for displacements*, and we will say that the slave displacements have been condensed.

By time-differentiating the previous equation, we obtain the more general velocity field that is admissible for the instantaneous constraints, i.e.[18]:

$$\dot{\boldsymbol{w}} = \boldsymbol{A}(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots)\,\dot{\boldsymbol{w}}_m \qquad [1.43]$$

which, therefore, represents a (linear) *constraint for velocities*; in it, by omitting the arguments:

$$\boldsymbol{A} := \frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}_m} + \frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}'_m}\frac{\partial}{\partial s} + \ldots \qquad [1.44]$$

is a linear differential operator, represented by a $N \times N_m$ matrix: we will call it the *velocity constraint operator*. Since, from equation [1.23], it is $\boldsymbol{D}_c\dot{\boldsymbol{w}} = \mathbf{0}$, $\forall \boldsymbol{w}_m$, it follows from equation [1.43], that $\boldsymbol{D}_c\boldsymbol{A} = \mathbf{0}$.

With equations [1.42], and [1.43], the strain–displacements relationship (upper part of equation [1.22]) transforms into:

$$\boldsymbol{\varepsilon}_u = \boldsymbol{\mathcal{E}}_u(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots) \qquad [1.45]$$

to be sided by geometric boundary conditions:

$$\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad \boldsymbol{\mathcal{W}}_{sH}(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots) = \breve{\boldsymbol{w}}_{sH} \qquad [1.46]$$

17. The high-order derivatives of $\boldsymbol{w}_m$ have been denoted by ellipsis. As an example of the appearance of these terms, the equation $u_s - u'_m - u'^2_s = 0$ admits the solution $u_s = u'_m + u''^2_m + \text{h.o.t.}$

18. Note that $\boldsymbol{A} := (\boldsymbol{I}_m, \boldsymbol{A}_s)$, $\boldsymbol{I}_m$ being the identity matrix of order $N_m$.

where the know-terms can be freely imposed. Similarly, the strain-rate-velocity relationship (upper part of equation [1.23]) becomes:

$$\dot{\boldsymbol{\varepsilon}}_u = \boldsymbol{D}_u \boldsymbol{A} \dot{\boldsymbol{w}}_m \qquad [1.47]$$

where $\boldsymbol{D}_u = \boldsymbol{D}_u \left( \boldsymbol{\mathcal{W}} \left( \boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots \right), \boldsymbol{\mathcal{W}}' \left( \boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots \right) \right) = \boldsymbol{D}_u \left( \boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots \right)$. We will refer to equations [1.45] and [1.47] as the *condensed kinematic relationships*.

REMARK 1.4. The condensation of the slave displacements leads to the appearance of the second- and high-order space-derivatives in the field, and first- and high-order space-derivatives at the boundaries.

### Condensation of the balance equations

Now, we address the problem of condensation of the reactive stresses by the power balance approach. First, we rewrite the VPP in the form [1.25] we already used in the mixed formulation:

$$\int_{\mathcal{S}} \sigma_u^T \boldsymbol{D}_u \dot{\boldsymbol{w}} ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \quad \forall \dot{\boldsymbol{w}} | \boldsymbol{D}_c \dot{\boldsymbol{w}} = \boldsymbol{0} \qquad [1.48]$$

Differently from that approach, however, we *will not introduce Lagrange multipliers* to express the geometrical constraints but, rather, we will use the master variables to identically satisfy them, via, $\dot{\boldsymbol{w}} = \boldsymbol{A} \dot{\boldsymbol{w}}_m$ (equation [1.43]). Since the reactive stresses do not appear in the VPP, the principle furnishes balance equations purely in the active stresses.

From a computational point of view, we find it conceptually clearer to reach the goal in two steps: (a) first, we integrate by parts equation [1.48] to free $\dot{\boldsymbol{w}}$ from the derivatives; and (b) then, we substitute the velocity constraint, and integrate again by parts, to free $\dot{\boldsymbol{w}}_m$ from the derivatives. By performing the first integration, we obtain:

$$\int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \left( \boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p} \right) ds + \sum_{H=A}^{B} \left[ \dot{\boldsymbol{w}}^T \left( \boldsymbol{\mathcal{D}}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{P} \right) \right]_H = 0 \quad \forall \dot{\boldsymbol{w}} | \boldsymbol{D}_c \dot{\boldsymbol{w}} = \boldsymbol{0} \qquad [1.49]$$

where we used the extended Green identity [1.8] for the $u$-parts of the operators and stress[19]. Then, in order to satisfy the constraints, we express the velocities in terms of

19. Note that equation [1.49] could be directly obtained as a linear combinations of the split balance equations [1.28] and boundary conditions [1.29], *simply by ignoring the reactive stresses*. Therefore, if we already know the equilibrium equations for an unconstrained model, we can follow this shortcut, thus avoiding the first integration by parts.

master velocities, namely $\dot{\boldsymbol{w}} = \boldsymbol{A}\dot{\boldsymbol{w}}_m$ in the field, and $\dot{\boldsymbol{w}} := (\dot{\boldsymbol{w}}_m, \dot{\boldsymbol{w}}_s)^T = \left(\dot{\boldsymbol{w}}_m, \dot{\boldsymbol{\mathcal{W}}}_s(\boldsymbol{w}_m, \ldots)\right)^T$ on the boundary, thus obtaining:

$$\int_{\mathcal{S}} (\boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p})^T \boldsymbol{A}\dot{\boldsymbol{w}}_m ds + \sum_{H=A}^{B} \left[\dot{\boldsymbol{w}}_m^T (\boldsymbol{\mathcal{D}}_{um}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_m) + \dot{\boldsymbol{\mathcal{W}}}_s^T (\boldsymbol{\mathcal{D}}_{us}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_s)\right]_H = 0 \quad \forall \dot{\boldsymbol{w}}_m \tag{1.50}$$

where the partitions $\boldsymbol{\mathcal{D}}_u^\star := (\boldsymbol{\mathcal{D}}_{um}^\star, \boldsymbol{\mathcal{D}}_{us}^\star)^T$, $\boldsymbol{P} = (\boldsymbol{P}_m, \boldsymbol{P}_s)^T$ have been introduced[20].

Now, a second integration by parts is needed, involving the $\boldsymbol{A}$ operator, whose relevant extended green identity is written as[21]:

$$\int_{\mathcal{S}} \boldsymbol{p}_c^T \boldsymbol{A}\dot{\boldsymbol{w}}_m ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}_m^T \boldsymbol{A}^\star \boldsymbol{p}_c ds + \sum_{H=A}^{B} \left[\dot{\boldsymbol{w}}_m^T \boldsymbol{\mathcal{A}}^\star \boldsymbol{p}_c\right]_H \tag{1.51}$$

where $\boldsymbol{A}^\star$(of dimensions $N_m \times N$) is the adjoint of $\boldsymbol{A}$, and $\boldsymbol{\mathcal{A}}_H^\star$ (also of dimensions $N_m \times N$) is the associated operator at the boundary, to be referred to as *the equilibrium condensation operators*. By remembering the expression [1.44] for the velocity constraint operator, we get[22]:

$$\begin{aligned} \boldsymbol{A}^\star &:= \left(\frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}_m} - \frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}'_m}\right) - \frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}'_m}\frac{\partial}{\partial s} + \ldots\right)^T \\ \boldsymbol{\mathcal{A}}_H^\star &:= \mp \left(\frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}'_m} + \ldots\right)_H^T \end{aligned} \tag{1.52}$$

With equation [1.51], the VPP [1.50] reads:

$$\begin{aligned} &\int_{\mathcal{S}} \dot{\boldsymbol{w}}_m^T \boldsymbol{A}^\star (\boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p}) ds + \sum_{H=A}^{B} \left[\dot{\boldsymbol{w}}_m^T \boldsymbol{\mathcal{A}}^\star (\boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p})\right]_H \\ &+ \sum_{H=A}^{B} \left[\dot{\boldsymbol{w}}_m^T (\boldsymbol{\mathcal{D}}_{um}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_m) + \dot{\boldsymbol{\mathcal{W}}}_s^T (\boldsymbol{\mathcal{D}}_{us}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_s)\right]_H = 0 \qquad \forall \dot{\boldsymbol{w}}_m \end{aligned} \tag{1.53}$$

20. Note that $\dot{\boldsymbol{\mathcal{W}}}_s = \frac{\partial \boldsymbol{\mathcal{W}}_s}{\partial \boldsymbol{w}_m}\dot{\boldsymbol{w}}_m + \ldots$ also depends on $\dot{\boldsymbol{w}}_m$.

21. Here $\boldsymbol{p}_c$ is a dummy variable assuming the meaning of "external constraint force" $\boldsymbol{p}_c := \boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p} = \boldsymbol{D}_c^\star \boldsymbol{\sigma}_c$.

22. Note the analogy between equations [1.52] and [1.9].

By taking into account that the velocities $\dot{\boldsymbol{w}}_m$ are arbitrary, the previous principle supplies the field equations:

$$\boldsymbol{A}^\star \boldsymbol{D}_u^\star \boldsymbol{\sigma}_u = \boldsymbol{A}^\star \boldsymbol{p} \qquad [1.54]$$

in which the two members represent Lagrange internal and external forces, respectively.

Consistently with the geometrical boundary conditions [1.46] (in which slave and master variables have been separated), the boundary terms in equation [1.53] also separate in:

$$\begin{aligned} &\left[\dot{\boldsymbol{w}}_m^T \left(\boldsymbol{\mathcal{A}}^\star \left(\boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p}\right) + \left(\boldsymbol{\mathcal{D}}_{um}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_m\right)\right)\right]_H = \mathbf{0} \\ &\left[\dot{\boldsymbol{\mathcal{W}}}_s^T \left(\boldsymbol{\mathcal{D}}_{us}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_s\right)\right]_H = \mathbf{0} \end{aligned} \qquad [1.55]$$

Equations [1.54] and [1.55] are the condensed equations sought for. The example of section 1.7 shows an application to a well-known linear problem of Timoshenko beam, with the purpose to corroborate the understanding of the theory illustrated here.

REMARK 1.5. The condensation of the reactive stresses leads to the appearance of space-derivatives of the loads, in the field equation [1.54] and to higher-order derivatives of the stresses in the equilibrium operator; moreover, it brings a contribution of the field load to the free boundaries (equation [1.55]).

REMARK 1.6. The condensed kinematic operator, $\boldsymbol{D}_u \boldsymbol{A}$ (equation [1.45]), and the condensed equilibrium operator, $\boldsymbol{A}^\star \boldsymbol{D}_u^\star$ (equation [1.54a], are mutually adjoint. Moreover, since $\boldsymbol{D}_c \boldsymbol{A} = \mathbf{0}$, then even $\boldsymbol{A}^\star \boldsymbol{D}_c^\star \equiv (\boldsymbol{D}_c \boldsymbol{A})^\star = \mathbf{0}$, thus explaining how $\boldsymbol{A}^\star$ annihilates $\boldsymbol{\sigma}_c$, and therefore the reactive stresses.

**The constrained Fundamental Problem: the displacement formulation**

By summarizing, the Fundamental Problem for the constrained beam, when formulated in terms of displacements $\boldsymbol{w}_m$ only, is governed by the following field equations:

– the condensed strain–displacement relationships [1.45];

– the condensed balance equations [1.54];

– the elastic law [1.30].

They are equipped with the alternative boundary conditions [1.55] and the geometric boundary condition [1.46]. By combining the field equations, we can express the balance equations in terms of the master displacements, namely:

$$\boldsymbol{A}^\star \boldsymbol{D}_u^\star \boldsymbol{E}_{uu} \boldsymbol{\mathcal{E}}_u = \boldsymbol{A}^\star \boldsymbol{p} \qquad [1.56]$$

The boundary conditions, when handled in the same way, read:

$$\begin{aligned}&[\boldsymbol{\mathcal{A}}^{\star}\boldsymbol{D}_u^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\boldsymbol{\mathcal{D}}_{um}^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u]_H=[\boldsymbol{P}_m+\boldsymbol{\mathcal{A}}^{\star}\boldsymbol{p}]_H\\&[\boldsymbol{\mathcal{D}}_{us}^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u]_H=[\boldsymbol{P}_s]_H\\&\boldsymbol{w}_{mH}=\breve{\boldsymbol{w}}_{mH},\quad \boldsymbol{\mathcal{W}}_{sH}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)=\breve{\boldsymbol{w}}_{sH}\end{aligned}\qquad [1.57]$$

**The linear theory**

If equations [1.56] and [1.57] are linearized around the reference configuration (and use is made of equation [1.47]), they read:

$$\boldsymbol{A}_0^{\star}\boldsymbol{L}_u\boldsymbol{A}_0\boldsymbol{w}_m=\boldsymbol{A}_0^{\star}\boldsymbol{p}_0\qquad [1.58]$$

together with:

$$\begin{aligned}&[\boldsymbol{\mathcal{A}}_0^{\star}\boldsymbol{L}_u\boldsymbol{A}_0+\boldsymbol{\mathcal{L}}_{um}\boldsymbol{A}_0]_H\,\boldsymbol{w}_m=[\boldsymbol{P}_{0m}+\boldsymbol{\mathcal{A}}_0^{\star}\boldsymbol{p}_0]_H\\&[\boldsymbol{\mathcal{L}}_{us}\boldsymbol{A}_0]_H\,\boldsymbol{w}_m=[\boldsymbol{P}_{0s}]_H\\&\boldsymbol{w}_{mH}=\breve{\boldsymbol{w}}_{mH},\quad [\boldsymbol{A}_{0s}\boldsymbol{w}_m]_H=\breve{\boldsymbol{w}}_{sH}\end{aligned}\qquad [1.59]$$

where the index 0 denotes evaluation at $\boldsymbol{w}_m=\boldsymbol{0}$, and moreover:

$$\begin{aligned}&\boldsymbol{L}_u:=\boldsymbol{D}_{0u}^{\star}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u},\qquad \boldsymbol{\mathcal{L}}_{umH}:=[\boldsymbol{\mathcal{D}}_{0um}^{\star}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}]_H\\&\boldsymbol{\mathcal{L}}_{usH}:=[\boldsymbol{\mathcal{D}}_{0us}^{\star}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}]_H\end{aligned}\qquad [1.60]$$

are condensed linear elastic operators, in the domain and at the boundary, with $\boldsymbol{L}_u$ self-adjoint (remember equation [1.41]). In the last of the boundary conditions [1.59], $\boldsymbol{\mathcal{W}}=\boldsymbol{A}_0\boldsymbol{w}_m+$ h.o.t. has been used by exploiting equation [1.43] with the partition $\boldsymbol{A}_0:=(\boldsymbol{I}_m,\boldsymbol{A}_{0s})^T$.

**Evaluation of the reactive stresses**

Differently from the mixed formulation, in which the reactive stresses are included in the set of the primary unknowns, in the displacement formulations, they have to be evaluated by the balance equations, *after* the boundary value problem in the master displacements has been solved. First, the active stresses $\boldsymbol{\sigma}_u=\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u$ are evaluated, and then the split balance equations [1.28] and [1.29] written in the form:

$$\begin{aligned}&\boldsymbol{D}_c^{\star}\boldsymbol{\sigma}_c=\boldsymbol{p}-\boldsymbol{D}_u^{\star}\boldsymbol{\sigma}_u\\&\boldsymbol{\mathcal{D}}_{cH}^{\star}\boldsymbol{\sigma}_c=\boldsymbol{P}_H-\boldsymbol{\mathcal{D}}_{uH}^{\star}\boldsymbol{\sigma}_u\end{aligned}\qquad [1.61]$$

where $\boldsymbol{\sigma}_c$ are the unknowns. These equations represent the equilibrium of an ideal "rigid skeleton" of the beam, able to exert only reactive stresses $\boldsymbol{\sigma}_c$, under the action of now *known* external and internal active forces. Such a problem, however, is overdetermined, since we have $N$ equilibrium equations in $M_c < N$ unknowns. Therefore, in order it admits a solution, the knownn terms must satisfy a compatibility condition, i.e. *they must be orthogonal to all the solutions of the adjoint homogeneous problem*[23]. Since the latter is just $\boldsymbol{D}_c\dot{\boldsymbol{w}} = \boldsymbol{0}$, the know-term must be orthogonal to $\dot{\boldsymbol{w}} = \boldsymbol{A}\dot{\boldsymbol{w}}_m,\ \forall\dot{\boldsymbol{w}}_m$, and therefore compatibility requires[24]:

$$\int_{\mathcal{S}} \left(\boldsymbol{p} - \boldsymbol{D}_u^{\star}\boldsymbol{\sigma}_u\right)^T \boldsymbol{A}\dot{\boldsymbol{w}}_m ds + \sum_{H=A}^{B} \left[\left(\boldsymbol{P} - \boldsymbol{\mathcal{D}}_u^{\star}\boldsymbol{\sigma}_u\right)^T \boldsymbol{A}\dot{\boldsymbol{w}}_m\right]_H = 0 \qquad \forall\dot{\boldsymbol{w}}_m \quad [1.62]$$

This expresses that *the difference between the virtual powers spent by the active stresses and the external forces in any admissible velocity field is zero*. But this equation is exactly equation [1.50], which has been already satisfied in formulating the problem. Therefore, equations [1.61], although overdetermined, are integrable because they are not linearly independent (see, again, the example of section 1.7).

