Chapter 1

The Loop-shaping Approach

1.1. Principle of the method

1.1.1. *Introduction*

The term "loop-shaping specification" denotes the practice of specifying the open-loop response of a servo-loop on the basis of a specification relating to several closed-loop transfers. The reason why we do this is that it is easier to work on a single transfer (the open-loop response) than on a multitude of transfers (the various loops, e.g. reference/error, disturbance/error, disturbance/control, etc.). In addition, the internal stability of the servo-loop (i.e. the stability of all the internal loops) can be guaranteed if the open loop response has certain characteristics (e.g. the Nyquist locus of the open loop in relation to point -1 with a monovariable system, or examination of the characteristic loci in the multivariable case). Hence, we can see the advantage of synthesis methods directly based on the open loop response, the frequency shape of which enables us to give the desired characteristics to the different loops.

1.1.2. *Sensitivity functions*

To illustrate the concept, the specification of the servo-loop's performances can be based on the arrangement shown in Figure 1.1, which includes:

– the model's *input* disturbances, Γ_1;

– the model's *output* disturbances, Γ_2;

– the reference signal or measuring noise, r;

– the value to be controlled, y, for which we have a measurement;

– the measuring error ε;

– the command u created by the controller $K(s)$, whose output disturbed by Γ_2 is really applied to the transfer function system $H(s)$.

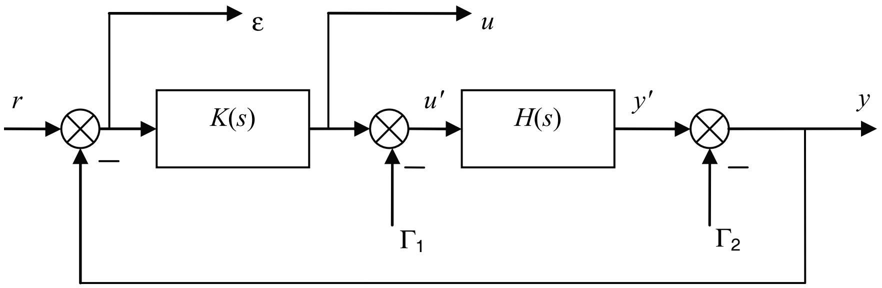

Figure 1.1. *General view of control system*

The task of an automation engineer is then to determine a controller $K(s)$ which, when looped with $H(s)$, minimizes the error ε at the cost of "reasonable" commands, with the looping being subject to the external inputs r, Γ_1 and Γ_2.

As regards the external inputs, the control and error signals are written as[1]:

$$\begin{aligned}
y(s) &= H_{r\to y}r(s) + H_{1\to y}\Gamma_1(s) + H_{2\to y}\Gamma_2(s) \\
\varepsilon(s) &= H_{r\to \varepsilon}r(s) + H_{1\to \varepsilon}\Gamma_1(s) + H_{2\to \varepsilon}\Gamma_2(s) \\
u(s) &= H_{r\to u}r(s) + H_{1\to u}\Gamma_1(s) + H_{2\to u}\Gamma_2(s)
\end{aligned}$$

Let us now detail the different transfers involved.

1.1.2.1. *Output sensitivity functions*

At the system's *output*, we can write:

$$\begin{aligned}
y(s) &= -\Gamma_2(s) + H\left(-\Gamma_1(s) + K\left(r(s) - y(s)\right)\right) \\
&= -\Gamma_2(s) - H\Gamma_1(s) + HKr(s) - HKy(s) \\
\left(I + HK\right)y(s) &= -\Gamma_2(s) - H\Gamma_1(s) + HKr(s) \\
y(s) &= \left(I + HK\right)^{-1} HKr(s) - \left(I + HK\right)^{-1} H\Gamma_1(s) - \left(I + HK\right)^{-1}\Gamma_2(s)
\end{aligned}$$

1 For ease of writing, the same letter-like symbols are used for temporal signals and their Laplace transforms, and the dependency on s of the transfers is usually omitted.

and:

$$\begin{aligned}\varepsilon(s) &= r(s) - y(s) \\ &= r(s) + \left(I + HK\right)^{-1} \Gamma_2(s) + \left(I + HK\right)^{-1} H.\Gamma_1(s) - \left(I + HK\right)^{-1} HKr(s) \\ &= \left(I - \left(I + HK\right)^{-1} HK\right) r(s) + \left(I + HK\right)^{-1} \Gamma_2(s) + \left(I + HK\right)^{-1} H\Gamma_1(s) \\ &= \left(I + HK\right)^{-1} \left(\left(I + HK\right) - HK\right) r(s) + \left(I + HK\right)^{-1} \Gamma_2(s) + \left(I + HK\right)^{-1} H.\Gamma_1(s) \\ &= \left(I + HK\right)^{-1} r(s) + \left(I + HK\right)^{-1} H\Gamma_1(s) + \left(I + HK\right)^{-1} \Gamma_2(s)\end{aligned}$$

In addition:

$$\begin{aligned}u(s) &= K\varepsilon(s) \\ &= K\left(I + HK\right)^{-1} r(s) + K\left(I + HK\right)^{-1} H\Gamma_1(s) + K\left(I + HK\right)^{-1} \Gamma_2(s)\end{aligned}$$

Denoting the output[2] sensitivity functions as follows:

$$S_y = (I + HK)^{-1}, \;\; T_y = (I + HK)^{-1} HK \qquad [1.1]$$

Thus we obtain:

$$\begin{aligned}y(s) &= T_y r(s) - S_y H\Gamma_1(s) - S_y \Gamma_2(s) \\ \varepsilon(s) &= S_y r(s) + S_y H\Gamma_1(s) + S_y \Gamma_2(s) \\ u(s) &= KS_y r(s) + KS_y H\Gamma_1(s) + KS_y \Gamma_2(s)\end{aligned}$$

As there is no reason for the product *KH* to be equal to *HK* in the MIMO case, we can obtain other expressions for the above signals.

1.1.2.2. *Input sensitivity functions*

At the system's *input*, we can write:

$$\begin{aligned}u(s) &= K\left(r(s) - \left(-\Gamma_2(s) + H\left(-\Gamma_1(s) + u(s)\right)\right)\right) \\ &= Kr(s) + K\Gamma_2(s) + KH\Gamma_1(s) - KHu(s) \\ &= \left(I + KH\right)^{-1} Kr(s) + \left(I + KH\right)^{-1} KH\Gamma_1(s) + \left(I + KH\right)^{-1} K\Gamma_2(s)\end{aligned}$$

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2 That is, when we open the loop at the level of the system input.

and:

$$\begin{aligned}u'(s) &= u(s) - \Gamma_1(s)\\ &= (I+KH)^{-1}Kr(s) + (I+KH)^{-1}K\Gamma_2(s) + (I+KH)^{-1}KH\Gamma_1(s) - \Gamma_1(s)\\ &= (I+KH)^{-1}Kr(s) + (I+KH)^{-1}K\Gamma_2(s) + \left((I+KH)^{-1}KH - I\right)\Gamma_1(s)\\ &= (I+KH)^{-1}Kr(s) + (I+KH)^{-1}K\Gamma_2(s) + (I+KH)^{-1}\left(KH - (I+KH)\right)\Gamma_1(s)\\ &= (I+KH)^{-1}Kr(s) - (I+KH)^{-1}\Gamma_1(s) + (I+KH)^{-1}K\Gamma_2(s)\end{aligned}$$

Furthermore:

$$\begin{aligned}y(s) &= Hu'(s) - \Gamma_2(s)\\ &= H(I+KH)^{-1}Kr(s) - H(I+KH)^{-1}\Gamma_1(s) + H(I+KH)^{-1}K\Gamma_2(s) - \Gamma_2(s)\\ &= H(I+KH)^{-1}Kr(s) - H(I+KH)^{-1}\Gamma_1(s) + \left(H(I+KH)^{-1}K - I\right)\Gamma_2(s)\end{aligned}$$

Finally:

$$\begin{aligned}\varepsilon(s) &= r(s) - y(s)\\ &= r(s) - H(I+KH)^{-1}Kr(s) + H(I+KH)^{-1}\Gamma_1(s) - \left(H(I+KH)^{-1}K - I\right)\Gamma_2(s)\\ &= \left(I - H(I+KH)^{-1}K\right)r(s) + H(I+KH)^{-1}\Gamma_1(s) - \left(H(I+KH)^{-1}K - I\right)\Gamma_2(s)\end{aligned}$$

