Chapter 1

Design at Serviceability Limit State (SLS)

1.1. Nomenclature

1.1.1. *Convention with the normal vector orientation*

The normal vector is chosen to be oriented toward the external part of the considered body. The usual conventions of mechanics of continuous media are chosen, leading to a positive stress for tension and a negative stress for compression.

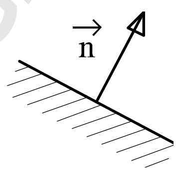


Figure 1.1. *Definition of the normal unit*

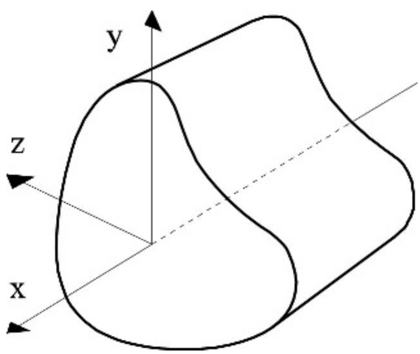
1.1.2. *Vectorial notation*

As opposite to the notation used for figures, where vectors are represented with an arrow, in the text, vectors are denoted by bold characters and its components have normal non-bold characters. As an example, we will have "$\mathbf{M} = M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}$".

1.1.3. *Part of the conserved reference section*

The conserved reference part of the beam used for the calculation of generalized stress in use of the static theorems is the "right" part.

1.1.4. *Frame*



- Representation of a cross-section.
- Origin of the frame: arbitrary point of the section.
- *x*-axis standing out.
- horizontal*z*-axis, leading to "negative" moments at the support level.
- *y*-axis defined from the orthonormal direct trihedron.

Figure 1.2. *Orientation of the frame for a general section*

1.1.5. *Compression stress $\sigma_{c,sup}$ in the most compressed fiber*

It is admitted that the neutral axis is located inside the cross-section, thus delimiting a tension zone and a compressed zone. This last assumption of a neutral axis inside the cross-section is no more valid when considering additional meaningful normal forces. Typically, under a positive moment (in span), the tension zone is located under the neutral axis, and the compression zone above the neutral axis, as shown in Figure 1.3. Obviously, in the presence of a negative bending moment, the tension zone and the compression zone are permuted with respect to the notation of Figure 1.3. The neutral axis as shown in Figure 1.3 allows the introduction of the concept of extremal compressed fiber, defined from the most distant parallel to the neutral axis belonging to the cross-section. The most compressed fiber in concrete is by definition the fiber associated with the minimal compressive stress in algebraic value, denoted by $\sigma_{c,\sup}$.

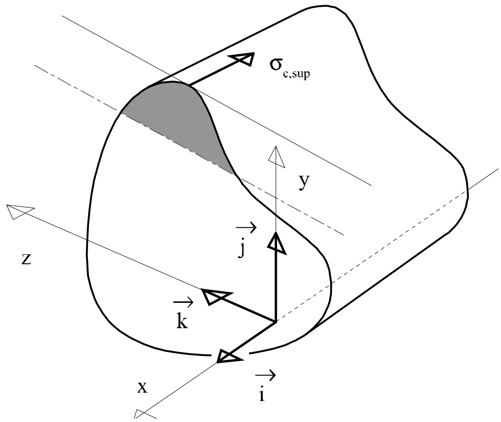


Figure 1.3. *Concept of extremal compressed fiber*

1.2. Bending behavior of reinforced concrete beams – qualitative analysis

1.2.1. *Framework of the study*

1.2.1.1. *Constitutive law of concrete*

The constitutive law of concrete is a strong unsymmetrical law in tension and in compression, both from the strength and the postfailure response, which is characterized by its ductility (see Figure 1.4). As a natural choice, the subscript c refers to concrete whereas the subscript s refers to the steel material. We adopt by $\sigma_{c,min}$ the extreme stress at the compression peak.

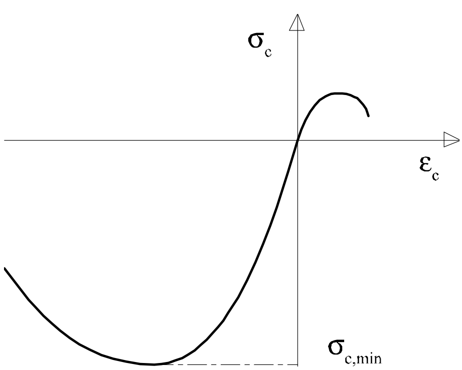


Figure 1.4. *Unsymmetrical response of concrete in uniaxial tension and compression*

1.2.1.2. *Beam theory in simple bending*

In this section, reinforced concrete beams in simple bending are studied (without axial forces), composed of a rectangular cross-section with a total height denoted by h and a width b. This section is reinforced by some steel reinforcement working in tension with a cross-section denoted by A_{s1} and by steel reinforcement working in compression with a cross-section denoted by A_{s2}. The center of gravity of the tensile reinforcement is at a distance d of the upper fiber, and one of the compression reinforcements is at a distance d' of the upper fiber of the cross-section.

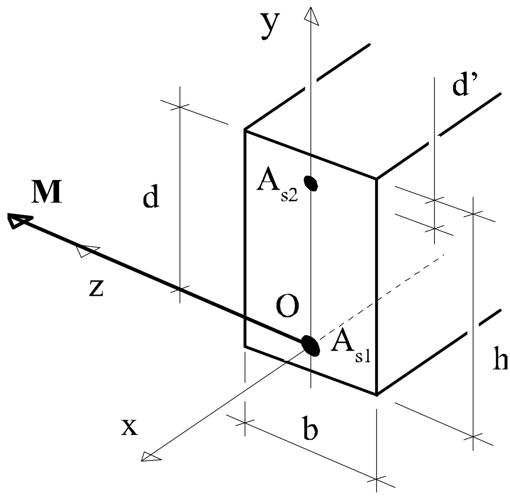


Figure 1.5. *Geometrical parameters of the reinforced cross-section; tensile and compression steel reinforcement*

In this case, again, it is implicitly accepted that the bending solicitation corresponds to a positive moment (the lower fibers are in tension with this convention, typically in span). Designing under negative moment (typically at support, for instance) is formally feasible by permuting the behavior of the cross-section. Furthermore, we can introduce the strain ε_{s1} as the strain of the tensile reinforcement with the largest tensile stress σ_{s1} and with the reinforcement area A_{s1}.

1.2.2. *Classification of cross-sectional behavior*

Three kinds of reinforced concrete beam responses can be distinguished, depending on the steel reinforcement density (Figure 1.6). These responses are explicitly detailed in the space of the bending moment with respect to the stress in the most compressed fiber in concrete.

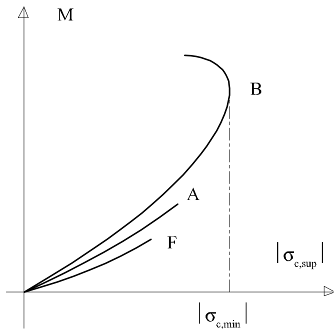


Figure 1.6. *Bending behavior of reinforced concrete beams with respect to the steel reinforcement density*

F: brittle response, which appears when the beam design does not respect the condition of non-brittleness.

A: beam with low steel reinforcement density, characterized by a global ductile response. Failure is induced by a large drawing of the tensile steel reinforcement. As discussed below, the letter *A* refers to Pivot A.

B: beam with high steel reinforcement density, characterized by the breaking up of the compressed part of the upper part concrete. The letter *B* refers to the behavior classified as Pivot B.

In the following, brittle reinforced concrete beams of type F will not be investigated.

1.2.3. *Parameterization of the response curves by the stress σ_{s1} of the most stressed tensile reinforcement*

In Figure 1.7, the response curves are parameterized by the stress σ_{s1} of the most stressed tensile reinforcement. When reading Figure 1.7 in the sense of increasing the bending moment (the solicitation), the stress value of the tensile steel reinforcement also increases. The stress in the compressed part of the concrete (typically in the upper part of the section for positive bending moment in span) is limited by a material characteristic value f_{cs} defined in the National Annexes to the Eurocode 2 - EC2 [EUR 04] (EC2 7.2.1). This material limitation is defined in the rules for reducing the longitudinal crack level in the tensile part of the cross-section, for reducing the microcacking phenomena of concrete in the compressed part and for limiting the time-dependent induced effects (including the specific creep effect).

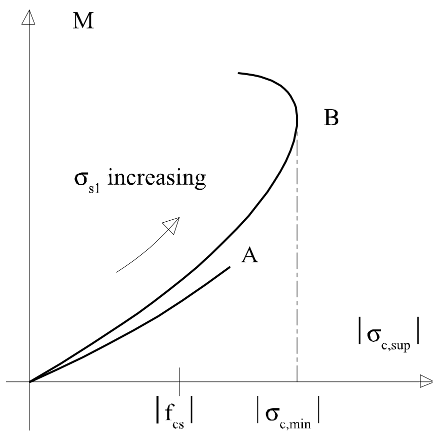


Figure 1.7. *Bending sectional behavior of reinforced concrete beams with respect to the steel reinforcement density; the response curves are parameterized by the stress level in the tensile steel reinforcement*

For the same bending moment, the share taken by tensile reinforcement is more important in case *A*. Consequently, the share taken by the concrete is less important; this generates therefore, in this material, a lower normal stress. This explains the relative position of two curves.

1.2.4. *Comparison of σ_{s1} of the tensile reinforcement for a given stress in the most compressed concrete fiber $\sigma_{c,sup}$*

In this section, the physics of the reinforced concrete beam behavior is discussed with respect to the steel reinforcement density. For a given value of the stress in the most compressed concrete fiber, bending moments for the two different types of reinforced concrete beams *A* and *B* are different (Figure 1.8). As the reinforced concrete beam of type *A* has a low density of tensile steel reinforcement, the following inequality holds: $M_B > M_A$.

The stress in the most compressed fiber in concrete is being fixed; the linear strain of the upper fibers is the same for the two kinds of reinforced concrete beams. As a result, it is necessary that the compressed part in the section of type *B* has to be larger than the one in the section of type *A*, in order to fulfill the moment inequality. The linear distribution of strains and the associated stress in the tensile reinforcement of section of type *A* are then larger than the section of type *B* (see Figure 1.8).

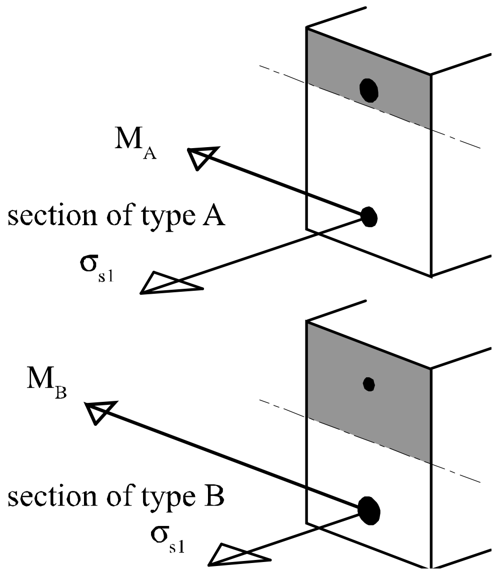


Figure 1.8. *Bending behavior of reinforced concrete beams with respect to the steel tensile reinforcement density; sections of type A and type B*

Reciprocally, if the stress value in the tensile steel reinforcement (in the lower part of the reinforced section) is the same for both types of the section, the section of type *A* will have a lower stress value (in absolute value) in the upper part of the compressed concrete in comparison to the section of type *B* (see Figure 1.9).

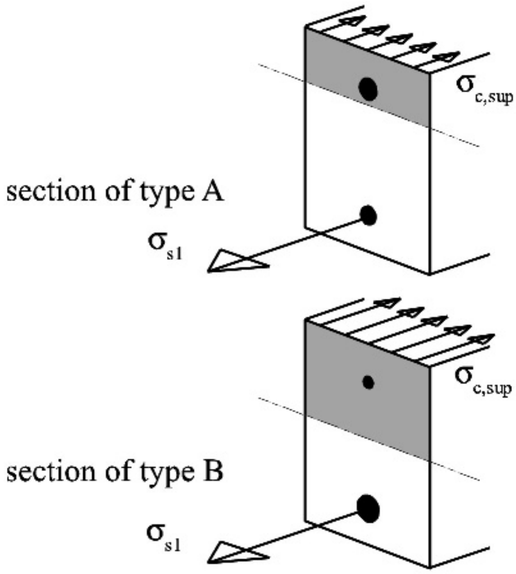


Figure 1.9. *Stress in the compression concrete block; sections of type A and type B*

1.2.5. *Comparison of the bending moments*

To reduce the non-elastic strain level (existence of permanent strain for a stress history ended by a vanishing stress value, for instance) leading to some possible large cracks or some possible structural member strains, the steel reinforcement tensile stress is limited to a material characteristic value f_{ss} that is also defined in the National Annexes to Eurocode 2 – EC2 (EC2 7.2.2).

For designing the structural members at the Serviceability Limit State (SLS), we should satisfy the following two simultaneous inequalities that have to be algebraically fulfilled (see also Figure 1.10 or 1.11):

$$\begin{cases} \sigma_{s1} \leq f_{ss} \\ \sigma_{c,\sup} \geq f_{cs} \end{cases} \qquad [1.1]$$

For reinforced concrete beams of kind *A*, slightly reinforced (or with a low tensile steel reinforcement density), the SLS is controlled by the drawing of the tensile steel reinforcement. The bending moment at the SLS is then calculated for:

$$\begin{cases} \sigma_{s1} = f_{ss} \\ \sigma_{c,\sup} \geq f_{cs} \end{cases} \qquad [1.2]$$

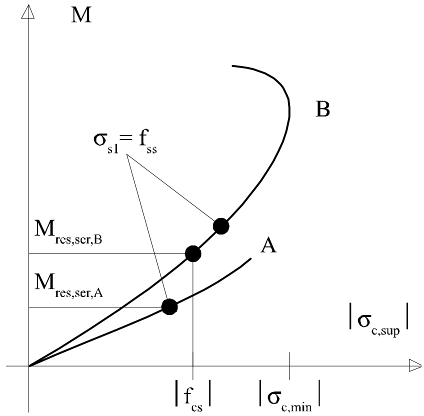


Figure 1.10. *Introduction to the concepts of pivots A and B at the serviceability limit state (SLS); in a Pivot B section, stress limitation in concrete occurs before stress limitation in steel*

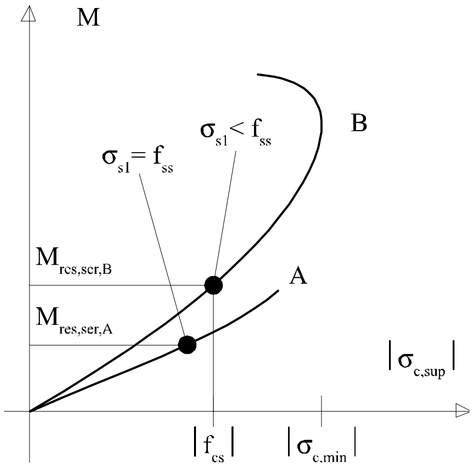


Figure 1.11. *Introduction to the concepts of pivots A and B at the serviceability limit state (SLS)*

For reinforced concrete beams of kind B, highly reinforced (or with a high tensile steel reinforcement density), the SLS is controlled by the compression in concrete, considered as unacceptable by the rules. The bending moment at the SLS is then calculated for:

$$\begin{cases} \sigma_{s1} \leq f_{ss} \\ \sigma_{c,\sup} = f_{cs} \end{cases} \tag{1.3}$$

1.3. Background on the concept of limit laws

1.3.1. *Limit law for material behavior*

1.3.1.1. *Definition*

A limit law, for a given material, is defined from the overall possible configurations reached by this material and is compatible with a given state criterion. A limit law, in its essence, is different from a constitutive law; the constitutive law implicitly contains all the successive temporal configurations before reaching the limit state. Therefore, in the concept of limit law, the notion of the stress or strain path is replaced by the notion of the limit state, which is the set of configurations compatible with a given state criterion.