## 1.4 Internally unconstrained prestressed beams

Usually, as already observed, the reference configuration $\boldsymbol{\varepsilon} = \boldsymbol{0}$ is assumed to be stress-free. However, there exist problems in which it is more convenient to refer to a configuration in which the body is solicited by time-independent preloads, causing a state of prestress. Buckling falls in this class of problems; another set of problems in which prestress is important concerns strings and cables.

Of course, a prestressed beam is also prestrained, this entailing a change of geometry with respect to its natural state. Thus, for example, if the prestress of straight beam is caused by an axial load, the beam is shortened with respect to its natural length; if the beam is transversely loaded, it is bent, twisted and shear-strained; if, for example, the beam has an initial uniform curvature, this is rendered non-uniform by preloads. Strictly speaking, such changes of geometry should be accounted for in analyzing the mechanical behavior of the beam, when incremental loads, additional with respect to preloads, are applied. Such an approach, however, would nullify the advantage to refer to a prestressed configuration because the relevant geometry would be more complex than the natural geometry. Therefore, in order to simplify the problem, *prestrains and deformations produced by preloads are neglected*, so that the reference prestressed configuration is confused with the

23. This property, known in functional analysis (Fredholm alternative), can be considered as a straightforward extension of the Rouché–Capelli theorem, holding in Algebra.

24. Note that we did not account for *any* geometric boundary condition in evaluating $\dot{\boldsymbol{w}}_m$, in order to make the treatment as general as possible.

natural configuration. In other words, the geometrical effects caused by the prestress are ignored.

### 1.4.1 The nonlinear theory

With these ideas in mind, let us consider a beam under time-independent *preloads* $\mathring{\boldsymbol{p}}(\boldsymbol{w})$, $\mathring{\boldsymbol{P}}_H(\boldsymbol{w})$, possibly dependent on the configuration, equilibrated with *prestresses* $\mathring{\boldsymbol{\sigma}}(s)$, and assume the equilibrium configuration as a *known* reference configuration. As a result, the following equations hold:

$$\begin{aligned} \boldsymbol{D}_0^\star \mathring{\boldsymbol{\sigma}} &= \mathring{\boldsymbol{p}}_0 \\ \boldsymbol{\mathcal{D}}_{0H}^\star \mathring{\boldsymbol{\sigma}} &= \mathring{\boldsymbol{P}}_{0H} \end{aligned} \qquad [1.63]$$

where the equilibrium operators and the loads have been evaluated at $\boldsymbol{w} \equiv \boldsymbol{0}$. Let us assume, then, that at time $t = 0$, *incremental loads* $\tilde{\boldsymbol{p}}(\boldsymbol{w})$, $\tilde{\boldsymbol{P}}_H(\boldsymbol{w})$ are applied to the beam. These loads bring the beam to occupy a new (possibly time-dependent) current configuration, described by the generalized displacements $\boldsymbol{w}(s,t)$, measured with respect to the reference configuration. Accordingly, kinematics is still governed by equations [1.1] and [1.2], that we repeat here:

$$\begin{aligned} \boldsymbol{\varepsilon} &= \boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right) \\ \boldsymbol{w}_H &= \breve{\boldsymbol{w}}_H \end{aligned} \qquad [1.64]$$

Similarly, the balance equations are still equations [1.11] and [1.12], but with *total loads* applied, accounting for the current values of the preloads:

$$\begin{aligned} \boldsymbol{D}^\star\left(\boldsymbol{w}, \boldsymbol{w}'\right)\boldsymbol{\sigma} &= \mathring{\boldsymbol{p}}(\boldsymbol{w}) + \tilde{\boldsymbol{p}}(\boldsymbol{w}) \\ \boldsymbol{\mathcal{D}}_H^\star\left(\boldsymbol{w}, \boldsymbol{w}'\right)\boldsymbol{\sigma} &= \mathring{\boldsymbol{P}}_H(\boldsymbol{w}) + \tilde{\boldsymbol{P}}_H(\boldsymbol{w}) \end{aligned} \qquad [1.65]$$

Concerning the elastic law, we have to modify the Hook law in order to get $\boldsymbol{\sigma} = \mathring{\boldsymbol{\sigma}}$ when $\boldsymbol{\varepsilon} = \boldsymbol{0}$. This is accomplished by considering an elastic potential represented by a *complete* quadratic polynomial[25]:

$$\phi(\boldsymbol{\varepsilon}) = \phi_0 + \mathring{\boldsymbol{\sigma}}^T\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{E}\boldsymbol{\varepsilon} \qquad [1.66]$$

Then, a linear non-homogeneous law follows from equation [1.14]:

$$\boldsymbol{\sigma} = \mathring{\boldsymbol{\sigma}} + \boldsymbol{E}\boldsymbol{\varepsilon} \qquad [1.67]$$

25. The constant $\phi_0$ is unessential, since a potential function is always defined to within a constant. Therefore, we will omit it further on.

The governing equations can be combined according to the displacement method, as done for the stress-free beam. The stress–displacement relationships then read:

$$\boldsymbol{\sigma} = \mathring{\boldsymbol{\sigma}} + \boldsymbol{E}\boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}') \tag{1.68}$$

and, as a result, the balance equations and the mechanical boundary conditions transform into:

$$\begin{aligned} &\boldsymbol{D}^{\star}\boldsymbol{E}\boldsymbol{\mathcal{E}} + (\boldsymbol{D}^{\star}\mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{p}}) = \tilde{\boldsymbol{p}} \\ &\boldsymbol{\mathcal{D}}_H^{\star}\boldsymbol{E}\boldsymbol{\mathcal{E}}_H + \left(\boldsymbol{\mathcal{D}}_H^{\star}\mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{P}}_H\right) = \tilde{\boldsymbol{P}}_H \end{aligned} \tag{1.69}$$

Equations [1.69] and the geometric boundary conditions [1.64b] constitute a nonlinear boundary value problem for the main unknowns $\boldsymbol{w}$.

REMARK 1.7. Equations [1.69] state that the incremental loads $\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{P}}_H$ are equilibrated not only by the incremental elastic forces, as happens in stress-free beams, but also by the imbalance between preloads and prestresses, which is *caused by the change of geometry*.

### 1.4.2 The linearized theory

Very often, in buckling problems, we are interested in determining the critical load only, or the response of the beam to *small* incremental loads, acting as disturbances/imperfections of the prestressed equilibrium configuration, mostly when the beam is close to the bifurcation. Similarly, in dynamics, we want to evaluate the frequencies of a prestressed beam or cable, or the response of the structure when *small* incremental loads externally excite the beam, especially when this is close to the resonance. In all these cases, the linearized version of equation [1.69] is sufficient to give an accurate response (i.e. to within the effects of the neglected prestrains), leading to a differential eigenvalue problem (for critical load or frequencies) or a non-homogeneous boundary value problem (for small incremental loads). The relevant framework is called *Linearized Theory*[26].

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26. We use the wordings *linearized* theory for prestressed beams, and *linear* theory for stress-free beams.

To linearize equations [1.69], we have to move one order ahead with respect to the series expansions [1.19][27]. Concerning the field equations, we have[28]:

$$\begin{aligned}&\boldsymbol{D}^{\star}\left(\boldsymbol{w},\boldsymbol{w}'\right)\mathring{\boldsymbol{\sigma}}=\boldsymbol{D}_{0}^{\star}\mathring{\boldsymbol{\sigma}}+\left(\frac{\partial\left(\boldsymbol{D}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}}\right)_{0}\boldsymbol{w}+\left(\frac{\partial\left(\boldsymbol{D}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}'}\right)_{0}\boldsymbol{w}'+\text{h.o.t.}\\&\mathring{\boldsymbol{p}}(\boldsymbol{w})=\mathring{\boldsymbol{p}}_{0}+\left(\frac{\partial\mathring{\boldsymbol{p}}}{\partial\boldsymbol{w}}\right)_{0}\boldsymbol{w}+\text{h.o.t.},\qquad\tilde{\boldsymbol{p}}(\boldsymbol{w})=\tilde{\boldsymbol{p}}_{0}+\text{h.o.t.}\end{aligned}\qquad[1.70]$$

in which we assumed $\tilde{\boldsymbol{p}}(\boldsymbol{w})$ and $\boldsymbol{w}$ small of the same order. Similarly, for the boundary conditions, we have:

$$\begin{aligned}&\boldsymbol{\mathcal{D}}_{H}^{\star}\left(\boldsymbol{w},\boldsymbol{w}'\right)\mathring{\boldsymbol{\sigma}}=\boldsymbol{\mathcal{D}}_{0H}^{\star}\mathring{\boldsymbol{\sigma}}+\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_{H}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}}\right)_{0}\boldsymbol{w}+\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_{H}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}'}\right)_{0}\boldsymbol{w}'+\text{h.o.t.}\\&\mathring{\boldsymbol{P}}_{H}(\boldsymbol{w})=\mathring{\boldsymbol{P}}_{0H}+\left(\frac{\partial\mathring{\boldsymbol{P}}_{H}}{\partial\boldsymbol{w}}\right)_{0}\boldsymbol{w}+\text{h.o.t.},\qquad\tilde{\boldsymbol{P}}_{H}(\boldsymbol{w})=\tilde{\boldsymbol{P}}_{0H}+\text{h.o.t.}\end{aligned}\qquad[1.71]$$

By retaining first-order terms only in the series expansions, and accounting for the equilibrium conditions [1.63] of the prestressed configuration, we obtain:

$$\begin{aligned}&\boldsymbol{L}\boldsymbol{w}+\boldsymbol{G}\boldsymbol{w}=\tilde{\boldsymbol{p}}_{0}\\&\boldsymbol{\mathcal{L}}_{H}\boldsymbol{w}+\boldsymbol{\mathcal{G}}_{H}\boldsymbol{w}=\tilde{\boldsymbol{P}}_{0}\end{aligned}\qquad[1.72]$$

where $\boldsymbol{L}$ and $\boldsymbol{\mathcal{L}}_H$ are the already introduced elastic stiffness operators of the linear theory (equation [1.21]), and:

$$\begin{aligned}&\boldsymbol{G}:=\left(\frac{\partial\left(\boldsymbol{D}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}}\right)_{0}+\left(\frac{\partial\left(\boldsymbol{D}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}'}\right)_{0}\frac{\partial}{\partial s}-\left(\frac{\partial\mathring{\boldsymbol{p}}}{\partial\boldsymbol{w}}\right)_{0}\\&\boldsymbol{\mathcal{G}}_{H}:=\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_{H}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}}\right)_{0}+\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_{H}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}'}\right)_{0}\frac{\partial}{\partial s}-\left(\frac{\partial\mathring{\boldsymbol{P}}_{H}}{\partial\boldsymbol{w}}\right)_{0}\end{aligned}\qquad[1.73]$$

are *geometric stiffness operators*, in the domain and on the boundary, respectively, accounting for prestress.

REMARK 1.8. The geometric stiffness accounts for the effect on the equilibrium of an *infinitely small change of geometry* of the beam, when it passes from the reference to an *adjacent* current configuration. The imbalance between prestresses and preloads, $\boldsymbol{D}^{\star}\left(\boldsymbol{w},\boldsymbol{w}'\right)\mathring{\boldsymbol{\sigma}}-\mathring{\boldsymbol{p}}\left(\boldsymbol{w}\right)$, when linearized, is just $\boldsymbol{G}\boldsymbol{w}$.

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27. From this circumstance, the linearized theory is also called the "second-order theory", this being a less precise wording, often used in technical circles.

28. Although $\mathring{\boldsymbol{\sigma}}$ is independent of $\boldsymbol{w}$, we prefer to differentiate the product $(\boldsymbol{D}^{\star}\mathring{\boldsymbol{\sigma}})$, to remember that $\boldsymbol{D}^{\star}$ operates on $\mathring{\boldsymbol{\sigma}}$, and, moreover, to avoid introducing the derivative of a matrix with respect to a vector.

## The geometric stiffness operator

To obtain an expression for the geometric stiffness operator [1.73a] in terms of strains, we use equation [1.9a] and obtain[29]:

$$\begin{aligned}\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}} &= \left(\frac{\partial\boldsymbol{\mathcal{E}}}{\partial\boldsymbol{w}}\right)^T\mathring{\boldsymbol{\sigma}} - \frac{\partial}{\partial s}\left(\frac{\partial\boldsymbol{\mathcal{E}}}{\partial\boldsymbol{w}'}\right)^T\mathring{\boldsymbol{\sigma}} - \left(\frac{\partial\boldsymbol{\mathcal{E}}}{\partial\boldsymbol{w}'}\right)^T\mathring{\boldsymbol{\sigma}}' \\ &= \sum_{i=1}^{M}\left(\left(\frac{\partial\mathcal{E}_i}{\partial\boldsymbol{w}} - \frac{\partial}{\partial s}\frac{\partial\mathcal{E}_i}{\partial\boldsymbol{w}'}\right)\mathring{\sigma}_i - \frac{\partial\mathcal{E}_i}{\partial\boldsymbol{w}'}\mathring{\sigma}_i'\right)\end{aligned} \qquad [1.74]$$

From this, we can evaluate the contributions to $\boldsymbol{G}\boldsymbol{w}$, namely:

$$\begin{aligned}\left(\frac{\partial\left(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}}\right)_0\boldsymbol{w} &= \sum_{i=1}^{M}\left(\left(\boldsymbol{A}_i\boldsymbol{w} - \boldsymbol{B}_i^T\boldsymbol{w}'\right)\mathring{\sigma}_i - \boldsymbol{B}_i^T\boldsymbol{w}\mathring{\sigma}_i'\right) \\ \left(\frac{\partial\left(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}'}\right)_0\boldsymbol{w}' &= \sum_{i=1}^{M}\left(\left(\boldsymbol{B}_i\boldsymbol{w}' - \boldsymbol{C}_i\boldsymbol{w}''\right)\mathring{\sigma}_i - \boldsymbol{C}_i\boldsymbol{w}'\mathring{\sigma}_i'\right)\end{aligned} \qquad [1.75]$$

where the following matrices of the second derivatives of $\mathcal{E}_i$, evaluated at $(\boldsymbol{w},\boldsymbol{w}') = (\boldsymbol{0},\boldsymbol{0})$, have been introduced[30]:

$$\boldsymbol{A}_i := \left(\frac{\partial^2\mathcal{E}_i}{\partial\boldsymbol{w}^2}\right)_0, \qquad \boldsymbol{B}_i := \left(\frac{\partial^2\mathcal{E}_i}{\partial\boldsymbol{w}\partial\boldsymbol{w}'}\right)_0, \qquad \boldsymbol{C}_i := \left(\frac{\partial^2\mathcal{E}_i}{\partial\boldsymbol{w}'^2}\right)_0 \qquad [1.77]$$

being $\boldsymbol{A}_i = \boldsymbol{A}_i^T$, $\boldsymbol{C}_i = \boldsymbol{C}_i^T$, while $\boldsymbol{B}_i^T = \left(\frac{\partial^2\mathcal{E}_i}{\partial\boldsymbol{w}'\partial\boldsymbol{w}}\right)_0 \neq \boldsymbol{B}_i$[31]. Hence:

$$\boldsymbol{G} = \sum_{i=1}^{M}\left(\left(\boldsymbol{A}_i\mathring{\sigma}_i - \boldsymbol{B}_i^T\mathring{\sigma}_i'\right) + \left(\boldsymbol{B}_i\mathring{\sigma}_i - \boldsymbol{B}_i^T\mathring{\sigma}_i - \boldsymbol{C}_i\mathring{\sigma}_i'\right)\frac{\partial}{\partial s} - \boldsymbol{C}_i\mathring{\sigma}_i\frac{\partial^2}{\partial s^2}\right) - \left(\frac{\partial\mathring{\boldsymbol{p}}}{\partial\boldsymbol{w}}\right)_0 \qquad [1.78]$$

The stiffness operator at the boundary can be obtained in a similar manner. By using equation [1.9] we have:

$$\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}} := \mp\left(\frac{\partial\boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right)}{\partial\boldsymbol{w}'}\right)_H^T\mathring{\boldsymbol{\sigma}} \qquad [1.79]$$

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29. Note that, e.g., $\left(\frac{\partial\boldsymbol{\mathcal{E}}}{\partial\boldsymbol{w}}\right)^T$ is the column-wise matrix $\left[\frac{\partial\mathcal{E}_1}{\partial\boldsymbol{w}}, \frac{\partial\mathcal{E}_2}{\partial\boldsymbol{w}}, \ldots, \frac{\partial\mathcal{E}_M}{\partial\boldsymbol{w}}\right]$, where the derivative of a scalar with respect to a vector denotes a column vector.

