1

Skin Friction Lines Pattern and Critical Points

This chapter introduces the basic physical features underlying the topological description of three-dimensional flows, such as the skin friction lines and the surface flow pattern revealed by surface flow visualizations. It is shown that the construction of the skin friction lines reveals the existence of critical points where the skin friction vector vanishes. The mathematical properties of these points are examined and their physical meaning emphasized.

1.1. Basic properties of the three-dimensional boundary layer

The notion of two-dimensional flow, and *a fortiori* of one-dimensional flow, is an approximation of reality allowing important mathematical simplifications. In addition, it is far easier to conceptualize and conceive a two-dimensional picture than a three-dimensional field. Even with the most modern approaches where the Navier–Stokes equations are solved by a numerical method, the cost of a three-dimensional calculation can be so high that a two-dimensional model must be considered. However, the real world being three-dimensional, this simplification is only justified for a limited number of practical situations (an axisymmetric propulsive nozzle, for example). This is particularly true when the flow separates, the field then nearly always adopting a three-dimensional organization.

The separation phenomenon is linked to the viscous nature of the fluid. In usual aerodynamic flows, where Reynolds numbers are very high (greater then 10^5), viscous effects are normally confined within a thin boundary layer in contact with the obstacle surface. In this boundary layer, the flow velocity changes from the outer value u_e to the value zero imposed by the no-slip condition at the wall. In two-dimensional flow, the evolution of the streamwise velocity component u through the boundary layer is expressed by the relation $u / u_e = f(y)$, where y is the distance from the wall (if the flow is attached, the normal velocity component v plays a minor role). In three dimensions, the velocity vector is no longer constrained to remain in a plane but can rotate of an angle β relative to its direction at the boundary-layer outer edge. It is usual to define the velocity evolution through a three-dimensional boundary layer in a local curvilinear coordinate system, where the longitudinal axis Ox is aligned with the outer velocity vector $\vec{V}_e$, the transverse axis Oz directly perpendicular to Ox and contained in a plane tangent to the obstacle surface, the third axis Oy being normal to the surface (see Figure 1.1). The velocity distribution through the boundary layer is defined by two functions (the normal component v not being considered):

– a profile for the streamwise velocity component: $u / u_e = f(y)$;

– a profile for the transverse (crosswise) component: $w / u_e = g(y)$.

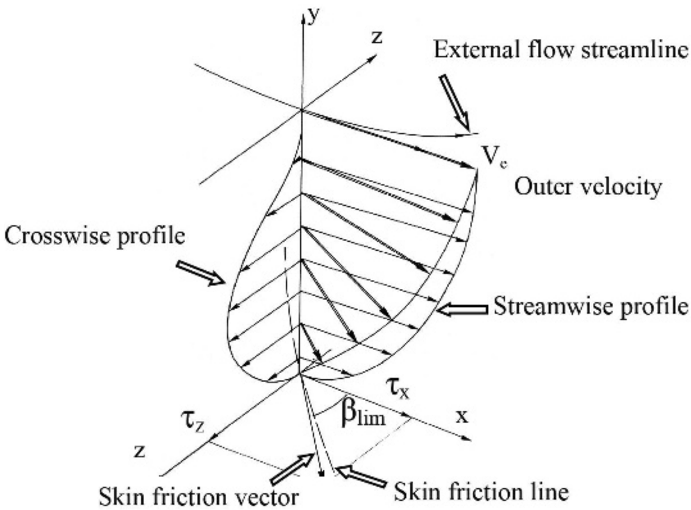


Figure 1.1. *Structure of a three-dimensional boundary layer*

In the chosen coordinate system, *w* is zero at the boundary-layer outer edge. The *w* profile characterizes the boundary-layer distortion; that is, the importance of the three-dimensional effect. This effect can be characterized by the distortion angle β defined by the relation:

$$\beta = \tan^{-1}(w/u)$$

On approaching the wall, the components *u* and *w* both tend to zero so that we can write the following first-order expansion:

$$u = y\left(\frac{\partial u}{\partial y}\right)_w \quad w = y\left(\frac{\partial w}{\partial y}\right)_w$$

Hence, the limit value for β at the wall is:

$$\beta_{\text{lim}} = \tan^{-1}\left[\left(\frac{\partial w}{\partial y}\right)_w \Big/ \left(\frac{\partial u}{\partial y}\right)_w\right] = \tan^{-1}\ (\tau_z/\tau_x)$$

In the framework of the classical boundary-layer approximation, the components of the skin friction vector at the wall are given by (assuming a Newtonian fluid, which is the case in classical aerodynamics):

$$\tau_x = \mu_w \left(\frac{\partial u}{\partial y} \right)_w \quad \tau_z = \mu_w \left(\frac{\partial w}{\partial y} \right)_w$$

The main property of a three-dimensional flow is the capacity for the boundary layer to develop a transverse component, whereas in two dimension it is constrained to remain in a plane. Under the action of various forces (adverse pressure gradient, in general), a boundary layer having initially two-dimensional properties can develop a transverse profile until a state called *separated* is reached. Existence of a third dimension offers to the flow the possibility to "escape" laterally when it is confronted with an adverse circumstance (most often a pressure rise). Understanding this point has a fundamental importance to thus understand the observed behaviors. In two dimension, the flow must remain in a plane, any possibility to escape in the transverse direction being forbidden for it. Consequently, the gradients associated with separation (compression and deceleration) are much more intense in two dimension than in three dimension. This fact can lead to modeling difficulties (that of turbulence in particular) in great part artificial, since the situations met in reality are rarely two dimensional.