1.3.1.2. *Application to the rectangular parabolic diagram, a limit law of the ultimate limit state*

In the case of a parabola–rectangle diagram applied to concrete modeling (one of the possible limit laws for concrete modeling at the ultimate limit state (ULS)), if the section reaches the concrete limit law, then the Bernoulli assumption leads to a nonlinear stress distribution along the cross-section, as shown by the stress distribution identified by number 5 in Figure 1.11.

In Figure 1.12, for the solicited bending moments $M_1 < M_2 < M_3 < M_4 < M_5 = M_{u,act}$, the stress distribution along the cross-section cannot be ranged as a parabola–rectangle law, for the bending moment M_1–M_4; for these solicitations, the stress response curves are transitory curves between the linear state and the limit state of the parabolic rectangular limit state.

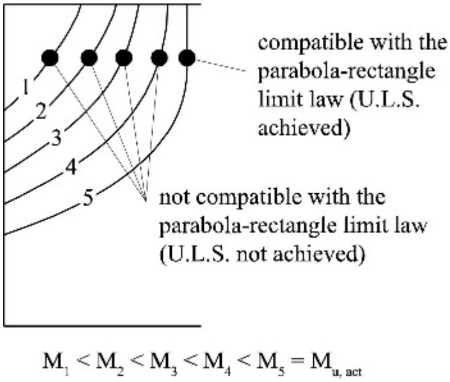


Figure 1.12. *Illustration of the concept of limit law – stress evolution inside the cross-section; Ultimate limit state parabolic-rectangular stress–strain relationship*

However, for the bending moment denoted by M_5 (ultimate limit acting bending moment), the limit state is reached, and the stress limit diagram is known.

1.3.2. *Example of limit laws in physics, case of the transistor*

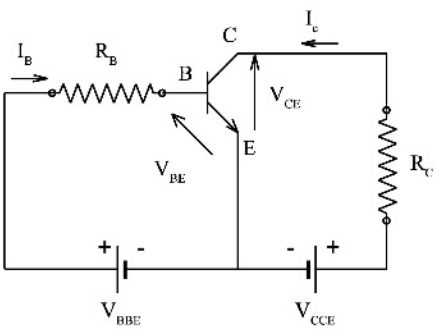


Figure 1.13. *Electronic example for limit laws in physics: the transistor*

Figure 1.13 shows the setting of a polarized transistor, which is often met in the field of electronics designs. The characteristics of the transistor are given in Figure 1.13. The linear part of Figure 1.14 with a softening slope contains the locus of the static point of this electronic system. In fact, the working point is located on this loading line.

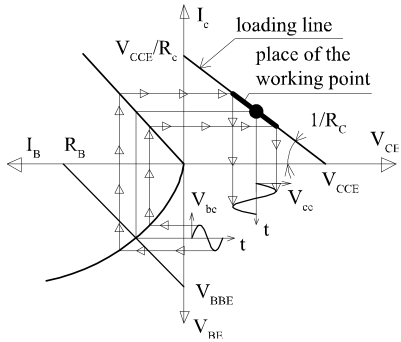


Figure 1.14. *Limit laws in electronics design*

This softening straight line is clearly a limit law: the path from the origin to reach this limit law is not known *a priori*, and only the limit state is of interest.

1.3.3. *Design of reinforced concrete beams in bending at the stress Serviceability Limit State*

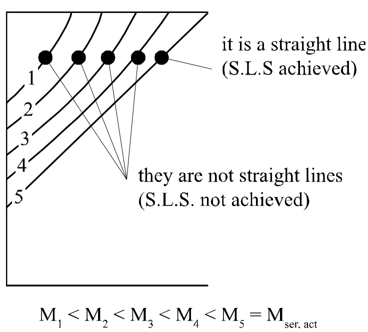


Figure 1.15. *Limit law at the stress serviceability limit state (SLS); stress distribution along the cross-section*

In the case of the limit behavior of concrete at the SLS, a section at its limit state will have a linear distribution of stress in the compressed part of the section (as a consequence of Bernoulli kinematics and elasticity behavior of each material at the SLS), the contribution of concrete in tension being neglected as a fundamental assumption. This “limit” behavior is visualized in Figure 1.15 by the linear stress distribution indicated by number 5.

For the external moments M_1, M_2, M_3, M_4 and M_5 with $M_1 < M_2 < M_3 < M_4 < M_5 = M_{ser,act}$, the stress distribution is theoretically not necessarily linear along the cross-section, for the moments from M_1–M_4; such curves are not known *a priori*. Only for the moment M_5 (limit moment at the SLS), the stress distribution at SLS is known and is linear. However, for this specific limit state, it can be also admitted in this case that the limit case (characterized by linear elasticity) is also reached for the transitory states.

1.4. Limit laws for steel and concrete at Serviceability Limit State

1.4.1. *Concrete at the cross-sectional SLS*

Concrete, at the cross-sectional SLS, is modeled by a linear elastic constitutive law in compression, with a stress limitation denoted by f_{cs}. The strength in tension is neglected (Figure 1.16) and its Young's modulus is denoted by E_c.

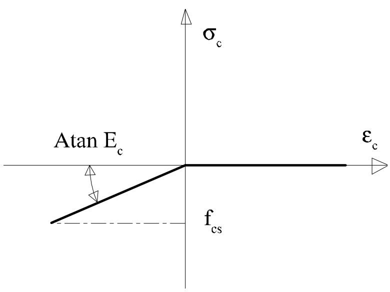


Figure 1.16. *Limit law of concrete at serviceability limit state*

1.4.2. *Steel at the cross-sectional SLS*

Steel at the cross-sectional SLS is also modeled by a linear elastic constitutive law, with a tension limit denoted by f_{ss} (Figure 1.17). A symmetrical behavior in tension and in compression is also accepted for modeling the behavior of the steel reinforcement, which means that the potential buckling of the steel reinforcement in the presence of a crushing phenomena in concrete is generally neglected, except in so far as the spacing of transverse reinforcement. As a result, for the compression steel reinforcement, stress in the absolute value is also limited by the same value f_{ss}. Young's modulus of the steel reinforcement is denoted by E_s.

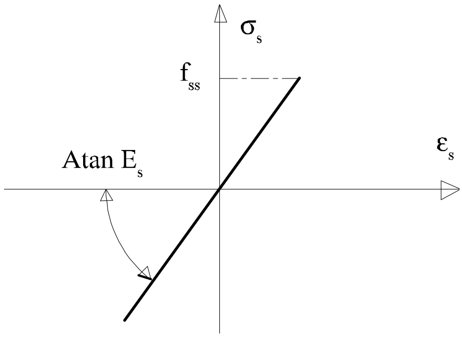


Figure 1.17. *Limit law of steel reinforcement at serviceability limit state*

1.4.3. *Equivalent material coefficient*

The equivalence coefficient (between steel and concrete) is by definition the ratio between the steel modulus and the concrete modulus $\alpha_e = E_s/E_c$. This ratio depends on the short-term or the long-term analysis considered at the SLS. As the Young's modulus is decreasing with the effects of time, the equivalence ratio tends to increase with the effects of time. This ratio is also sometimes denoted by the equivalence coefficient "*n*" in the earlier textbooks on reinforced concrete design. A typical order of magnitude for this ratio is $\alpha_e = 15$.

1.5. Pivots notion and equivalent stress diagram

1.5.1. *Frame and neutral axis*

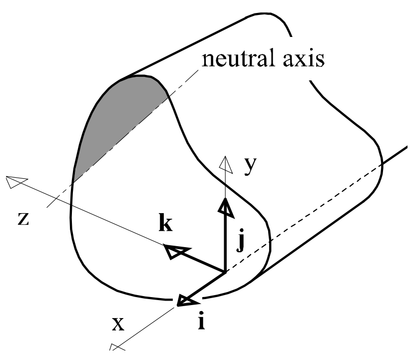


Figure 1.18. *Neutral axis in a general section*

Let us consider a beam in which the direct orthonormal frame *O, x, y,* and *z* is defined. The beam is solicitated by an external screw given at the SLS. The frame is in conformity with section 1.1.3. or 1.1.4. Under the action of the external forces, a neutral axis can be defined (Figure 1.18). It is admitted that the direction of the neutral axis is already known. In the opposite case, some symmetrical considerations have to be added in the reasoning or we can proceed by iterations. The neutral axis separates the tension and the compression parts of the section.

Each of the "n" steel bars of the reinforced section will be identified by the subscript q with $q \in [1;n]$. Let $\boldsymbol{P_q}$ be the vector whose origin is denoted by O and with an extremity defined at the center of gravity of the qth reinforcement bar. The oriented angle between the axis z and the neutral axis is denoted by ω.

Let d_q be the distance between the center of gravity of the *qth* reinforcement bar and the parallel to the neutral axis assigned to pass through the most compressed fiber. Among the "n" steel bars of the reinforced section, the most tensioned reinforcement bar is identified by its subscript r and its cross-sectional area A_{sr} with $r \in [1;n]$. The effective height "d_r" of the cross-section can now be defined as the distance between the center of gravity of the most tensioned reinforcement bar, and the parallel to the neutral axis passing through the most compressed fiber (Figure 1.19).

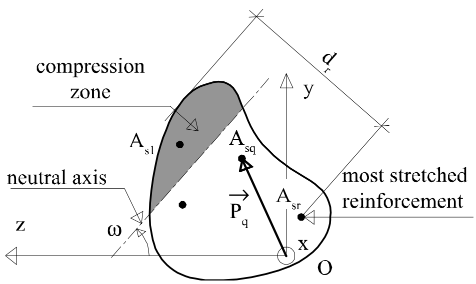


Figure 1.19. *Definition of the notion of effective height d_r*

1.5.2. *Conservation of planeity of a cross-section*

The conservation law of planeity of a cross-section during the deformation process allows us to relate due to some geometrical properties, the strain of each constituent of the cross-section, the compressed part of the concrete and each steel reinforcement bar. The ordinate axis "*u*" can be defined from a straight line perpendicular to the neutral axis, and oriented from the less compressed to the most compressed concrete fibers. The origin of this axis is located at the basis of this axis, positioned from the projection of the origin point *O* on this line. The stress field in this section is a function of this parameter "*u*". The following scalar variable can be introduced from $u_q = \mathbf{P_q} \cdot (\mathbf{j} \cos\ \omega + \mathbf{k}\ \sin \omega)$, where the point denotes the scalar product (dot product) between two vectors (Figure 1.20).

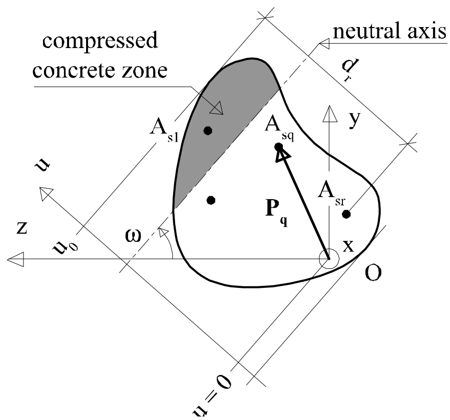


Figure 1.20. *Parameterization of the stress field with the variable u*

The neutral axis is identified by the value $u_r + (1-\alpha)d_r$ for the variable u, where α is a dimensionless distance characterizing the position of the neutral axis. α is a dimensionless parameter normalized by the total height of the cross-section. The most compressed concrete fiber is associated with the projection value u_0 for the variable u; the equality $u_0 = d_q + u_q$ is exact whatever the considered subscript $q \in [1;n]$. The strain ε_{sq} in each reinforcement bar and the strain $\varepsilon_c(u)$ in concrete are related with each other by the relationship induced by the planeity conservation of the cross-section:

$$\frac{\varepsilon_{s1}}{d_1 - \alpha d_r} = \frac{\varepsilon_{s2}}{d_2 - \alpha d_r} = \ldots = \frac{\varepsilon_{sq}}{d_q - \alpha d_r} = \ldots$$

$$= \frac{\varepsilon_{sr}}{d_r - \alpha d_r} = \ldots = \frac{\varepsilon_{sv}}{d_v - \alpha d_r} = \frac{-\varepsilon_c(u)}{-u + u_r + (1 - \alpha)d_r} \qquad [1.4]$$

1.5.3. *Planeity conservation law in term of stress*

From equation [1.4], the "limit" laws associated with each material (namely elasticity for each material constituents), at the SLS, allow us to obtain a relationship between the stress in the steel reinforcement bars and the stress in the compressed concrete part along the cross-section. The following equalities finally hold between the stress in the q reinforcement bars σ_{sq} and the stress distribution in the compressed part of concrete $\sigma_c(u)$:

$$\frac{\sigma_{s1} / \alpha_e}{d_1 - \alpha d_r} = \frac{\sigma_{s2} / \alpha_e}{d_2 - \alpha d_r} = \ldots = \frac{\sigma_{sq} / \alpha_e}{d_q - \alpha d_r} = \ldots = \frac{\sigma_{sr} / \alpha_e}{d_r - \alpha d_r} = \ldots = \frac{\sigma_{sv} / \alpha_e}{d_v - \alpha d_r} = \frac{-\sigma_c(u)}{-u + u_r + (1 - \alpha)d_r} \qquad [1.5]$$

By setting k the proportionality factor of this last equality (proportional to the curvature), a series of affine relationships for the stress value in both the tensile and the compression steel reinforcement bars, as for the compressed part of concrete is obtained, which can be again written as:

$$\sigma_{sq} / \alpha_e = k(d_q - \alpha d_r) \forall q \in [1;t] \qquad [1.6]$$

The calculated stress in the most compressed fiber of concrete is given by:

$$\sigma_c(u) = k[-u + u_r + (1-\alpha)d_r] \qquad [1.7]$$

It is worth mentioning that the characteristic limit values in both the steel and concrete part of the composite section are defined in a algebraic manner, with the mechanics of continuous media convention, that is f_{ss} is positive and f_{cs} is negative. At the limit state formally, the following equalities can be viewed to be valid only in the case $\sigma_{sr} = f_{ss}$ or $\sigma_c(u_0) = f_{cs}$ (see section 1.2.5). We can also consider that the limit state is always reached in elasticity, at least at the SLS.