30. Note that these are sub-matrices of the Hessian of $\mathcal{E}_i$ at the origin, once the variables have been ordered as $(\boldsymbol{w},\boldsymbol{w}')$:

$$\boldsymbol{H}_i^0 = \begin{pmatrix}\boldsymbol{A}_i & \boldsymbol{B}_i \\ \boldsymbol{B}_i^T & \boldsymbol{C}_i\end{pmatrix} \qquad [1.76]$$

31. Indeed, $\boldsymbol{B}_i = \left[\frac{\partial^2\mathcal{E}_i}{\partial u_j\partial u_k'}\right]_0$, $\boldsymbol{B}_i^T = \left[\frac{\partial^2\mathcal{E}_i}{\partial u_j'\partial u_k}\right]_0$ with $j$ being the row and $k$ the column.

from which:

$$\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}}\right)_{0H}\boldsymbol{w}=\mp\sum_{i=1}^{M}\boldsymbol{B}_i^T\boldsymbol{w}\mathring{\sigma}_i,\qquad\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}'}\right)_{0H}\boldsymbol{w}'=\mp\sum_{i=1}^{M}\boldsymbol{C}_i\boldsymbol{w}'\mathring{\sigma}_i' \quad [1.80]$$

and therefore, from equation [1.73b], we finally get:

$$\boldsymbol{\mathcal{G}}_H=\mp\sum_{i=1}^{M}\left(\boldsymbol{B}_i^T\mathring{\sigma}_i+\boldsymbol{C}_i\mathring{\sigma}_i\frac{\partial}{\partial s}\right)-\left(\frac{\partial\mathring{\boldsymbol{P}}}{\partial\boldsymbol{w}}\right)_{0H} \quad [1.81]$$

## 1.5 Internally constrained prestressed beams

We consider again a prestressed beam, but refer ourselves to an internally constrained model, so that all the aspects illustrated in the previous sections are involved in this more complex problem. As for the stress-free beam, we want to tackle both the mixed and displacement formulations, and as for the prestressed beam, we want to develop models in the nonlinear and linearized frameworks. Therefore, four different models are illustrated further on.

### 1.5.1 The nonlinear mixed formulation

Let us assume that the beam is in equilibrium under time-independent but configuration-dependent preloads $\mathring{\boldsymbol{p}}(\boldsymbol{w})$, $\mathring{\boldsymbol{P}}_H(\boldsymbol{w})$, and prestresses $\mathring{\boldsymbol{\sigma}}$, and ignore any deformation of the beam, so that the equilibrium configuration is confused with the reference configuration. Prestresses can be either of active or reactive type, i.e. $\mathring{\boldsymbol{\sigma}}=(\mathring{\boldsymbol{\sigma}}_u,\mathring{\boldsymbol{\sigma}}_c)^T$, and we assume they have been already determined from a prestress analysis. As a result, the equilibrium equations [1.63] hold, with the partition introduced:

$$\begin{aligned}&\boldsymbol{D}_{0u}^{\star}\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{D}_{0c}^{\star}\mathring{\boldsymbol{\sigma}}_c=\mathring{\boldsymbol{p}}_0\\&\boldsymbol{\mathcal{D}}_{0uH}^{\star}\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{\mathcal{D}}_{0cH}^{\star}\mathring{\boldsymbol{\sigma}}_c=\mathring{\boldsymbol{P}}_{0H}\end{aligned} \quad [1.82]$$

and where the index 0 denotes evaluation at the trivial configuration.

Let us consider, then, incremental loads $\tilde{\boldsymbol{p}}(\boldsymbol{w})$, $\tilde{\boldsymbol{P}}_H(\boldsymbol{w})$ applied to the beam at $t=0$. They cause the beam to assume a current unknown configuration, in which unconstrained and constrained strains are related to displacements by equations [1.22], which we repeat here:

$$\begin{aligned}\boldsymbol{\varepsilon}_u&=\boldsymbol{\mathcal{E}}_u(\boldsymbol{w},\boldsymbol{w}')\\\boldsymbol{0}&=\boldsymbol{\mathcal{E}}_c(\boldsymbol{w},\boldsymbol{w}')\end{aligned} \quad [1.83]$$

The balance equations [1.65], relevant to the current configuration, in the split form read:

$$\begin{aligned} &\boldsymbol{D}_u^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{\sigma}_u+\boldsymbol{D}_c^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{\sigma}_c=\mathring{\boldsymbol{p}}(\boldsymbol{w})+\tilde{\boldsymbol{p}}(\boldsymbol{w})\\ &\boldsymbol{\mathcal{D}}_{uH}^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{\sigma}_a+\boldsymbol{\mathcal{D}}_{cH}^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{\sigma}_c=\mathring{\boldsymbol{P}}_H(\boldsymbol{w})+\tilde{\boldsymbol{P}}_H(\boldsymbol{w}) \end{aligned} \qquad [1.84]$$

The constitutive law is non-homogeneous, as equation [1.67], but it concerns only the active stresses, as equation [1.30], namely:

$$\boldsymbol{\sigma}_u=\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{E}_{uu}\boldsymbol{\varepsilon}_u \qquad [1.85]$$

By using the previous equations, we write the balance equations and the mechanical boundary conditions in terms of displacements and incremental reactive stresses:

$$\tilde{\boldsymbol{\sigma}}_c:=\boldsymbol{\sigma}_c-\mathring{\boldsymbol{\sigma}}_c \qquad [1.86]$$

Moreover, we append to them the constraints and the geometric boundary conditions. Hence, the final boundary value problem consists of the following field equations:

$$\begin{aligned} &\boldsymbol{D}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\boldsymbol{D}_c^\star\tilde{\boldsymbol{\sigma}}_c+\left(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}}-\mathring{\boldsymbol{p}}\right)=\tilde{\boldsymbol{p}}\\ &\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{0} \end{aligned} \qquad [1.87]$$

and the boundary conditions[32]:

$$\begin{aligned} &\boldsymbol{\mathcal{D}}_{uH}^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_{uH}+\boldsymbol{\mathcal{D}}_{cH}^\star\tilde{\boldsymbol{\sigma}}_c+\left(\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}}-\mathring{\boldsymbol{P}}_H\right)=\tilde{\boldsymbol{P}}_H\\ &\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H \end{aligned} \qquad [1.88]$$

Comparison with equations [1.37] and [1.38], relevant to the unprestressed beam, highlights the contribution of the prestress.

### 1.5.2 The linearized mixed formulation

To linearize equations [1.87a] and [1.88a], we use the series expansions [1.70] and [1.71] and assume incremental reactive stresses and loads to be small first-order quantities; moreover, we replace the constraint condition by its first-order approximation. Thus, we get the linearized equations (compare them with

32. Note that the prestresses have been merged, via $\boldsymbol{D}_u^\star\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{D}_c^\star\mathring{\boldsymbol{\sigma}}_c=\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}}$ and $\boldsymbol{\mathcal{D}}_{uH}^\star\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{\mathcal{D}}_{cH}^\star\mathring{\boldsymbol{\sigma}}_c=\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}}$, since they are known terms in this analysis.

equations [1.39] and [1.40], relevant to the linear theory of the unprestressed beam, and with equations [1.72], relevant to the linearized theory of the prestressed unconstrained beam):

$$\left(\begin{pmatrix} \boldsymbol{L}_u & \boldsymbol{D}_{0c}^{\star} \\ \boldsymbol{D}_{0c} & \boldsymbol{0} \end{pmatrix} + \begin{pmatrix} \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\right) \begin{pmatrix} \boldsymbol{w} \\ \tilde{\boldsymbol{\sigma}}_c \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{p}}_0 \\ \boldsymbol{0} \end{pmatrix} \qquad [1.89]$$

with the mechanical boundary conditions:

$$\boldsymbol{\mathcal{L}}_{uH}\boldsymbol{w} + \boldsymbol{\mathcal{G}}_H\boldsymbol{w} + \boldsymbol{\mathcal{D}}_{0cH}^{\star}\tilde{\boldsymbol{\sigma}}_c = \tilde{\boldsymbol{P}}_{0H} \qquad [1.90]$$

where $\boldsymbol{L}_u,\ \boldsymbol{\mathcal{L}}_{uH}$ are defined in equations [1.41], and $\boldsymbol{G},\ \boldsymbol{\mathcal{G}}_H$ in equations [1.73].

### 1.5.3 The nonlinear displacement formulation

If we follow the displacement formulation, we have to condense slave displacements and reactive stresses, as we did for the stress-free beam. Kinematics is governed by the condensed strain–displacement relationships [1.45] and geometric boundary conditions [1.46], i.e.:

$$\begin{aligned} \boldsymbol{\varepsilon}_u &= \boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right) \\ \boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, &\quad \boldsymbol{\mathcal{W}}_{sH}\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right) = \breve{\boldsymbol{w}}_{sH} \end{aligned} \qquad [1.91]$$

Equilibrium is governed by equations [1.54] and boundary conditions by equations [1.55], provided total loads $\boldsymbol{p} := \mathring{\boldsymbol{p}} + \tilde{\boldsymbol{p}},\ \boldsymbol{P}_H := \mathring{\boldsymbol{P}}_H + \tilde{\boldsymbol{P}}_H$ are taken into account, i.e.:

$$\boldsymbol{A}^{\star}\boldsymbol{D}_u^{\star}\boldsymbol{\sigma}_u = \boldsymbol{A}^{\star}\left(\mathring{\boldsymbol{p}} + \tilde{\boldsymbol{p}}\right) \qquad [1.92]$$

and:

$$\begin{aligned} &\left[\left(\boldsymbol{\mathcal{A}}^{\star}\left(\boldsymbol{D}_u^{\star}\boldsymbol{\sigma}_u - \mathring{\boldsymbol{p}} - \tilde{\boldsymbol{p}}\right) + \left(\boldsymbol{\mathcal{D}}_{um}^{\star}\boldsymbol{\sigma}_u - \mathring{\boldsymbol{P}}_m - \tilde{\boldsymbol{P}}_m\right)\right)\right]_H = \boldsymbol{0} \\ &\left[\left(\boldsymbol{\mathcal{D}}_{us}^{\star}\boldsymbol{\sigma}_u - \mathring{\boldsymbol{P}}_s - \tilde{\boldsymbol{P}}_s\right)\right]_H = \boldsymbol{0} \end{aligned} \qquad [1.93]$$

where all the operators and loads depend on $\boldsymbol{w}_m$ and its derivatives, via $\boldsymbol{w} = \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right)$ (equation [1.42]). Finally, the constitutive law is given by equation [1.85], i.e.:

$$\boldsymbol{\sigma}_u = \mathring{\boldsymbol{\sigma}}_u + \boldsymbol{E}_{uu}\boldsymbol{\varepsilon}_u \qquad [1.94]$$

Combination of the previous field equations leads to (compare them with equation [1.56], where prestress was absent):

$$\boldsymbol{A}^{\star}\boldsymbol{D}_u^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{A}^{\star}\left(\boldsymbol{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{p}}\right) = \boldsymbol{A}^{\star}\tilde{\boldsymbol{p}} \qquad [1.95]$$

and the relevant boundary conditions (compare with [1.57]):

$$\begin{aligned}
&\left[\boldsymbol{\mathcal{A}}^\star \boldsymbol{D}_u^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{\mathcal{D}}_{um}^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{\mathcal{A}}^\star\left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{p}}\right) + \left(\boldsymbol{\mathcal{D}}_{um}^\star \mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}_m\right)\right]_H \\
&= \left[\tilde{\boldsymbol{P}}_m + \boldsymbol{\mathcal{A}}^\star \tilde{\boldsymbol{p}}\right]_H \\
&\left[\boldsymbol{\mathcal{D}}_{us}^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \left(\boldsymbol{\mathcal{D}}_{us}^\star \mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}_s\right)\right]_H = \left[\tilde{\boldsymbol{P}}_s\right]_H \\
&\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad \boldsymbol{\mathcal{W}}_{sH}\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right) = \breve{\boldsymbol{w}}_{sH}
\end{aligned} \qquad [1.96]$$

Once the problem has been solved, the total reactive stresses follow from the not-condensed equilibrium equations.

REMARK 1.9. Equations [1.95] and [1.96], as equations [1.69], contain unbalanced preloads–prestress forces. These, however, differently from the previous formulations, are expressed in terms of active forces only, since premultiplication by $\boldsymbol{A}^\star$ filters the reactive contributions.

### 1.5.4 The linearized displacement formulation

Linearization of equations [1.95] and [1.96] calls for using series expansions of all operators and loads. Elastic terms and incremental loads can been dealt with as for the unprestressed beam (equations [1.58] and [1.59]). Additional geometric terms arise from imbalanced prestresses and preloads, requiring expansion of the $u$-parts of the equilibrium operators,$\boldsymbol{D}_u^\star$, $\boldsymbol{\mathcal{D}}_u^\star = \left(\boldsymbol{\mathcal{D}}_{um}^\star, \boldsymbol{\mathcal{D}}_{us}^\star\right)^T$ and of the preloads $\mathring{\boldsymbol{p}}$. Now, as we observed, these quantities depend on the master displacements via the constraints, e.g. $\boldsymbol{D}_u^\star = \boldsymbol{D}_u^\star\left(\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right), \boldsymbol{\mathcal{W}}'\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right)\right)$; thus, we find it more convenient to first expand them with respect to $\boldsymbol{w}, \boldsymbol{w}'$, and then to use $\boldsymbol{w} = \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right) = \boldsymbol{A}_0\boldsymbol{w}_m + \text{h.o.t.}$, namely:

$$\begin{aligned}
\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u &= \boldsymbol{D}_{0u}^\star \mathring{\boldsymbol{\sigma}}_u + \left(\frac{\partial\left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}}\right)_0 \boldsymbol{w} + \left(\frac{\partial\left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}'}\right)_0 \boldsymbol{w}' + \text{h.o.t.} \\
&= \boldsymbol{D}_{0u}^\star \mathring{\boldsymbol{\sigma}}_u + \left[\left(\frac{\partial\left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial\left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s}\right] \boldsymbol{A}_0 \boldsymbol{w}_m + \text{h.o.t.}
\end{aligned} \qquad [1.97]$$

Similarly, we obtain:

$$\begin{aligned}
&\boldsymbol{\mathcal{D}}_{uH}^\star \mathring{\boldsymbol{\sigma}}_u = \boldsymbol{\mathcal{D}}_{0uH}^\star \mathring{\boldsymbol{\sigma}}_u + \left[\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_{uH}^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_{uH}^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}_m'}\right)_0 \frac{\partial}{\partial s}\right] \boldsymbol{A}_0 \boldsymbol{w}_m + \text{h.o.t.} \\
&\mathring{\boldsymbol{p}} = \mathring{\boldsymbol{p}}_0 + \left(\frac{\partial \mathring{\boldsymbol{p}}}{\partial \boldsymbol{w}}\right)_0 \boldsymbol{A}\boldsymbol{w}_m + \text{h.o.t.}, \quad \mathring{\boldsymbol{P}}_H(\boldsymbol{w}) = \mathring{\boldsymbol{P}}_{0H} + \left(\frac{\partial \mathring{\boldsymbol{P}}_H}{\partial \boldsymbol{w}}\right)_0 \boldsymbol{A}_0 \boldsymbol{w}_m + \text{h.o.t.}
\end{aligned} \qquad [1.98]$$

Hence, the field equations are linearized as follows (compare them with equation [1.58]):

$$\boldsymbol{A}_0^\star \boldsymbol{L}_u \boldsymbol{A}_0 \boldsymbol{w}_m + \boldsymbol{A}_0^\star \boldsymbol{G}_u \boldsymbol{A}_0 \boldsymbol{w}_m = \boldsymbol{A}_0^\star \tilde{\boldsymbol{p}}_0 \tag{1.99}$$

and the boundary conditions as (compare them with equation [1.59]):

$$\begin{aligned} &\left[\boldsymbol{\mathcal{A}}_0^\star \boldsymbol{L}_u \boldsymbol{A}_0 + \boldsymbol{\mathcal{L}}_{um} \boldsymbol{A}_0 + \boldsymbol{\mathcal{A}}_0^\star \boldsymbol{G}_u \boldsymbol{A}_0 + \boldsymbol{\mathcal{G}}_{um} \boldsymbol{A}_0\right]_H \boldsymbol{w}_m = \left[\tilde{\boldsymbol{P}}_{0m} + \boldsymbol{\mathcal{A}}_0^\star \tilde{\boldsymbol{p}}_0\right]_H \\ &\left[\boldsymbol{\mathcal{L}}_{us} \boldsymbol{A}_0 + \boldsymbol{\mathcal{G}}_{us} \boldsymbol{A}_0\right]_H \boldsymbol{w}_m = \left[\tilde{\boldsymbol{P}}_{0s}\right]_H \\ &\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad \left[\boldsymbol{A}_{0s} \boldsymbol{w}_m\right]_H = \breve{\boldsymbol{w}}_{sH} \end{aligned} \tag{1.100}$$

where the condensed elastic stiffness operators $\boldsymbol{L}_u, \boldsymbol{\mathcal{L}}_{umH}, \boldsymbol{\mathcal{L}}_{usH}$ have already been defined (equations [1.60]), and the condensed geometric stiffness operators are:

$$\begin{aligned} \boldsymbol{G}_u &:= \left(\frac{\partial \left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial \left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s} - \left(\frac{\partial \mathring{\boldsymbol{p}}}{\partial \boldsymbol{w}}\right)_0 \\ \boldsymbol{\mathcal{G}}_{uH} &:= \left(\frac{\partial \left(\boldsymbol{\mathcal{D}}_{uH}^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial \left(\boldsymbol{\mathcal{D}}_{uH}^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s} - \left(\frac{\partial \mathring{\boldsymbol{P}}_H}{\partial \boldsymbol{w}}\right)_0 + \dots \end{aligned} \tag{1.101}$$

with $\boldsymbol{\mathcal{G}}_{uH} := (\boldsymbol{\mathcal{G}}_{um}, \boldsymbol{\mathcal{G}}_{us})_H^T$. The latter are therefore the $u$-part of the operators $\boldsymbol{G}$, $\boldsymbol{\mathcal{G}}_H$ of the unconstrained beam, defined in equations [1.73], and also appearing in the mixed formulation for the constrained beam.