In what follows, we will analyse the separation phenomenon in three-dimensional flows. The two-dimensional case will be reconsidered in Chapter 6 to realize that description of these flows in fact leads to a far greater conceptual complication. We will restrict ourselves to steady flows whose properties (in particular, the velocity field) are independent of time. As in reality flows are most often turbulent – hence unsteady – so we will consider the field resulting from some kind of time averaging for the fluctuating quantities. Although in great part artificial, this assumption is the assumption given by classical measurement techniques, visualization methods (except short-time techniques) and calculations using the time-averaged Navier–Stokes equations (the so-called Reynolds Averaged Navier-Stokes or RANS approach). The considerations that follow can be applied to an unsteady flow by considering an instantaneous picture of the flow field, as the picture given by particle image velocimetry (PIV).

The theory presented later is not a predictive theory since it applies to a vector field given either by experiment or by calculation. It is a descriptive theory used as a support to a consistent description of the field; that is, free of features in contradiction with elementary topological rules.

1.2. Skin friction lines and surface flow pattern

Usually, separation in three-dimensional flows is defined by considering the properties of the flow at the surface of the obstacle. Then, we can introduce concepts allowing a rational definition of separation having the advantage of being experimentally observed. Then, we will see how it is possible to go to the flow field structure in space from its trace on the surface.

Let us consider a body delimited by a surface (S) and define an orthonormal local system $(\vec{n}, \vec{i}, \vec{j})$ made up of the unit vector $\vec{n}$ normal to the surface and two unit vectors $\vec{i}$ and $\vec{j}$ contained in the plane tangent to (S) at the considered point. Let us designate by (x, y, z) the corresponding coordinates, y being the distance normal to the surface. The fluid flowing past (S) exerts on a surface element dS of (S) a force $\vec{F} = \vec{P} dS$, where $\vec{P}$ is the vector tension that is decomposed in the form:

$$\vec{P} = -p\vec{n} + \vec{\tau}$$

The term $-p\vec{n}$ is the normal action of pressure and $\vec{\tau}$ is the shear stress tangential action. For non-viscous fluids, tension reduces to its normal component. The shear stress $\vec{\tau}$ can be written in the form:

$$\vec{\tau} = \tau_x \vec{i} + \tau_z \vec{j}$$

Thus, in a three-dimensional flow, the wall shear stress is a vector, whereas in two dimensions it is considered as a scalar quantity. The shear stress exerted on the surface (*S*) of the body constitutes a vector field, the problem being to determine the lines of force or trajectories of this vector field. As for streamlines, these lines are defined as the solution curves of the differential system:

$$\frac{dx}{\tau_x(x,z)} = \frac{dz}{\tau_z(x,z)} = dt \qquad [1.1]$$

where t is an integration parameter. We call *skin friction lines* the curves solution of the system [1.1], such curves having the property to be tangent to the local skin friction vector at the contact point. Sometimes, we use the terminology wall streamline (or limit streamline), which is defined as the limit position of a streamline when its distance to the surface tends to zero. In fact, this concept is fictitious since the velocity is null on the surface. As demonstrated above, the limit position of the velocity vector $\vec{V}$ coincides with the shear stress vector, hence:

$$u(x,z,y) = \left(\frac{\partial u}{\partial y}\right)_w y \approx \tau_x y$$

$$w(x,z,y) = \left(\frac{\partial w}{\partial y}\right)_w y \approx \tau_z y$$

Consequently, on approaching the body surface, the streamlines tend to the lines of force of the vector field $\vec{\tau}_w$;

that is, to the skin friction lines. Although the two concepts are equivalent, it is more convenient to use the concept of skin friction lines that can be constructed without difficulty (in principle) from the vectors field $\vec{\tau}_w$ determined from either measurements or calculations. Furthermore, the traces observed on a model covered with a viscous film can be identified with skin friction lines if some precautions are taken in the interpretation of the picture. Figure 1.2 shows the skin friction line pattern visualized with a viscous product on a flattened cylindrical ellipsoid tested in a subsonic wind tunnel.

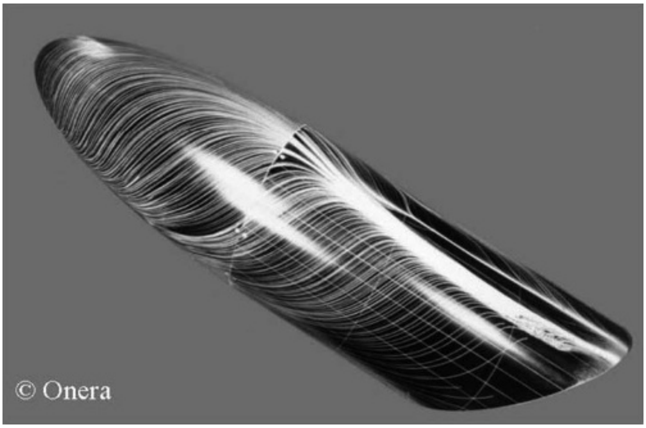


Figure 1.2. *Visualization of the skin friction line pattern over a flattened cylindrical ellipsoid*

We will call surface-flow pattern the shear stress vector field on the surface of a body and the skin friction lines covering the body. The terminology is questionable since there is no flow on the surface but it is commonly used so we will comply with that habit.

Let us consider the vorticity vector $\vec{\Omega}$ whose components are:

$$\omega_x = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}, \omega_z = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}, \omega_y = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$$

On a wall:

$$\omega_x = -\left(\frac{\partial w}{\partial y}\right)_w, \omega_z = \left(\frac{\partial u}{\partial y}\right)_w, \omega_y = 0$$

Hence the classical results:

The vorticity $\vec{\Omega}_w$ on a body is tangent to the surface and perpendicular to the shear stress vector $\vec{\tau}_w$. The lines of force of vorticity (vortex lines) on the surface are the orthogonal trajectories of the skin friction lines.