By setting the following as input sensitivity functions[3]:

$$S_u = (I+KH)^{-1},\ \ T_u = (I+KH)^{-1}KH \qquad [1.2]$$

then:

$$\begin{aligned}y(s) &= HS_uKr(s) - HS_u\Gamma_1(s) + (HS_uK - I)\Gamma_2(s)\\ \varepsilon(s) &= (I - HS_uK)r(s) + HS_u\Gamma_1(s) - (HS_uK - I)\Gamma_2(s)\\ u(s) &= S_uKr(s) + T_u\Gamma_1(s) + S_uK\Gamma_2(s)\end{aligned}$$

Hence, finally, we obtain:

$$\begin{array}{lll} H_{r\to y} = T_y = HS_uK & H_{1\to y} = -S_yH = -HS_u & H_{2\to y} = -S_y = HS_uK - I\\ H_{r\to\varepsilon} = S_y = I - HS_uK & H_{1\to\varepsilon} = S_yH = HS_u & H_{2\to\varepsilon} = S_y = I - HS_uK\\ H_{r\to u} = KS_y = S_uK & H_{1\to u} = KS_yH = T_u & H_{2\to u} = KS_y = S_uK\end{array}$$

3 That is, when we open the loop at the system's input.

From this, we can draw the following fundamental relations:

$$\left.\begin{array}{l} T_y = HS_u K \\ S_y H = HS_u \\ S_y + T_y = I \end{array}\right\} \qquad \left.\begin{array}{l} T_u = KS_y H \\ S_u K = KS_y \\ S_u + T_u = I \end{array}\right\} \qquad [1.3]$$

It should be noted that in the case of positive output feedback, we repeat all the previous steps, replacing *r* with -*r* and *K* with -*K*, from which we draw the following relations:

$$\left.\begin{array}{l} T_y = -HS_u K \\ S_y H = HS_u \\ S_y + T_y = I \end{array}\right\} \qquad \left.\begin{array}{l} T_u = -KS_y H \\ S_u K = KS_y \\ S_u + T_u = I \end{array}\right\}$$

where:

$$S_u = (I - KH)^{-1}, \;\; T_u = -\left(I - KH\right)^{-1} KH$$
$$S_y = (I - HK)^{-1}, \;\; T_y = -(I - HK)^{-1} HK$$

1.1.3. *Declination of performance objectives*

In view of the previous developments, by frequency modeling only the direct (*S*) and complementary (*T*) sensitivity functions, we are therefore able to model all the closed-loop transfers, because they depend only on these functions. Thus, the work on many transfers can be assimilated to work on the two sensitivity functions *S* and *T*.

In addition:

– when $\underline{\sigma}\left(HK\right) >> 1$ or when $\underline{\sigma}\left(KH\right) >> 1$ (which can happen, particularly in low frequencies in the presence of integrators in the control law), then:

$$\overline{\sigma}\left(S_y\right) \approx \overline{\sigma}\left(\left(HK\right)^{-1}\right) = \frac{1}{\underline{\sigma}\left(HK\right)}, \;\; T_y \approx I$$

$$\overline{\sigma}\left(S_u\right) \approx \overline{\sigma}\left(\left(KH\right)^{-1}\right) = \frac{1}{\underline{\sigma}\left(KH\right)}, \;\; T_u \approx I$$

and[4]:

$$\begin{aligned}
&\bar{\sigma}(S_y H) \approx \frac{\bar{\sigma}(H)}{\underline{\sigma}(HK)} \\
&\bar{\sigma}(KS_y) \approx \frac{\bar{\sigma}(K)}{\underline{\sigma}(HK)} \leq \frac{\bar{\sigma}(K)}{\underline{\sigma}(H).\underline{\sigma}(K)} \approx \frac{1}{\underline{\sigma}(H)} \\
&\bar{\sigma}(H.S_u) \approx \frac{\bar{\sigma}(H)}{\underline{\sigma}(KH)} \\
&\bar{\sigma}(S_u K) \approx \frac{\bar{\sigma}(K)}{\underline{\sigma}(KH)} \leq \frac{\bar{\sigma}(K)}{\underline{\sigma}(K).\underline{\sigma}(H)} \approx \frac{1}{\underline{\sigma}(H)}
\end{aligned}$$

Then, in relation to the open-loop response in the model's output, we obtain:

$$\begin{aligned}
&\bar{\sigma}(H_{r\to y}) = \bar{\sigma}(H_{1\to u}) \approx 1 \\
&\bar{\sigma}(H_{r\to u}) = \bar{\sigma}(H_{2\to u}) \approx \frac{1}{\underline{\sigma}(H)} \\
&\bar{\sigma}(H_{1\to y}) = \bar{\sigma}(H_{1\to \varepsilon}) \approx \frac{\bar{\sigma}(H)}{\underline{\sigma}(HK)} \\
&\bar{\sigma}(H_{2\to y}) = \bar{\sigma}(H_{r\to \varepsilon}) = \bar{\sigma}(H_{2\to \varepsilon}) \approx \frac{1}{\underline{\sigma}(HK)}
\end{aligned}$$

but also for the input ones:

$$\begin{aligned}
&\bar{\sigma}(H_{r\to y}) = \bar{\sigma}(H_{1\to u}) \approx 1 \\
&\bar{\sigma}(H_{r\to u}) = \bar{\sigma}(H_{2\to u}) \approx \frac{1}{\underline{\sigma}(H)} \\
&\bar{\sigma}(H_{1\to y}) = \bar{\sigma}(H_{1\to \varepsilon}) \approx \frac{\bar{\sigma}(H)}{\underline{\sigma}(KH)} \\
&\bar{\sigma}(H_{2\to y}) = \bar{\sigma}(H_{r\to \varepsilon}) \approx \bar{\sigma}(H_{2\to \varepsilon}) \approx \frac{1}{\underline{\sigma}(HK)}
\end{aligned}$$

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4 Remember that:

$$\underline{\sigma}(B)\underline{\sigma}(C) \leq \underline{\sigma}(BC) \leq \bar{\sigma}(BC) \leq \bar{\sigma}(B)\bar{\sigma}(C)$$

$$\text{if } A^{-1} \text{ exists, } \bar{\sigma}(A) = \frac{1}{\underline{\sigma}(A^{-1})} \Rightarrow \frac{\bar{\sigma}(A)}{\underline{\sigma}(A)} = \frac{1}{\underline{\sigma}(A^{-1})\underline{\sigma}(A)} \geq 1$$

Thus, by giving the open loop a high gain through its singular values, the automation engineer can favor the performance of looping in relation to the external inputs by way of all the transfers relating to the error ε ($H_{1\to\varepsilon}, H_{2\to\varepsilon}$ and $H_{r\to\varepsilon}$) but has no flexibility on transfers relating to the control signal which do not depend on the open-loop response.

– when $\bar{\sigma}(HK) << 1$ or when $\bar{\sigma}(KH) << 1$, (which can happen, particularly in high frequencies in the presence of roll-off filters in the control law), then:

$$\bar{\sigma}(T_y) \approx \bar{\sigma}(HK),\ S_y \approx I$$
$$\bar{\sigma}(T_u) \approx \bar{\sigma}(KH),\ S_u \approx I$$

and:

$$\bar{\sigma}(KS_y) \approx \bar{\sigma}(K) \approx \frac{\bar{\sigma}(HK)}{\underline{\sigma}(H)}$$
$$\bar{\sigma}(S_y H) \approx \bar{\sigma}(H)$$
$$\bar{\sigma}(S_u K) \approx \bar{\sigma}(K) \approx \frac{\bar{\sigma}(KH)}{\underline{\sigma}(H)}$$
$$\bar{\sigma}(HS_u) \approx \bar{\sigma}(H)$$

Then, in relation to the open loop at the model's output, we obtain:

$$\bar{\sigma}(H_{r\to y}) = \bar{\sigma}(H_{1\to u}) \approx \bar{\sigma}(HK)$$
$$\bar{\sigma}(H_{r\to u}) = \bar{\sigma}(H_{2\to u}) \approx \frac{\bar{\sigma}(HK)}{\underline{\sigma}(H)}$$
$$\bar{\sigma}(H_{1\to y}) = \bar{\sigma}(H_{1\to\varepsilon}) \approx \bar{\sigma}(H)$$
$$\bar{\sigma}(H_{r\to\varepsilon}) = \bar{\sigma}(H_{2\to y}) = \bar{\sigma}(H_{2\to\varepsilon}) \approx 1$$

but also at the input:

$$\bar{\sigma}(H_{r\to y}) = \bar{\sigma}(H_{1\to u}) \approx \bar{\sigma}(KH)$$
$$\bar{\sigma}(H_{r\to u}) = \bar{\sigma}(H_{2\to u}) \approx \frac{\bar{\sigma}(KH)}{\underline{\sigma}(H)}$$
$$\bar{\sigma}(H_{1\to y}) = \bar{\sigma}(H_{1\to\varepsilon}) \approx \bar{\sigma}(H)$$
$$\bar{\sigma}(H_{r\to\varepsilon}) = \bar{\sigma}(H_{2\to y}) \approx \bar{\sigma}(H_{2\to\varepsilon}) \approx 1$$

Thus, by giving the open loop a low gain through its singular values, the automation engineer can favor the command of the looping (in terms of power consumption) in relation to the external inputs by way of all the transfers relating to the control signal u ($H_{1\to u}$, $H_{2\to u}$ and $H_{r\to u}$) but has no flexibility on transfers relating to the error which do not depend on the open-loop response.