We finally obtain in Pivot A, with $\sigma_{sr} = f_{ss}$:

$$\sigma_{sq} / \alpha_e = k(d_q - \alpha d_r) \forall q \in [1;t] \text{ and } \sigma_{sr} / \alpha_e = f_{ss} / \alpha_e \text{ for } q = r \qquad [1.8a]$$

$$\sigma_c(u) = k[-u + u_r + (1-\alpha)d_r] \qquad [1.8b]$$

In Pivot B, the stress relationships are obtained from $\sigma_c(u_0) = f_{cs}$:

$$\sigma_{sq} / \alpha_e = k(d_q - \alpha d_r) \forall q \in [1; t] \quad [1.9a]$$

$$\sigma_c(u) = k[-u + u_r + (1-\alpha)d_r] \quad [1.9b]$$

1.5.4. *Introduction to pivot concepts*

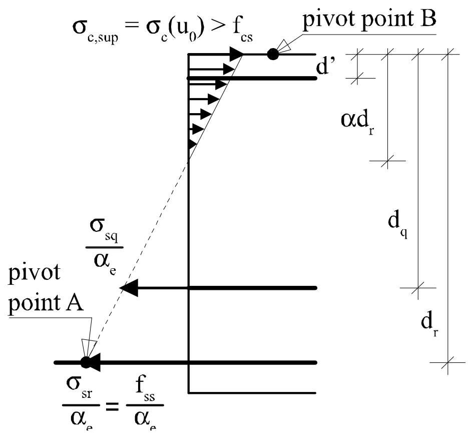


Figure 1.21. *Definition of Pivot A;* $\alpha_e = E_s/E_c$

Designing by *h* the distance between the two extremal parallels to the neutral axis (*h* is typically the height of the cross-section), the series of equalities, equation [1.8], can be presented as:

$$\sigma_{sq}/\alpha_e = k(d_q - \alpha d_r) \ \forall \ q \in [1; v \ ]$$
$$\text{and } \sigma_{sr}/\alpha_e = f_{ss} / \alpha_e \text{ for } q = r \quad [1.10a]$$

$$\sigma_c(u) = k[-u + u_r + (1-\alpha)d_r]$$
$$\text{and } 0 > \sigma_c(u) > f_{cs} \forall u \in [u_0 - h; \ u_0] \quad [1.10b]$$

This system of equations can be interpreted as the analytical expression of a plane whose trace in the space *{u,x}* is the straight line appearing in Figure 1.21. This line crosses the fixed point corresponding to the most tensioned steel reinforcement with an equivalent tensile stress equal to f_{ss}/α_e. This point is called Pivot A (Figure 1.21).

The series of equalities, equation [1.9], can be written as:

$$\sigma_{sq} / \alpha_e = k(d_q - \alpha d_r)\ \forall q \in [1;v]$$
$$\text{and } \sigma_{sr} / \alpha_e < f_{ss} / \alpha_e \text{ for } q = r \qquad [1.11a]$$

$$\sigma_c(u) = k[-u + u_r + (1-\alpha)d_r] \text{ and } 0 > \sigma_c(u_0) = f_{cs} \qquad [1.11b]$$

Again, this system of equations can interpreted as the analytical expression of a plane whose trace in the space *{u,x}* is the straight line appearing in Figure 1.22. This line crosses the fixed point, corresponding to the most compressed fiber with an equivalent compression stress equal to f_{cs}. This point is called Pivot B.

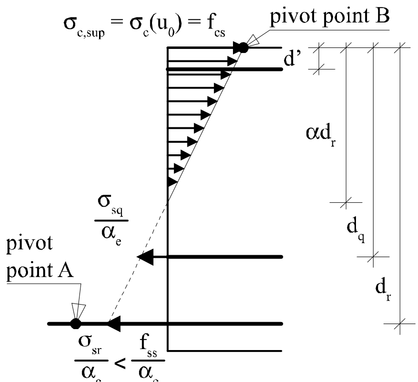


Figure 1.22. *Definition of Pivot B;* $\alpha_e = E_s/E_c$

These fixed points play the role of a turning point or pivot point, around which the equivalent stress lines turn, and define the profile of equivalent stress along the cross-section, which is the stress for the concrete part in compression, and the equivalent or homogenized stress for the steel reinforcement bars, typically the stress divided by the equivalent coefficient α_e.

1.5.5. *Pivot rules*

The planeity conservation assumption at SLS, which is in fact a fundamental kinematics assumption, leads to an equivalent linear stress diagram that contains one of the two Pivot points:

– the point "Pivot A" that corresponds to an equivalent stress f_{ss}/α_e of the most tensioned steel reinforcement bars;

– the point "Pivot B" that corresponds to a compression stress f_{cs} in the most compressed fiber in concrete.

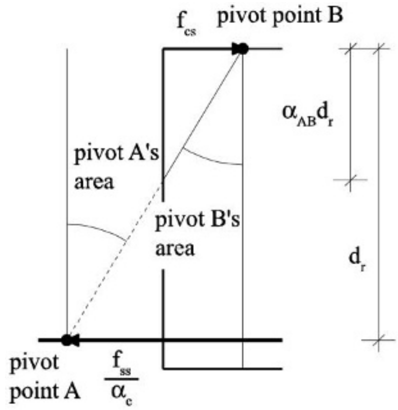


Figure 1.23. *Working zone of the composite cross-section in Pivot A and Pivot B*

This straight line exactly corresponds to the compression stress field in the compressed part of concrete, whereas the stress in the steel reinforcement bars, in tension or in compression, are deduced from the equivalent stress line by introducing the equivalence coefficient $1/\alpha_e$. At the cross-sectional SLS, the diagram of equivalent stresses should cross one of the two pivot points (Pivot A and Pivot B) and should also respect the limit stress requirement for the extremal point at the other pivot point (see Figure 1.23). The limit case between the two pivots is referred as Pivot AB, and is characterized by the limit value for the relative position of the neutral axis α_{AB} defined from the upper fiber of the cross-section as $\alpha_{\mathrm{AB}} = \alpha_e f_{cs}/(\alpha_e f_{cs} - f_{ss})$.

1.6. Dimensionless coefficients

1.6.1. *Goal*

To characterize the static equilibrium equations of the cross-section, it is necessary to integrate both the stress field and the moment in the compressed part of concrete, defined at a given point of the cross-section. This calculation will lead to the determination of the internal forces screw applied to concrete. The total internal screw will then be deduced from the concrete screw, by adding the tensile and the compression steel reinforcement screw evaluated at the same point. The calculation of internal and external screws, both with equilibrium equations will be written, as much as possible, in a dimensionless format, to avoid dimension confusion and try to give the more general presentation. Dimensionless numbers are also important to classify the sectional behavior, in its more general formulation.

1.6.2. *Total height of the cross-section*

As indicated in Part 1.5.4, the total height of the cross-section is denoted by h (h is the distance between two parallels to the neutral axis passing through the extremity of the cross-section). For a rectangular or a T-beam, h has its usual definition and is the height of the cross-section.

1.6.3. *Relative position of the neutral axis*

In section 1.5.2, the position of the neutral axis was identified from the following equation $u = u_r + (1 - \alpha)\, d_r$, where α is the relative height of the neutral axis. Even if the understanding of this relative position is quite intuitive as explained in Figure 1.24, a rigorous mathematical definition can be given based on geometrical arguments. The height parameter d_r can be defined from the distance between two lines parallel to the neutral axis, one passing through the most compressed fiber and the other crossing the most tensioned steel reinforcement. The distance between the most compressed fiber and the neutral axis is denoted by αd_r (Figure 1.24).

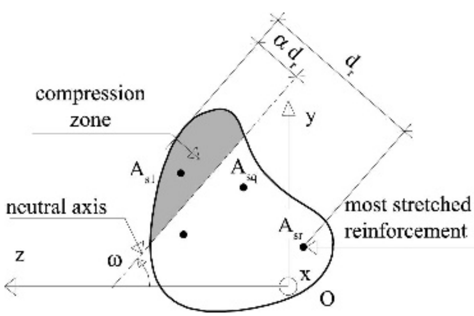


Figure 1.24. *Definition of the relative height of the neutral axis*

1.6.4. *Shape filling coefficient*

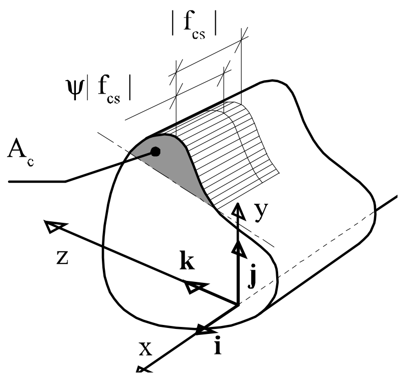


Figure 1.25. *Definition of the shape filling coefficient* ψ

The screw of internal forces applied to concrete is calculated for the normal force N_c, by integration of the compression stress over the domain of the compressed concrete denoted by D_c:

$$\boldsymbol{N_c} = \iint_{D_C} \sigma_C \,\boldsymbol{i}.dy.dz = \boldsymbol{i}.\iint_{D_C} \sigma_C .dy.dz \qquad [1.12]$$

The resultant can be written in the following form: $\boldsymbol{N_c} = N_c\,\boldsymbol{i}$. It is worth mentioning that the stress function $\sigma_c(u)$ is a continuous function within the compression domain D_c. By virtue of the mean value theorem, it can be rigorously deduced that there exists a characteristic variable u^* belonging to the domain of variation of u, such as:

$$N_c = \sigma(u^*)\iint_{D_C} dy.dz \qquad [1.13]$$

The integral in such a definition represents the area of the compression domain D_c, which can be denoted by A_c. The shape filling coefficient ψ can now be defined from this mean value:

$$\psi = \frac{N_c}{f_{cs}\,A_c} = \frac{\sigma(u^*)}{f_{cs}} \qquad [1.14]$$

The meaning of the shape filling coefficient can be more easily understood from considerations of the volumes highlighted in Figure 1.25, and the intrinsic relationship between the actual stress volume and the equivalent uniform stress volume. The shape filling coefficient is the ratio between the volume of a cylinder with a basis equal to D_c and a height equal to $\sigma(u^*)$ and the volume of another cylinder of the same basis but with height equal to f_{cs}. It can be remarked that the volume of a cylinder with a basis equal to D_c and with height $\sigma(u^*)$ is equal to N_c. The stress $\sigma(u^*)$ is called the mean stress.

1.6.5. *Dimensionless formulation for the position of the center of pressure*

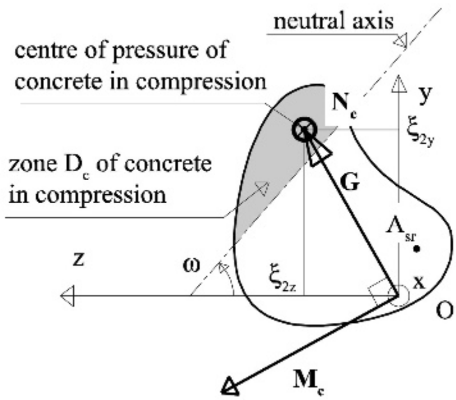


Figure 1.26. *Notion of center of pressure*

The internal force of the internal screw including the normal force in concrete $\boldsymbol{N_c}$ has been evaluated previously; the internal moment needs now to be calculated for complete equilibrium determination of the cross-section. By designing D_c, the concrete part in compression, the screw of resultant moment $\boldsymbol{M_c}$ calculated, for instance, at the origin of the frame will be given by:

$$\begin{aligned} \boldsymbol{M_c} &= \iint_{D_C} (y\boldsymbol{j} + z\boldsymbol{k}) \wedge (\sigma_c.\boldsymbol{i}).dy.dz \\ &= \boldsymbol{j}.\iint_{D_C} \sigma_c.z.dy.dz - \boldsymbol{k}.\iint_{D_C} \sigma_c.y.dy.dz \end{aligned} \qquad [1.15]$$

Such an expression of the resultant moment, evaluated at the origin of the frame, depends on the stress function $\sigma_c(y,z)$ that possesses a constant sign, in the concrete domain in compression D_c. This stress function has lower and upper bounds, thus also including the vector norm of $y\boldsymbol{j} + z\boldsymbol{k}$ that parameterizes the point of the studied compression domain. The conditions of application of the second theorem of the generalized mean value are satisfied, and then we can conclude the existence of a vector $\boldsymbol{G} = \xi_{2y}\boldsymbol{j} + \xi_{2z}\boldsymbol{k}$ whose extremity is inside the compression domain D_c (Figure 1.26), and such that:

$$\begin{aligned}\boldsymbol{M}_c &= \iint_{D_c} (y\boldsymbol{j} + z\boldsymbol{k}) \wedge (\sigma_c.\boldsymbol{i}).dy.dz \\ &= \boldsymbol{G}.\iint_{D_c} \sigma_c.\boldsymbol{i}.dy.dz = \boldsymbol{G} \wedge \boldsymbol{N}_c\end{aligned} \quad [1.16]$$

The origin of the frame for calculation of this screw can be chosen to coincide with the center of gravity of the most tensioned steel reinforcement bar (A_{sr}). By dividing the different length (associated with the level arm of the equivalent torque calculation) ξ_{2y} and ξ_{2z} by d_r, the dimensionless parameters δ_{gy} and δ_{gz} appear as:

$$\xi_{2y} = (1-\alpha\delta_{gy})d_r \text{ and } \xi_{2z} = (1-\alpha\delta_{gz})d_r \quad [1.17]$$

These size parameters are called the dimensionless coefficient of the center of pressure. Using some symmetry or mechanical considerations, it is possible to calculate these dimensionless parameters by engineering rules. However, in the more general case, for unsymmetrical cross-section, a mathematical integral calculation is often needed.