1.3. Critical points of the skin friction line pattern

1.3.1. *General solution and the eigenvalue problem*

As seen in the above section, the skin friction lines on a body are solutions of the system [1.1] and, in general, only one skin friction line goes through one point on the surface defined, either by an equation of the form:

$$f(x, z) = Cst.$$

or by the parametric form:

$$x = x(t)$$

$$z = z(t)$$

where t is a parameter such that the skin friction line is traveled by making t vary from $-\infty$ to $+\infty$. In practice, it is exceptional to obtain an analytical definition of the skin friction lines; they must be determined by a step-by-step progress through the skin friction vector field $\vec{\tau}$, as it is done to build streamlines in a velocity field.

The points that do not satisfy the general rule for the tracing of the skin friction lines are the *critical points* (or singular points) of the system [1.1] where simultaneously:

$$\tau_x(x,z) = 0, \tau_z(x,z) = 0$$

To study the skin friction lines behavior near a critical point P_0 of coordinates (x_0, z_0) we make a first-order expansion in the vicinity of P_0:

$$\tau_x = \left(\frac{\partial \tau_x}{\partial x}\right)_{P_0} (x - x_0) + \left(\frac{\partial \tau_x}{\partial z}\right)_{P_0} (z - z_0)$$

$$\tau_z = \left(\frac{\partial \tau_z}{\partial x}\right)_{P_0} (x - x_0) + \left(\frac{\partial \tau_z}{\partial z}\right)_{P_0} (z - z_0)$$

In what follows, it is assumed that the first-order derivatives of the components τ_x and τ_z are non-zero, such a circumstance necessitating higher order expansions. In addition, for the sake of simplicity, the origin of the coordinate system is placed at the critical point P_0, which does not restrict the generality of the problem (this simply consists of making the change of variables $x' = x - x_0$, $z' = z - z_0$; however, to keep simpler notations we will not use primed quantities). At the point P_0, the system [1.1] is written as:

$$\frac{dx}{\left(\frac{\partial \tau_x}{\partial x}\right)_{P_0} x + \left(\frac{\partial \tau_x}{\partial z}\right)_{P_0} z} = \frac{dz}{\left(\frac{\partial \tau_z}{\partial x}\right)_{P_0} x + \left(\frac{\partial \tau_z}{\partial z}\right)_{P_0} z}$$

If λ and μ are two constants, we can write:

$$\frac{dx}{\left(\frac{\partial \tau_x}{\partial x}\right)_{P_0} x + \left(\frac{\partial \tau_x}{\partial z}\right)_{P_0} z} = \frac{dz}{\left(\frac{\partial \tau_z}{\partial x}\right)_{P_0} x + \left(\frac{\partial \tau_z}{\partial z}\right)_{P_0} z}$$

$$= \frac{\lambda dx + \mu dz}{\lambda \left[\left(\frac{\partial \tau_x}{\partial x}\right)_{P_0} x + \left(\frac{\partial \tau_x}{\partial z}\right)_{P_0} z\right] + \mu \left[\left(\frac{\partial \tau_z}{\partial x}\right)_{P_0} x + \left(\frac{\partial \tau_z}{\partial z}\right)_{P_0} z\right]}$$

A solution of the system is looked for by writing the above expression in the form of a logarithmic derivative:

$$\frac{\lambda dx + \mu dz}{\lambda \left[\left(\frac{\partial \tau_x}{\partial x}\right)_{P_0} x + \left(\frac{\partial \tau_x}{\partial z}\right)_{P_0} z\right] + \mu \left[\left(\frac{\partial \tau_z}{\partial x}\right)_{P_0} x + \left(\frac{\partial \tau_z}{\partial z}\right)_{P_0} z\right]} = \frac{d(\lambda dx + \mu dz)}{S(\lambda x + \mu z)}$$

where S is a constant. If this form is possible, then by putting:

$$\frac{d(\lambda dx + \mu dz)}{S(\lambda x + \mu z)} = -dt$$

we have:

$$\lambda x + \mu z = A \exp(-St)$$

where t is the integration variable and A is a constant.

For the logarithmic form to be possible, we must simultaneously have:

$$\left[\left(\frac{\partial \tau_x}{\partial x}\right)_{P_0} - S\right]\lambda + \left(\frac{\partial \tau_z}{\partial x}\right)_{P_0} \mu = 0$$

$$\left(\frac{\partial \tau_x}{\partial z}\right)_{P_0} \lambda + \left[\left(\frac{\partial \tau_z}{\partial z}\right)_{P_0} - S\right]\mu = 0$$

This linear algebraic system, homogeneous for S, admits non-trivial solutions if its determinant is non-zero; hence the condition on S:

$$\begin{vmatrix} \left(\frac{\partial \tau_x}{\partial x}\right)_{P_0} - S & \left(\frac{\partial \tau_z}{\partial x}\right)_{P_0} \\ \left(\frac{\partial \tau_x}{\partial z}\right)_{P_0} & \left(\frac{\partial \tau_z}{\partial z}\right)_{P_0} - S \end{vmatrix} = 0$$

The skin friction lines behavior in the vicinity of the critical point is dictated by the nature of the solutions for the above second-degree equations for S. Before continuing the discussion, let us introduce the Jacobian matrix:

$$F = \begin{vmatrix} \left(\frac{\partial \tau_x}{\partial x}\right)_{P_0} & \left(\frac{\partial \tau_z}{\partial x}\right)_{P_0} \\ \left(\frac{\partial \tau_x}{\partial z}\right)_{P_0} & \left(\frac{\partial \tau_z}{\partial z}\right)_{P_0} \end{vmatrix}$$

and put:

$$p \equiv -\text{ trace of } F = -\left[\left(\frac{\partial \tau_x}{\partial x}\right)_{P_0} + \left(\frac{\partial \tau_z}{\partial z}\right)_{P_0}\right]$$

$$q \equiv \text{determinant of } F = \left(\frac{\partial \tau_x}{\partial x}\right)_{P_0}\left(\frac{\partial \tau_z}{\partial z}\right)_{P_0} - \left(\frac{\partial \tau_z}{\partial x}\right)_{P_0}\left(\frac{\partial \tau_x}{\partial z}\right)_{P_0}$$