Thus, the "loop-shaping" approach consists of appropriate modeling of the singular values of the open loop response so as to guarantee certain specifications relating to $S_{u,y}$ and $T_{u,y}$, which themselves are derived from the declination of the set of specifications relating to all the loop transfers. Indeed, it is simpler to model a single transfer function (the open loop) than a multitude of closed-loop transfers.

The approach is particularly advantageous for the robustness of any looping that is subject to external inputs. Indeed:

– desensitizing the error ε to disturbances consists of giving $S_{u,y}$ a rejection behavior on the frequency dynamics of these disturbances, i.e. giving the singular values of the open loop a high gain within the same frequency range;

– desensitizing the control signal u to disturbances consists of giving $T_{u,y}$ a rejection behavior on the frequency dynamics of these disturbances, i.e. giving the singular values of the open loop a low gain within the same frequency range.

Disturbance rejection problems can therefore, theoretically, be easily solved by including low-pass and high-pass filters in the shape of the open-loop response, depending on which axis of synthesis is favored (control signal or performance), which renders this approach particularly attractive in the world of industry.

1.1.4. *Declination of the robustness objectives*

It is necessary to combine the concept of robustness with the above declination of performance objectives when the looped system is subject to uncertainties (neglected time-constants, parametric variations, etc.).

Consider the standard arrangement as shown in Figure 1.2. Analysis of the stability robustness of the system $M(s)$, subject to unstructured uncertainties $\Delta(s)$, can be performed with the help of the small-gain theorem [ZHO 96]. Considering that $M(s) \in RH_\infty$ and $\Delta(s) \in RH_\infty$, the system is stable for any $\Delta(s)$ such that $\|\Delta(s)\|_\infty \leq \gamma^{-1}$ iff:

$$\|M(s)\|_\infty \leq \gamma$$

The uncertainty block $\Delta(s)$ is therefore seen as being an internal disturbance which could destabilize the system, whereas in the previous section the signals r, Γ_1 and Γ_2 were external disturbances. A similar declination methodology can be established to complement the previous one.

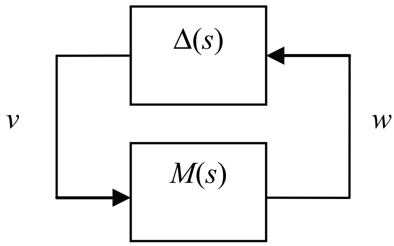

Figure 1.2. *Standard form for robustness analysis*

Six types of representation of unstructured uncertainties are usually employed:

– Direct multiplicative uncertainty:

- at input (Figure 1.3):

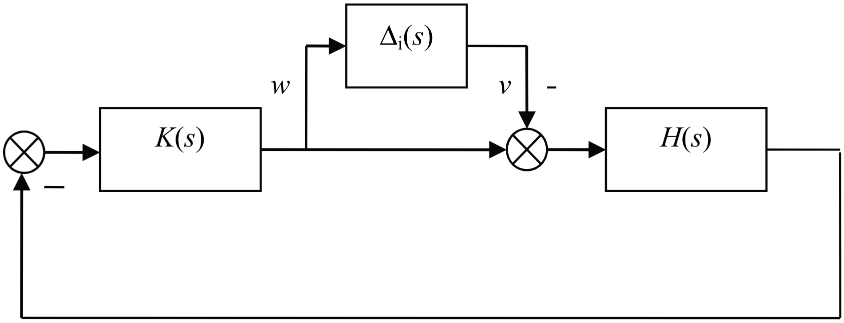

Figure 1.3. *Direct multiplicative uncertainty at input*

In this case, we achieve a representation similar to Figure 1.2 by using the relations:

$$w = T_u v = KS_y Hv \ ;$$

- at output (Figure 1.4):

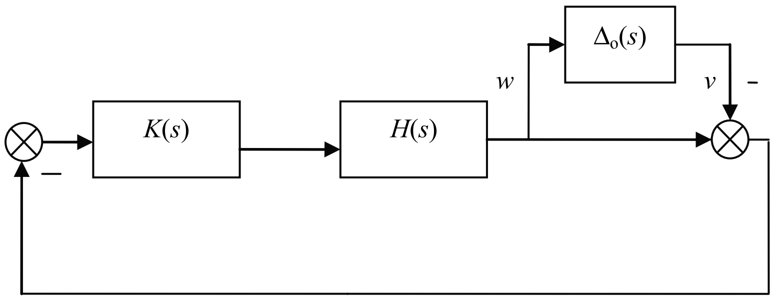

Figure 1.4. *Direct multiplicative uncertainty at output*

In this case:

$$w = T_y v = HS_u K v$$

According to the small-gain theorem, with this type of representation of uncertainties, the looped system is internally stable if:

$$\bar{\sigma}(T_y) < \frac{1}{\bar{\sigma}(\Delta_0)}$$

$$\bar{\sigma}(T_u) < \frac{1}{\bar{\sigma}(\Delta_i)}$$

When $\bar{\sigma}(HK) << 1$ or when $\bar{\sigma}(KH) << 1$ (which can happen, particularly in high frequencies in the presence of roll-off filters in the control law), then:

$$\bar{\sigma}(T_y) \approx \bar{\sigma}(HK),\ S_y \approx I$$

$$\bar{\sigma}(T_u) \approx \bar{\sigma}(KH),\ S_u \approx I$$

In this case, the condition of stability robustness in relation to uncertainties represented in direct multiplicative form can be taken into account in the loop-shaping specification as follows:

$$\bar{\sigma}(HK) < \frac{1}{\bar{\sigma}(\Delta_0)}$$

$$\bar{\sigma}(KH) < \frac{1}{\bar{\sigma}(\Delta_i)}$$

This enables us to position the roll-off necessary for the open loop to ensure the condition of stability robustness.

– Inverse multiplicative uncertainty:

- at input (Figure 1.5):

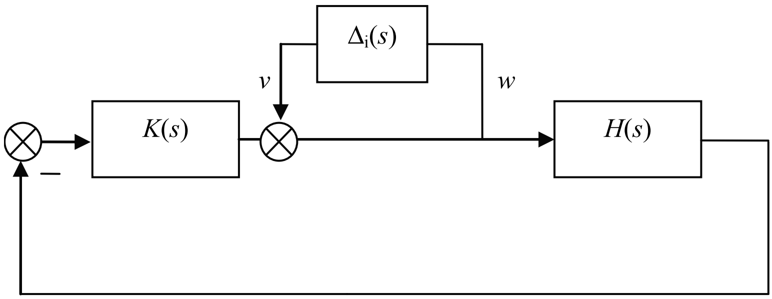

Figure 1.5. *Inverse multiplicative uncertainty at input*

In this case:

$$w = S_u v \;;$$

- at output (Figure 1.6):