1.7. Equilibrium and resolution methodology

1.7.1. *Equilibrium equations*

The screw of internal forces can be decomposed into two internal screws, one related to the reinforcement steel bars and the other to the concrete domain in compression. When considering only the internal screw related to the reinforcement steel bars, both in tension and in compression, the normal force resultant $\boldsymbol{N}_s$ and the internal torque resultant $\boldsymbol{M}_s$ are calculated as:

$$\boldsymbol{N}_s = \sum_{q=1}^{n} A_{sq}\sigma_{sq}\,\boldsymbol{i} \text{ (or } N_s = \sum_{q=1}^{n} A_{sq}\sigma_{sq}\text{)}$$

$$\text{and } \boldsymbol{M}_s = \sum_{q=1}^{n} A_{sq}\sigma_{sq}\boldsymbol{P_q}\wedge\boldsymbol{i} \qquad [1.18]$$

The cross-sectional mechanics equilibrium in bending, at SLS, can then be written in the following form (Figure 1.27):

$$\boldsymbol{M}_{act} = \boldsymbol{M}_s + \boldsymbol{G}\wedge\boldsymbol{N}_c \text{ and } N_{act} = N_s + \psi.A_c.f_{cs} \qquad [1.19]$$

The moment equilibrium equation can be also rewritten as:

$$\boldsymbol{M}_{act} = \boldsymbol{M}_s + \psi(1-\alpha\delta_{gz})d_rA_cf_{cs}\boldsymbol{j} - \psi(1-\alpha\delta_{gy})d_rA_cf_{cs}\boldsymbol{k}$$

$$\text{with } \boldsymbol{M}_s = \sum_{q=1}^{n} A_{sq}\sigma_{sq}\boldsymbol{P_q}\wedge\boldsymbol{i} \text{ and } N_s = \sum_{q=1}^{n} A_{sq}\sigma_{sq} \qquad [1.20]$$

The normal force equilibrium equation is simply reduced to the following scalar equation:

$$N_{act} = N_s + \psi.A_c.f_{cs} \qquad [1.21]$$

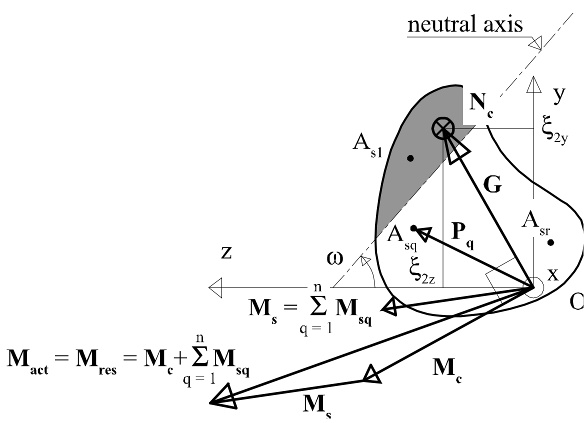


Figure 1.27. *Equilibrium of a general composite section; calculation of the screw of internal action*

1.7.2. *Discussion on the resolution of equations with respect to the number of unknowns*

It has been shown, from the application of the conservation principle of planeity coupled with the notions of limit laws and pivot concepts, that the stress distribution inside the cross-section can be considered as a discretized function of the following characteristic parameters α, $d_q(\forall q \in [1;n])$, f_{ss} or f_{cs}. The equilibrium equations give a system composed of two vectorial equations (three scalar equations) in the general case: $\boldsymbol{M}_{act} = \boldsymbol{M}_s + \boldsymbol{G} \wedge \boldsymbol{N}_c$ and $N_{act} = N_s + N_c$. The mathematical problem, in conformity with the specifications at the SLS, is a well-posed problem only if the number of unknowns is also equal to three in the general case. These unknown variables are of different types and are typically of geometrical nature: steel reinforcement area, position of the neutral axis, etc.

– *General case*: three linearly independent scalar equations;

(two moment scalar equations and one normal force resultant equation).

– *Symmetrical case*: two linearly independent scalar equations;

(one moment scalar equation and one normal force resultant equation).

For symmetrical sections, with symmetrical steel reinforcement, and with symmetrical loading (in this case ω is known), the three-parameter problem is reduced to a two-parameter problem associated with two equations: one scalar moment equation and one resultant equation. Different engineering problems can be envisaged, for reinforced concrete design, which can be illustrated through the following examples:

– Design of a unique unknown steel reinforcement area (in tension or in compression) whose center of gravity is known.

One of the steel reinforcement area A_{si} being unknown (in tension or in compression), two additional parameters have to be chosen, in the unsymmetrical case, such as the relative height of the neutral axis α and the neutral axis direction (angle ω). We have then to solve a problem of three equations with three unknowns A_{si}, ω and α. For a symmetrical section, the two unknown parameters can be the steel reinforcement area and the relative position of the neutral axis α.

– Design of a unique unknown steel reinforcement area (in tension or in compression) with a given neutral axis (related to the variable ω or to the variable α).

One of the steel reinforcement area A_{si} being unknown (in tension or in compression), two additional parameters have to be chosen, in the unsymmetrical case, such as the two spatial coordinates of the considered steel reinforcement. For a symmetrical section, the two unknown parameters can be the steel reinforcement area and the ordinate of the center of gravity of the steel reinforcement. A current case is the case of a symmetrical section, where the neutral axis is fixed at the boundary of Pivot A and Pivot B, to avoid the design of the cross-section at Pivot B (in order to limit the steel reinforcement quantity). In such a case, the two unknown parameters can be the areas of the tension and compression steel reinforcement, for a given fixed location of these steel reinforcements.

1.7.3 *Reduced moments*

From the equilibrium moment equation, we have $\boldsymbol{M}_{act} = \boldsymbol{M}_s + [\xi_{2z}.\boldsymbol{j} - d_r(1-\alpha.\delta_{gy}).\boldsymbol{k}].\psi.A_c.f_{cs}$, which, by dividing each member by the moment quantity $-d_r.A_c.f_{cs}/f(\alpha)$ where $f(\alpha)$ is a function of the dimensionless parameter α, can be formulated as:

$$\frac{-f(\alpha).\boldsymbol{M}_{act}}{d.A_c.f_{cs}} = \frac{-f(\alpha).\boldsymbol{M}_s}{d.A_c.f_{cs}} - \frac{f(\alpha).\psi.\xi_{2z}.\boldsymbol{j}}{d} + f(\alpha).\psi.(1-\alpha.\delta_g).\boldsymbol{k}. \qquad [1.22]$$

From this dimensionless equilibrium equation, the dimensionless reduced moment can be introduced:

$$\mu_{act,ser} = \frac{-f(\alpha).\boldsymbol{M}_{act}}{d.A_c.f_{cs}} \text{ and } \mu_{res,ser} = f(\alpha).\psi.(1-\alpha.\delta_{gy}) \qquad [1.23]$$

with $\mu_{act,ser}$ is the acting reduced moment and $\mu_{res,ser}$ is the reduced concrete contribution moment.

The function $f(\alpha)$ can be chosen such as the reduced acting moment which does not contain explicitly the variable α; it is quite efficient to take $f(\alpha) = d_r.A_c/(d_r^2.b)$ where b is a characteristic length of the cross-section. For instance, for a rectangular section solicited in its symmetry plane, the natural choice $f(\alpha) = \alpha$ leads to the following acting reduced moment:

$$\mu_{act} = \frac{-M_{act}}{b.d_r^2.f_{cs}} \qquad [1.24]$$

and the reduced concrete contribution moment is written as:

$$\mu_{res} = \alpha.\psi.(1-\alpha.\delta_{gy}) \qquad [1.25]$$

Another example can illustrate the role played by the function $f(\alpha)$ to highlight the role of the dimensionless parameters. Let us take the example of a T-section, solicited in its plan of symmetry. It is possible to choose the function $f(\alpha)$ is such a way:

$$f(\alpha) = (\frac{b}{b_w}-1).\frac{h_0}{d_r} + \alpha \qquad [1.26]$$

leading to:

$$\mu_{act} = \frac{-M_{act}}{b_w.d_r^2.f_{cs}} \qquad [1.27]$$

The reduced bending moment is then formulated as:

$$\mu_{res} = [(\frac{b}{b_w} - 1).\frac{h_0}{d_r} + \alpha].\psi.(1-\alpha.\delta_{gy}) \qquad [1.28]$$

Another choice for the function $f(\alpha)$ in the case of the T-section would be based on:

$$f(\alpha) = (1 - \frac{b_w}{b}).\frac{h_0}{d_r} + \alpha \qquad [1.29]$$

With this alternative of the function $f(\alpha)$, the reduced bending moment is normalized with respect to the width of the concrete slab:

$$\mu_{act} = \frac{-M_{act}}{b.d_r^2.f_{cs}} \qquad [1.30]$$

The reduced bending moment is then formulated as:

$$\mu_{res} = [(1 - \frac{b_w}{b}).\frac{h_0}{d_r} + \alpha\frac{b_w}{b}].\psi.(1-\alpha.\delta_{gy}) \qquad [1.31]$$

In the symmetrical loading case with symmetrical cross-section, we finally obtain, for the reduced moment equation:

$$\frac{-f(\alpha).M_{act}}{d_r.A_c.f_{cs}} = \frac{-f(\alpha).M_s}{d_r.A_c.f_{cs}} + f(\alpha).\psi.(1-\alpha\delta_{gy}.) \qquad [1.32]$$

In this relationship, it is convenient to define the physical meaning of each term:

$\frac{-f(\alpha).M_{act}}{d.A_c.f_{cs}} = \mu_{act,ser}$ reduced acting moment

$\frac{-f(\alpha).M_s}{d.A_c.f_{cs}} = \mu_{res,ser,s}$ reduced steel reinforcement moment

$f(\alpha).\psi.(1-\alpha.\delta_g) = \mu_{res,ser,c}$ reduced concrete contribution moment

The total moment equilibrium equation is simply obtained by the sum of each contribution of the steel and the concrete part of the reinforced concrete section:

$$\mu_{act,ser} = \mu_{res,ser,s} + \mu_{res,ser,c} \qquad [1.33]$$

1.7.4. *Case of a rectangular section*

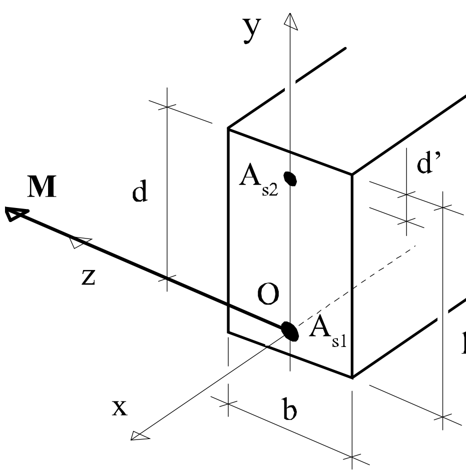


Figure 1.28. *Characteristics of a rectangular reinforced concrete section*

Using the notations of Figure 1.28, the two equilibrium equations both for the normal force and the bending moment (only two equations for symmetrical reasons) are written as:

$$N_{act} = A_{s1}\sigma_{s1} + A_{s2}\sigma_{s2} + \alpha\psi bdf_{cs} \quad [1.34]$$

$$M_{act} = -A_{s2}\sigma_{s2}(d - d') - \alpha\psi(1 - \alpha\delta_g)bd^2 f_{cs} \quad [1.35]$$

1.8. Case of pivot A for a rectangular section

1.8.1. *Studied section*

The case of a rectangular cross-section reinforced by both tensile (area A_{s1}) and compressive (area A_{s2}) steel reinforcements symmetrically disposed along the median axis y is now considered. The origin of the frame is located at the center of gravity of the tensile steel reinforcement. The geometrical characteristics of the cross-section are denoted by b, h, d and d' and are illustrated in Figure 1.28. The concrete and steel material stress limits are given at the SLS by f_{ss} for the steel reinforcement, and f_{cs} for the concrete, the equivalence coefficient being classically denoted by α_e. The acting solicitation is evaluated at the point O and is characterized by two components, the acting bending moment $\boldsymbol{M}_{act} = M_{act}\boldsymbol{k}$ and the normal force resultant $\boldsymbol{N}_{act} = N_{act}\cdot\boldsymbol{i}$. The unknown of the problem are, respectively, the tensile reinforcement area A_{s1} and the relative height of the neutral axis α.

1.8.2. *Shape filling coefficient*

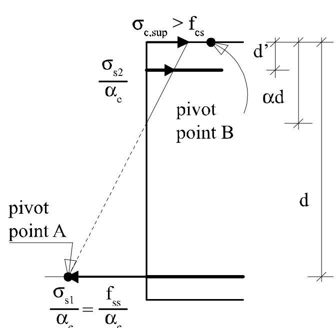


Figure 1.29. *Stress diagram in the homogenized section of the reinforced concrete section for Pivot A*

The variable α varies in the interval $[0;\alpha_e f_{cs}/(\alpha_e f_{cs}-f_{ss})]$, which depends on both the elastic and the stress limit values at the SLS. Stresses inside the composite cross-section are related by the equalities:

$$\frac{-\sigma_{s2}}{\alpha d - d'} = \frac{f_{ss}}{d - \alpha d} = \frac{-\alpha_e.\sigma_{csup}}{\alpha d} \qquad [1.36]$$

The stress diagram in the homogenized section is shown in Figure 1.29. By definition, the normal force in the compressed part of concrete is calculated as $N_c = \alpha\,\psi.b.d\,f_{cs}.$, which is also equivalent after calculation to $N_c = 0{,}5.\sigma_{c,sup}.\alpha.b.d$. The shape filling coefficient ψ is then simply expressed by:

$$\psi = \frac{-\alpha}{2\alpha_e(1-\alpha)}\cdot\frac{f_{ss}}{f_{cs}} \qquad [1.37]$$

1.8.3. *Dimensionless coefficient related to the center of pressure*

At the SLS, the pressure center of the compressed part of concrete is at a distance $\alpha d/3$ of the most compressed fiber (the upper fiber of concrete), leading to the dimensionless coefficient $\delta_g = 1/3$ (Figure 1.30).