The equation giving the eigenvalues S_1 and S_2 can be written in the condensed form as:

$$S^2 + pS + q = 0 \qquad [1.2]$$

Hence the solutions:

$$S_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

To any eigenvalue $S_{1,2}$ corresponds an *eigenvector* of components $\left[\lambda_{1,2}, \mu_{1,2}\right]$. The eigenvalues S_1 and S_2 depend only on p and q, which means that the Jacobian matrix is the only function of the local properties of the surface flow at the critical point, such properties (the derivatives of the shear stress components at P_0) resulting from a Navier–Stokes calculation or experiments. The above classical analysis allows us to solve the indeterminate form taken by the system [1.1] at the critical point when the shear stress vector vanishes. It allows the construction of the skin friction lines in the vicinity of P_0. The study of the solution near a critical point leads to an eigenvalue problem (here S_1 and S_2) similar to the problem encountered in linear stability analyses. The roots S_1 and S_2 are the eigenvalues of the Jacobian matrix and the associated solutions the eigenvectors of this matrix.

To S_1 and S_2 correspond an infinity of solutions for which λ and μ are proportional, we can choose:

$$\lambda_1 = -\left(\frac{\partial \tau_z}{\partial x}\right)_{P_0}, \qquad \mu_1 = \left(\frac{\partial \tau_x}{\partial x}\right)_{P_0} - S_1 \qquad \text{for} \quad S_1$$

and

$$\lambda_2 = \left(\frac{\partial \tau_z}{\partial z}\right)_{P_0} - S_2, \qquad \mu_2 = \left(\frac{\partial \tau_x}{\partial z}\right)_{P_0} \qquad \text{for} \quad S_2$$

Thus, the solution determining the skin friction lines will be of the form:

$$\lambda_1 x + \mu_1 z = A_1 \exp(-S_1 t)$$

$$\lambda_2 x + \mu_2 z = A_2 \exp(-S_2 t) \qquad [1.3]$$

where A_1 and A_2 are up to now undetermined constants. The solution of system [1.3] in term of (x, z) gives:

$$x(t) = \frac{A_1 \mu_2 \exp(-S_1 t) - A_2 \mu_1 \exp(-S_2 t)}{\lambda_1 \mu_2 - \lambda_2 \mu_1}$$

$$z(t) = -\frac{A_1 \lambda_2 \exp(-S_1 t) - A_2 \lambda_1 \exp(-S_2 t)}{\lambda_1 \mu_2 - \lambda_2 \mu_1} \qquad [1.4]$$

The slope of the skin friction lines at point P_0 is given by:

$$\frac{dz}{dx} = -\frac{\lambda_2 - \dfrac{S_2 A_2}{S_1 A_1} \lambda_1 \exp\left[-(S_2 - S_1)t\right]}{\mu_2 - \dfrac{S_2 A_2}{S_1 A_1} \mu_1 \exp\left[-(S_2 - S_1)t\right]}$$

The shape of the solutions depends on the nature (real or complex) and sign of the eigenvalues at point P_0. The solution behavior can be discussed by considering the plane $[p,q]$ where the parabola (P) of equation (see Figure 1.3) is plotted:

$$q = \frac{p^2}{4}$$

The surface flow at P_0 has an "image" in this plane so that according to the location of this image relative to the parabola (P) and to the axes $p = 0$ and $q = 0$, the skin friction lines will have specific behaviors.

Considering the expression of vorticity given earlier, we immediately conclude:

Vorticity being zero where the shear stress vector vanishes, the critical points of the skin friction line pattern are also critical points for the vorticity-line pattern on the wall.

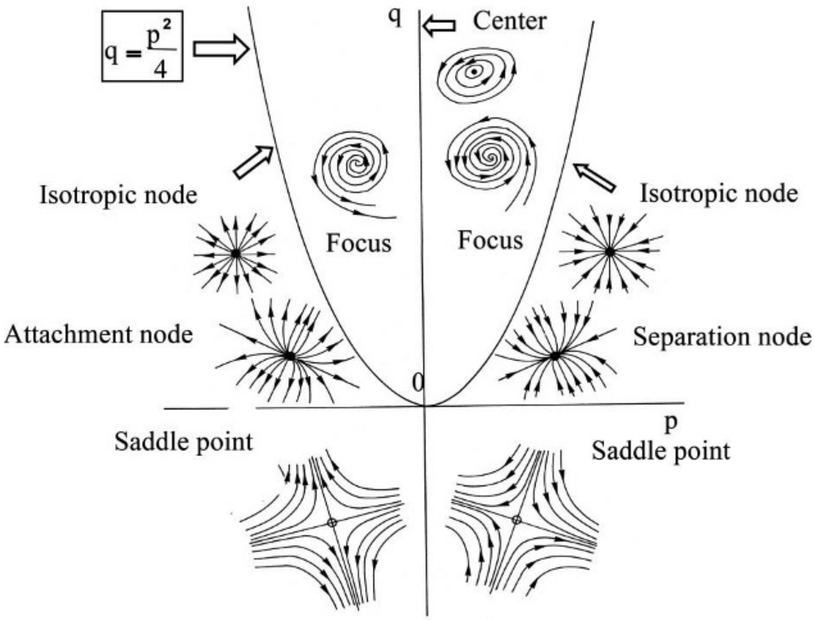


Figure 1.3. *Critical points classification in the plane* $[p, q]$

1.3.2. *The different critical points*

1.3.2.1. *Critical point of the node type*

In the present case, the two roots S_1 and S_2 are real, distinct with the same sign, this occurs if:

$$p^2 - 4q > 0, q > 0$$

The representative point in the plane $[p,q]$ is below the parabola (the roots being real) and above the axis $q = 0$ (since they have the same sign).