## 1.6 The variational formulation

In the previous sections, we formulated the Fundamental Problem of beam mechanics, via the *power balance approach*, based on the VPP, which provided the field equations and the alternative boundary conditions. We also mentioned the possibility of achieving the same goal by the *force balance approach* (when the beam is locally rigid), based on the application of the linear and angular momentum principles. There exists, however, a third method, which is called the *variational approach*, which we want to discuss here with a little detail.

A variational principle states that the solution to a given field problem renders stationary a (properly built-up) functional in its domain, i.e. in the space of functions from which the functional depends. The stationary condition, provided by the variational calculus, is a *differential* equation, which is called the *Eulerian equation* of the variational problem. In elastostatics, when the Fundamental Problem is formulated in the context of the displacement method, the proper functional is the *total potential energy* (TPE), which is a scalar function of the *admissible* vector displacement field (i.e. compatible with the external and, possibly, internal constraints). With each arbitrarily chosen admissible displacement field, a scalar value of TPE is associated; by the stationary condition, we look for the particular vector field (possibly not unique) which makes the TPE "locally flat" in its neighborhood. This means that a first-order perturbation of the field that solves the Eulerian equations produces a second- or high-order perturbation in the value assumed by the TPE. The Eulerian equations supplied by the TPE principle are the balance equations and the mechanical boundary conditions we derived in alternate procedures, but, differently from those, *directly expressed in terms of displacements*.

The TPE principle, however, being related to an energy, only works for conservative systems[33]. When the beam is elastic (and therefore it cannot dissipate energy), we just have to assume that the external loads are conservative. However, if the request of conservativeness strongly limits the applicability of the variational approach, another circumstance mitigates this drawback, namely: *the first variation of the TPE is found to coincide with the VWP expression*, with stresses expressed in terms of displacements. Differently from the TPE principle, the VPP holds for any system, conservative and not, so that the varied form of TPE can be used as a method to automatically derive the VWP, to be applied, for example, to a non-conservative case. In this form, the Variational principle is said to be *extended*.

In this section, we will go over all the problems we studied in this chapter, by reobtaining known results via the variational approach.

### 1.6.1 The total potential energy principle

We define the TPE functional, $\Pi\left[\boldsymbol{w}\right]$, whose domain $\mathscr{U}$ is the space of the kinematically admissible displacements $\boldsymbol{w}$, as:

$$\Pi\left[\boldsymbol{w}\right] := U\left[\boldsymbol{w}\right] - W\left[\boldsymbol{w}\right] \qquad [1.102]$$

where $U\left[\boldsymbol{w}\right]$ is the elastic potential energy and $-W\left[\boldsymbol{w}\right]$ is the force potential energy (equal to the external work $W\left[\boldsymbol{w}\right]$, changed in sign, spent from the forces to bring the beam from the reference to the current configuration). Here:

$$\begin{aligned} U\left[\boldsymbol{w}\right] &:= \int_{\mathcal{S}} \phi(\boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right))ds \\ W\left[\boldsymbol{w}\right] &:= -\int_{\mathcal{S}} \psi(\boldsymbol{w})ds - \sum_{H=A}^{B} \Psi(\boldsymbol{w}_H) \end{aligned} \qquad [1.103]$$

33. Follower forces, therefore, are excluded.

where $\phi(\boldsymbol{\varepsilon})$ is the density of the elastic potential energy of the beam, $\psi(\boldsymbol{w}), \Psi(\boldsymbol{w}_H)$ are the potential energies of the forces and $\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}')$ expresses admissibility of strains and displacements (equation [1.1]).

The total potential energy principle (TPEP) states that the displacement field $\boldsymbol{w}$ that solves the elastic problem makes $\Pi[\boldsymbol{w}]$ stationary, i.e.:

$$\delta\Pi[\boldsymbol{w}] = 0 \qquad \forall\delta\boldsymbol{w} \in \mathscr{U} \tag{1.104}$$

Equivalently, we can say that *among all the kinematically admissible displacement fields, the ones also equilibrated render stationary the TPE*. By using the variational calculus, we find:

$$\begin{aligned}\delta\Pi[\boldsymbol{w}] &:= \int\limits_{\mathcal{S}} \left(\frac{\partial\phi}{\partial\boldsymbol{\varepsilon}}\right)^T \delta\boldsymbol{\varepsilon} ds + \int\limits_{\mathcal{S}} \left(\frac{\partial\psi}{\partial\boldsymbol{w}}\right)^T \delta\boldsymbol{w} ds + \sum_{H=A}^{B} \left(\frac{\partial\Psi}{\partial\boldsymbol{w}_H}\right)^T \delta\boldsymbol{w}_H \\ &= \int\limits_{\mathcal{S}} \boldsymbol{\sigma}^T \delta\boldsymbol{\varepsilon} ds - \int\limits_{\mathcal{S}} \boldsymbol{p}^T \delta\boldsymbol{w} ds - \sum_{H=A}^{B} \boldsymbol{P}_H^T \delta\boldsymbol{w}_H = 0 \quad \forall(\delta\boldsymbol{w}, \delta\boldsymbol{\varepsilon}) \,|\, \delta\boldsymbol{\varepsilon} = \boldsymbol{D}\delta\boldsymbol{w}\end{aligned} \tag{1.105}$$

where we accounted for the elastic law [1.14], the definition of force potential energies, $\boldsymbol{p} := -\partial\psi(\boldsymbol{w})/\partial\boldsymbol{w}$, $\boldsymbol{P}_H := -\partial\Psi(\boldsymbol{w}_H)/\partial\boldsymbol{w}_H$[34], and, finally, for the kinematic constraint [1.5] linking variations of strains and displacements. We, however, observe that equation [1.105] coincides with the VWP, equation [1.13], in which the stresses are expressed in terms of strains via the elastic law, and these, in turn, in terms of displacements, via the strain–displacement relationships. Therefore, *the TPE principle and the VWP are equivalent for conservative systems* and lead to the same balance equations (and boundary conditions). If, in contrast, forces are not conservative, then the extended form of the principle (i.e. the last line of equation [1.105]) also holds.

**Dynamical systems: the Hamilton principle**

When inertia forces have to be taken into account, we can either apply the d'Alembert principle, by including inertial effects in the external forces, or use the *Hamilton principle*. When specialized to the problem at hand, the principle states that *the true evolution $\boldsymbol{w}_\star(s,t)$ of a beam makes stationary the functional*:

$$\mathcal{H}[\boldsymbol{w}] := \int_{t_1}^{t_2} (T[\boldsymbol{w}] - \Pi[\boldsymbol{w}])\, dt \tag{1.106}$$

34. The minus sign denotes a decrement of energy when the force spends a positive work.

*in the space of all the kinematically admissible motions* $\boldsymbol{w}(s,t)$ *which bring the beam from a specified state* $\boldsymbol{w}(s,t_1)$ *to a specified state* $\boldsymbol{w}(s,t_2)$*, where* $t_1, t_2$ *are two selected times.* Here, $T$ is the *kinetic energy* of the beam, and $\Pi := U - W$ is the TPE, already introduced. The Variational principle therefore requires that:

$$\delta\mathcal{H} := \delta\int_{t_1}^{t_2}(T-\Pi)\,dt = 0 \qquad \forall\delta\boldsymbol{w}|\delta\boldsymbol{w}(s,t_1) = \delta\boldsymbol{w}(s,t_2) = \mathbf{0} \quad [1.107]$$

Its varied form:

$$\int_{t_1}^{t_2}(\delta T - \delta U + \delta W)\,dt = 0 \qquad \forall\delta\boldsymbol{w}|\delta\boldsymbol{w}(s,t_1) = \delta\boldsymbol{w}(s,t_2) = \mathbf{0} \quad [1.108]$$

is called the *extended hamilton principle*; it holds even for non-conservative forces (e.g. for visco-elastic or externally damped beams). When kinetic effects are negligible, the Hamilton principle reduces to the TPE principle.

### 1.6.2 Unconstrained beams

If we limit ourselves to linear elastic material, we have $\phi(\boldsymbol{\varepsilon}) = {}^1\!/_2\boldsymbol{\mathcal{E}}^T\boldsymbol{E}\boldsymbol{\mathcal{E}}$ (equation [1.15]); moreover, if the external forces are dead loads $\boldsymbol{p}$, $\boldsymbol{P}_H$ (i.e. independent of $\boldsymbol{w}$), then, to within an inessential constant, is $\psi(\boldsymbol{w}) := -\boldsymbol{p}\boldsymbol{w}$, $\Psi(w_H) := -\boldsymbol{P}_H\boldsymbol{w}_H$, so that:

$$\Pi[\boldsymbol{w}] := \frac{1}{2}\int_{\mathcal{S}}\boldsymbol{\mathcal{E}}^T(\boldsymbol{w},\boldsymbol{w}')\,\boldsymbol{E}\boldsymbol{\mathcal{E}}(\boldsymbol{w},\boldsymbol{w}')\,ds - \int_{\mathcal{S}}\boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H \quad [1.109]$$

By equating to zero the first variation, and observing that, for the symmetry of $\boldsymbol{E}$, it is $\delta\phi = \boldsymbol{\mathcal{E}}^T\boldsymbol{E}\delta\boldsymbol{\mathcal{E}} = (\boldsymbol{E}\boldsymbol{\mathcal{E}})^T\delta\boldsymbol{\mathcal{E}}$, where $\delta\boldsymbol{\mathcal{E}} = \boldsymbol{D}\delta\boldsymbol{w}$, equation [1.5], we have:

$$\begin{aligned}\delta\Pi[\boldsymbol{w}] &= \int_{\mathcal{S}}(\boldsymbol{E}\boldsymbol{\mathcal{E}})^T\boldsymbol{D}\delta\boldsymbol{w}ds - \int_{\mathcal{S}}\delta\boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\delta\boldsymbol{w}_H^T\boldsymbol{P}_H \\ &= \int_{\mathcal{S}}\delta\boldsymbol{w}^T(\boldsymbol{D}^\star\boldsymbol{E}\boldsymbol{\mathcal{E}} - \boldsymbol{p})\,ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T(\boldsymbol{\mathcal{D}}^\star\boldsymbol{E}\boldsymbol{\mathcal{E}} - \boldsymbol{P})\right]_H = 0 \qquad \forall\delta\boldsymbol{w}\end{aligned} \quad [1.110]$$

where we used the extended Green identity [1.8]. From equation [1.110], the balance equations [1.18] follow.

If we linearize the strain–displacement relationship, by taking $\boldsymbol{\mathcal{E}} = \boldsymbol{D}_0\boldsymbol{w}$, then, after integration by parts, the equilibrium operators $\boldsymbol{D}_0^\star$ and $\boldsymbol{\mathcal{D}}_{0H}^\star$ appear, so that the balance equations [1.20] of the linear theory are recovered.

### 1.6.3 Constrained beams

When internal constraints exist, of type $\boldsymbol{\varepsilon}_c = \mathbf{0}$, the displacements $\boldsymbol{w}$ are no longer free, but they have to satisfy auxiliary equations $\boldsymbol{\mathcal{E}}_c(\boldsymbol{w}, \boldsymbol{w}') = \mathbf{0}$, which restrict the space $\mathscr{U}$ of the kinematically admissible displacements. To account for constraints, we can follow two different strategies, already discussed with reference to the VPP approach, i.e.: (a) to use Lagrange multipliers, according to the *mixed formulation*; and (b) to refer to master variables, identically satisfying the constraints, according to the *displacement formulation*. We briefly illustrate both the approaches.

#### The mixed formulation

By following the Lagrange multiplier technique, we modify the TPE functional (equation [1.109]) by adding to it a zero-quantity, namely the auxiliary conditions multiplied by unknown functions $\boldsymbol{\lambda} = \boldsymbol{\lambda}(s)$[35]; the modified TPE, therefore, reads:

$$\check{\Pi}[\boldsymbol{w}, \boldsymbol{\lambda}] := \Pi_u[\boldsymbol{w}] + \Pi_\lambda[\boldsymbol{w}, \boldsymbol{\lambda}] \qquad [1.111]$$

where $\Pi_u[\boldsymbol{w}]$ is the TPE of the unconstrained beam when $\boldsymbol{\varepsilon}_c = \mathbf{0}$, and $\Pi_\lambda[\boldsymbol{w}, \boldsymbol{\lambda}]$ the "work of the Lagrange multipliers in the zero-strains", namely:

$$\begin{aligned}\Pi_u[\boldsymbol{w}] &:= \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^T(\boldsymbol{w}, \boldsymbol{w}')\, \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u(\boldsymbol{w}, \boldsymbol{w}')\, ds - \int_{\mathcal{S}} \boldsymbol{w}^T \boldsymbol{p}\, ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T \boldsymbol{P}_H \\ \Pi_\lambda[\boldsymbol{w}, \boldsymbol{\lambda}] &:= \int_{\mathcal{S}} \boldsymbol{\lambda}^T \boldsymbol{\mathcal{E}}_c(\boldsymbol{w}, \boldsymbol{w}')\, ds\end{aligned} \qquad [1.112]$$

The variation of the first contribution, by remembering equation [1.110], is:

$$\delta\Pi_u[\boldsymbol{w}] = \int_{\mathcal{S}} \delta\boldsymbol{w}^T\left(\boldsymbol{D}_u^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \boldsymbol{p}\right) ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\left(\boldsymbol{\mathcal{D}}^\star{}_u \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \boldsymbol{P}\right)\right]_H \qquad [1.113]$$

The variation of the second contribution, since $\delta\left(\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c\right) = \delta\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c + \boldsymbol{\lambda}^T\delta\boldsymbol{\mathcal{E}}_c$, reads:

$$\begin{aligned}\delta\Pi_\lambda[\boldsymbol{w}, \boldsymbol{\lambda}] &= \int_{\mathcal{S}}\left(\delta\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c + \boldsymbol{\lambda}^T\boldsymbol{D}_c\delta\boldsymbol{w}\right) ds \\ &= \int_{\mathcal{S}}\left(\delta\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c + \delta\boldsymbol{w}^T\boldsymbol{D}_c^\star\boldsymbol{\lambda}\right) ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\boldsymbol{\mathcal{D}}_c^\star\boldsymbol{\lambda}\right]_H\end{aligned} \qquad [1.114]$$

35. The *constrained* problem "finds the function $u(s)$ which makes the functional $I[u(s)] := \int_a^b \mathcal{L}(u(s), u'(s))\, ds$ stationary, under the differential constraints $f_i(u(s), u'(s)) = 0,\ i = 1, \ldots, n$", is equivalent to the *unconstrained* problem: "it finds the functions $u(s)$ and $\lambda_i(s)$ that makes stationary the modified functional $\check{I}[u(s), \lambda_i(s)] := \int_a^b \mathcal{L}(u(s), u'(s))\, ds + \sum_{i=1}^n \int_a^b \lambda_i(s) f_i(u(s), u'(s))\, ds$".

having accounted for $\delta\boldsymbol{\mathcal{E}}_c = \boldsymbol{D}_c\delta\boldsymbol{w}$ and performed an integration by parts according to equation [1.8].