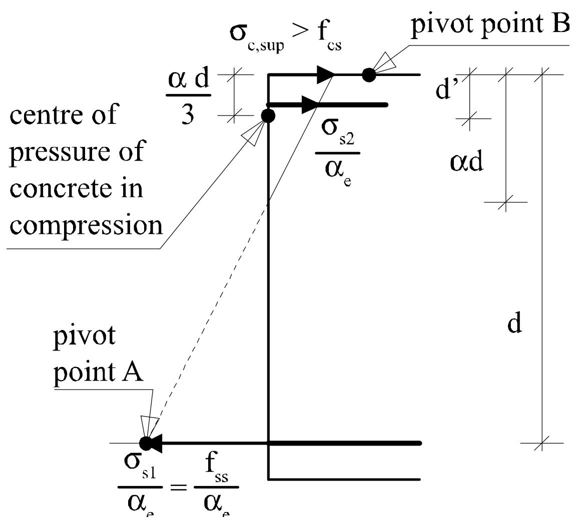


Figure 1.30. *Position of the center of pressure at the serviceability limit state; Pivot A*

1.8.4. *Equations formulation*

A third-order polynomial equation of the unknown neutral axis position has to be solved at the SLS. This polynomial equation is obtained from writing the equilibrium equations in bending moment and normal forces, coupled with the kinematics assumptions of the cross-section and the elastic behavior of the steel and concrete constituents of the composite cross-section:

$$N_{act} = A_{s1}\sigma_{s1} + A_{s2}\sigma_{s2} + \alpha\psi b d f_{cs} \tag{1.38}$$

$$M_{act} = -A_{s2}\sigma_{s2}(d - d') - \alpha\psi(1-\alpha\delta_g) b d^2 f_{cs} \tag{1.39}$$

The stress in the compression steel reinforcement σ_{s2} also depends on the unknown α.

$$\sigma_{s2} = \frac{-f_{ss}.(\alpha d - d')}{d - \alpha d} \tag{1.40}$$

It means that only the second bending moment, equation [1.39], contains one single unknown α out of the two unknowns of the design problem, respectively, the tensile reinforcement area A_{s1} and the relative height of the neutral axis α. As the filling shape coefficient ψ is given by:

$$\psi = \frac{-\alpha}{2\alpha_e(1-\alpha)} \cdot \frac{f_{ss}}{f_{cs}} \tag{1.41}$$

and by setting the section equilibrium $M = M_{ser}$, equation of the equilibrium bending moment finally leads to:

$$M = \frac{A_{s2}(d-d')(\alpha d-d')f_{ss}}{d(1-\alpha)} + \frac{\alpha^2(3-\alpha)}{6\alpha_e(1-\alpha)} b d^2 f_{ss} \tag{1.42}$$

We recognize a third-order polynomial equation:

$$\alpha^3 - 3\alpha^2 + 6\alpha_e\left(\frac{-M}{bd^2 f_{cs}}\frac{f_{cs}}{f_{ss}} - \frac{(d-d')A_{s2}}{bd^2}\right)\alpha +$$

$$6\alpha_e\left(\frac{M}{bd^2 f_{cs}}\frac{f_{cs}}{f_{ss}} + \frac{d'(d-d')A_{s2}}{bd^3}\right) = 0 \tag{1.43}$$

Some other dimensionless coefficients can be introduced as:

$$\delta' = \frac{d'}{d}, \rho'_{cs} = \frac{A_{s2}}{bd} \text{ and } \mu_{act} = \frac{-M}{bd^2 f_{cs}} \qquad [1.44]$$

We note that all these dimensionless coefficients δ', ρ'_{cs} and μ_{act} are positive numbers. The third-order polynomial equation can finally be written as:

$$\alpha^3 - 3\alpha^2 + 6\alpha_e[(f_{cs}/f_{ss})\mu_{act} - (1-\delta')\rho'_{cs}]\alpha +$$
$$6\alpha_e[-(f_{cs}/f_{ss})\mu_{act} + \delta'(1-\delta')\rho'_{cs}] = 0 \qquad [1.45]$$

Algebraically, this third-order equation can easily be solved using Cardano's method (see Appendix 1).

1.8.5. *Resolution*

Following Cardano's method, such a third-order polynomial equation can be presented in a canonical format using canonical parameters:

$$p = -3 - 6\alpha_e[(-f_{cs}/f_{ss})\mu_{act} + (1-\delta')\rho\delta'_{cs}] \text{ and}$$
$$q = -2 - 6\alpha_e(1-\delta')^2\rho'_{cs} \qquad [1.46]$$

Both canonical parameters p and q are negative numbers. The solution α can be extracted from Cardano's formula as one of the three real equations, whose solution of interest, for physical reasons, is:

$$\alpha = 1 + 2\sqrt{\frac{-p}{3}} \cos\left[\frac{Arc\cos\left(-\frac{q}{2}\left|\frac{p}{3}\right|^{-1.5}\right) - 2\pi}{3}\right] \qquad [1.47]$$

which can be equivalently written as:

$$\alpha = 1 + 2\sqrt{\frac{-p}{3}} \cos\left[\frac{Arc\cos\left(\frac{3q}{2p}\sqrt{\frac{3}{-p}}\right) - 2\pi}{3}\right] \qquad [1.48]$$

If there is no compression steel reinforcement (A_{s2} = 0), the neutral axis equation to be solved is still a third-order polynomial equation that is now simplified in:

$$\alpha^3 - 3\alpha^2 - 6\alpha_e\ \mu_{act}\ (\frac{-f_{cs}}{f_{ss}})\ \alpha + 6\alpha_e\ \mu_{act}(\frac{-f_{cs}}{f_{ss}})\ = 0 \qquad [1.49]$$

The canonical parameters in this particular case (A_{s2} = 0) are also simplified:

$$p = -3 - 6\alpha_e(-f_{cs} / f_{ss})\mu_{act} \text{ and } q = -2 \qquad [1.50]$$

In the absence of compression steel reinforcement (A_{s2} = 0), the solution for the position of neutral axis α in Pivot A is then calculated as:

$$\alpha = 1 + 2\sqrt{1 + 2\alpha_e\left(-\frac{f_{cs}}{f_{ss}}\right)\mu_{act}} \times \cos\left[\frac{Arc\cos\left[\left(1 + 2\alpha_e\left(-\frac{f_{cs}}{f_{ss}}\right)\mu_{act}\right)^{-1.5}\right] - 2\pi}{3}\right] \qquad [1.51]$$

Once the neutral axis position α is calculated, the axial force equilibrium equation gives the tensile steel area:

$$A_{s1} = \frac{N_{act}}{f_{ss}} + \frac{\alpha - \delta'}{1 - \alpha}A_{s2} + \frac{\alpha^2 bd}{2\alpha_e(1 - \alpha)} \qquad [1.52]$$

In the absence of compression steel reinforcement (A_{s2} = 0), the tensile steel area in Pivot A is given by:

$$A_{s1} = \frac{N_{act}}{f_{ss}} + \frac{\alpha^2 bd}{2\alpha_e(1 - \alpha)} \qquad [1.53]$$

The stress SLS has been then solved in Pivot A.

1.9. Case of pivot B for a rectangular section

1.9.1. *Studied section*

The case of a rectangular cross-section reinforced by both tensile (area As_1) and compressive (area As_2) steel reinforcements symmetrically disposed along the median axis y is considered, but now the section is ruled by Pivot B. The geometrical characteristics of the cross-section are also denoted by b, h, d and d', and are illustrated in Figure 1.28. The concrete and steel material stress limits are given at the SLS by f_{ss} for the steel reinforcement, and f_{cs} for the concrete, the equivalence coefficient being classically denoted by α_e. The acting solicitation is evaluated at the point O, and is characterized by two components, the acting bending moment $\boldsymbol{M}_{act} = M_{act}\boldsymbol{k}$ and the normal force resultant $\boldsymbol{N}_{act} = N_{act}.\boldsymbol{i}$. The unknown of the problem are, respectively, the tensile reinforcement area A_{s1} and the relative height of the neutral axis α.

1.9.2. *Shape filling coefficient*

In a case of simple bending solicitation (without combined axial forces), the variable α varies in the interval $[\alpha_e f_{cs}/(\alpha_e f_{cs} - f_{ss});\ h/d]$, which depends on both the elastic and the stress limit values at the SLS. In case of bending with combined axial forces, the neutral axis can be located outside the cross-section. However, in the simple bending case, the neutral axis is inside the cross-section, which gives the upper bound variation of the dimensionless coefficient α. By definition, the normal force in the compression part N_c is calculated from:

$$N_c = \psi . f_{cs} .\alpha.b.d \qquad [1.54]$$

As the stress diagram is linear at the SLS, it can be easily deduced that the filling shape coefficient ψ in Pivot B is equal to $\psi = 1/2$, as $N_c = 0{,}5.\alpha.f_{cs}.b.d$.

1.9.3. *Dimensionless coefficient related to the center of pressure*

At the SLS, the pressure center of the compressed part of concrete is at a distance $\alpha d/3$ of the most compressed fiber (the upper fiber of concrete), leading to the dimensionless coefficient of $\delta_g = 1/3$ (Figure 1.31).

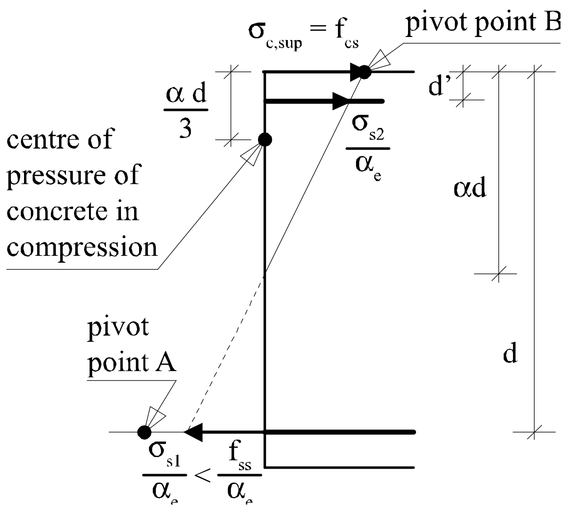


Figure 1.31. *Position of the center of pressure at the serviceability limit state; Pivot B*

1.9.4. *Equations formulation*

In the more general case, at Pivot B, a third-order polynomial equation of the unknown neutral axis position α has also to be solved at the SLS. This polynomial equation is obtained from writing the equilibrium equations in moment and normal forces, coupled with the kinematics assumptions of the cross-section and the elastic behavior of the steel and concrete constituents of the composite cross-section:

$$N_{act} = A_{s1}\sigma_{s1} + A_{s2}\sigma_{s2} + \alpha\psi bdf_{cs} \qquad [1.55]$$

$$M_{act} = -A_{s2}\sigma_{s2}(d-d') - \alpha\psi(1-\alpha\delta_g)bd^2 f_{cs} \qquad [1.56]$$

The stress in the compression steel reinforcement σ_{s2} also depends on the unknown α.

$$\sigma_{s2} = \frac{\alpha_e f_{cs}(\alpha d - d')}{\alpha d} \qquad [1.57]$$

It is worth mentioning that the stress expression in the compression steel reinforcement is different in Pivot A and Pivot B. Inserting the value of the shape filling coefficient ψ at Pivot B, ψ = ½, and the equilibrium moment equation is finally given by:

$$M = -A_{s2}\frac{(d\text{-}d')(\alpha d\text{-}d')}{\alpha d}\alpha_e f_{cs} - \frac{\alpha(3 - \alpha)}{6}bd^2 f_{cs} \qquad [1.58]$$

This equation is equivalent to the third-order polynomial equation, which allows us to calculate algebraically the position of the neutral axis at Pivot B:

$$\alpha^3 - 3\alpha^2 + 6\left(\frac{-M}{bd^2 f_{cs}} - \frac{(d\text{-}d')\ \alpha_e A_{s2}}{bd^2}\right)\alpha + \frac{6\alpha_e d'(d\text{-}d')A_{s2}}{bd^3} = 0 \qquad [1.59]$$

The dimensionless coefficients can be introduced as:

$$\delta' = \frac{d'}{d},\ \rho_{cs} = \frac{\alpha_e A_{s2}}{bd} \text{ and } \mu_{act} = \frac{-M}{bd^2 f_{cs}} \qquad [1.60]$$

We note that all these dimensionless coefficients δ', ρ'_{cs} and μ_{act} are also positive numbers. The third-order polynomial equation can finally be written as:

$$\alpha^3 + 3\alpha^2 + 6\ [\mu_{act} - (1-\delta')\rho_{cs}]\alpha + 6\delta'(1-\delta')\rho_{cs} = 0 \qquad [1.61]$$

In this case, again, this third-order equation can be easily algebraically solved using Cardano's method (see Appendix 1). It can be outlined that the third-order equation in Pivot B is different from the third-order equation in Pivot A, and the two resolutions are indeed two different mathematical problems.

1.9.5. *Resolution*

Following Cardano's method, such a third-order polynomial equation can be presented in canonical format using the canonical parameters:

$$p = 6[\mu_{act} - \rho_{cs}(1-\delta')] - 3 \text{ and } q = 6[\mu_{act} - \rho_{cs}(1-\delta')^2] - 2 \qquad [1.62]$$

Both canonical parameters p and q are generally negative numbers. The solution α can be extracted from Cardano's formula as:

$$\alpha = 1 + 2\sqrt{\frac{-p}{3}} \cos\left[\frac{Arc\cos\left(\frac{3q}{2p}\sqrt{\frac{3}{-p}}\right) - 2\pi}{3}\right] \qquad [1.63]$$

If there is no compression steel reinforcement ($A_{s2} = 0$), the neutral axis equation to be solved is now a second-order polynomial equation which can be written as:

$$\alpha^2 - 3\alpha + 6\mu_{act} = 0 \qquad [1.64]$$

whose solution is well known in reinforced concrete designs:

$$\alpha = 1{,}5(1 - \sqrt{1 - \frac{8}{3}\mu_{act}}). \qquad [1.65]$$

Once the neutral axis position α has been calculated, the axial force equilibrium equation gives the tensile steel area:

$$A_{s1} = \frac{-\alpha N_{act}}{\alpha_e f_{cs}(1-\alpha)} + \frac{\alpha - \delta'}{1 - \alpha} A_{s2} + \frac{\alpha^2 bd}{2\alpha_e(1-\alpha)} \qquad [1.66]$$

In the absence of compression steel reinforcement ($A_{s2} = 0$), the tensile steel area in Pivot B is given by:

$$A_{s1} = \frac{-\alpha N_{act}}{\alpha_e f_{cs}(1-\alpha)} + \frac{\alpha^2 bd}{2\alpha_e(1-\alpha)} \qquad [1.67]$$

The stress SLS has been then solved in Pivot B.

1.9.6. *Synthesis*

In Pivot A, the calculation of the position of the neutral axis at the stress SLS needs the resolution of a third-order polynomial equation, whatever compression steel reinforcements are incorporated in the cross-section

(A_{s2} = 0 or A_{s2}≠ 0). In Pivot B, the calculation of the position of the neutral axis at the stress SLS needs the resolution of a polynomial third-order polynomial equation, when compression steel reinforcement are incorporated in the cross-section (A_{s2}≠ 0). However, only in Pivot B, the third-order equation degenerates to a second-order polynomial equation in absence of compression steel reinforcement (A_{s2} = 0).

1.10. Examples – bending of reinforced concrete beams with rectangular cross-section

1.10.1. *A design problem at SLS – exercise*

1.10.1.1. *Structural problem*

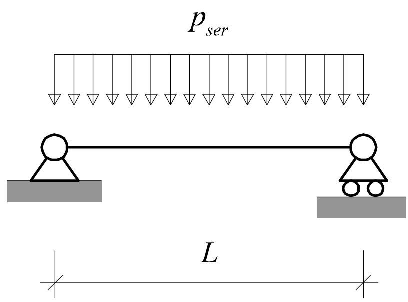


Figure 1.32. *Simply supported reinforced concrete beam*

As a first application of the presented theory, we would like to design the reinforcement of a given beam with a given rectangular cross-section (see Figures 1.32 and 1.33). This is a simply supported beam of length L = 6 m.