Let us examine the solution behavior in the vicinity of P_0 by assuming $S_2 > S_1$. The relations [1.4] show that all the skin friction lines go through P_0 which is reached when $t \to \infty$. Indeed, we can easily verify that $x(t)$ and $z(t)$ simultaneously tend to 0 when $t \to \infty$. In addition, all the skin friction lines have the same slope at P_0, which is given by:

$$\left.\frac{dz}{dx}\right|_{P_0} = -\frac{\lambda_2}{\mu_2}$$

Let us consider the lines corresponding to $A_1 = 0$ and $A_2 = 0$. If $A_1 = 0$:

$$x(t) = -\frac{A_2\mu_1 \exp(-S_2 t)}{\lambda_1\mu_2 - \lambda_2\mu_1} \qquad z(t) = \frac{A_1\lambda_2 \exp(-S_2 t)}{\lambda_1\mu_2 - \lambda_2\mu_1}$$

Then: $\left.\dfrac{dz}{dx}\right|_{P_0} = -\dfrac{\lambda_1}{\mu_1}$

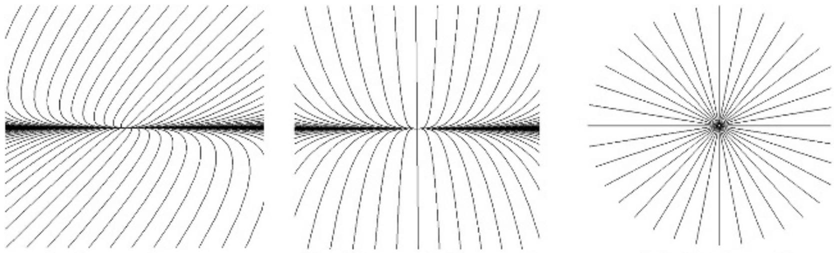
If $A_2 = 0$:

$$x(t) = \frac{A_1\mu_2 \exp(-S_1 t)}{\lambda_1\mu_2 - \lambda_2\mu_1} \qquad z(t) = -\frac{A_1\mu_2 \exp(-S_1 t)}{\lambda_1\mu_2 - \lambda_2\mu_1}$$

and:

$$\left.\frac{dz}{dx}\right|_{P_0} = -\frac{\lambda_2}{\mu_2}$$

Consequently, at P_0 all the skin friction lines are tangent to the line corresponding to $A_2 = 0$, except the line for which $A_1 = 0$. In other words, at the critical point all the skin friction lines have a common tangent, except for one of them. The surface flow in the vicinity of P_0 is represented in Figure 1.4(a). The two eigenvectors are orthogonal for the node shown in Figure 1.4(b).



a) eigenvectors $(\lambda_1 = 4, \lambda_2 = 0)$ b) orthogonal eigenvectors $(\lambda_1 = 100, \lambda_2 = 0)$ c) isotropic node $(S_1 = S_2)$

Figure 1.4. *Critical point of the node type (* $S_1 = 4, S_2 = 0$ *)*

Such a critical point is called a *node*. When t varies from $-\infty$ to $+\infty$, the skin friction lines run starting from the critical point (if A_1 and A_2 are negative). Such a node is an *attachment node*. If the skin friction lines run toward the critical point (A_1 and A_2 are positive), the node is a *separation node*. If A_1 and A_2 are negative, then p is positive so that the representative points in the plane [p,q] are on the left-hand side of the axis $q = 0$. For A_1 and A_2 positive, the image points are on the right-hand side of this axis.

Figure 1.5 shows a surface flow visualization of an attachment node at the nose of a blunted body having the shape of a flattened ellipsoid.

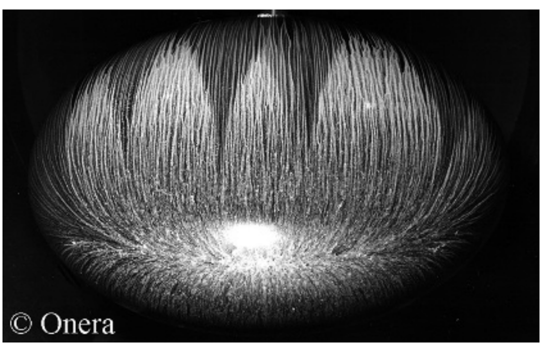


Figure 1.5. *Attachment node at the nose of a blunt body*

Special case: $S_1 = S_2$

The skin friction lines are defined by:

$$x(t) = \frac{A_1\mu_2 - A_2\mu_1}{\lambda_1\mu_2 - \lambda_2\mu_1}\exp(-S_1 t)$$

$$z(t) = -\frac{A_1\lambda_2 - A_2\lambda_1}{\lambda_1\mu_2 - \lambda_2\mu_1}\exp(-S_1 t)$$

Hence:

$$\frac{dz}{dx} = \frac{A_1\lambda_2 - A_2\lambda_1}{A_2\mu_1 - A_1\mu_2}$$

All the skin friction lines go through the critical point P_0 where they all have a different slope. Such a critical point is called an *isotropic node*, the corresponding pattern being shown in Figure 1.4(c). According to the sign of S_1, the skin friction lines are traveled either from P_0 (case of an attachment isotropic node) or to P_0 (case of a separation isotropic node). In the plane $[p,q]$, isotropic nodes (which correspond to double solutions of the eigenvalue equation) have images located on the parabola (P).