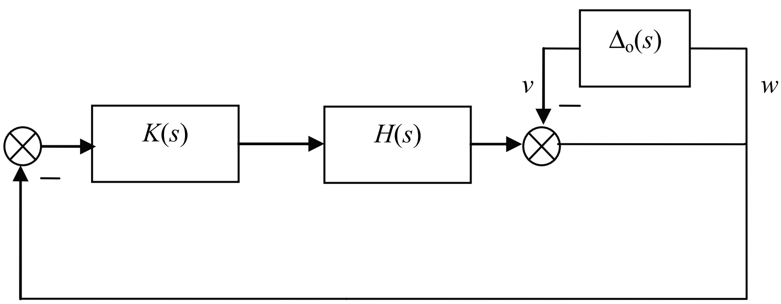

Figure 1.6. *Inverse multiplicative uncertainty at output*

In this case:

$$w = S_y v$$

According to the small-gain theorem, with this type of representation of uncertainties, the looped system is internally stable if:

$$\overline{\sigma}\left(S_y\right) < \frac{1}{\overline{\sigma}(\Delta_0)}$$

$$\overline{\sigma}(S_u) < \frac{1}{\overline{\sigma}(\Delta_i)}$$

When $\underline{\sigma}(HK) >> 1$ or when $\underline{\sigma}(KH) >> 1$ (which can happen, particularly at low frequencies in the presence of integrators in the control law), then:

$$\overline{\sigma}\left(S_y\right) \approx \overline{\sigma}\left((HK)^{-1}\right) = \frac{1}{\underline{\sigma}(HK)}, \quad T_y \approx I$$

$$\overline{\sigma}(S_u) \approx \overline{\sigma}\left((KH)^{-1}\right) = \frac{1}{\underline{\sigma}(KH)}, \quad T_u \approx I$$

In this case, the condition of stability robustness in relation to uncertainties represented in inverse multiplicative form can be taken into account in the loop-shaping specification as follows:

$$\frac{1}{\underline{\sigma}(HK)} < \frac{1}{\overline{\sigma}(\Delta_0)} \Leftrightarrow \underline{\sigma}(HK) > \overline{\sigma}(\Delta_0)$$

$$\frac{1}{\underline{\sigma}(KH)} < \frac{1}{\overline{\sigma}(\Delta_i)} \Leftrightarrow \underline{\sigma}(KH) > \overline{\sigma}(\Delta_i)$$

– Direct additive uncertainty (Figure 1.7):

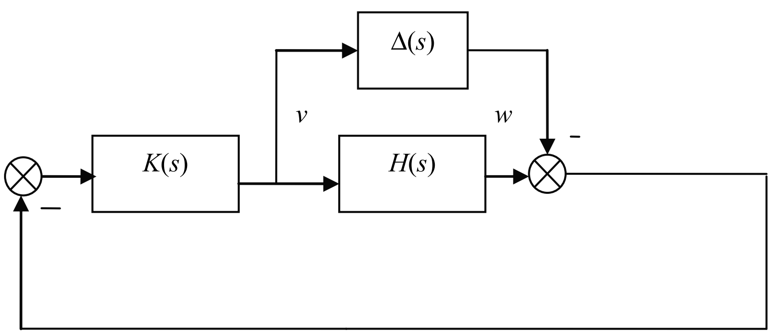

Figure 1.7. *Direct additive uncertainty*

In this case:

$$w = S_u K v = K S_y v$$

When $\overline{\sigma}(HK) << 1$ or when $\overline{\sigma}(KH) << 1$, then:

$$\overline{\sigma}(KS_y) \approx \frac{\overline{\sigma}(HK)}{\underline{\sigma}(H)},\ S_y \approx I$$

$$\overline{\sigma}(S_u K) \approx \frac{\overline{\sigma}(KH)}{\underline{\sigma}(H)},\ S_u \approx I$$

In this case, the condition of stability robustness in relation to uncertainties represented in direct additive form can be taken into account in the loop-shaping specification as follows:

$$\overline{\sigma}(HK) < \frac{\underline{\sigma}(H)}{\overline{\sigma}(\Delta)}$$

$$\overline{\sigma}(KH) < \frac{\underline{\sigma}(H)}{\overline{\sigma}(\Delta_i)}$$

– Inverse additive uncertainty (Figure 1.8):

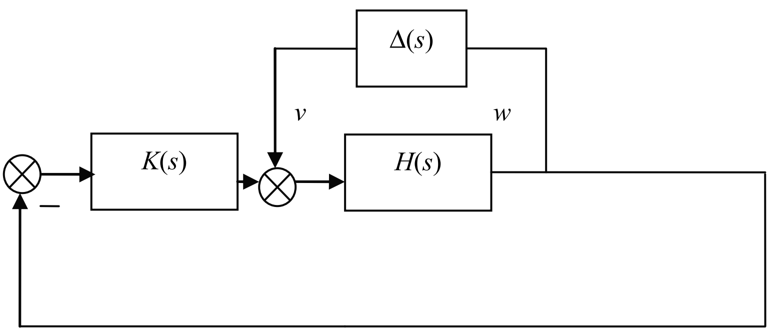

Figure 1.8. *Inverse additive uncertainty*

In this case:

$$w = S_y H v = H S_u v$$

When $\underline{\sigma}(HK) >> 1$ or when $\underline{\sigma}(KH) >> 1$ (which can happen, particularly at low frequencies in the presence of integrators in the control law), then:

$$\overline{\sigma}\left(S_y H\right) \approx \frac{\overline{\sigma}(H)}{\underline{\sigma}(HK)},\ T_y = I$$

$$\overline{\sigma}\left(H S_u\right) \approx \frac{\overline{\sigma}(H)}{\underline{\sigma}(KH)},\ T_u = I$$

In this case, the condition of stability robustness in relation to uncertainties represented in inverse additive form can be taken into account in the loop-shaping specification as follows:

$$\frac{\overline{\sigma}(H)}{\underline{\sigma}(HK)} < \frac{1}{\overline{\sigma}(\Delta)} \Leftrightarrow \underline{\sigma}(HK) > \frac{\overline{\sigma}(\Delta)}{\overline{\sigma}(H)}$$

$$\frac{\overline{\sigma}(H)}{\underline{\sigma}(KH)} < \frac{1}{\overline{\sigma}(\Delta)} \Leftrightarrow \underline{\sigma}(KH) > \frac{\overline{\sigma}(\Delta)}{\overline{\sigma}(H)}$$

Thus, we have demonstrated how unstructured uncertainties can be taken into account in the loop-shaping approach, which is a welcome addition to the declination of performances presented above, with the simplicity of the approach also being an asset: modeling a single transfer (the open-loop response) enables us to implement loops that perform in relation to external signals as well as that are robust in relation to uncertainties.

1.2. Generalized phase and gain margins

Continuing with the open-loop response modeling approach, we can seek to extend the concepts of gain and phase margins to the case of multivariable systems.

1.2.1. *Phase and gain margins at the model's output*

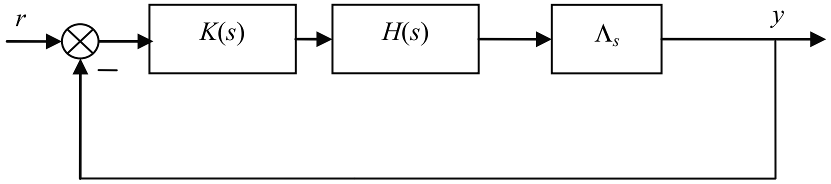

Figure 1.9. *Generalized phase and gain margins at the model's output*

What are the maximum acceptable variations in gain and phase at the model's output which would destabilize the loop shown in Figure 1.9, in which we consider:

$$\Lambda_s = \text{diag}\left(k_i e^{j\phi_i},\ i = 1,..,p\right)$$

Nominally:

$$\left.\begin{aligned} k_i &= 0 \\ \phi_i &= 0 \end{aligned}\right\} \Rightarrow \Lambda_s = I$$

We can represent this uncertainty in direct multiplicative form at output (Figure 1.4).

Thus, we have:

$$\Lambda_s = I - \Delta_o \Rightarrow \Delta_o = I - \Lambda_s = \text{diag}\left(1 - k_i e^{j\phi_i},\ i = 1,..,p\right)$$

We set:

$$\alpha_1 = \frac{1}{\left\|T_y(s)\right\|_\infty}$$

Hence, according to the small-gain theorem, the system is stable if:

$$\left\|\Delta_o\right\|_\infty < \alpha_1 \Leftrightarrow \left|1 - k_i e^{j\phi_i}\right| < \alpha_1$$

When $\phi_i = 0$, this condition leads to:

$$1 - \alpha_1 < k_i < 1 + \alpha_1$$

Similarly, when $k_i = 1$:

$$\left|1 - e^{j\phi_i}\right| < \alpha_1 \Leftrightarrow \left|e^{-j\frac{\phi_i}{2}} - e^{j\frac{\phi_i}{2}}\right| < \alpha_1 \Leftrightarrow 2\left|\sin\left(\frac{\phi_i}{2}\right)\right| < \alpha_1$$

so:

$$|\phi_i| < 2\arcsin\left(\frac{\alpha_1}{2}\right)$$

In addition, the uncertainty can be represented in inverse multiplicative form at output (Figure 1.6).