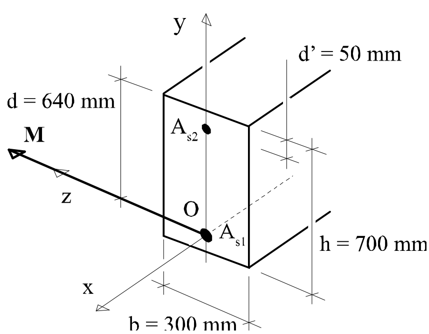


Figure 1.33. *Characteristics of the rectangular cross-section*

The geometrical characteristics are given in Figure 1.33 and detailed below:

$$b = 300\text{ mm};\ h = 700\text{ mm};\ d = 640\text{ mm};\ d' = 50\text{ mm} \qquad [1.68]$$

For engineering design, the order of magnitude for both the position of the center of gravity of the tensile steel reinforcement and the compression steel reinforcement is usually taken as:

$$d \sim 0.9\ h;\ d' \sim 0.1\ h \qquad [1.69]$$

which is almost satisfied in our present case.

1.10.1.2. *Material parameters*

The reinforced concrete section is composed of *B500B* steel bars and a *C25/30* type of concrete. In the nomenclature of steel classification *B500B*, the first *B* stands for steel for reinforced concrete. The next three digits represent the specified characteristic value of upper yield strength f_{yk} = 500 MPa. The last symbol *B* stands for ductility class. In the nomenclature of concrete classification *C25/30*, the first *C* stands for concrete. The next digits are related to characteristic strengths. The first characteristic compression strength is the cylinder characteristic strength f_{ck} = 25MPa, whereas the second characteristic compression strength is the cubic characteristic strength. Hence, we calculate the limit stresses at the SLS as:

$$\begin{aligned} &f_{ss} = 0.8 f_{yk} = 0.8 \times 500 = 400\text{ MPa and} \\ &f_{cs} = -0.6 f_{ck} = -0.6 \times 25 = -15\text{ MPa} \end{aligned} \qquad [1.70]$$

The equivalence coefficient is given for this problem as: α_e = 15. The steel Young's modulus is equal to *Es* = 200,000 MPa.

There could be some discussions on the calculation of the concrete Young modulus E_c. From EC2 (Eurocode 2) [EUR 04], we have (in MPa):

$$E_c = \frac{E_{cm}}{1 + \varphi(\infty, t_0)} \text{ with } E_{cm} = 22{,}000\left(\frac{f_{cm}}{10}\right)^{0.3} \text{ and } f_{cm} = f_{ck} + 8 \qquad [1.71]$$

where φ is the creep coefficient (time-dependent effects) and the subscript m represents the mean value of the considered variable. The order of magnitude of φ typically varies between 1 and 2. The formula in this equation is not homogeneous from a dimensional point of view, except for the numbers 22,000, 10 and 8 can be viewed as specific stress values given in megapascals (MPa). We calculate for the mean value of the concrete Young modulus:

$$E_{cm} = 22{,}000\left(\frac{33}{10}\right)^{0.3} = 31475.81\,\text{MPa} \qquad [1.72]$$

For a specific concrete, the calculation of the equivalent coefficient α_e depends on the creep coefficient φ. For a given value of the equivalent coefficient α_e, the creep coefficient can be calculated as:

$$\phi(\infty, t_0) = \frac{\alpha_e E_{cm}}{E_s} - 1 = \frac{15 \times 31{,}475.81}{200{,}000} - 1 = 1.361 \qquad [1.73]$$

It means, in this example, that the value of $\alpha_e = 15$ corresponds to a creep coefficient φ of approximately 1.361. The determination of the equivalence coefficient is still debated, actually (see also [THO 09] or [PAI 09], for instance), and the typical value of $\alpha_e = 15$, which was the usual value employed in the old French rules BAEL 91, which is still relevant to use with the new EC2 rules.

More generally, the creep coefficient depends on the type of concrete and on the type of analysis, namely the short-term or the long-term analyses.

1.10.1.3. *Loading parameters*

The simply supported reinforced concrete beam is loaded by some uniform distributed loads:

– permanent loads (without the own weight): $g_1 = 24.75$ kN/m;

– variable loads: $q = 20$ kN/m.

The volumetric density of the reinforced concrete beam can be taken as $\overline{\omega} = 25\,\text{kN/m}^3$. The permanent loads are then calculated as the sum of g_1 and the own weight g_2:

$$g = g_1 + g_2 = g_1 + \overline{\omega} bh = 24.75 + 25 \times 0.3 \times 0.7 = 30\,\text{kN/m} \qquad [1.74]$$

The combination of action at the SLS is usually one for the permanent load and one for the variable load, leading to the serviceability distributed load:

$$p_{ser} = g + q = 30 + 20 = 50\,\text{kN/m} \qquad [1.75]$$

The critical section of this problem is the one where the bending moment is maximum, that is at mid-span, leading to the bending solicitation for designing at the SLS:

$$M_{ser} = p_{ser} \frac{L^2}{8} = 50 \times \frac{6^2}{8} = 225\,\text{kN.m} \qquad [1.76]$$

1.10.1.4. *Steel reinforcement at SLS*

The exercice consists of designing the steel reinforcement of the reinforced concrete beam at the Stress SLS for this solicitation M_{ser}= 225 kN.m, and for a higher solicitation:

– M_{ser} = 225 kN.m;

– M_{ser} = 405 kN.m.

We first start from the first solicitation M_{ser} = 225 kN.m.

1.10.2. *Resolution in Pivot A – M_{ser} = 225 kN.m*

As presented in section 1.5.5, the limit value of the neutral axis position at the boundary of the two pivots, Pivot A and Pivot B, is given from:

$$\alpha_{AB} = \frac{\alpha_e}{\alpha_e - \dfrac{f_{ss}}{f_{cs}}} = \frac{15}{15 + \dfrac{400}{15}} = \frac{9}{25} = 0.36 \qquad [1.77]$$

The bending moment solicitation at the boundary of Pivot A and Pivot B can be calculated from:

$$M_{AB} = -N_{c,AB} \times \left(d - \frac{\alpha_{AB} d}{3} \right) = -bd^2 f_{cs} \frac{\alpha_{AB}}{2} \left(1 - \frac{\alpha_{AB}}{3} \right)$$

$$= 0.3 \times 0.64^2 \times 15 \times \frac{0.36}{2} \times \left(1 - \frac{0.36}{3} \right) = 0.29196 \text{ MN.m} \quad [1.78]$$

It is also possible to introduce the dimensionless reduced moment at the boundary between Pivot A and Pivot B as:

$$\mu_{AB} = \frac{M_{AB}}{-bd^2 f_{cs}} = \frac{\alpha_{AB}}{2} \left(1 - \frac{\alpha_{AB}}{3} \right)$$

$$= \frac{0.36}{2} \times \left(1 - \frac{0.36}{3} \right) = 0.1584 \quad [1.79]$$

It can be easily checked, for this solicitation, that Pivot A rules the behavior of the cross-section at SLS as:

$$M_{ser} = 0.225 \text{ MN.m} \le M_{AB} = 0.29196 \text{ MN.m or}$$

$$\mu_{ser} = \frac{M_{ser}}{-bd^2 f_{cs}} = 0.122 \le \mu_{AB} = 0.1584 \quad [1.80]$$

Then, the dimensionless position of the neutral axis given by parameter α should be the solution of the following third-order polynomial equation:

$$\alpha^3 - 3\alpha^2 - 6\alpha_e \mu_{act} \left(\frac{-f_{cs}}{f_{ss}} \right) \alpha + 6\alpha_e \mu_{act} \left(\frac{-f_{cs}}{f_{ss}} \right) = 0, \text{ or}$$

$$\alpha^3 - 3\alpha^2 - 0.41175\alpha + 0.41175 = 0 \quad [1.81]$$

The canonical parameters needed for the resolution of this third-order polynomial equation by Cardano's method are then calculated:

$$p = -3 - 6\alpha_e (-f_{cs} / f_{ss}) \mu_{act} = -3.41175 \text{ and } q = -2 \quad [1.82]$$

The solution for the position of neutral axis α in Pivot A, for this cross-section, is calculated as:

$$\alpha = 1 + 2\sqrt{\frac{-p}{3}}\cos\left[\frac{Arc\cos\left(\frac{3q}{2p}\sqrt{\frac{3}{-p}}\right) - 2\pi}{3}\right] \text{ or}$$

$$\alpha = 1 + 2\sqrt{\frac{3.41175}{3}}\cos\left[\frac{0.601393 - 2\pi}{3}\right] = 0.322737 \qquad [1.83]$$

Finally, the tensile steel reinforcement area is obtained from equilibrium of the normal force component as:

$$A_{s1} = \frac{\alpha^2 bd}{2\alpha_e(1-\alpha)} = \frac{\alpha^2 \times 0.3 \times 0.64}{2 \times 15 \times (1-\alpha)} = 9.84289 \times 10^{-4} \text{ m}^2 \qquad [1.84]$$

The tensile steel reinforcement area of 9.843 cm^2 is needed to verify the stress SLS of this cross-section for the bending solicitation M_{ser} = 225 kN.m. Using the abacus of Appendix 2, we can choose 5ϕ16 (10.053 cm^2), which has to be an upper bound of the minimum tensile steel area of 9.843 cm^2. An example of steel reinforcement location is presented in Figure 1.34.

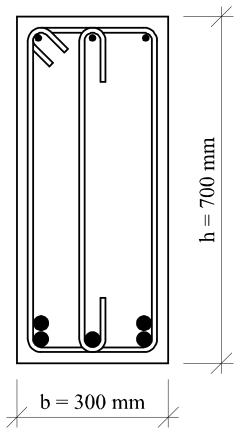


Figure 1.34. *5ϕ16 tensile steel reinforcement for M_{ser} = 225 kN.m at serviceability limit state*

1.10.3. *Resolution in Pivot B – M_{ser} = 405 kN.m*

By comparing the reduced moment μ_{ser} with the reduced frontier moment between Pivot A and Pivot B, $\mu_{AB} = 0.1584$, we see that for this solicitation M_{ser} = 405 kN.m, the section will behave in Pivot B:

$$\mu_{ser} = \frac{M_{ser}}{-bd^2 f_{cs}} = 0.2197 \geq \mu_{AB} = 0.1584 \qquad [1.85]$$

In Pivot B, and without additional compression steel reinforcement, the position of the neutral axis is obtained from resolution of a second-order polynomial equation whose root of interest is:

$$\alpha = \frac{3}{2}\left[1 - \sqrt{1 - \frac{8}{3}\mu_{ser}}\right] = \frac{3}{2}\left[1 - \sqrt{1 - \frac{8}{3} \times 0.2197}\right] = 0.53478 \qquad [1.86]$$

Finally, the tensile steel reinforcement area is obtained from equilibrium of the normal force component as:

$$A_{s1} = \frac{\alpha^2 bd}{2\alpha_e (1-\alpha)} = \frac{\alpha^2 \times 0.3 \times 0.64}{2 \times 15 \times (1-\alpha)} = 39.34 \times 10^{-4} \text{ m}^2 \qquad [1.87]$$

The tensile steel reinforcement area of 39.34 cm^2 is needed to verify the stress SLS of this cross-section for the bending solicitation M_{ser} = 405 kN.m. Using the abacus of Appendix 2, we can choose 5ϕ32 (40.212 cm^2), which has to be an upper bound of the minimum tensile steel area of 39.34 cm^2.

It can be seen that for a solicitation M_{ser} = 405 kN.m less than twice the other solicitation M_{ser} = 225 kN.m, the steel reinforcement quantity has been multiplied by approximately four (from approximately 10 cm^2 to 40 cm^2). Even if physically possible, it is generally not interesting to design a section at Pivot B for economical reasons. This can be explained by Figure 1.35, where the variation of the steel reinforcement quantity measured by ρ_{s1} is compared in Pivot A and Pivot B without compression steel reinforcement (A_{s2} = 0).

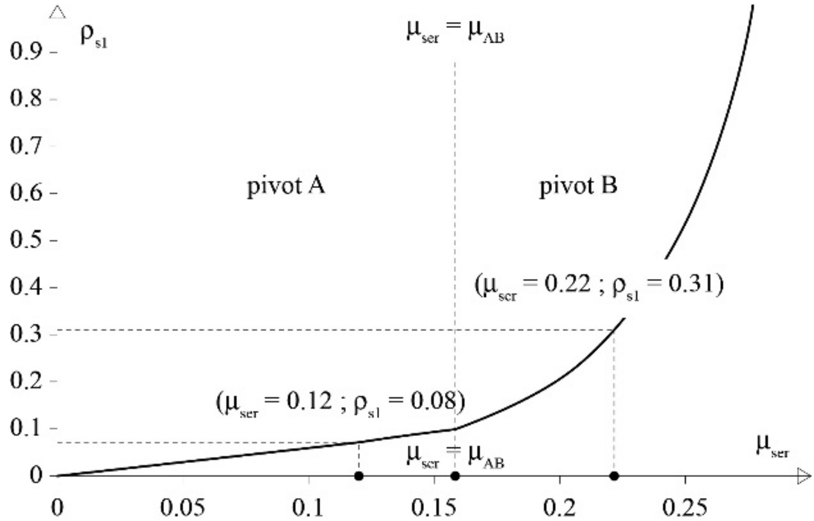


Figure 1.35. *Evolution of the dimensionless steel quantity ρ_{s1} with respect to the solicitation expressed with the reduced moment μ_{ser}*

In Pivot A, we have the following relationships for the calculation of the dimensionless steel reinforcement area ρ_{s1}:

$$\rho_{s1} = \frac{\alpha_e A_{s1}}{bd} = \frac{\alpha^2}{2(1-\alpha)} \text{ and } \mu_{ser} = \frac{3\alpha^2 - \alpha^3}{6\alpha_e\left(-\frac{f_{cs}}{f_{ss}}\right)(1-\alpha)} \qquad [1.88]$$

In Pivot B, we have the following relationships for the calculation of the dimensionless steel reinforcement area:

$$\rho_{s1} = \frac{\alpha_e A_{s1}}{bd} = \frac{\alpha^2}{2(1-\alpha)} \quad \text{and} \quad \mu_{ser} = \frac{3\alpha - \alpha^2}{6} \qquad [1.89]$$

Figure 1.35 shows the change of the slope of the steel area variation with respect to the solicitation represented by the reduced bending moment.

To avoid a design in Pivot B with too much steel reinforcement, it is suggested to add some compression steel reinforcement and to design the section at the boundary between the two pivots: Pivot AB.