1.3.2.2. *Critical point of the saddle point type*

Now, the two roots S_1 and S_2 are real, with opposite signs and such that (to fix ideas): $S_1 < 0 < S_2$. Let us recall the following expressions for the solution curves:

$$x(t) = \frac{A_1\mu_2}{\lambda_1\mu_2 - \lambda_2\mu_1}\exp(-S_1 t) - \frac{A_2\mu_1}{\lambda_1\mu_2 - \lambda_2\mu_1}\exp(-S_2 t)$$

$$z(t) = \frac{-A_1\lambda_2}{\lambda_1\mu_2 - \lambda_2\mu_1}\exp(-S_1 t) + \frac{A_2\lambda_1}{\lambda_1\mu_2 - \lambda_2\mu_1}\exp(-S_2 t)$$

which can be written in the form:

$$x(t) = \frac{1}{\lambda_1\mu_2 - \lambda_2\mu_1} \exp(-S_1 t)\left\{A_1\mu_2 - A_2\mu_1 \exp\left[-(S_2 - S_1)t\right]\right\}$$

$$z(t) = \frac{-1}{\lambda_1\mu_2 - \lambda_2\mu_1} \exp(-S_1 t)\left\{A_1\lambda_2 - A_2\lambda_1 \exp\left[-(S_2 - S_1)t\right]\right\}$$

The following behaviors can be verified:

when: $t \to +\infty, \exp(-S_1 t) \to +\infty \quad \text{and} \quad \exp(-S_2 t) \to 0$

Hence $x(t)$ and $z(t)$ tend to $+\infty$.

when: $t \to -\infty, \exp(-S_1 t) \to 0 \quad \text{and} \quad \exp(-S_2 t) \to +\infty$

So that $x(t)$ and $z(t)$ still tend to $+\infty$.

In the general case, *the critical point is never reached*. If the parameter t varies from $-\infty$ to $+\infty$ a skin friction line runs from a point infinitely far from P_0 when tending to P_0; thereafter, it continues to a point infinitely far when moving away from P_0.

Let us consider the case: $A_1 = 0$ or $A_2 = 0$.

If $A_1 = 0$:

$$x(t) = -\frac{A_2\mu_1}{\lambda_1\mu_2 - \lambda_2\mu_1} \exp(-S_2 t)$$

$$z(t) = \frac{A_2\lambda_1}{\lambda_1\mu_2 - \lambda_2\mu_1} \exp(-S_2 t)$$

Thus, simultaneously $x(t)$ and $z(t) \to 0$ when $t \to +\infty$. The solution curve passes through P_0 where its slope is:

$$\frac{dz}{dx} = -\frac{\lambda_1}{\mu_1}$$

If $A_2 = 0$:

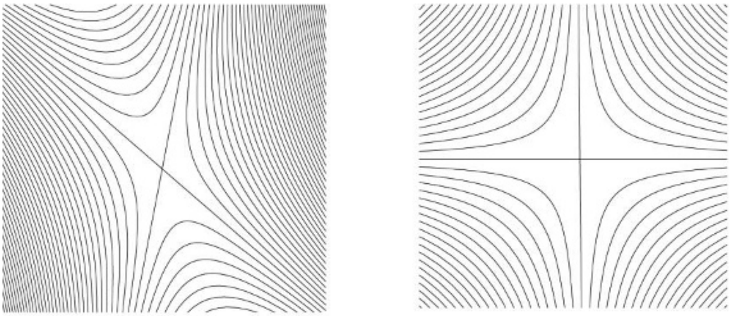
$$x(t) = \frac{A_1\mu_2}{\lambda_1\mu_2 - \lambda_2\mu_1}\exp(-S_1 t)$$

$$z(t) = \frac{-A_1\mu_2}{\lambda_1\mu_2 - \lambda_2\mu_1}\exp(-S_1 t)$$

We see that $x(t)$ and $z(t) \to 0$ when $t \to +\infty$: the critical point is reached, the slope at P_0 being:

$$\frac{dz}{dx} = -\frac{\lambda_2}{\mu_2}$$

Only two skin friction lines run through P_0; all the other skin friction lines avoid the critical point adopting the shape of a hyperbolic curve, as shown in Figure 1.6(a).



a) eigenvectors $(\lambda_1 = -6, \lambda_2 = 1)$ b) orthogonal eigenvectors $(\lambda_1 = 0, \lambda_2 = 100)$

Figure 1.6. *Critical point of the saddle point type* $(S_1 = -1,\ S_2 = 1)$

Such a critical point is called a *saddle point*. For the saddle point drawn in Figure 1.6(b), the two eigenvectors are orthogonal. If the coefficient q in equation [1.2] is negative, the roots S_1 and S_2 have different signs. The representative point in the plane $[p,q]$ lies below the axis $q=0$ in Figure 1.3. Figure 1.7 shows a saddle point observed in the surface pattern of a flow separating in front of an obstacle.