Thus, we have:

$$\Lambda_s = \left(I + \Delta_o\right)^{-1} \Leftrightarrow \Lambda_s^{-1} = I + \Delta_0 \Leftrightarrow \Delta = \Lambda_s^{-1} - I = \operatorname{diag}\left(\frac{1}{k_i} e^{-j\phi_i} - 1,\ i = 1,..,p\right)$$

We set:

$$\alpha_2 = \frac{1}{\left\| S_y(s) \right\|_\infty}$$

Hence, according to the small-gain theorem, the system is stable if:

$$\left\| \Delta_0 \right\|_\infty < \alpha_2 \Leftrightarrow \left| \frac{1}{k_i} e^{-j\phi_i} - 1 \right| < \alpha_2$$

When $\phi_i = 0$, we obtain $\left| \frac{1}{k_i} - 1 \right| < \alpha_2$, which leads to:

$$\frac{1}{1+\alpha_2} < k_i < \frac{1}{1-\alpha_2}$$

When $k_i = 1$, $\left| e^{-j\phi_i} - 1 \right| < \alpha_2 \Rightarrow \left| e^{j\frac{\phi_i}{2}} - e^{-j\frac{\phi_i}{2}} \right| < \alpha_2 \Rightarrow 2\left| \sin\left(\frac{\phi_i}{2}\right) \right| < \alpha_2$; thus:

$$\left| \phi_i \right| < 2 \arcsin\left(\frac{\alpha_2}{2}\right)$$

1.2.2. *Phase and gain margins at the model's input:*

We now look for the maximum acceptable variations in gain and phase at the model's input which would destabilize the loop shown in Figure 1.10, in which we consider:

$$\Lambda_u = \operatorname{diag}\left(k_i e^{j\phi_i},\ i = 1,..,m\right)$$

By setting:

$$\beta_1 = \frac{1}{\left\| T_u(s) \right\|_\infty}$$

$$\beta_2 = \frac{1}{\left\| S_u(s) \right\|_\infty}$$

A process strictly similar to the one outlined above establishes the following conditions:

$$\boxed{\begin{array}{l} 1-\beta_1 < k_i < 1+\beta_1 \\ \left|\phi_i\right| < 2\arcsin\left(\dfrac{\beta_1}{2}\right) \end{array}}$$

and:

$$\boxed{\begin{array}{l} \dfrac{1}{1+\beta_2} < k_i < \dfrac{1}{1-\beta_2} \\ \left|\phi_i\right| < 2\arcsin\left(\dfrac{\beta_2}{2}\right) \end{array}}$$

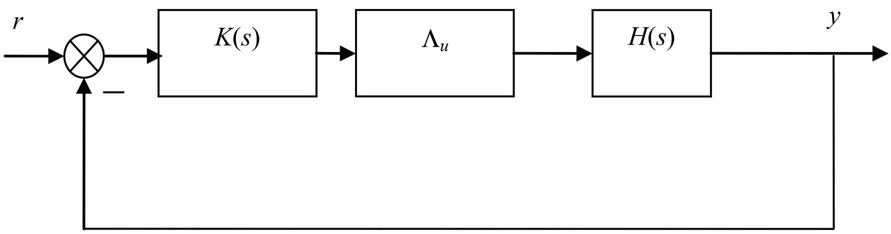

Figure 1.10. *Generalized phase and gain margins at the model's input*

1.3. Limitations inherent to bandwidth

We shall now speak of a limitation inherent to the bandwidth attainable when the system in question has a certain number of unstable poles p_i and/or a certain number of unstable zeros z_i. [SKO 01] shows that the attainable bandwidth ω_{BP} must be such that:

$$\boxed{\max_{\mathrm{Re}(p_i)>0}\left(\left|p_i\right|\right) < \omega_{BP} < \max_{\mathrm{Re}(z_i)>0}\left(\left|z_i\right|\right)} \quad [1.4]$$

1.4. Examples

Below, we give a few examples of typical variants of the loop-shaping technique. For simplicity's sake, we shall work with a monovariable system. This being the case, it is clear that:

$$S_u = S_y = S$$
$$T_u = T_y = T$$

1.4.1. *Example 1: sinusoidal disturbance rejection*

Suppose we wish to set the value y at 0 and we assume that the loop is subject to a disturbance at the model's output $\Gamma_2(s)$ represented as a sinusoidal signal of amplitude Γ_0 and frequency ω_0 (e.g. local mechanical deformation, etc.). The aim is to determine a loop-shaping specification for the servo-loop; therefore, we shall focus successively on two areas: performance and then command.

Two axes for synthesis may be envisaged, depending on the "high-level constraints":

– first case: it is of crucial importance, when good performance is required, to desensitize the error ε, which must remain below a certain value $\varepsilon_{\max}$.

If we set the following for the open-loop response: $H_{BO}(s) = H(s)K(s)$, then $H_{2\to\varepsilon}(s) = S(s) = \dfrac{1}{1+H_{BO}(s)}$ must exhibit rejection behavior in ω_0, which is possible if the open loop is high-gain at this frequency, because in this case: $S(j\omega_0) \approx \dfrac{1}{H_{BO}(j\omega_0)}$. The specification on the error thus imposes the gain of the open loop in ω_0, because:

$$S(j\omega_0) \approx \frac{\Gamma_0}{\left|H_{BO}(j\omega_0)\right|} < \varepsilon_{\max} \Rightarrow \left|H_{BO}(j\omega_0)\right| > \frac{\Gamma_0}{\varepsilon_{\max}}$$

This enables us to determine the level of gain needed for the open-loop response to lend S the desired depth of rejection in ω_0 (see Figure 1.11).

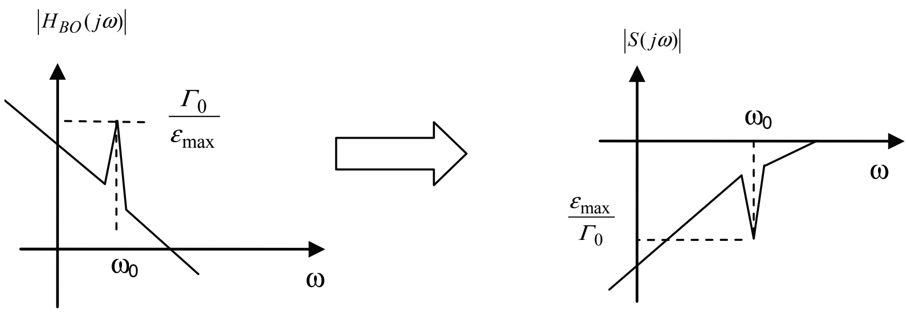

Figure 1.11. *Loop-shape for sinusoidal disturbance rejection on performance*

Note that the actuator therefore needs to be chosen such that it can respond to a command equal to:

$$u_{max} = |K(j\omega_0)|\varepsilon_{max}$$

– second case: it is crucial to protect the integrity of the system and therefore to desensitize the command signal u, which must never surpass a given value u_{max}.

Thus, $\frac{u(s)}{\Gamma_2(s)} = K(s)S(s) = \frac{K(s)}{1+H_{BO}(s)}$ must exhibit rejection behavior in ω_0, which is possible if the open loop is low-gain, because in this case: $K(j\omega_0)S(j\omega_0) \approx K(j\omega_0) = \frac{H_{BO}(j\omega_0)}{H(j\omega_0)} \approx \frac{T(j\omega_0)}{H(j\omega_0)}$. Again, the specification on the control signal imposes the gain of the open loop in ω_0 because:

$$\Gamma_0 \left| \frac{H_{BO}(j\omega_0)}{H(j\omega_0)} \right| < u_{max} \Rightarrow |H_{BO}(j\omega_0)| < u_{max} \frac{|H(j\omega_0)|}{\Gamma_0}$$

This enables us to determine the level of gain needed for the open-loop response to lend T the desired depth of rejection in ω_0 (see Figure 1.12).