The Variational principle finally reads:

$$\begin{aligned}\delta\breve{\Pi}\left[\boldsymbol{w},\boldsymbol{\lambda}\right] &= \int_{\mathcal{S}} \left[\delta\boldsymbol{w}^T\left(\boldsymbol{D}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{D}_c^\star\boldsymbol{\lambda} - \boldsymbol{p}\right) + \delta\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c\right] ds \\ &+ \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\left(\boldsymbol{\mathcal{D}}^\star{}_u\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{\mathcal{D}}_c^\star\boldsymbol{\lambda} - \boldsymbol{P}\right)\right]_H = 0 \qquad \forall\left(\delta\boldsymbol{w},\delta\lambda\right)\end{aligned} \tag{1.115}$$

from which the constrained elastic problem, equations [1.37] and [1.38], follows, with $\boldsymbol{\lambda} \equiv \boldsymbol{\sigma}_c$.

If we linearize the strain–displacement relationship, by taking $\boldsymbol{\mathcal{E}}_u = \boldsymbol{D}_{0u}\boldsymbol{w}$, $\boldsymbol{\mathcal{E}}_c = \boldsymbol{D}_{0c}\boldsymbol{w}$, then, after integration by parts, the equilibrium operators $\boldsymbol{D}_{0u}^\star$, $\boldsymbol{D}_{0c}^\star$ and $\boldsymbol{\mathcal{D}}_{0uH}^\star$, $\boldsymbol{\mathcal{D}}_{0cH}^\star$ appear, so that the balance equations [1.39] and [1.40] of the linear theory are recovered.

## The displacement formulation

Instead of using Lagrange multipliers, we consider a TPE reduced to the unconstrained contribution $\Pi_u\left[\boldsymbol{w}\right]$ (equation [1.112a]), whose domain $\mathscr{U}_m := \{\boldsymbol{w}|\boldsymbol{w}\in\mathscr{U},\, \boldsymbol{w} = \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)\}$ is a *subset* of $\mathscr{U}$, where $\boldsymbol{w}_m$ are master variables identically satisfying the constraints, i.e. $\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{\mathcal{W}},\frac{\partial}{\partial s}\boldsymbol{\mathcal{W}}\right) = \boldsymbol{0}$, $\forall\boldsymbol{w}_m$. Therefore, the TPE is sided by constraints as follows:

$$\begin{aligned}&\Pi_u\left[\boldsymbol{w}\right] := \frac{1}{2}\int_{\mathcal{S}}\boldsymbol{\mathcal{E}}_u^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds - \int_{\mathcal{S}}\boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H \\ &\quad\boldsymbol{w} = \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)\end{aligned} \tag{1.116}$$

Constraints [1.116b] could be easily accounted for by direct substitution in equation [1.116a], by leading to a new functional $\Pi_u\left[\boldsymbol{w}_m\right] := \Pi_u\left[\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)\right]$ in which $\boldsymbol{w}_m$ are free variables. However, we find it more convenient first to perform the variation $\delta\Pi_u\left[\boldsymbol{w}\right]$ (already performed in equation [1.113]) and then to substitute the constraints, both in the arguments (e.g. $\boldsymbol{\mathcal{E}}_u = \boldsymbol{\mathcal{E}}_u\left(\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right),\frac{\partial}{\partial s}\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)\right)$), and in the variation, i.e. $\delta\boldsymbol{w} = \boldsymbol{A}\delta\boldsymbol{w}_m$ (having used equation [1.43], multiplied by $dt$). In so doing, we obtain:

$$\begin{aligned}\delta\Pi_u\left[\boldsymbol{w}_m\right] &= \int_{\mathcal{S}}\left(\boldsymbol{D}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \boldsymbol{p}\right)^T\left(\boldsymbol{A}\delta\boldsymbol{w}_m\right)ds \\ &+ \sum_{H=A}^{B}\left[\left(\boldsymbol{A}\delta\boldsymbol{w}_m\right)^T\left(\boldsymbol{\mathcal{D}}^\star{}_u\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \boldsymbol{P}\right)\right]_H = 0 \qquad \forall\delta\boldsymbol{w}_m\end{aligned} \tag{1.117}$$

However, this is just equation [1.50], with the active stresses expressed in terms of strains. Therefore, by performing similar steps, the Fundamental Problem equations [1.56], [1.57] are recovered.

If we linearize kinematics, by taking $\boldsymbol{\mathcal{E}}_u = \boldsymbol{D}_{0u}\boldsymbol{w}$, $\boldsymbol{A} = \boldsymbol{A}_0 = (\boldsymbol{I}_m, \boldsymbol{A}_{0s})^T$, then, after integration by parts, the operators $\boldsymbol{D}^\star_{0u}, \boldsymbol{A}_0$ and $\boldsymbol{\mathcal{D}}^\star_{0H}, \boldsymbol{\mathcal{A}}^\star_{0H}$ appear, so that the balance equations [1.59] of the linear theory are recovered.

### 1.6.4 Unconstrained prestressed beams

When preloads and prestresses act on the beam, the TPE [1.102] must accordingly be modified. By remembering expression [1.66] of the elastic potential and considering total dead loads, we have:

$$\Pi\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] := \Pi\left[\boldsymbol{w}\right] + \mathring{\Pi}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] \qquad [1.118]$$

where $\Pi\left[\boldsymbol{w}\right]$ is the TPE of the unprestressed beam (equation [1.109], with incremental loads $\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{P}}_H$ replacing $\boldsymbol{p}, \boldsymbol{P}_H$), and:

$$\mathring{\Pi}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] := \int_{\mathcal{S}} \mathring{\boldsymbol{\sigma}}^T \boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right) ds - \int_{\mathcal{S}} \boldsymbol{w}^T \mathring{\boldsymbol{p}} ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T \mathring{\boldsymbol{P}}_H \qquad [1.119]$$

is the contribution of prestresses and preloads. $\delta\left(\mathring{\boldsymbol{\sigma}}^T\boldsymbol{\mathcal{E}}\right) = \mathring{\boldsymbol{\sigma}}^T \boldsymbol{D}\delta\boldsymbol{w}$, we have, after integration by parts:

$$\delta\mathring{\Pi}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] := \int_{\mathcal{S}} \delta\boldsymbol{w}^T\left(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{p}}\right) ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\left(\boldsymbol{\mathcal{D}}^\star\mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{P}}\right)\right]_H \qquad [1.120]$$

The Variational principle $\delta\Pi\left[\boldsymbol{w}\right] + \delta\mathring{\Pi}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] = 0$ then leads to the balance equations [1.69], where the elastic and incremental load terms spring from the first contribution (see equation [1.110]), and prestress and preload terms stem from the second.

#### The linearized theory

The variational formulation is often followed in literature in the context of the linearized theory of prestressed beams (under conservative loads). The main idea of the method consists of assuming a *quadratic* polynomial expression for the TPE, in order to get equilibrium equations *linear* in the displacements (given that the variation entails a lowering of 1 in the polynomial degree). Therefore, we write the strains by series expansions, as the sum of linear and quadratic contributions in the displacements, namely:

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(1)}\left(\boldsymbol{w},\boldsymbol{w}'\right) + \boldsymbol{\mathcal{E}}^{(2)}\left(\boldsymbol{w},\boldsymbol{w}'\right) + \text{h.o.t.} \qquad [1.121]$$

and, moreover, we assume the preloads as $\mathrm{O}(1)$-quantities and the incremental load as $\mathrm{O}(\boldsymbol{w})$-quantities. Hence, the TPE [1.118] reads as:

$$\Pi\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] = \Pi^{(1)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] + \Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] + \text{h.o.t.} \qquad [1.122]$$

where, by omitting the arguments:

$$\begin{aligned} \Pi^{(1)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] &:= \int_{\mathcal{S}} \mathring{\boldsymbol{\sigma}}^T \boldsymbol{\mathcal{E}}^{(1)} ds - \int_{\mathcal{S}} \boldsymbol{w}^T \mathring{\mathbf{p}} ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T \mathring{\boldsymbol{P}}_H \\ \Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] &:= \int_{\mathcal{S}} \left(\frac{1}{2}\boldsymbol{\mathcal{E}}^{(1)T}\boldsymbol{E}\boldsymbol{\mathcal{E}}^{(1)} + \mathring{\boldsymbol{\sigma}}^T \boldsymbol{\mathcal{E}}^{(2)}\right) ds - \int_{\mathcal{S}} \boldsymbol{w}^T \tilde{\mathbf{p}} ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T \tilde{\boldsymbol{P}}_H \end{aligned} \tag{1.123}$$

are first- and second-order terms, respectively. However, we observe that $\delta\Pi^{(1)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] = 0 \ \forall\delta\boldsymbol{w}$, since it expresses the total virtual work spent by the equilibrated prestresses and preloads, acting in the reference configuration, in the kinematically admissible infinitesimal strains $\boldsymbol{\mathcal{E}}^{(1)}$ and infinitely small displacements $\boldsymbol{w}$. Hence, the first-order term of the potential energy is not essential, and we can assume, after truncation, $\Pi\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] \equiv \Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right]$.

The first- and second-order parts of the strain components read:

$$\begin{aligned} \mathcal{E}_i^{(1)} &:= \left(\frac{\partial \mathcal{E}_i}{\partial \boldsymbol{w}}\right)_0^T \boldsymbol{w} + \left(\frac{\partial \mathcal{E}_i}{\partial \boldsymbol{w}'}\right)_0^T \boldsymbol{w}' \\ \mathcal{E}_i^{(2)} &:= \frac{1}{2}\left(\boldsymbol{w}^T \boldsymbol{A}_i \boldsymbol{w} + 2\boldsymbol{w}^T \boldsymbol{B}_i \boldsymbol{w}' + \boldsymbol{w}'^T \boldsymbol{C}_i \boldsymbol{w}'\right) \end{aligned} \tag{1.124}$$

where we used the positions [1.77]. When the variational principle $\delta\Pi^{(2)} = 0, \ \forall\delta\boldsymbol{w}$ is invoked, the first, third and fourth addenda in equation [1.123b] lead, after straightforward (and therefore omitted) calculations, to the familiar terms of the linear theory of the stress-free beam. In contrast, we focus the attention on the second term, whose variation reads:

$$\begin{aligned} \delta\int_{\mathcal{S}} \mathring{\boldsymbol{\sigma}}^T \boldsymbol{\mathcal{E}}^{(2)} ds &= \sum_{i=1}^{M} \int_{\mathcal{S}} \mathring{\sigma}_i \delta\mathcal{E}_i^{(2)} ds \\ &= \sum_{i=1}^{M} \int_{\mathcal{S}} \mathring{\sigma}_i \left(\delta\boldsymbol{w}^T \boldsymbol{A}_i \boldsymbol{w} + \delta\boldsymbol{w}^T \boldsymbol{B}_i \boldsymbol{w}' + \boldsymbol{w}^T \boldsymbol{B}_i \delta\boldsymbol{w}' + \delta\boldsymbol{w}'^T \boldsymbol{C}_i \boldsymbol{w}'\right) ds \\ &= \sum_{i=1}^{M} \int_{\mathcal{S}} \delta\boldsymbol{w}^T \left[\left(\boldsymbol{A}_i \boldsymbol{w} + \boldsymbol{B}_i \boldsymbol{w}'\right)\mathring{\sigma}_i - \left(\boldsymbol{B}_i^T \boldsymbol{w}\mathring{\sigma}_i + \boldsymbol{C}_i \boldsymbol{w}'\mathring{\sigma}_i\right)'\right] ds + \\ &+ \left[\delta\boldsymbol{w}^T \left(\boldsymbol{B}_i^T \boldsymbol{w}\mathring{\sigma}_i + \boldsymbol{C}_i \boldsymbol{w}'\mathring{\sigma}_i\right)\right]_A^B \end{aligned} \tag{1.125}$$

where we accounted for the symmetry of $\boldsymbol{A}_i, \boldsymbol{C}_i$ and integrated by parts. By remembering equations [1.78] and [1.81], we can write:

$$\delta\int_{\mathcal{S}} \mathring{\boldsymbol{\sigma}}^T \boldsymbol{\mathcal{E}}^{(2)} ds = \int_{\mathcal{S}} \delta\boldsymbol{w}^T \boldsymbol{G}\boldsymbol{w} ds + \sum_{H=A}^{B} \left[\delta\boldsymbol{w}^T \boldsymbol{\mathcal{G}}\boldsymbol{w}\right]_H \tag{1.126}$$

to within, of course, the effects of the follower preloads, absent here. Therefore, the variational principle leads to balance equation [1.72].

REMARK 1.10. The two contributions under the integral sign in $\Pi^{(2)}$ represent, in order: (a) the elastic potential of a stress-free beam, when kinematics is linearized; and (b) *the work spent by the prestress in the second-order part of the strain–displacement relationship*. While the first term behaves as the progenitor of the linear elastic stiffnesses $\boldsymbol{L}$ and $\boldsymbol{\mathcal{L}}_H$, the second term is the progenitor of the geometric stiffnesses $\boldsymbol{G}$ and $\boldsymbol{\mathcal{G}}_H$.

## 1.6.5 Constrained prestressed beams

### The nonlinear mixed formulation

We already introduced in equation [1.112] a modified TPE $\check{\Pi}[\boldsymbol{w},\boldsymbol{\lambda}]$ for unprestressed beams, able to account for the constraints $\boldsymbol{\mathcal{E}}_c(\boldsymbol{w},\boldsymbol{w}')=\boldsymbol{0}$ via the Lagrange multipliers. Now, we just have to update the expression of the elastic potential to include the contribution of prestress, and to add the potential of the preloads, as we did in equation [1.118]. Therefore, we have:

$$\check{\Pi}[\boldsymbol{w},\boldsymbol{\lambda};\mathring{\boldsymbol{\sigma}}_u] := \Pi_u[\boldsymbol{w}] + \Pi_\lambda[\boldsymbol{w},\boldsymbol{\lambda}] + \mathring{\Pi}[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}_u] \qquad [1.127]$$

where $\Pi_u[\boldsymbol{w}]$ is the TPE of the unconstrained beam when $\boldsymbol{\varepsilon}_c = \boldsymbol{0}$ (equation [1.112a), $\Pi_\lambda[\boldsymbol{w},\boldsymbol{\lambda}]$ is the work of the Lagrange multipliers on the constrained zero-strains (equation [1.112b]), and the additional term is[36]:

$$\mathring{\Pi}[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}_u] := \int_{\mathcal{S}} \mathring{\boldsymbol{\sigma}}_u^T \boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right) ds - \int_{\mathcal{S}} \boldsymbol{w}^T \mathring{\boldsymbol{p}} ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T \mathring{\boldsymbol{P}}_H \qquad [1.128]$$

The variations of the first two contributions are given by equations [1.113] and [1.114]; the variation of the third contribution is:

$$\delta\mathring{\Pi}[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}] := \int_{\mathcal{S}} \delta\boldsymbol{w}^T\left(\boldsymbol{D}_u^\star\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{p}}\right) ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\left(\boldsymbol{\mathcal{D}}^\star{}_u\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}\right)\right]_H \qquad [1.129]$$

Hence, the variational principle reads:

$$\begin{aligned}\delta\check{\Pi}[\boldsymbol{w},\boldsymbol{\lambda};\mathring{\boldsymbol{\sigma}}_u] = &\int_{\mathcal{S}} \delta\boldsymbol{w}^T\left(\boldsymbol{D}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \tilde{\boldsymbol{p}}\right) ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\left(\boldsymbol{\mathcal{D}}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \tilde{\boldsymbol{P}}\right)\right]_H \\ &+ \int_{\mathcal{S}}\left[\delta\boldsymbol{w}^T\left(\boldsymbol{D}_u^\star\mathring{\boldsymbol{\sigma}}_u + \boldsymbol{D}_c^\star\boldsymbol{\lambda} - \mathring{\boldsymbol{p}}\right) + \delta\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c\right] ds + \\ &+ \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\left(\boldsymbol{\mathcal{D}}_u^\star\mathring{\boldsymbol{\sigma}}_u + \boldsymbol{\mathcal{D}}_c^\star\boldsymbol{\lambda} - \mathring{\boldsymbol{P}}\right)\right]_H = 0 \quad \forall(\delta\boldsymbol{w},\delta\boldsymbol{\lambda})\end{aligned} \qquad [1.130]$$

From the latter, the boundary value problems [1.87] and [1.88] follow, if $\boldsymbol{\lambda} := \mathring{\boldsymbol{\sigma}}_c + \tilde{\boldsymbol{\sigma}}_c$ is taken.