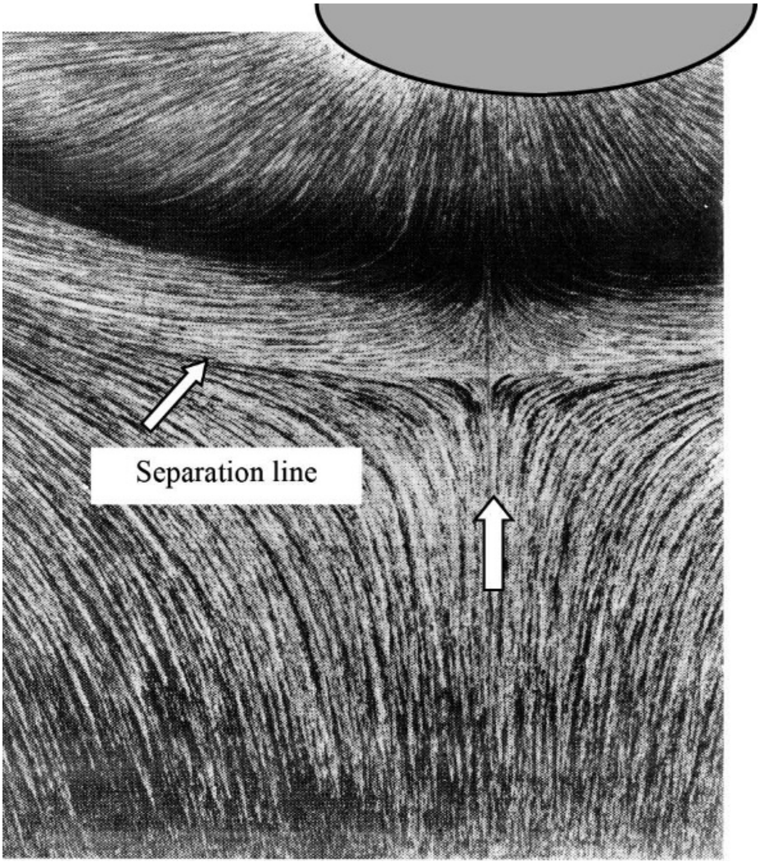


Figure 1.7. *Saddle point linked to separation in front of a cylindrical obstacle ([EAS 68])*

1.3.2.3. *Critical point of the focus type*

If the discriminant in equation [1.2] is negative, the eigenvalues are a complex conjugate. Then we can write:

$$S_1 = S' + iS''$$

$$S_2 = S' - iS''$$

S_1 correspond to:

$$\lambda_1 = \lambda_1^{'} + i\lambda_1^{''} \quad \text{and} \quad \mu_1 = \mu_1^{'} + i\mu_1^{''}$$

and S_2 to:

$$\lambda_2 = \lambda_2^{'} + i\lambda_2^{''} \quad \text{and} \quad \mu_2 = \mu_2^{'} + i\mu_2^{''}$$

Let us consider the root $S = S' + iS''$ to which are associated $\lambda = \lambda'' + i\lambda''$ and $\mu = \mu' + i\mu''$, the index 1 (or 2) being omitted for sake of simplicity. The integration constant A being written in the form:

$$A = \exp(\alpha + i\beta)$$

We have from the system [1.3]:

$$(\lambda' + i\lambda'')x + (\mu' + i\mu'')z = \exp\left[-(S' + iS'')t + \alpha + i\beta\right]$$

Hence, by separating the real and imaginary parts:

$$\lambda' x + \mu' z = \cos(S'' t - \beta)\exp(-S' t + \alpha)$$

$$\lambda'' x + \mu'' z = -\sin(S'' t - \beta)\exp(-S' t + \alpha)$$

Then:

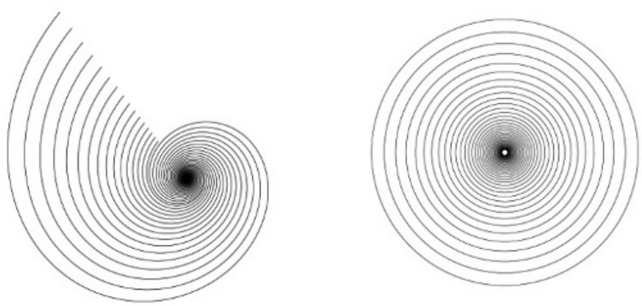
$$x(t) = \frac{\mu''\cos(S''t - \beta)\exp(-S't + \alpha) + \mu'\sin(S''t - \beta)\exp(-S't + \alpha)}{\lambda'\mu'' - \lambda''\mu'}$$

$$z(t) = \frac{\lambda''\cos(S''t-\beta)\exp(-S't+\alpha)+\lambda'\sin(S''t-\beta)\exp(-S't+\alpha)}{\lambda'\mu''-\lambda''\mu'}$$

If we assume that S' is strictly positive $\left(S'>0\right)$, then it is easy to verify that $x(t)$ and $z(t)$ tend to zero when $t \to +\infty$. The critical point P_0 is reached by all the solution curves. Let us examine the slope:

$$\frac{dz}{dx} = -\frac{(S''\mu'-S'\mu'')\cos(S''t-\beta)-(S''\mu''+S'\mu')\sin(S''t-\beta)}{(S''\lambda'-S'\lambda'')\cos(S''t-\beta)-(S''\lambda''+S'\lambda')\sin(S''t-\beta)}$$

At P_0, $dz\,/\,dx$ takes different values according to the value of the constant β: the skin friction lines have no common tangent. Because of the presence of periodic functions damped by exponentials, the skin friction lines end at the critical point P_0 after spiralling around it, as shown in Figure 1.8(a). Such a critical point is called a *focus*.



a) critical point for $S_{1,2} = 1 \pm 4i$ b) degenerated focus or center $\left(S_{1,2} = \pm 4i\right)$

Figure 1.8. *Critical point of the focus type*

A focus around which the solution curves roll up is sometimes called a stable focus. In fluid mechanics, such behavior is associated with a vortex, or more exactly with a vortical surface resulting from separation.