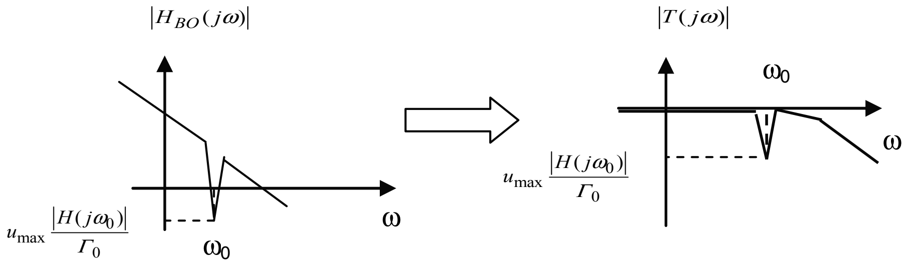

Figure 1.12. *Loop-shape for sinusoidal disturbance rejection on control*

Note that the system therefore needs to be designed so that it can deal with an error at least equal to:

$$\varepsilon_{max} = \frac{u_{max}}{|K(j\omega_0)|}$$

1.4.2. *Example 2: reference tracking and friction rejection*

Now suppose that we wish to track a reference signal $r(s)$ with maximum velocity Ω_{max} and maximum acceleration γ_{max} and we assume that the servo-loop is subject to a disturbance at the model's input $\Gamma_1(s)$ as a value step Γ_0 (e.g. dry friction). The closed-loop must be able to track $r(s)$ with an error less than ε_{max} at all times. We seek to determine:

– the low-frequency loop-shaping specification;

– the medium-frequency loop-shaping specification (nominal bandwidth, gain margin, etc.).

The performance specification is expressed in mathematical terms as:

$$\varepsilon(t) < \varepsilon_{max},\ \forall t > 0 \Rightarrow |\varepsilon(j\omega)| < \left|\frac{\varepsilon_{max}}{s}\right|_{s=j\omega} \Rightarrow |Sr(s) + SH\Gamma_1(s)|_{s=j\omega} < \left|\frac{\varepsilon_{max}}{s}\right|_{s=j\omega}$$

The reference and the disturbance taken into consideration here are low-frequency signals, which means that S must have little gain at low frequencies so as to desensitize the error, and that therefore the open loop must be high-gain at low frequencies so that the loop can track $r(s)$ in the presence of $\Gamma_1(s)$. In this case:

$$S(s) \approx \frac{1}{H_{BO}(s)}$$

The external inputs are written:

$$r(t) = \Omega_{\max} t + \frac{\gamma_{\max}}{2} t^2,\ \forall t > 0 \Rightarrow r(s) = \frac{\Omega_{\max}}{s^2} + \frac{\gamma_{\max}}{s^3}$$

$$\Gamma_1(t) = \Gamma_0,\ \forall t > 0 \Rightarrow \Gamma_1(s) = \frac{\Gamma_0}{s}$$

Thus we have:

$$\varepsilon(t) < \varepsilon_{\max},\ \forall t > 0$$

$$\Rightarrow |\varepsilon(j\omega)| < \left|\frac{\varepsilon_{\max}}{s}\right|_{s=j\omega}$$

$$\Rightarrow \left| S\left(\frac{\Omega_{\max}}{s^2} + \frac{\gamma_{\max}}{s^3}\right) + SH.\frac{\Gamma_0}{s}\right|_{s=j\omega} < \left|\frac{\varepsilon_{\max}}{s}\right|_{s=j\omega}$$

This means that:

$$\left|\frac{1}{H_{BO}(s)}\left(\frac{\Omega_{\max}}{s^2} + \frac{\gamma_{\max}}{s^3} + H(s)\frac{\Gamma_0}{s}\right)\right|_{s=j\omega} < \left|\frac{\varepsilon_{\max}}{s}\right|_{s=j\omega}$$

This imposes a low-frequency shape on the open loop:

$$\boxed{|H_{BO}(j\omega)| > \left|\frac{\Omega_{\max} s + \gamma_{\max} + H(s) s^2 \Gamma_0}{\varepsilon_{\max} s^2}\right|_{s=j\omega}}$$

For instance, for a load: $H(s) = \dfrac{1}{Js^2}$. We then obtain:

$$\boxed{|H_{BO}(j\omega)| > \left|\frac{J\Omega_{\max} s + J\gamma_{\max} + \Gamma_0}{J\varepsilon_{\max} s^2}\right|_{s=j\omega}}$$

This can enable us to tune the Proportional-Integrator filter used in the modeling of the open-loop response (the number of integrators needed can be determined by the final value theorem (FVT) on the static error). It should be noted that in this case, the open loop does indeed present a high gain value at low frequency to reject the static disturbance and track the reference signal (Figure 1.13).

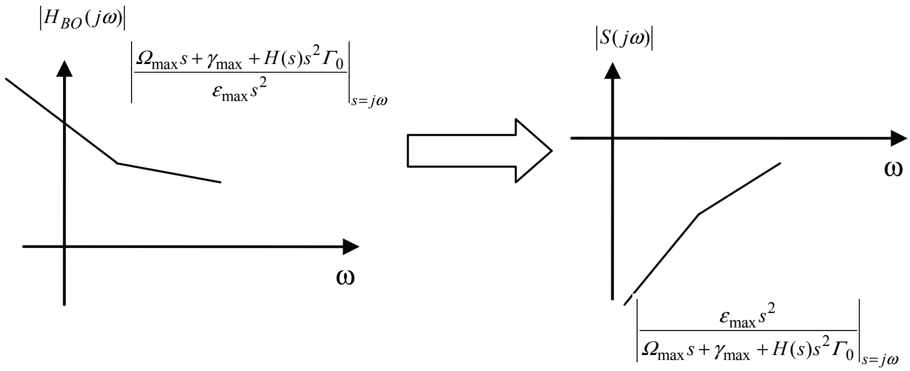

Figure 1.13. *Loop-shape for reference tracking and friction rejection*

Finally, we can look for the nominal bandwidth to be specified for the open loop. It is reasonable to assume that the open loop will behave like a phase lead $\frac{1+aTs}{1+Ts}$ in the vicinity of the bandwidth; in order to have a minimum phase margin $\Delta\Phi_{\min}$, this phase lead must be such that:

$$a = \frac{1+\sin\left(\Delta\Phi_{\min}\right)}{1-\sin\left(\Delta\Phi_{\min}\right)}$$

For instance, $a = 4.5$ for a phase margin of 40° (Figure 1.14).

The frequency which corresponds to this maximum phase is:

$$\omega_{BP} = \frac{1}{T\sqrt{a}}$$

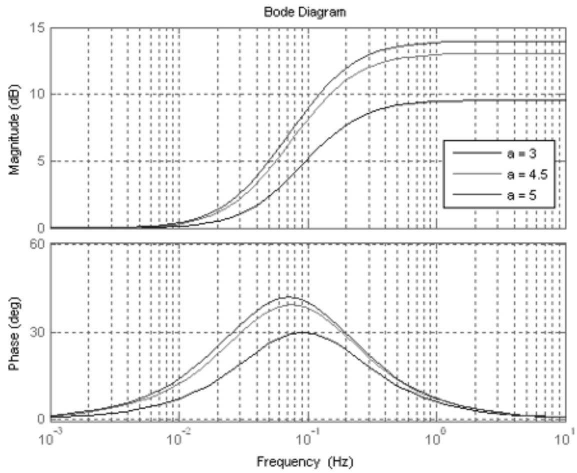

Figure 1.14. *Phase lead centered on* $\omega_{BP} = \dfrac{1}{T\sqrt{a}}$ *for T=1*

and at this frequency, the phase lead has the following gain:

$$\left|\frac{1+aTj\omega_{BP}}{1+Tj\omega_{BP}}\right| = \left|\frac{1+aTj\dfrac{1}{T\sqrt{a}}}{1+Tj\dfrac{1}{T\sqrt{a}}}\right| = \left|\frac{1+aj\dfrac{1}{\sqrt{a}}}{1+j\dfrac{1}{\sqrt{a}}}\right| = \sqrt{\frac{1+\dfrac{a^2}{a}}{1+\dfrac{1}{a}}} = \sqrt{\frac{1+a}{1+\dfrac{1}{a}}} = \sqrt{a}$$

$\omega_{BP} = \dfrac{1}{T\sqrt{a}}$ is the desired "nominal" bandwidth of the servo-loop for which the phase is maximum.