36. This is analogous to that in equation [1.119], relevant to the unconstrained beam, but it is limited to the admissible strains.

### The linearized mixed formulation

According to the linearized theory, we have to retain in equation [1.127] the second-order terms only. Since $\boldsymbol{\lambda} = \mathring{\boldsymbol{\sigma}}_c + \tilde{\boldsymbol{\sigma}}_c$ is a sum of a zero-th order and a first-order term, then $\boldsymbol{\lambda}^T \boldsymbol{\mathcal{E}}_c^{(2)} = \mathring{\boldsymbol{\sigma}}_c^T \boldsymbol{\mathcal{E}}_c^{(2)} + \tilde{\boldsymbol{\sigma}}_c^T \boldsymbol{\mathcal{E}}_c^{(1)} + \text{h.o.t.}$; therefore[37]:

$$\begin{aligned} \breve{\Pi}^{(2)}\left[\boldsymbol{w}, \tilde{\boldsymbol{\sigma}}_c; \mathring{\boldsymbol{\sigma}}\right] := & \int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^{(1)T}(\boldsymbol{w}, \boldsymbol{w}') \boldsymbol{E}_{uu} \boldsymbol{\mathcal{E}}_u^{(1)}(\boldsymbol{w}, \boldsymbol{w}') ds \\ & - \int_{\mathcal{S}} \boldsymbol{w}^T \tilde{\boldsymbol{p}} ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T \tilde{\boldsymbol{P}}_H + \int_{\mathcal{S}} \left( \mathring{\boldsymbol{\sigma}}^T \boldsymbol{\mathcal{E}}^{(2)} + \tilde{\boldsymbol{\sigma}}_c \boldsymbol{D}_{0c} \boldsymbol{w} \right) ds \end{aligned} \qquad [1.131]$$

where we accounted for $\boldsymbol{\mathcal{E}}_c^{(1)} = \boldsymbol{D}_{0c} \boldsymbol{w}$ and we merged two terms.

The variational principle, by remembering equations [1.126], therefore reads:

$$\begin{aligned} \delta \breve{\Pi}^{(2)}\left[\boldsymbol{w}, \tilde{\boldsymbol{\sigma}}_c; \mathring{\boldsymbol{\sigma}}\right] = & \int_{\mathcal{S}} \delta \boldsymbol{w}^T \left( \boldsymbol{L}_u \boldsymbol{w} - \tilde{\boldsymbol{p}} \right) ds + \sum_{H=A}^{B} \left[ \delta \boldsymbol{w}^T \left( \boldsymbol{\mathcal{L}}_u \boldsymbol{w} - \tilde{\boldsymbol{P}} \right) \right]_H \\ & + \int_{\mathcal{S}} \delta \boldsymbol{w}^T \boldsymbol{G} \boldsymbol{w} ds + \sum_{H=A}^{B} \left[ \delta \boldsymbol{w}^T \boldsymbol{\mathcal{G}} \boldsymbol{w} \right]_H \\ & + \int_{\mathcal{S}} \left( \delta \tilde{\boldsymbol{\sigma}}_c^T \boldsymbol{\mathcal{E}}_c + \delta \boldsymbol{w}^T \boldsymbol{D}_{0c}^{\star} \tilde{\boldsymbol{\sigma}}_c \right) ds + \sum_{H=A}^{B} \delta \boldsymbol{w}_H^T \boldsymbol{\mathcal{D}}_{0cH}^{\star} \tilde{\boldsymbol{\sigma}}_c \quad \forall \left( \delta \boldsymbol{w}, \delta \tilde{\boldsymbol{\sigma}}_c \right) \end{aligned} \qquad [1.132]$$

from which equations [1.89] and [1.90] are recovered, together with the constraint equation.

### The nonlinear displacement formulation

We write the TPE $\Pi_u[\boldsymbol{w}]$ with the geometrical constraint appended, as we did for the unprestressed beam (equation [1.116]), but we add the prestress contribution $\mathring{\Pi}[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}]$ (equation [1.128]):

$$\begin{aligned} \Pi\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] := & \Pi_u\left[\boldsymbol{w}\right] + \mathring{\Pi}\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] \\ \boldsymbol{w} = & \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right) \end{aligned} \qquad [1.133]$$

37. Note that the free variables are now $\boldsymbol{w}, \tilde{\boldsymbol{\sigma}}_c$.

By following the same steps of the unprestressed case, we perform the variation, substitute $\delta\boldsymbol{\mathcal{E}}_u = \boldsymbol{D}_u \delta\boldsymbol{w}$ and integrate by parts, to obtain:

$$\delta\Pi\left[\mathbf{w};\mathring{\boldsymbol{\sigma}}\right] = \int_{\mathcal{S}} \delta\mathbf{w}^T\left(\mathbf{D}_u^{\star}\mathbf{E}_{uu}\boldsymbol{\mathcal{E}}_u - \tilde{\mathbf{p}}\right)ds + \sum_{H=A}^{B}\left[\delta\mathbf{w}^T\left(\boldsymbol{\mathcal{D}}_u^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \tilde{\mathbf{P}}\right)\right]_H$$
$$+\int_{\mathcal{S}} \delta\mathbf{w}^T\left(\mathbf{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u - \mathring{\mathbf{p}}\right)ds + \sum_{H=A}^{B}\left[\delta\mathbf{w}^T\left(\boldsymbol{\mathcal{D}}_u^{\star}\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}\right)\right]_H = 0 \qquad \forall\delta\boldsymbol{w} = \boldsymbol{A}\delta\boldsymbol{w}_m$$

[1.134]

Then, we substitute the constraint, both in the arguments and in the variation (i.e. $\delta\boldsymbol{w} = \boldsymbol{A}\left(\boldsymbol{w}_m, \boldsymbol{w}_m'\right)\delta\boldsymbol{w}_m$), thus obtaining:

$$\delta\Pi\left[\boldsymbol{w}_m;\mathring{\boldsymbol{\sigma}}\right] = \int_{\mathcal{S}} \left(\left(\boldsymbol{D}_u^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \tilde{\boldsymbol{p}}\right) + \left(\boldsymbol{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{p}}\right)\right)^T\left(\boldsymbol{A}\delta\boldsymbol{w}_m\right)ds$$
$$+\sum_{H=A}^{B}\left[\left(\boldsymbol{A}\delta\boldsymbol{w}_m\right)^T\left(\boldsymbol{\mathcal{D}^{\star}}_u\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \left(\boldsymbol{\mathcal{D}^{\star}}_u\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}\right)\right)\right]_H = 0 \qquad \forall\delta\boldsymbol{w}_m$$

[1.135]

By integrating by parts with the aid of equation [1.51], equation [1.95] is recovered, with the boundary conditions [1.96].

## The linearized displacement formulation

When the TPE [1.133a] is truncated at the second order, and the constraint [1.133b] is linearized, they become:

$$\Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}_u\right] := \int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^{(1)T}(\boldsymbol{w},\boldsymbol{w}')\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u(\boldsymbol{w},\boldsymbol{w}')ds$$
$$-\int_{\mathcal{S}} \boldsymbol{w}^T\tilde{\boldsymbol{p}}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\tilde{\boldsymbol{P}}_H + \int_{\mathcal{S}} \mathring{\boldsymbol{\sigma}}_u^T\boldsymbol{\mathcal{E}}_u^{(2)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds \qquad [1.136]$$
$$\boldsymbol{w} = \boldsymbol{A}_0\boldsymbol{w}_m$$

The variational principle reads:

$$\delta\Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}_u\right] = \int_{\mathcal{S}} \delta\boldsymbol{w}^T\left(\boldsymbol{L}_u\boldsymbol{w} - \tilde{\boldsymbol{p}}\right)ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\left(\boldsymbol{\mathcal{L}}_u\boldsymbol{w} - \tilde{\boldsymbol{P}}\right)\right]_H$$
$$+\int_{\mathcal{S}} \delta\boldsymbol{w}^T\boldsymbol{G}_u\boldsymbol{w}ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\boldsymbol{\mathcal{G}}_u\boldsymbol{w}\right]_H = 0 \qquad \forall\delta\boldsymbol{w} = \boldsymbol{A}_0\delta\boldsymbol{w}_m \qquad [1.137]$$

where, concerning the geometric term, we used equation [1.126] and exploited similarity between the definitions [1.78], [1.81] for $\boldsymbol{G}$, $\boldsymbol{\mathcal{G}}_H$, and definitions [1.101] for $\boldsymbol{G}_u$, $\boldsymbol{\mathcal{G}}_{uH}$.

By substituting the constraint, namely $\boldsymbol{w} = \boldsymbol{A}_0\boldsymbol{w}_m,\ \delta\boldsymbol{w} = \boldsymbol{A}_0\delta\boldsymbol{w}_m$, we obtain:

$$\begin{aligned}&\int_{\mathcal{S}} (\boldsymbol{A}_0\delta\boldsymbol{w}_m)^T \left((\boldsymbol{L}_u + \boldsymbol{G}_u)(\boldsymbol{A}_0\boldsymbol{w}_m) - \tilde{\boldsymbol{p}}\right) ds \\ &+ \sum_{H=A}^{B} \left[(\boldsymbol{A}_0\delta\boldsymbol{w}_m)^T \left((\boldsymbol{\mathcal{L}}_u + \boldsymbol{\mathcal{G}}_u)(\boldsymbol{A}_0\boldsymbol{w}_m) - \tilde{\boldsymbol{P}}\right)\right]_H = 0 \qquad \forall\delta\boldsymbol{w}_m\end{aligned} \tag{1.138}$$

Then, by integrating by parts (with the help of the linearized version of equation [1.51], i.e. for $\boldsymbol{A}$ replaced by $\boldsymbol{A}_0$), and by splitting the boundary terms into master and slave contributions, i.e. by letting $\boldsymbol{A}_0 = (\boldsymbol{I}_m, \boldsymbol{A}_{0s})^T$, we get:

$$\begin{aligned}&\int_{\mathcal{S}} \delta\boldsymbol{w}_m^T \boldsymbol{A}_0^\star \left((\boldsymbol{L}_u + \boldsymbol{G}_u)(\boldsymbol{A}_0\boldsymbol{w}_m) - \tilde{\boldsymbol{p}}\right) ds \\ &+ \sum_{H=A}^{B} \left[\delta\boldsymbol{w}_m^T \boldsymbol{\mathcal{A}}_0^\star \left((\boldsymbol{L}_u + \boldsymbol{G}_u)(\boldsymbol{A}_0\boldsymbol{w}_m) - \tilde{\boldsymbol{p}}\right)\right]_H \\ &+ \sum_{H=A}^{B} \left[\delta\boldsymbol{w}_m^T \left((\boldsymbol{\mathcal{L}}_{um} + \boldsymbol{\mathcal{G}}_{um})(\boldsymbol{A}_0\boldsymbol{w}_m) - \tilde{\boldsymbol{P}}_m\right)\right]_H \\ &+ \sum_{H=A}^{B} \left[(\boldsymbol{A}_{0s}\delta\boldsymbol{w}_m)^T \left((\boldsymbol{\mathcal{L}}_{us} + \boldsymbol{\mathcal{G}}_{us})(\boldsymbol{A}_0\boldsymbol{w}_m) - \tilde{\boldsymbol{P}}_s\right)\right]_H = 0 \qquad \forall\delta\boldsymbol{w}_m\end{aligned} \tag{1.139}$$

From this form, the field equations [1.99] and the boundary conditions [1.100] follow.

## 1.7 Example: the linear Timoshenko beam

Let us consider the unconstrained linear model of the Timoshenko beam, undergoing transverse displacements and rotations only, governed by the following equations:

$$\begin{aligned}&\begin{pmatrix}\gamma\\ \kappa\end{pmatrix} = \begin{pmatrix}u' - \theta\\ \theta'\end{pmatrix}, \quad \begin{pmatrix}-\partial_s & 0\\ -1 & -\partial_s\end{pmatrix}\begin{pmatrix}T\\ M\end{pmatrix} = \begin{pmatrix}p\\ c\end{pmatrix}, \\ &\begin{pmatrix}T\\ M\end{pmatrix} = \begin{pmatrix}GA_t & 0\\ 0 & EJ\end{pmatrix}\begin{pmatrix}\gamma\\ \kappa\end{pmatrix}\end{aligned} \tag{1.140}$$

with the boundary conditions (a geometrical one excludes the dual mechanical):

$$\begin{pmatrix}u_H\\ \theta_H\end{pmatrix} = \begin{pmatrix}\breve{u}_H\\ \breve{\theta}_H\end{pmatrix}, \qquad \begin{pmatrix}\mp 1 & 0\\ 0 & \mp 1\end{pmatrix}\begin{pmatrix}T_H\\ M_H\end{pmatrix} = \begin{pmatrix}P_H\\ C_H\end{pmatrix}, \qquad H = A, B \tag{1.141}$$

The previous equations are, in order: the (infinitesimal) strain–displacement relationships; the equilibrium equations (in the reference configuration); and the elastic law. Here,

$\boldsymbol{\varepsilon} = (\gamma, \kappa)^T$ are generalized strains, shear–strain and curvature, respectively; $\boldsymbol{w} = (u, \theta)^T$ are generalized displacements, transverse displacement and rotation, respectively; $\boldsymbol{\sigma} = (T, M)^T$ are generalized stresses, shear–force and bending moment, respectively; $\boldsymbol{p} = (p, c)^T$ are external forces, transverse and couples; and $GA_t$, $EJ$ are elastic stiffnesses. Moreover, $\breve{\boldsymbol{w}}_H = \left(\breve{u}_H, \breve{\theta}_H\right)^T$ are prescribed displacements/rotations and $\boldsymbol{P}_H = (P_H, C_H)^T$ are prescribed forces/couples at ends. The minus/plus identity matrices are the boundary equilibrium operator $\boldsymbol{\mathcal{D}}_H^\star$. The strain-rate-velocity relationships read:

$$\begin{pmatrix} \dot{\gamma} \\ \dot{\kappa} \end{pmatrix} = \begin{pmatrix} \partial_s & -1 \\ 0 & \partial_s \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{\theta} \end{pmatrix} \quad [1.142]$$

which defines the operator $\boldsymbol{D} \equiv \boldsymbol{D}_0$, adjoint of the equilibrium operator $\boldsymbol{D}^\star \equiv \boldsymbol{D}_0^\star$ appearing in equation [1.140b].

For this model, we want to enforce the constraint condition $\gamma = 0$ (unshearable beam) and derive the displacement formulation (Euler–Bernoulli beam). Then, we want to find the reactive stress $T$.

The admissible strain is $\kappa$, the constrained strain is $\gamma$; consistently, $T$ is the reactive stress and $M$ is the active stress. In the constraint equation $u' - \theta = 0$, we chose the non-differentiated variable $\theta$ as slave variable and, consequently, $u$ as master variable. Accordingly, $\boldsymbol{D}_u = (0, \partial_s)$, $\boldsymbol{D}_c = (\partial_s, -1)$ and $\boldsymbol{D}_u^\star = (0, -\partial_s)^T$, $\boldsymbol{D}_c^\star = (-\partial_s, -1)^T$. From the constraint, we get $\theta = u'$, and therefore $\boldsymbol{\mathcal{W}} = (u, u')^T$ (equation [1.42]); by time-differentiating it, we obtain (equation [1.43]):

$$\begin{pmatrix} \dot{u} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 1 \\ \partial_s \end{pmatrix} \dot{u} \quad [1.143]$$

which defines the velocity constraint operator $\boldsymbol{A} := (1, \partial_s)^T$. Note that $\boldsymbol{D}_c\boldsymbol{A} = \partial_s - \partial_s = 0$.