1.10.4. *Resolution in pivot AB*

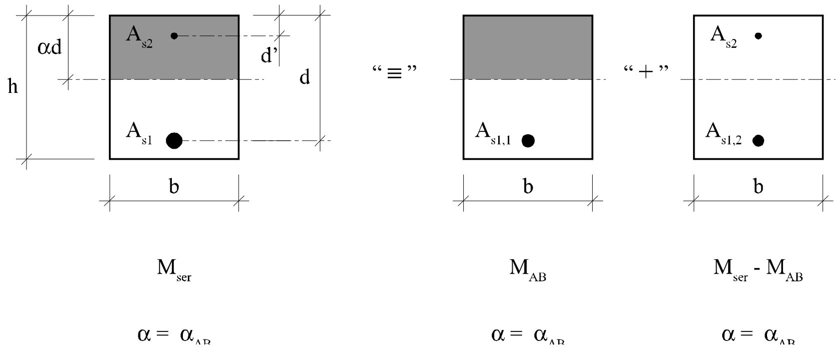


Figure 1.36. *Decomposition of the cross-section into two parts; Design at Pivot AB*

We choose to add some additional compression steel reinforcement with a section area A_{s2}, located at a distance d' from the upper fiber of the cross-section (d'≈0.1 h; in the present case, we choose d' = 5 cm). In order to keep the design problem as a well-posed mathematical problem, we have to fix an additional parameter by adding the unknown of the compression steel reinforcement area. Following the previous analysis, it has been shown that we have interest to fix the position of the neutral axis in order to behave in Pivot A. It is chosen to fix the position of the neutral axis such as:

$$\alpha = \alpha_{AB} \quad \Rightarrow \quad \begin{cases} \sigma_{s1} = f_{ss} \\ \sigma_{c,\sup} = f_{cs} \end{cases} \qquad [1.90]$$

The section is decomposed into parts as shown in Figure 1.36, one section with the compressed part of the concrete and only the tensile reinforcement, whereas the other section is composed of some tensile and compression steel reinforcement. The total area of tensile steel reinforcement is equal to:

$$A_{s1} = A_{st1} + A_{st2} \qquad [1.91]$$

The calculation of the tensile steel reinforcement in the first part of the cross-section (see Figure 1.36) is obtained from the normal force equilibrium equation, having in mind that this cross-section is solicitated by a fictitious bending moment equal to M_{AB}:

$$A_{st1} = \alpha_{AB} \frac{bd}{2}\left(-\frac{f_{cs}}{f_{ss}}\right) = \frac{\alpha_{AB}{}^2 bd}{2\alpha_e\left(1-\alpha_{AB}\right)} \Rightarrow A_{st1}$$

$$= 0.36 \times \frac{0.3\times 0.64}{2} \times \frac{15}{400} = 12.96\times 10^{-4}\,\text{m}^2 \qquad [1.92]$$

Considering the second part of the cross-section (see also Figure 1.36), the stress in the compression steel reinforcement is calculated from the compatibility of strain in the deformed cross-section:

$$\sigma_{s2} = \alpha_e f_{cs}\left(1-\frac{d'}{\alpha_{AB} d}\right) \Rightarrow$$

$$\sigma_{s2} = -15\times 15\times\left(1-\frac{0.05}{0.36\times 0.64}\right) = -176.17\text{ MPa} \qquad [1.93]$$

The moment equilibrium equation written at the center of gravity of the tensile steel reinforcement induces:

$$A_{s2} = \frac{M_{ser} - M_{AB}}{-\sigma_{s2}\left(d-d'\right)} \Rightarrow$$

$$A_{s2} = \frac{0.405\times 10^6 - 0.292\times 10^6}{176.17\times 10^6\times\left(0.64-0.05\right)} = 10.87\times 10^{-4}\,\text{m}^2 \qquad [1.94]$$

The normal force equilibrium equation gives the tension steel reinforcement:

$$f_{ss} A_{st1} + \sigma_{s2} A_{st2} = 0 \Rightarrow A_{st2} = A_{s2}\frac{-\sigma_{s2}}{f_{ss}}$$

$$= 10.87\times 10^{-4}\times\frac{176.17}{400} = 4.79\times 10^{-4}\,\text{m}^2 \qquad [1.95]$$

We finally obtain for this cross-section designed at Pivot AB by adding some compression steel reinforcement: A_{s1} = 12.96 + 4.79 = 17.75 cm^2 and A_{s2} = 10.87 cm^2. We can take 6ϕ20 (18.85 cm^2) for the tensile steel reinforcement, and 2ϕ20 + 1ϕ25 (11.19 cm2) for the compression steel reinforcement. The total steel quantity for this alternative less consuming design is 30.04 cm^2 (with $A_{s2} \neq 0$) compared to 40.21cm^2 obtained before in Pivot B without the use of compression steel reinforcement (with $A_{s2} = 0$). In this case, adding some compression steel reinforcement allows us to gain approximately 25% of the total steel reinforcement.

In fact, the interest of adding compression steel reinforcement can be clearly seen in Figure 1.37 for sufficiently large bending solicitation $\mu_{ser} \geq \mu_{AB}$. In the case where compression steel reinforcements are added to the upper part of the cross-section and the section is designed at Pivot AB, the total dimensionless steel area ρ_s is given by:

$$\rho_s = \frac{\alpha_{AB}^2}{2(1-\alpha_{AB})} + \frac{\mu_{ser} - \mu_{AB}}{\left(1 - \dfrac{d'}{\alpha_{AB} d}\right)\left(1 - \dfrac{d'}{d}\right)} \left[1 + \alpha_e \left(\frac{-f_{cs}}{f_{ss}}\right)\left(1 - \frac{d'}{\alpha_{AB} d}\right)\right]$$

$$\text{with } \rho_s = \frac{\alpha_e}{bd}(A_{s1} + A_{s2}) \qquad [1.96]$$

As shown by Figure 1.37, adding some compression steel reinforcement in this case makes the variation of the steel quantity linear with respect to the bending solicitation, whereas a design at Pivot B without additional compression steel reinforcement leads to a strongly nonlinear curve, which may overestimate significantly the steel quantity needed at SLS.

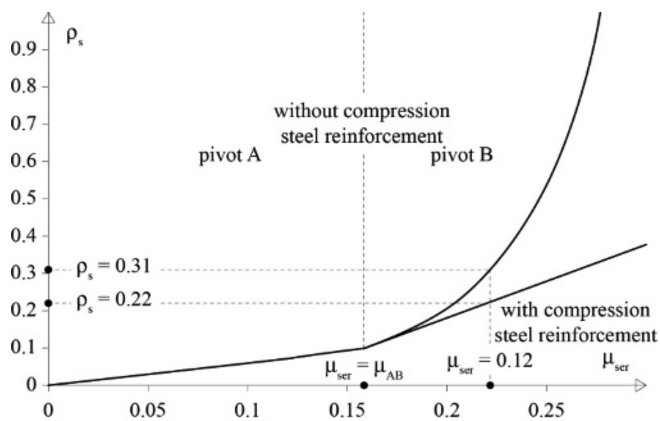


Figure 1.37. *Evolution of the total dimensionless steel quantity ρ_s with respect to the solicitation expressed with the reduced moment μ_{ser}*

1.10.5. *Design of a reinforced concrete section, an optimization problem*

In fact, designing the section at Pivot AB at SLS may lead to the optimized solution with respect to the total steel quantity used in the reinforced section. It can be shown that the position of the neutral axis (see Chapter 2 on the so-called "static moment equation") is obtained from a second-order polynomial equation written as:

$$\alpha = -(\rho_{s1} + \rho_{s2}) + \sqrt{(\rho_{s1} + \rho_{s2})^2 + 2(\rho_{s1} + \delta' \rho_{s2})} \qquad [1.97]$$

where the dimensionless coefficient $\delta' = d'/d$ has been used. We can introduce the change of variable:

$$\rho_{s1} = \rho_s \cos^2\theta \text{ and } \rho_{s2} = \rho_s \sin^2\theta \qquad [1.98]$$

It is easy to check that $\rho_S = \rho_{s1} + \rho_{s2}$ and $\tan^2\theta = \rho_{s2}/\rho_{s1}$. The optimization consists of finding the optimized value θ so that the SLS capacity reduced moment μ is maximized for a given quantity of steel reinforcement ρ_s. This optimization problem is illustrated in Figure 1.38 for the problem with geometrical parameters defined in Figure 1.33.

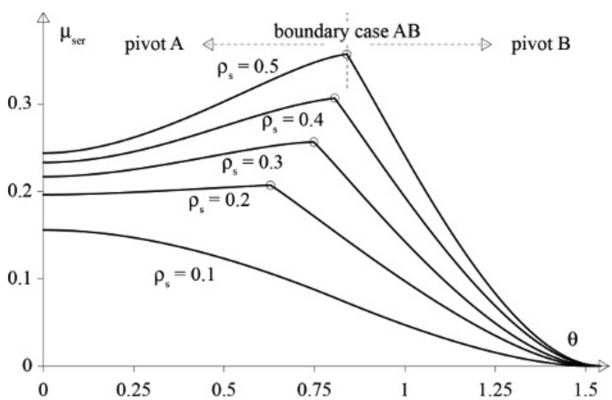


Figure 1.38. *Optimization of the reduced moment μ_{ser} with respect to the steel reinforcement ratio; $\rho_{s2}/\rho_{s1} = \tan^2\theta$ for a given quantity $\rho_S \in \{0.1; 0.2; 0.3; 0.4; 0.5\}$; $\delta' = 0.078125$; B500B steel bars and C25/30 type of concrete*

The optimization problem can be explicitly formulated from the analytical expression of the neutral axis position and the reduced moment parameters. The position of the neutral axis is then expressed with respect to the new optimization variables:

$$\alpha(\rho_s,\theta) = -\rho_s + \sqrt{\rho_s^2 + 2\rho_s\left(\cos^2\theta + \delta'\sin^2\theta\right)} \quad [1.99]$$

The reduced moment in Pivot A, for $\alpha(\rho_s,\theta) \leq \alpha_{AB}$, is given by:

$$\mu(\rho_s,\theta) = \frac{\alpha^2(3-\alpha)}{6\alpha_e(1-\alpha)}\left(\frac{f_{ss}}{-f_{cs}}\right) + \frac{\rho_s\sin^2\theta}{\alpha_e}\frac{(1-\delta')(\alpha-\delta')}{1-\alpha}\left(\frac{f_{ss}}{-f_{cs}}\right) \quad [1.100]$$

The reduced moment in Pivot B, for $\alpha(\rho_s,\theta) \geq \alpha_{AB}$, is given by:

$$\mu(\rho_s,\theta) = \rho_s\left(\sin^2\theta\right)(1-\delta')\frac{\alpha-\delta'}{\alpha} + \frac{\alpha}{2}\left(1-\frac{\alpha}{3}\right) \quad [1.101]$$

As shown by Figure 1.38, the optimization problem is a singular optimization problem and the optimization parameter is associated with Pivot AB, leading to the transcendental equation:

$$\alpha(\rho_s,\theta) = -\rho_s + \sqrt{\rho_s^2 + 2\rho_s\left(\cos^2\theta + \delta'\sin^2\theta\right)} = \alpha_{AB} \quad [1.102]$$

which can be analytically solved from:

$$\theta = \text{arc}\cos\sqrt{\frac{\alpha_{AB}^2 + 2(\alpha_{AB}-\delta')\rho_s}{2\rho_s(1-\delta')}} \text{ for}$$

$$\rho_s \geq \frac{\alpha_{AB}^2}{2(1-\alpha_{AB})} = 0.10125 \quad [1.103]$$

For $\mu_{ser} \leq \mu_{AB} = 0.1584$, that is for a Pivot A section with only tensile steel reinforcement, the best solution is obtained for $\theta = 0$, that is, there is no need to add some compression steel reinforcement in the reinforced concrete section.

However, it can be shown that designing the section at SLS at the boundary between Pivot A and Pivot B for $\mu_{ser} \geq \mu_{AB}$ is not necessarily the best solution for all the range of parameters. The optimization problem coincides with the Pivot AB design in the problem handled in Figure 1.38, based on *B500B* steel bars and the *C25/30* type of concrete, and with $\delta' = 0.078125$. However, for the same materials but changing the value of the geometrical ratio δ' up to 0.15, it can be seen in Figures 1.39 and 1.40 that the optimized solution is different from the optimized solutionof Pivot AB.

For instance, this geometrical ratio $\delta' = 0.15$ corresponds to the reinforced section based on the following parameters:

b (m)... 0.5
d′ (m).. 0.06
d (m)... 0.4
h (m)... 0.46
f_{ss} (MPa).................................... 400
f_{cs} (MPa)................................... −15
M_{ser} (MN.m)............................. 0.22
α_e..15 [1.104]

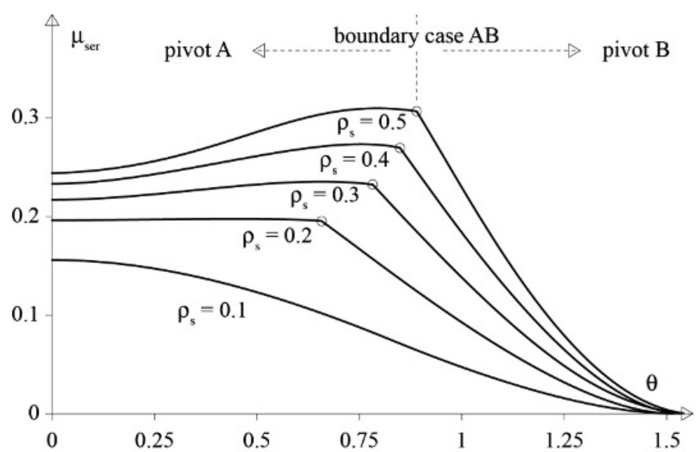


Figure 1.39. *Optimization of the reduced moment μ_{ser} with respect to the steel reinforcement ratio; $\rho_{s2} / \rho_{s1} = \tan^2 \theta$ for a given quantity $\rho_s \in \{0.1;0.2;0.3;0.4;0.5\}$; $\delta' = 0.15$; B500B steel bars and C25/30 type of concrete*

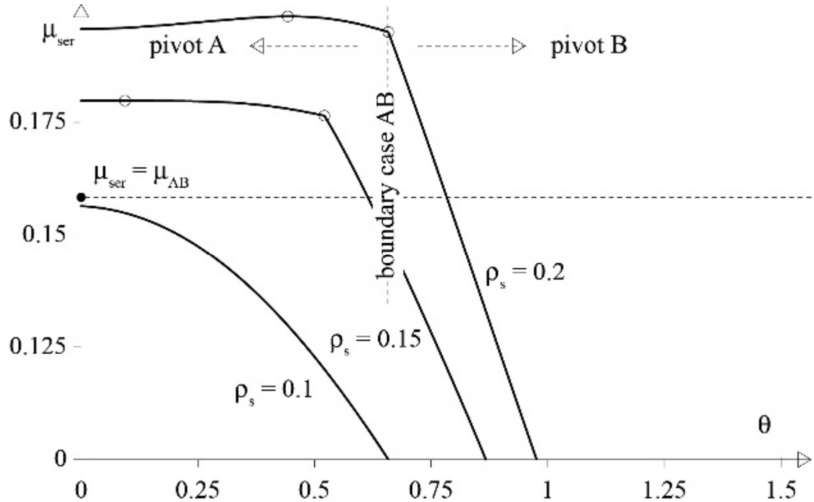


Figure 1.40. *Optimization of the reduced moment μ_{ser} with respect to the steel reinforcement ratio; $\rho_{s2} / \rho_{s1} = \tan^2 \theta$ for a given quantity $\rho_s \in \{0.1;0.15;0.2\}$; $\delta' = 0.15$; B500B steel bars and C25/30 type of concrete*

As an example related to the sensitivity analysis shown by Figures 1.39 and 1.40, we give below an example of an reinforced concrete section in Pivot B with a total steel quantity lesser than the one associated with the design in Pivot AB. For this solicitation $\mu_{ser} = 0.183$, the design at Pivot AB leads to:

$$\alpha = \alpha_{AB} = 0.36; A_{s2} = 6.705\text{ cm}^2; A_{s1} = 15.700\text{ cm}^2;$$
$$A_s = 22.405\text{ cm}^2 \text{ and then } \rho_s = 0.168 \quad [1.105]$$

whereas the optimized solution, for the same solicitation $\mu_{ser} = 0.183$ is given in Pivot B, for:

$$\alpha_{opt} = 0.416 > \alpha_{AB} = 0.36; A_{s2} = 1.065\text{ cm}^2; A_{s1} = 20.181\text{ cm}^2;$$
$$A_s = 21.246\text{ cm}^2 \text{ and then } \rho_s = 0.159 \quad [1.106]$$

In this case, the optimized solution allows the gain of more than 5% of steel area with respect to the design at Pivot AB.