However, at low frequency:

$$|H_{BO}(j\omega)| > \left|\frac{J\Omega_{\max}s + J\gamma_{\max} + \Gamma_0}{J\varepsilon_{\max}s^2}\right|_{s=j\omega}$$

For the sake of continuity, it is reasonable to assume that the open loop will therefore behave as follows until the nominal bandwidth is reached:

$$\left|H_{BO}(j\omega)\right| \to \left|\left(\frac{J\Omega_{\max} s + J\gamma_{\max} + \Gamma_0}{J\varepsilon_{\max} s^2}\right)\left(\frac{1+aTs}{1+Ts}\right)\right|_{s=j\omega}$$

At the bandwidth ω_{BP}, the open loop is unitary; the nominal bandwidth is approximated by the relation:

$$\left|\frac{J\Omega_{\max} j\omega_{BP} + J\gamma_{\max} + \Gamma_0}{J\varepsilon_{\max}\omega_{BP}^2}\right|\left|\frac{1+aTj\omega_{BP}}{1+Tj\omega_{BP}}\right| = \left|\frac{J\Omega_{\max} j\omega_{BP} + J\gamma_{\max} + \Gamma_0}{J\varepsilon_{\max}\omega_{BP}^2}\right|\sqrt{a} = 1$$

This relation by itself imposes the minimum value for the phase margin, because it is based on a hypothesis about the phase margin:

$$a = \frac{1+\sin\left(\Delta\Phi_{\min}\right)}{1-\sin\left(\Delta\Phi_{\min}\right)}$$

For instance, if we look again at the case of pure inertia:

$$H(s) = \frac{1}{Js^2}$$

Hence, in the absence of a reference signal, i.e. with only dry friction rejection:

$$\left|\frac{\Gamma_0}{\varepsilon_{\max}\omega_{BP}^2}\right|\sqrt{a} = 1 \Leftrightarrow \omega_{BP} = a^{1/4}\sqrt{\frac{\Gamma_0}{J\varepsilon_{\max}}}$$

Finally, we need to specify the minimum gain margin (the phase margin having already been set at 40°). Strictly speaking, an examination of the parametric dispersion of $H(s)$, followed by the use of the small-gain theorem, should give us a good idea of the minimum gain margin.

For the sake of simplicity, it may be specified that the Nyquist locus of the open loop remains at a distance from the point -1 at least equal to 0.5, which means that the Nichols locus of the open loop cannot be inside the 6 dB circle, i.e. the minimum gain margin is 4 dB.

Taking account of the fact that the direct chain has a static gain G_0 which can vary from δG %, ($G_0 \rightarrow G_0(1+\delta G)$), it is then specified for the minimum gain margin:

$$\Delta G_{\min} = 4\ dB + 20\log(1+\delta G)$$

1.4.3. *Example 3: issue of flexible modes and high-frequency disturbances*

Consider $H(s)$, a system comprising a distribution of flexible modes (succession of resonances/antiresonances): the frequencies of the resonances are denoted as ω_S; the frequencies of the antiresonant modes are denoted as $\omega_{\overline{S}}$. We wish to set the value y at 0. The loop is subject to a disturbance at the model's input $\Gamma_1(s)$, the spectrum of which is centered on ω_S and $\omega_{\overline{S}}$; to compound the issue, the servo-loop is also subject to a white noise-type disturbance at the model's output $\Gamma_2(s)$ whose frequency spectrum naturally encapsulates ω_S and $\omega_{\overline{S}}$. By studying the impact of disturbances likely to excite the flexible modes of H, we aim to put forward a loop-shaping declination in line with the following synthesis logic:

– primary objective: regarding the high-frequency disturbance at the model's output $\Gamma_2(s)$, to favor the command signal which will not be unduly stimulated.

– secondary objective: regarding the disturbance at the model's input $\Gamma_1(s)$, to examine the two possible directions for synthesis: performance and then command.

We begin by writing the closed-loop transfers, u and ε in terms of the sensitivity functions S and T; here we have two sources of excitation for the servo-loop:

$$\varepsilon(s) = SH\Gamma_1(s) + S\Gamma_2(s) = S\left(H\Gamma_1(s) + \Gamma_2(s)\right)$$
$$u(s) = T\Gamma_1(s) + KS\Gamma_2(s) = T\left(\Gamma_1(s) + \frac{\Gamma_2(s)}{H}\right)$$

First, we shall examine the primary objective.

Primary objective: processing of Γ_2

For this disturbance, the control signal should not be unnecessarily excited. However, in relation to $\Gamma_2(s)$:

$$u(s) = KS\Gamma_2(s) = \frac{T}{H}\Gamma_2(s)$$

In order to satisfy the objective, therefore, we need the following to be true:

$$\left|\frac{T}{H}\Gamma_2\right| << u_{\max}$$

We note that H has resonant/antiresonant modes; the antiresonant modes for H therefore become the resonant modes for $1/H$. Hence, we can see that the antiresonant modes of H might excite the control signal: in order to avoid this, the open loop must exhibit powerful rejection of antiresonant frequencies. Indeed, in this case:

$$\left|\frac{T(j\omega_{\bar{S}})}{H(j\omega_{\bar{S}})}\right| \approx \left|\frac{H_{BO}(j\omega_{\bar{S}})}{H(j\omega_{\bar{S}})}\right| << 1$$

Thus, for this particular frequency (Figure 1.15):

$$\left|H_{BO}(j\omega_{\bar{S}})\frac{\Gamma_2}{H(j\omega_{\bar{S}})}\right| << u_{\max} \Rightarrow \left|H_{BO}(j\omega_{\bar{S}})\right| << \left|\frac{H(j\omega_{\bar{S}}).u_{\max}}{\Gamma_2}\right|$$

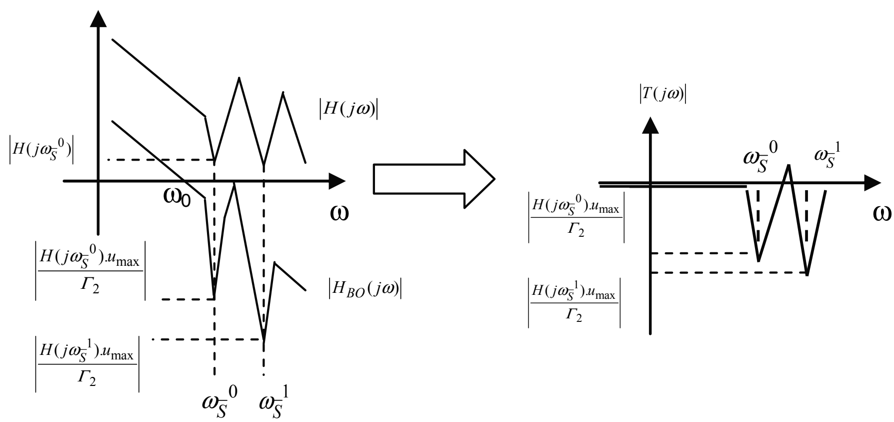

Figure 1.15. *Loop-shape for processing of Γ_2*

Secondary objective: processing of Γ_1

Γ_1 might excite the flexible modes; in this case we need to:

– either attenuate its effect on the stabilization error (favor the performance);

– or attenuate its effect on the power consumption (favor the control signal, same compromise as in example 1).

– First case: it is of crucial importance, when good performance is required, to desensitize the error ε, which must remain below a certain low value ε_{max}.

If we impose high-gain behavior on the open loop at the frequency of the flexible modes ω_S, then:

$$|\varepsilon| \approx \frac{|H(j\omega_S)\Gamma_1 + \Gamma_2|}{|H_{BO}(j\omega_S)|} << 1$$

$$|u| \approx \left|\Gamma_1 + \frac{\Gamma_2}{H(j\omega_S)}\right|$$

In this case, the residual destabilization in response to Γ_1 but also in response to Γ_2 can be minimized; the necessary consumption to minimize the impact of the excited flexible modes on the error may therefore be significant on the controller output. On the other hand, this objective may be contradictory to the loop-shape established to deal with Γ_2, because the primary objective in relation to Γ_2 (white noise) dealt with the excitation of the control signal.

In any case, the gain of the open-loop response can be set at that frequency ω_S to obtain an error less than ε_{max}. Indeed:

$$\frac{|H(j\omega_S).\Gamma_1 + \Gamma_2|}{|H_{BO}(j\omega_S)|} < \varepsilon_{max} \Rightarrow |H_{BO}(j\omega_s)| > \frac{|H(j\omega_S)\Gamma_1 + \Gamma_2|}{\varepsilon_{max}}$$

and it would be helpful to design the actuator so that $\left|\Gamma_1 + \frac{\Gamma_2}{H(j\omega_S)}\right| < u_{max}$ (Figure 1.16).