The condensed strain–displacement relationships [1.45] and the condensed strain-rate-velocities [1.47], read:

$$\kappa = \left(u'\right)', \qquad \dot{\kappa} = (0, \partial_s) \begin{pmatrix} 1 \\ \partial_s \end{pmatrix} \dot{u} = \dot{u}'' \quad [1.144]$$

while the geometric boundary conditions [1.46] are:

$$\begin{pmatrix} u_H \\ u'_H \end{pmatrix} = \begin{pmatrix} \breve{u}_H \\ \breve{\theta}_H \end{pmatrix} \quad [1.145]$$

To build up the extended Green identity for the velocity constraint operator, equation [1.51], we take a dummy vector $\boldsymbol{p}_c = (p_c, c_c)^T$, perform the scalar product $\boldsymbol{p}_c^T \boldsymbol{A} \dot{\boldsymbol{w}}_m = p_c\dot{u} + c_c\dot{u}'$, and integrate by parts over $\mathcal{S}$ to free the velocities from the space-derivatives, thus obtaining:

$$\int_{\mathcal{S}} \left(p_c\dot{u} + c_c\dot{u}'\right) ds = \int_{\mathcal{S}} \dot{u}\left(p_c - c_c'\right) ds + \left[\dot{u}c_c\right]_A^B \quad [1.146]$$

Therefore, the equilibrium condensation operators are (check equations [1.52]):

$$\boldsymbol{A}^\star = \begin{pmatrix} 1 & -\partial_s \end{pmatrix}, \qquad \boldsymbol{\mathcal{A}}_A^\star = \begin{pmatrix} 0, & -1 \end{pmatrix}, \qquad \boldsymbol{\mathcal{A}}_B^\star = \begin{pmatrix} 0, & 1 \end{pmatrix} \qquad [1.147]$$

Note that $\boldsymbol{A}^\star \boldsymbol{D}_c^\star = -\partial_s + \partial_s = 0$.

To condense the equilibrium equations, we could use the previous operators directly in equations [1.54] and [1.55]. However, for illustrative purposes, we restart the whole procedure. First, we write the VPP in the form [1.48], by using $\dot{\kappa} = \dot{\theta}'$:

$$\begin{aligned} &\int M\dot{\theta}' ds = \int_{\mathcal{S}} \left(p\dot{u} + c\dot{\theta}\right) ds + P_A\dot{u}_A + P_B\dot{u}_B + C_A\dot{\theta}_A + C_B\dot{\theta}_B \\ &\quad \forall \left(\dot{u}, \dot{\theta}\right) |\dot{\theta} = \dot{u}' \end{aligned} \qquad [1.148]$$

Note that *the constraint has not been substituted, yet*! Then, we perform a first integration by parts:

$$\begin{aligned} &\int_{\mathcal{S}} \left[-p\dot{u} - \left(M' + c\right)\dot{\theta}\right] ds - P_A\dot{u}_A - P_B\dot{u}_B \\ &\quad - \left(M_A + C_A\right)\dot{\theta}_A - \left(-M_B + C_B\right)\dot{\theta}_B = 0 \qquad \forall \left(\dot{u}, \dot{\theta}\right) |\dot{\theta} = \dot{u}' \end{aligned} \qquad [1.149]$$

Only after that, we substitute the constraint:

$$\begin{aligned} &\int_{\mathcal{S}} \left[-p\dot{u} - \left(M' + c\right)\dot{u}'\right] ds - P_A\dot{u}_A - P_B\dot{u}_B \\ &\quad - \left(M_A + C_A\right)\dot{u}_A' - \left(-M_B + C_B\right)\dot{u}_B' = 0 \qquad \forall \dot{u}' \end{aligned} \qquad [1.150]$$

This equation is in the form of equation [1.50], where the first two boundary terms refer to the master variable, and the last two to the slave variable, expressed in terms of the master one. A second integration by parts leads to:

$$\begin{aligned} &\int_{\mathcal{S}} \left[-p + (M' + c)'\right] \dot{u} ds - \left[(M' + c)\dot{u}\right]_A^B - P_A\dot{u}_A - P_B\dot{u}_B \\ &\quad - \left(M_A + C_A\right)\dot{u}_A' - \left(-M_B + C_B\right)\dot{u}_B' = 0 \qquad \forall \dot{u}' \end{aligned} \qquad [1.151]$$

Because of the arbitrariness of $\dot{u}$, we get:

$$\begin{aligned} &M'' = p - c' \\ &(M_A' + c_A - P_A)\dot{u}_A = 0, \qquad \left(-M_A - C_A\right)\dot{u}_A' = 0 \\ &(-M_B' - c_B - P_B)\dot{u}_B = 0, \quad \left(M_B - C_B\right)\dot{u}_B' = 0 \end{aligned} \qquad [1.152]$$

Therefore, the couple density $c$ contributes to the translational equilibrium, in the field and at the boundaries (i.e. they enter the “master part” of the equation, not the “slave part”, as stated by equations [1.56] and [1.96]).

If, e.g., the beam is clamped at $A$ ($\dot{u}_A = 0,\ \dot{u}'_A = 0$) and free at $B$ ($\dot{u}_B \neq 0,\ \dot{u}'_B \neq 0$), the mechanical conditions are $-\left(M'_B + c_B + P_B\right) = 0$, $M_B - C_B = 0$. By using kinematics and the elastic law, we have $M = EJu''$, from which:

$$\begin{aligned} &EJu'''' = p - c' \\ &u_A = \breve{u}_A, \qquad\qquad u'_A = \breve{\theta}_A \\ &-\left(EIu'''_B + c_B + P_B\right) = 0, \quad EJu''_B - C_B = 0 \end{aligned} \qquad [1.153]$$

Once the problem has been solved, the field balance equation [1.140b] read as equations [1.61a]:

$$\begin{aligned} &T' = -p \\ &T = -c - EJu''' \end{aligned} \qquad [1.154]$$

These are *not* independent, because of [1.153a]. From either of them, the reactive stress $T$ is drawn.

## 1.8 Summary

In this chapter, we formulated a 1D beam metamodel, i.e. an ensemble of property and rules that each specific model, to be developed in the following chapters, must obey. It calls for analyzing: (a) kinematics, (b) dynamics, and (c) rheology of the model.

We started, in section 1.2, analyzing internally unconstrained beams, by defining column-vectors of unknown generalized displacements, strains and stresses, and known generalized field forces and boundary forces (generally non-conservative), as well as boundary displacements. We separately addressed kinematics, dynamics and rheology.

Concerning kinematics, we first discussed *locally rigid/non-rigid beams*, as 1D bodies not-endowed/endowed with kinematic descriptors able to account for the "change of shape" of the point (typically the deformation of the underlying cross-section). We introduced *nonlinear* strain–displacement relationships, whose time-differentiation led to *linear* strain-rate-velocity relationships, which define the *(differential) kinematic operator*. Since this is configuration dependent, it differs from that of the linear theory, which is evaluated at the reference configuration. Differential relationships are sided by algebraic *geometric boundary conditions*, prescribing displacements at the ends of the beam.

Concerning dynamics, we derived balance (or equilibrium) equations, and *mechanical (or natural) boundary conditions*, via VPP. This states that an equality must hold between the powers spent by forces on virtual velocities, on one side, and stresses on virtual strain-rates, on the other side, when arbitrary strain-rates and velocities are assigned to the body, provided they are respectful of the kinematic constraints. The VPP provides differential balance equations, governed by an *equilibrium operator*, and algebraic boundary conditions, both linear in the stresses (but nonlinear in the displacements). The VPP can also be read as an extended Green identity, which states that the equilibrium operator is the *adjoint* of the kinematic operator, and that mechanical boundary conditions are the adjoint of the geometrical boundary conditions. Such a property is known as *duality property;* differently from the linear theory, this holds in the current (not in the reference!) configuration, to which the virtual motion is superimposed. If the beam is locally rigid, the balance equations can also be interpreted (or alternatively derived) as the cardinal equation of motion (or equilibrium) of an infinitesimal segment of the beam, and the mechanical boundary conditions as the equality of the emerging stresses to the forces applied to the boundary.

Concerning rheology, we limited ourselves to *hyperelastic materials* (often called although improperly, elastic), for which stresses at a point at an instant not only depend on strains at the same point at the same instant, as occurs for simply elastic materials, but, moreover, the stresses spend a deformation work over the strains, which is independent of the strain-path. Therefore, hyperelasticity is synonymous of conservativeness (i.e. lack of dissipation) of the material. It entails the existence of an *elastic potential*, function of the strains, from which stresses are derived by differentiation. Here, *linear* hyperelastic materials were considered only, for which stresses and strains are proportional, by the way of an *elastic matrix*.

The equations of the problem were combined according to the *displacement method*, which consists of expressing the balance equations in terms of the displacements only, which are therefore the main unknown of the problem. Linearization of these equations around the reference configuration supplies the familiar *(tangent) stiffness operators* (in the domain and at the boundary) of the linear theory.

In section 1.3 we considered internally constrained beams, in which one or more of the strains are prescribed to identically vanish along the beam. The constraints call for splitting the generalized strain vector into an *unconstrained (or admissible) part*, collecting the non-zero strains, and a *constrained part*, collecting the vanishing strains. Accordingly, the generalized stress vector was split ino the *active part*, and a (maybe, partially) *reactive part*, concerning the stresses spending power on the admissible and constrained strains, respectively. Because of the reactive character of part of the stresses, the elastic law only involves active stresses and admissible strains. The equations of the constrained problem were combined according to two different philosophies: (a) *the mixed formulation*, in which displacements and reactive stresses were assumed as the main variables; and (b) *the displacement formulation*, in which the equations were further manipulated to eliminate reactive stresses.

In the mixed formulation, the VPP must account for the prescribed internal constraints. These are introduced by the *Lagrange multipliers* technique, which, in the case studied here, just assumes the physical meaning of reactive stresses. The balance equations supplied by the VPP contain active as well reactive stresses, only the former being expressible in terms of displacements via the elastic law and the unconstrained strain–displacement relationships. The increased number of unknowns, however, is balanced by the nonlinear constraint relationships (i.e. the conditions of vanishing of the restrained strains), which must be appended to the balance equations.

The task of the displacement formulation consists of eliminating the reactive stresses from the equation of motion, and, moreover, to express them in terms of a reduced set of *free* displacement variables, able to describe the most general configuration of the body compatible with the constraints. The goal is similar to that of the analytical mechanics, in which we want to write the Lagrange equations of motion in terms of Lagrange parameters only. To this end, the constraint equations are solved (when possible, and maybe by a perturbation method) to express a set of *slave variables* as function of the remaining *master (or free) variables*. The relationship linking all displacements to the master displacements is called the *constraint for displacements*. By using it, the (active) strain–displacement relationships and the geometric boundary conditions are expressed in terms of master variables only, this operation being referred to as the *condensation of the kinematic equations*. When the constraints for displacements are time-differentiated, linear *constraints for velocities* are obtained (although nonlinear in the displacements, since referred to the current configuration). These relationships define a *(differential) velocity constraint operator*, which plays an important role in the formulation. To filter reactive stresses, we used the VPP, in which the velocity constraints were directly substituted (and not accounted for via Lagrange multipliers, as done in the mixed formulation!). The procedure leads to balance equations which are linear combinations of the original equation, able to automatically filter the reactive stresses. The linear operator acting on them is called the *equilibrium condensation operator*, which turns out to be the adjoint of the velocity constraint operator. When we combine the condensed kinematic and equilibrium equations, and we make use of the elastic law, final equilibrium equations, pure in the master variables, are obtained. Reactive stresses, if of interest, can be derived after having solved the elastic problem, by resorting to the non-condensed balance equations. Although they appear in an over-determined form, they can be solved, since the relevant compatibility condition is satisfied by the VPP itself!

In sections 1.4 and 1.5, we studied *prestressed beams*. These are bodies subjected to time-independent preloads which bring the beam into a prestressed configuration, which is taken as reference configuration, in place of the natural one. After that, incremental loads, possibly time-dependent, act on the beam, by bringing it into the current configuration. The main difference with the formulation of the stress-free beams relies on the elastic law, which becomes linear but non-homogeneous, to account for prestresses when the incremental strains are zero. To simplify the analysis, prestrains are usually neglected, i.e. the beam is assumed to undergo a prestress by keeping its original geometry. For these beams the linear approximation (according to the *linearized theory*) is of remarkable importance in the technical applications, since it allows us to solve important problems such as: to find the critical value of the load in buckling problems; to evaluate the eigenfrequencies of strings and cables; to determine the response of prestressed beam/cables to *small incremental loads*, and/or imperfections; i.e. solving *linear* problems in which, however, the *geometric stiffness*, related to the prestress, plays a non-negligible role. If the beam is internally unconstrained, the prestress simply adds an extra-term to the nonlinear equilibrium equation, with respect to the unprestressed case. If, in contrast, the beam is internally constrained, we have to distinguish: (a) in the mixed formulation, the incremental reactive stress also appears among the unknowns, while the prestress contributes to the stiffness of the beam; and (b) in the displacement formulation, all the reactive stresses, pre-existing and incremental, are filtered, so that only the active prestress appears in the stiffness.

In closing the chapter, all the previous models were reformulated by an alternative approach, the *variational formulation*. This consists of enforcing the stationary condition of the TPE functional, over the domain of the admissible displacements. Internal constraints can also be taken into account by introducing Lagrange multipliers. The approach only requires analyzing kinematics and elasticity, and furnishes the balance equations directly in terms of displacements, and, possibly, reactive stresses. As a drawback, it can only be used for conservative forces. Remarkably, its varied form is just the virtual work principle, which in contrast holds for any type of force. The linearized theory also admits a variational formulation, when the forces are conservative, in which only the second-order part of the TPE is retained. In particular, the geometric stiffness comes out of the work spent by the prestresses in the second-order part of the strains.

Tables 1.1 and 1.2 summarize the main results of the analysis carried out in the chapter. They report the solving equations for (a) the nonlinear Fundamental Problem, (b) for the linear/linearized problem, and (c) the relevant expressions for the TPE, for all cases examined: unconstrained/constrained, unprestressed/prestressed beams and, when appropriated, mixed/displacement formulations. The tables make it easy to compare formulas, and to appreciate the contributions of reactive stresses and/or prestresses.