If S' is strictly negative $\left(S' < 0\right)$, $x(t)$ and $z(t) \rightarrow 0$ when $t \rightarrow -\infty$ and $x(t)$ and $z(t) \rightarrow +\infty$ when $t \rightarrow -\infty$: the solution curves unroll from the critical point that is then called an instable focus. The denominations stable or instable focus pertain to the dynamic systems theory. We should avoid deducing any conclusion relative to the hydrodynamic stability of a flow (that is a vortex breakdown). In the case of a focus $q > p^2 / 4$, so that the representative point in the [*p,q*] plane is inside the parabola of Figure 1.3.

In the case $S' = 0$, we have:

$$x(t) = \frac{\exp(\alpha)\left[\mu"\cos(S"t-\beta)+\mu'\sin(S"t-\beta)\right]}{\lambda'\mu"-\lambda"\mu'}$$

$$x(t) = \frac{\exp(\alpha)\left[\lambda"\cos(S"t-\beta)+\lambda'\sin(S"t-\beta)\right]}{\lambda'\mu"-\lambda"\mu'}$$

The skin friction lines are closed curves having an elliptical shape that encircles a limit curve reducing to a point coincident with $.\alpha \rightarrow -\infty.$ For the other solution lines, P_0 is never reached. Such a point, which is a special kind of degenerated focus, is called a *center* (see Figure 1.8(b)). In the plane [*p,q*], centers correspond to points located on the axis $p = 0$. Figure 1.9 shows two foci forming on the rear window of an automobile.

Other critical points corresponding to special situations in the plane [*p,q*] are considered in Chapter 6 devoted to the two-dimensional case reconsidered within the framework of the critical point theory.

The various critical points used to study the topology of separated three-dimensional flows are represented in Figure 1.10. The sense along which the skin friction lines are run allows giving the critical point physical signification by linking it either to an attachment or a separation. These notions are clarified in the following chapter.

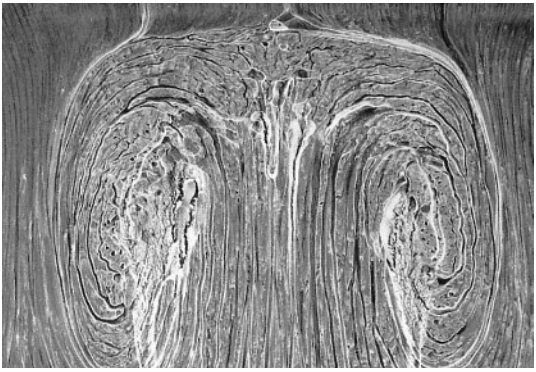


Figure 1.9. *Two-foci combination on an automobile rear window*

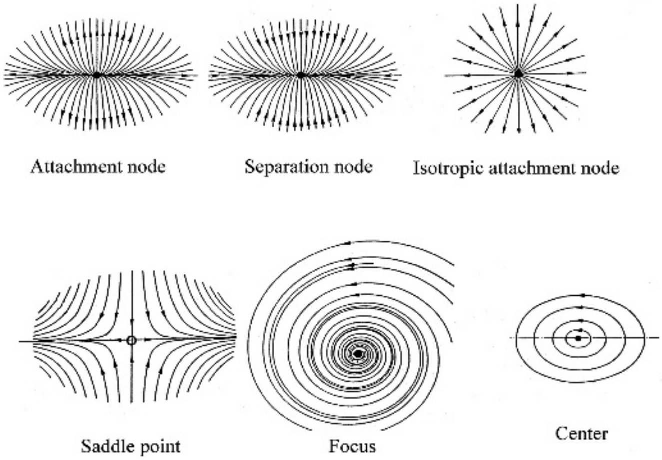


Figure 1.10. *The main critical points used for the topological description of three-dimensional flows*

1.4. Critical points of the wall vorticity lines

The critical points of the shear stress and vorticity fields are coincident since they simultaneously vanish as seen in section 1.2. Hence, the previous analysis can be applied to the vector field $\vec{\Omega}_w$, the orthogonal property of the skin friction and vorticity lines allowing us to deduce the nature of the critical point for the vector field $\vec{\Omega}_w$ from that of the vector field $\vec{\tau}_w$. Consequently, the following results are given without demonstration. As shown in Figure 1.11:

– to a node for $\vec{\tau}_w$ corresponds a center for $\vec{\Omega}_w$. If the node is isotropic, the curves constituting the center for $\vec{\Omega}_w$ are circles;

– to a saddle point for $\vec{\tau}_w$ corresponds for $\vec{\Omega}_w$ a saddle point having undergone a rotation;

– to a focus for $\vec{\tau}_w$ corresponds a focus for $\vec{\Omega}_w$;

– to a center for $\vec{\tau}_w$ corresponds a node for $\vec{\Omega}_w$.

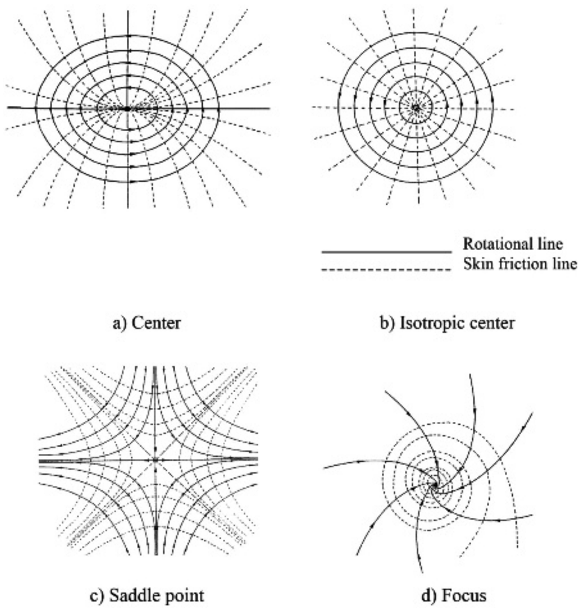


Figure 1.11. *Critical points for the rotational field at the wall*