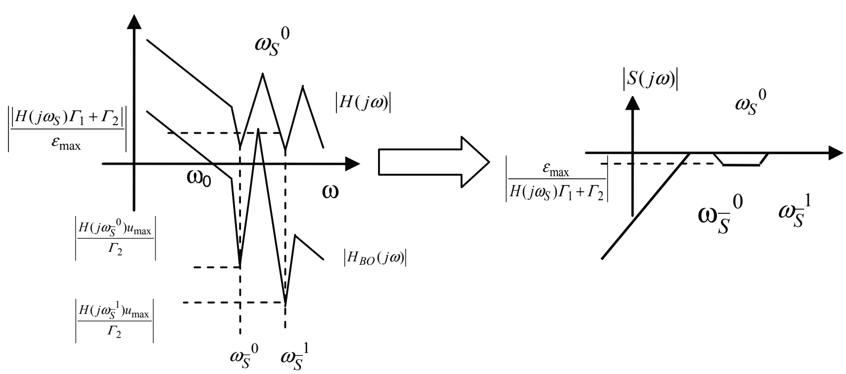

Figure 1.16. *Loop-shape for the processing of Γ_1 favoring the performance*

– Second case: it is crucially important to protect the integrity of the system, and thus to desensitize the control signal u, which must remain below a given value $u_{\max}$.

If we impose low-gain behavior on the open loop at the frequency of the flexible mode in question, then:

$$|\varepsilon| \approx |H(j\omega_S)\Gamma_1 + \Gamma_2|$$

$$|u| \approx \left| H_{BO}(j\omega_S)\left(\Gamma_1 + \frac{\Gamma_2}{H(j\omega_S)}\right)\right| << 1$$

$$|u'| \approx \left| -\Gamma_1 + H_{BO}(j\omega_S)\frac{\Gamma_2}{H(j\omega_S)}\right| \approx \Gamma_1$$

In this case, the error is entirely subject to the effect of the flexible modes in response to Γ_1, and also experiences Γ_2, but the consumption is very low, which is also consistent with the loop-shape established to deal with Γ_2 . Thus, we have (see Figure 1.17):

$$\left| H_{BO}(j\omega_S)\left(\Gamma_1 + \frac{\Gamma_2}{H(j\omega_S)}\right)\right| < u_{\max} \Rightarrow |H_{BO}(j\omega_s)| < \frac{u_{\max}}{\left|\Gamma_1 + \frac{\Gamma_2}{H(j\omega_S)}\right|}$$

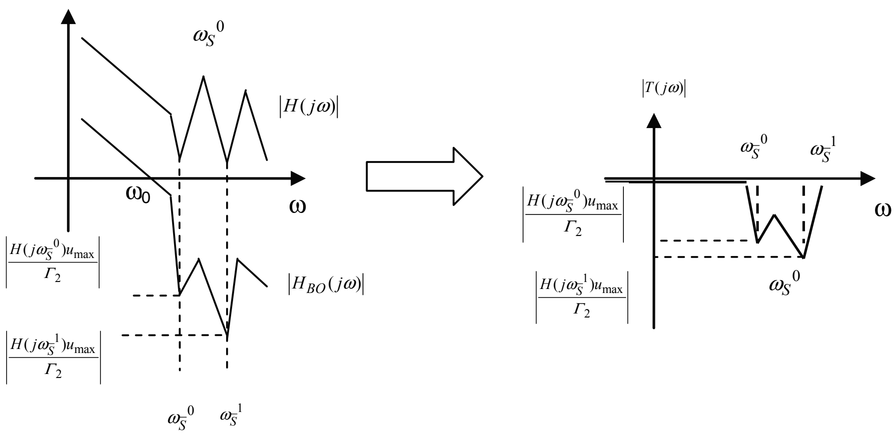

Figure 1.17. *Loop-shape for the processing of Γ_1 favoring the control signal*

1.4.4. *Example 4: stability robustness in relation to system uncertainties*

Consider the representation of a looped uncertain system (here in the form of multiplicative uncertainty at input), see Figure 1.18.

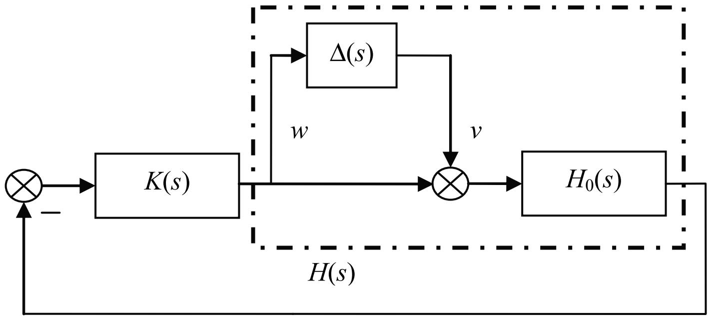

Figure 1.18. *Looping of an uncertain system*

Knowing a nominal model $H_0(s)$ for synthesis, we wish to calculate the controller $K(s)$ such that the looping remains unconditionally stable for any given variation $\Delta(s)$.

$$H(s) = \left(1 + \Delta(s)\right) H_0(s) \Leftrightarrow \Delta(s) = \frac{H(s) - H_0(s)}{H_0(s)}$$

$\Delta(s)$ therefore represents a modeling error around the nominal plant, or parametric variations. It has been shown that:

$$w(s) = \frac{K(s)H_0(s)}{1 + K(s)H_0(s)} v(s) = T(s)v(s)$$

Because of the small-gain theorem, we can state that the system $H(s)$ looped by $K(s)$ will remain stable for any variation $\Delta(s)$ if:

$$\forall \omega, \; \overline{\sigma}\left(T(j\omega)\right) < \underline{\sigma}\left(\Delta(j\omega)^{-1}\right)$$

and particularly at high frequency, when the open loop is usually low-gain:

$$\overline{\sigma}\left(H_{BO}(j\omega)\right) << 1 \Rightarrow \overline{\sigma}\left(H_{BO}(j\omega)\right) < \underline{\sigma}\left(\Delta(j\omega)^{-1}\right)$$

This enables us to estimate the roll-off necessary for the open loop to satisfy the condition of stability robustness, which constitutes the loop-shaping declination of the open loop at high frequency.

In the example illustrated in Figure 1.19, the synthesis model $H_0(s)$ is low-order. There is a relative modeling error $\Delta(s)$ between the real-world physical system $H(s)$ and its synthesis model.

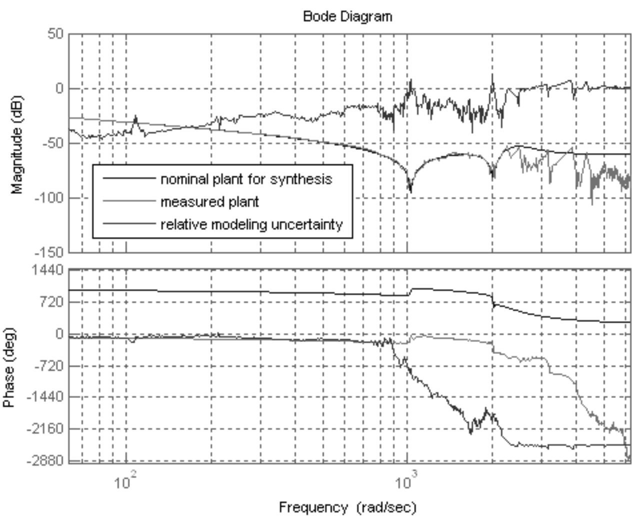

Figure 1.19. *Nominal system and relation modeling error*

Because the relative modeling error $\Delta(s)$ can be bounded, the system will be stable if the previous condition is satisfied.

1.5. Conclusion

Frequency specification of the different closed-loop transfers to satisfy requirements of performances and robustness can be viewed as a specification of the singular values of the open loop. The specification will therefore be presented in the following form:

$$
\boxed{\begin{array}{c}
\underline{\sigma}\left(H_{BO}(j\omega)\right)_{BF} > \left|W_{BF}(j\omega)\right| \\
\omega_{BP} > \omega_{BP\,\min} \\
\Delta\Phi > \Delta\Phi_{\min} \\
\Delta G > \Delta G_{\min} \\
\overline{\sigma}\left(H_{BO}(j\omega)\right)_{HF} < \left|W_{HF}(j\omega)\right|
\end{array}}
$$

This specification is obtained by examining the direction of work to be favored for each type of disturbance on its proper dynamic frequency range: the performance in terms of error, or integrity of the control signal. In general, it can be established that:

– favoring the integrity of the control signal will impose a low-gain function *T*, and therefore a low-gain open-loop response at the frequency in question;

– favoring the performance will impose a low-gain function *S*, and therefore a high-gain open-loop response at the frequency in question.

This specification can also be performed robustly by taking account of the uncertainties of the system.