| Unconstrained & Unprestressed beams | |
|---|---|
| Nonlinear Problem | Linear Problem |
| $\boldsymbol{D}^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{p}$<br>$\boldsymbol{\mathcal{D}}_H^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}_H\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{P}_H$<br>$\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H$ | $\boldsymbol{L}\boldsymbol{w}=\boldsymbol{p}_0$<br>$\boldsymbol{\mathcal{L}}_H\boldsymbol{w}=\boldsymbol{P}_{0H}$<br>where: $\boldsymbol{L}:=\boldsymbol{D}_0^\star\boldsymbol{E}\boldsymbol{D}_0,\quad \boldsymbol{\mathcal{L}}_H:=\boldsymbol{\mathcal{D}}_{0H}^\star\boldsymbol{E}\boldsymbol{D}_{0H}$ |
| **Constrained & Unprestressed beams: Mixed Formulation** | |
| Nonlinear Problem | Linear Problem |
| $\boldsymbol{D}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\boldsymbol{D}_c^\star\boldsymbol{\sigma}_c=\boldsymbol{p}$<br>$\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{0}$<br>$\boldsymbol{\mathcal{D}}_{uH}^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_{uH}+\boldsymbol{\mathcal{D}}_{cH}^\star\boldsymbol{\sigma}_c=\boldsymbol{P}_H$<br>$\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H$ | $\begin{pmatrix}\boldsymbol{L}_u & \boldsymbol{D}_{0c}^\star\\ \boldsymbol{D}_{0c} & \boldsymbol{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{w}\\ \boldsymbol{\sigma}_c\end{pmatrix}=\begin{pmatrix}\boldsymbol{p}_0\\ \boldsymbol{0}\end{pmatrix}$<br>$\boldsymbol{\mathcal{L}}_{uH}\boldsymbol{w}+\boldsymbol{\mathcal{D}}_{0cH}^\star\boldsymbol{\sigma}_c=\boldsymbol{P}_{0H}$<br>$\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H$<br>where: $\boldsymbol{L}_u:=\boldsymbol{D}_{0u}^\star\boldsymbol{E}_{uu}\boldsymbol{D}_{0u},\quad \boldsymbol{\mathcal{L}}_{uH}:=\left[\boldsymbol{\mathcal{D}}_{0u}^\star\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}\right]_H$ |
| **Constrained & Unprestressed beams: Displacement Formulation** | |
| Nonlinear Problem | Linear Problem |
| $\boldsymbol{A}^\star\boldsymbol{D}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u=\boldsymbol{A}^\star\boldsymbol{p}$<br>$\left[\boldsymbol{\mathcal{A}}^\star\boldsymbol{D}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\boldsymbol{\mathcal{D}}_{um}^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\right]_H=\left[\boldsymbol{P}_m+\boldsymbol{\mathcal{A}}^\star\boldsymbol{p}\right]_H$<br>$\left[\boldsymbol{\mathcal{D}}_{us}^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\right]_H=\left[\boldsymbol{P}_s\right]_H$<br>$\boldsymbol{w}_{mH}=\breve{\boldsymbol{w}}_{mH},\quad \boldsymbol{\mathcal{W}}_{sH}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)=\breve{\boldsymbol{w}}_{sH}$ | $\boldsymbol{A}_0^\star\boldsymbol{L}_u\boldsymbol{A}_0\boldsymbol{w}_m=\boldsymbol{A}_0^\star\boldsymbol{p}_0$<br>$\left[\boldsymbol{\mathcal{A}}_0^\star\boldsymbol{L}_u\boldsymbol{A}_0+\boldsymbol{\mathcal{L}}_{um}\boldsymbol{A}_0\right]_H\boldsymbol{w}_m=\left[\boldsymbol{P}_{0m}+\boldsymbol{\mathcal{A}}_0^\star\boldsymbol{p}_0\right]_H$<br>$\left[\boldsymbol{\mathcal{L}}_{us}\boldsymbol{A}_0\right]_H\boldsymbol{w}_m=\left[\boldsymbol{P}_{0s}\right]_H$<br>$\boldsymbol{w}_{mH}=\breve{\boldsymbol{w}}_{mH},\quad \left[\boldsymbol{A}_{0s}\boldsymbol{w}_m\right]_H=\breve{\boldsymbol{w}}_{sH}$<br>where: $\boldsymbol{L}_u:=\boldsymbol{D}_{0u}^\star\boldsymbol{E}_{uu}\boldsymbol{D}_{0u},\ \boldsymbol{\mathcal{L}}_{umH}:=\left[\boldsymbol{\mathcal{D}}_{0um}^\star\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}\right]_H$<br>$\boldsymbol{\mathcal{L}}_{usH}:=\left[\boldsymbol{\mathcal{D}}_{0us}^\star\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}\right]_H$ |
| **Unconstrained & Prestressed beams** | |
| Nonlinear Problem | Linearized Problem |
| $\boldsymbol{D}^\star\boldsymbol{E}\boldsymbol{\mathcal{E}}+\left(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}}-\mathring{\boldsymbol{p}}\right)=\tilde{\boldsymbol{p}}$<br>$\boldsymbol{\mathcal{D}}_H^\star\boldsymbol{E}\boldsymbol{\mathcal{E}}_H+\left(\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}}-\mathring{\boldsymbol{P}}_H\right)=\tilde{\boldsymbol{P}}_H$<br>$\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H$ | $\boldsymbol{L}\boldsymbol{w}+\boldsymbol{G}\boldsymbol{w}=\tilde{\boldsymbol{p}}_0$<br>$\boldsymbol{\mathcal{L}}_H\boldsymbol{w}+\boldsymbol{\mathcal{G}}_H\boldsymbol{w}=\tilde{\boldsymbol{P}}_0$<br>$\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H$<br>where: $\boldsymbol{L}:=\boldsymbol{D}_0^\star\boldsymbol{E}\boldsymbol{D}_0,\quad \boldsymbol{\mathcal{L}}_H:=\boldsymbol{\mathcal{D}}_{0H}^\star\boldsymbol{E}\boldsymbol{D}_{0H}$<br>$\boldsymbol{G}:=\left(\frac{\partial\left(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}}\right)_0+\left(\frac{\partial\left(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}'}\right)_0\frac{\partial}{\partial s}-\left(\frac{\partial\mathring{\boldsymbol{p}}}{\partial\boldsymbol{w}}\right)_0$<br>$\boldsymbol{\mathcal{G}}_H:=\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}}\right)_0+\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}'}\right)_0\frac{\partial}{\partial s}-\left(\frac{\partial\mathring{\boldsymbol{P}}_H}{\partial\boldsymbol{w}}\right)_0$ |

Table 1.1: The Fundamental Problem: nonlinear and linear/linearized equations

| Constrained & Prestressed beams: Mixed Formulation | |
|---|---|
| Nonlinear Problem | Linearized Problem |
| $\boldsymbol{D}_u^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{D}_c^\star \tilde{\boldsymbol{\sigma}}_c + (\boldsymbol{D}^\star \mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{p}}) = \tilde{\boldsymbol{p}}$<br>$\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w},\boldsymbol{w}'\right) = \boldsymbol{0}$<br>$\boldsymbol{\mathcal{D}}_{uH}^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_{uH} + \boldsymbol{\mathcal{D}}_{cH}^\star \tilde{\boldsymbol{\sigma}}_c + \left(\boldsymbol{\mathcal{D}}_H^\star \mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{P}}_H\right) = \tilde{\boldsymbol{P}}_H$<br>$\boldsymbol{w}_H = \breve{\boldsymbol{w}}_H$ | $\left(\begin{pmatrix} \boldsymbol{L}_u & \boldsymbol{D}_{0c}^\star \\ \boldsymbol{D}_{0c} & \boldsymbol{0} \end{pmatrix} + \begin{pmatrix} \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\right)\begin{pmatrix} \boldsymbol{w} \\ \tilde{\boldsymbol{\sigma}}_c \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{p}}_0 \\ \boldsymbol{0} \end{pmatrix}$<br>$\boldsymbol{\mathcal{L}}_{uH}\boldsymbol{w} + \boldsymbol{\mathcal{G}}_H \boldsymbol{w} + \boldsymbol{\mathcal{D}}_{0cH}^\star \tilde{\boldsymbol{\sigma}}_c = \tilde{\boldsymbol{P}}_{0H}$<br>$\boldsymbol{w}_H = \breve{\boldsymbol{w}}_H$<br>where: $\boldsymbol{L}_u := \boldsymbol{D}_{0u}^\star \boldsymbol{E}_{uu}\boldsymbol{D}_{0u}$,<br>$\boldsymbol{\mathcal{L}}_{uH} := [\boldsymbol{\mathcal{D}}_{0u}^\star \boldsymbol{E}_{uu}\boldsymbol{D}_{0u}]_H$<br>$\boldsymbol{G} := \left(\frac{\partial(\boldsymbol{D}^\star \mathring{\boldsymbol{\sigma}})}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial(\boldsymbol{D}^\star \mathring{\boldsymbol{\sigma}})}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s} - \left(\frac{\partial \mathring{\boldsymbol{p}}}{\partial \boldsymbol{w}}\right)_0$<br>$\boldsymbol{\mathcal{G}}_H := \left(\frac{\partial(\boldsymbol{\mathcal{D}}_H^\star \mathring{\boldsymbol{\sigma}})}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial(\boldsymbol{\mathcal{D}}_H^\star \mathring{\boldsymbol{\sigma}})}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s} - \left(\frac{\partial \mathring{\boldsymbol{P}}_H}{\partial \boldsymbol{w}}\right)_0$ |

| Constrained & Prestressed beams: Displacement Formulation | |
|---|---|
| Nonlinear Problem | Linearized Problem |
| $\boldsymbol{A}^\star \boldsymbol{D}_u^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{A}^\star\left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{p}}\right) = \boldsymbol{A}^\star \tilde{\boldsymbol{p}}$<br>$\Big[\boldsymbol{\mathcal{A}}^\star \boldsymbol{D}_u^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{\mathcal{D}}_{um}^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{\mathcal{A}}^\star\left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{p}}\right)$<br>$+\left(\boldsymbol{\mathcal{D}}_{um}^\star \mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}_m\right)\Big]_H = \left[\tilde{\boldsymbol{P}}_m + \boldsymbol{\mathcal{A}}^\star \tilde{\boldsymbol{p}}\right]_H$<br>$\left[\boldsymbol{\mathcal{D}}_{us}^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \left(\boldsymbol{\mathcal{D}}_{us}^\star \mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}_s\right)\right]_H = \left[\tilde{\boldsymbol{P}}_s\right]_H$<br>$\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad \boldsymbol{\mathcal{W}}_{sH}\left(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots\right) = \breve{\boldsymbol{w}}_{sH}$ | $\boldsymbol{A}_0^\star \boldsymbol{L}_u \boldsymbol{A}_0 \boldsymbol{w}_m + \boldsymbol{A}_0^\star \boldsymbol{G}_u \boldsymbol{A}_0 \boldsymbol{w}_m = \boldsymbol{A}_0^\star \tilde{\boldsymbol{p}}_0$<br>$[\boldsymbol{\mathcal{A}}_0^\star \boldsymbol{L}_u \boldsymbol{A}_0 + \boldsymbol{\mathcal{L}}_{um}\boldsymbol{A}_0 + \boldsymbol{\mathcal{A}}_0^\star \boldsymbol{G}_u \boldsymbol{A}_0 + \boldsymbol{\mathcal{G}}_{um}\boldsymbol{A}_0]_H\, \boldsymbol{w}_m$<br>$= \left[\tilde{\boldsymbol{P}}_{0m} + \boldsymbol{\mathcal{A}}_0^\star \tilde{\boldsymbol{p}}_0\right]_H$<br>$[\boldsymbol{\mathcal{L}}_{us}\boldsymbol{A}_0 + \boldsymbol{\mathcal{G}}_{us}\boldsymbol{A}_0]_H\, \boldsymbol{w}_m = \left[\tilde{\boldsymbol{P}}_{0s}\right]_H$<br>$\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad [\boldsymbol{A}_{0s}\boldsymbol{w}_m]_H = \breve{\boldsymbol{w}}_{sH}$<br>where: $\boldsymbol{L}_u := \boldsymbol{D}_{0u}^\star \boldsymbol{E}_{uu}\boldsymbol{D}_{0u}$,<br>$\boldsymbol{\mathcal{L}}_{umH} := [\boldsymbol{\mathcal{D}}_{0um}^\star \boldsymbol{E}_{uu}\boldsymbol{D}_{0u}]_H$<br>$\boldsymbol{\mathcal{L}}_{usH} := [\boldsymbol{\mathcal{D}}_{0us}^\star \boldsymbol{E}_{uu}\boldsymbol{D}_{0u}]_H$<br>$\boldsymbol{G}_u := \left(\frac{\partial(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u)}{\partial \boldsymbol{w}}\right)_0$<br>$+\left(\frac{\partial(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u)}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s} - \left(\frac{\partial \mathring{\boldsymbol{p}}}{\partial \boldsymbol{w}}\right)_0$<br>$\boldsymbol{\mathcal{G}}_{uH} := \left(\frac{\partial(\boldsymbol{\mathcal{D}}_{uH}^\star \mathring{\boldsymbol{\sigma}}_u)}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial(\boldsymbol{\mathcal{D}}_{uH}^\star \mathring{\boldsymbol{\sigma}}_u)}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s}$<br>$-\left(\frac{\partial \mathring{\boldsymbol{P}}_H}{\partial \boldsymbol{w}}\right)_0 + \ldots$ |

Table 1.1: (Continued) The Fundamental Problem: nonlinear and linear/linearized equations

| Unconstrained & Unprestressed beams | |
|---|---|
| Nonlinear Problem | Linear Problem |
| $\Pi\left[\boldsymbol{w}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H$ | $\Pi\left[\boldsymbol{w}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}^{(1)T}\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}^{(1)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H$<br>where: $\boldsymbol{\mathcal{E}}^{(1)} = \boldsymbol{D}_0\boldsymbol{w}$ |
| **Constrained & Unprestressed beams: Mixed Formulation** | |
| Nonlinear Problem | Linear Problem |
| $\breve{\Pi}\left[\boldsymbol{w},\boldsymbol{\lambda}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H + \int_{\mathcal{S}}\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$ | $\breve{\Pi}\left[\boldsymbol{w},\boldsymbol{\lambda}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^{(1)T}\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u^{(1)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H + \int_{\mathcal{S}}\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c^{(1)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>where: $\boldsymbol{\mathcal{E}}_u^{(1)} = \boldsymbol{D}_{0u}\boldsymbol{w}, \quad \boldsymbol{\mathcal{E}}_c^{(1)} = \boldsymbol{D}_{0c}\boldsymbol{w}$ |
| **Constrained & Unprestressed beams: Displacement Formulation** | |
| Nonlinear Problem | Linear Problem |
| $\Pi_u\left[\boldsymbol{w}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H$<br>$\boldsymbol{w} = \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}'_m,\ldots\right)$ | $\Pi_u\left[\boldsymbol{w}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^{(1)T}\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u^{(1)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H, \quad \boldsymbol{w} = \boldsymbol{A}_0\boldsymbol{w}_m$<br>where: $\boldsymbol{\mathcal{E}}_u^{(1)} = \boldsymbol{D}_{0u}\boldsymbol{w}$ |
| **Unconstrained & Prestressed beams** | |
| Nonlinear Problem | Linearized Problem |
| $\Pi\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H + \int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}^T\boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$-\int_{\mathcal{S}} \boldsymbol{w}^T\mathring{\boldsymbol{p}}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\mathring{\boldsymbol{P}}_H$ | $\Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] := \int_{\mathcal{S}}\left(\frac{1}{2}\boldsymbol{\mathcal{E}}^{(1)T}\boldsymbol{E}\boldsymbol{\mathcal{E}}^{(1)} + \mathring{\boldsymbol{\sigma}}^T\boldsymbol{\mathcal{E}}^{(2)}\right)ds$<br>$-\int_{\mathcal{S}} \boldsymbol{w}^T\tilde{\boldsymbol{p}}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\tilde{\boldsymbol{P}}_H$ |

Table 1.2: The Variational formulation: the EPT functional for nonlinear and linear/linearized theories

<table>
<tr><th colspan="2">Constrained & Prestressed beams: Mixed Formulation</th></tr>
<tr><td>Nonlinear Problem</td><td>Linearized Problem</td></tr>
<tr><td>$\breve{\Pi}\left[\boldsymbol{w},\boldsymbol{\lambda};\mathring{\boldsymbol{\sigma}}_u\right] := \frac{1}{2}\int_{\mathcal{S}}\boldsymbol{\mathcal{E}}_u^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$-\int_{\mathcal{S}}\boldsymbol{w}^T\boldsymbol{p}ds-\sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H+\int_{\mathcal{S}}\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$+\int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}_u^T\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds-\int_{\mathcal{S}}\boldsymbol{w}^T\mathring{\boldsymbol{p}}ds$<br>$-\sum_{H=A}^{B}\boldsymbol{w}_H^T\mathring{\boldsymbol{P}}_H$</td><td>$\breve{\Pi}^{(2)}\left[\boldsymbol{w},\tilde{\boldsymbol{\sigma}}_c;\mathring{\boldsymbol{\sigma}}\right] := \int_{\mathcal{S}}\boldsymbol{\mathcal{E}}_u^{(1)T}(\boldsymbol{w},\boldsymbol{w}')\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u^{(1)}(\boldsymbol{w},\boldsymbol{w}')ds$<br>$-\int_{\mathcal{S}}\boldsymbol{w}^T\tilde{\boldsymbol{p}}ds-\sum_{H=A}^{B}\boldsymbol{w}_H^T\tilde{\boldsymbol{P}}_H$<br>$+\int_{\mathcal{S}}\left(\mathring{\boldsymbol{\sigma}}^T\boldsymbol{\mathcal{E}}^{(2)}+\tilde{\boldsymbol{\sigma}}_c\boldsymbol{\mathcal{E}}_c^{(1)}\right)ds$</td></tr>
</table>

<table>
<tr><th colspan="2">Constrained & Prestressed beams: Displacement Formulation</th></tr>
<tr><td>Nonlinear Problem</td><td>Linearized Problem</td></tr>
<tr><td>$\Pi\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] := \frac{1}{2}\int_{\mathcal{S}}\boldsymbol{\mathcal{E}}_u^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$-\int_{\mathcal{S}}\boldsymbol{w}^T\boldsymbol{p}ds-\sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H$<br>$+\int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}_u^T\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds-\int_{\mathcal{S}}\boldsymbol{w}^T\mathring{\boldsymbol{p}}ds-\sum_{H=A}^{B}\boldsymbol{w}_H^T\mathring{\boldsymbol{P}}_H$<br>$\boldsymbol{w}=\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}'_m,\ldots\right)$</td><td>$\Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}_u\right] := \int_{\mathcal{S}}\boldsymbol{\mathcal{E}}_u^{(1)T}(\boldsymbol{w},\boldsymbol{w}')\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u(\boldsymbol{w},\boldsymbol{w}')ds$<br>$-\int_{\mathcal{S}}\boldsymbol{w}^T\tilde{\boldsymbol{p}}ds-\sum_{H=A}^{B}\boldsymbol{w}_H^T\tilde{\boldsymbol{P}}_H+\int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}_u^T\boldsymbol{\mathcal{E}}_u^{(2)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$<br>$\boldsymbol{w}=\boldsymbol{A}_0\boldsymbol{w}_m$</td></tr>
</table>

Table 1.2: (Continued) The Variational formulation: the EPT functional for nonlinear and linear/linearized theories