1.10.6. *General design at Serviceability Limit State with tensile and compression steel reinforcements*

In this section, the design of a reinforced concrete section at SLS with both tensile and compression steel reinforcements is presented. The section is not necessarily designed at Pivot AB, as developed in section 1.10.4. In the general case, the section is decomposed into two parts (see Figure 1.41).

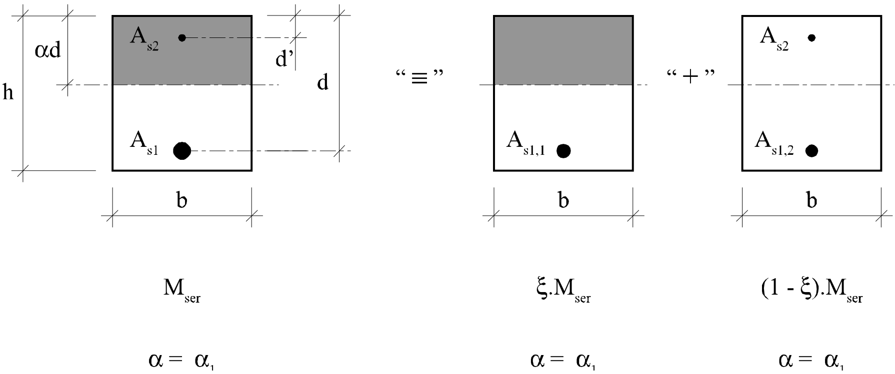


Figure 1.41. *Decomposition of the cross-section into two parts; general design*

It is assumed to write the total bending solicitation into two terms:

$$M_{ser} = M_1 + M_2 \text{ with } M_1 = \xi M_{ser} \text{ and } M_2 = (1-\xi) M_{ser} \qquad [1.107]$$

where ξ is a free dimensionless parameter that fixes the neutral axis position. For instance, if the neutral axis position is fixed at $\alpha = \alpha_{AB}$, then the parameter ξ is equal to $\xi_{AB} = M_{AB}/M_{ser}$. But this last case is of course a particular case. The design of the first fictitious section with only tensile steel reinforcements is first presented (solicitation $M_1 = \xi M_{ser}$). For a given value of the dimensionless coefficient ξ, the steel quantity $A_{s1,1}$ is calculated for the reduced moment $\mu_{ser,1} = \dfrac{-\xi M_{ser}}{b.d^2.f_{cs}}$. If $\mu_{ser,1} \le \mu_{AB}$, the calculation of the dimensionless neutral axis position α_1 is computed from the cubic equation:

$$\alpha_1^3 - 3\alpha_1^2 - 6\alpha_e \mu_{ser,1}\left(\frac{-f_{cs}}{f_{ss}}\right)\alpha_1 + 6\alpha_e \mu_{ser,1}\left(\frac{-f_{cs}}{f_{ss}}\right) = 0 \quad [1.108]$$

leading to the root:

$$\alpha_1 = 1 + 2\sqrt{1+2\alpha_e\left(-\frac{f_{cs}}{f_{ss}}\right)\mu_{ser,1}}\cos\left\{\frac{\mathrm{Arc\ cos}\left[1+2\alpha_e\left(-\frac{f_{cs}}{f_{ss}}\right)\mu_{ser,1}\right]^{-1.5} - 2\pi}{3}\right\} \qquad [1.109]$$

If $\mu_{ser,1} \geq \mu_{AB}$, the calculation of the dimensionless neutral axis position α_1 is computed from the second-order polynomial equation:

$$\alpha_1^2 - 3\alpha_1 + 6\mu_{ser,1} = 0 \qquad [1.110]$$

leading to the solution:

$$\alpha_1 = 1{,}5\left(1 - \sqrt{1 - \frac{8}{3}\mu_{ser,1}}\right) \qquad [1.111]$$

In both cases, in Pivot A or Pivot B, the tensile steel area of section $A_{s1,1}$ is obtained from:

$$A_{s1,1} = \frac{\alpha_1^2 bd}{2\alpha_e(1 - \alpha_1)} \qquad [1.112]$$

Now considering the second fictitious section with some tensile steel reinforcement (area $A_{s1,2}$), compression steel reinforcements (area A_{s2}), without any concrete in compression and solicited by the moment $M_2=(1-\xi)M_{ser}$. If α_1 corresponds to Pivot A, then the stresses in the tensile and compression steel reinforcements are calculated from:

$$\sigma_{s2} = -\frac{\alpha - \delta'}{1 - \alpha} f_{ss}$$

$$\sigma_{s1,1} = \sigma_{s1,2} = f_{ss} \qquad [1.113]$$

The bending moment equilibrium equation can be written at the center of the gravity of each steel reinforcement:

$$-A_{s2}\sigma_{s2}(1-\delta')d = (1-\xi)M_{ser}$$
$$A_{s1,2}\sigma_{s1,2}(1-\delta')d = (1-\xi)M_{ser} \qquad [1.114]$$

It is worth mentioning that a combination of each of these equations is in fact equivalent to the normal force equilibrium equation $A_{s1,2}\sigma_{s1} + A_{s2}\sigma_{s2} = 0$. The tensile and compression steel area can be finally obtained from:

$$A_{s2} = \frac{(1 - \xi)M_{ser}(1 - \alpha_1)}{(1 - \delta')(\alpha_1 - \delta')\, d\, f_{ss}}$$

$$A_{s1,2} = \frac{(1 - \xi)M_{ser}}{(1 - \delta')\, d\, f_{ss}} \text{ and } A_{s1,1} = \frac{\alpha_1^2 bd}{2\alpha_e(1 - \alpha_1)}$$

$$A_{s1} = \frac{(1 - \xi)M_{ser}}{(1 - \delta')d\, f_{ss}} + \frac{\alpha_1^2 bd}{2\alpha_e(1 - \alpha_1)} \qquad [1.115]$$

If α_1 corresponds to Pivot B, then the stresses in the steel reinforcements are obtained from:

$$\sigma_{s2} = \frac{\alpha_1 - \delta'}{\alpha_1}\alpha_e f_{cs}$$

$$\sigma_{s1,1} = \sigma_{s1,2} = -\frac{1 - \alpha_1}{\alpha_1}\alpha_e f_{cs} \qquad [1.116]$$

Also, using the bending moment equilibrium equation [1.114], the steel reinforcement areas can be calculated in Pivot B from:

$$A_{s2} = -\frac{\alpha_1(1 - \xi)M_{ser}}{(1 - \delta')(\alpha_1 - \delta')\, d\, \alpha_e f_{cs}}$$

$$A_{s1,2} = -\frac{\alpha_1(1 - \xi)M_{ser}}{(1 - \delta')(1 - \alpha_1)\, d\, \alpha_e f_{cs}} \text{ and } A_{s1,1} = \frac{\alpha_1^2 bd}{2\alpha_e(1 - \alpha_1)}$$

$$A_{s1} = -\frac{\alpha_1(1 - \xi)M_{ser}}{(1 - \delta')(1 - \alpha_1)\, d\, \alpha_e f_{cs}} + \frac{\alpha_1^2 bd}{2\alpha_e(1 - \alpha_1)} \qquad [1.117]$$

Using these equations, it would be possible to compute the total steel area $A_{s1} + A_{s2}$ with respect to the dimensionless free parameter ξ. It is also possible to optimize the design of the reinforced concrete section with respect to the total steel area by studying the evolution of the total steel area $A_{s1} + A_{s2}$ with respect to the dimensionless position of the neutral axis α_1. In Pivot A, the dimensionless compression steel area ρ_{s2} is obtained from:

$$\rho_{s2} = \frac{\alpha_1^3 - 3\alpha_1^2 + 6\alpha_e\,(f_{cs}/f_{ss})\mu_{ser}\alpha_1 - 6\alpha_e\,(f_{cs}/f_{ss})\,\mu_{ser}}{6(1-\delta')(\,\alpha_1 - \delta')} \qquad [1.118]$$

and the total dimensionless steel area ρ_s is then equal to:

$$\rho_s = \frac{\alpha_1^3 - 3\alpha_1^2 + 6\alpha_e\,(f_{cs}/f_{ss})\mu_{ser}\alpha_1 - 6\alpha_e\,(f_{cs}/f_{ss})\,\mu_{ser}}{6(1 - \alpha_1)(\,\alpha_1 - \delta')} + \frac{\alpha_1^2}{2(1 - \alpha_1)} \qquad [1.119]$$

In Pivot B, the dimensionless compression steel area ρ_{s2} is obtained from:

$$\rho_{s2} = \frac{\alpha_1^3 - 3\alpha_1^2 + 6\,\mu_{ser}\alpha_1}{6(1-\delta')\,(\alpha_1 - \delta')} \qquad [1.120]$$

and the total dimensionless steel area ρ_s is then equal to:

$$\rho_s = \frac{\alpha_1^3 - 3\alpha_1^2 + 6\,\mu_{ser}\alpha_1}{6(1 - \alpha_1)\,(\alpha_1 - \delta')} + \frac{\alpha_1^2}{2(1 - \alpha_1)} \qquad [1.121]$$

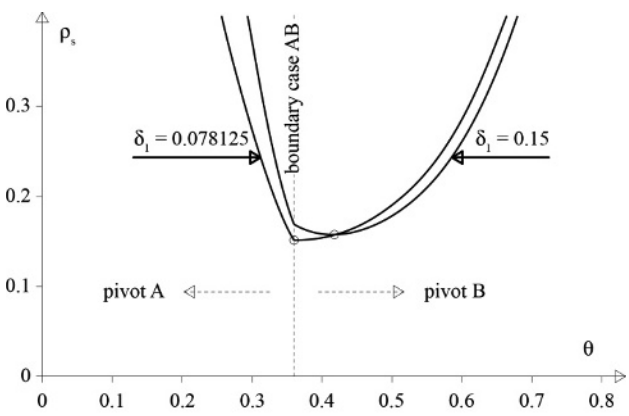


Figure 1.42. *Evolution of the steel quantity with respect to the dimensionless position of the neutral axis; B500B steel bars and C25/30 type of concrete;* $\mu_{ser} = 0.183; \delta' \in \{0.078125; 0.15\}$

The optimization problem consists of finding the smallest steel quantity ratio ρ_s with respect to α, for a given reduced moment $\mu_{ser} = 0.183$. It is clearly shown in Figure 1.42 that the optimization problem is sensible regarding the value of the parameter $\delta' = d'/d$. For the parameter $\delta' = 0.078125$ (problem analyzed in Figure 1.38), it is confirmed that the optimization solution is the solution associated with the design at Pivot AB, whereas for $\delta' = 0.15$ (problem analyzed in Figures 1.39 and 1.40), the optimized solution is different from the solution at the boundary between Pivot A and Pivot B. The optimal solution $\alpha_{opt} = 0.416 > \alpha_{AB} = 0.36$ is obtained in this case for the branch ruled by Pivot B, from the quartic equation:

$$\frac{\partial \rho_s}{\partial \alpha}(\alpha = \alpha_{opt}) = 0 \quad \Rightarrow$$

$$\alpha_{opt}^4 - 2(1+\delta')\alpha_{opt}^3 - \frac{3}{4}(-6\delta' - \delta'^2 - 1 + 2\mu_{ser})\alpha_{opt}^2 -$$
$$\frac{3}{2}\delta'(1+\delta')\alpha_{opt} + \frac{3}{2}\delta'\mu_{ser} = 0 \qquad [1.122]$$

where ρ_s is given by equation [1.121] in Pivot B. For this value of $\delta' = 0.15$, we find $\alpha_{opt} \geq \alpha_{AB}$ and $\alpha = \alpha_{opt}$ is the solution of the constrained optimization problem. On the other hand, if the solution of the quartic equation [1.122] was less than α_{AB}, then the optimized solution would be the one of Pivot AB, as already mentioned for the case with $\delta' = 0.078125$.

1.11. Reinforced concrete beams with T-cross-section

1.11.1. *Introduction*

A T-cross-section is analyzed where both compression (with area A_{s2}) and tensile (with area A_{s1}) steel bars reinforce the composite cross-section (see Figure 1.43).

The geometry of the cross-section is characterized by the different length parameters b, b_w, h, h_0, d and d', where b is the width of the concrete slab, h_0 is the depth of the flange (slab) thickness, and the width of the web is denoted by b_w. The position of the neutral axis is as usual, characterized by αd from the upper fiber of the cross-section.

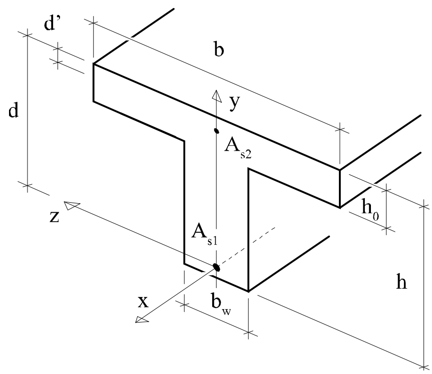


Figure 1.43. *Geometry of a T-cross-section*

If the depth of the compression block is within the flanged portion of the beam, that is the neutral axis depth αd is less than the flange (slab) thickness h_0, measured from the top of the slab ($\alpha d < h_0$), then the section can be calculated as an "equivalent" rectangular cross-section with the width equal to b (as tensile concrete contribution is neglected in the analysis). Then, we find again the configuration previously investigated, for the design of reinforced concrete beams with rectangular cross-section at the SLS. However, when the depth of the compression block is larger than the flange (slab) thickness, the neutral axis is located in the web of the T-cross-section, and the calculation has to be based on the T-cross-section calculation, as detailed in this section.

The concrete and steel material stress limits are given at the SLS by f_{ss} for the steel reinforcement, and f_{cs} for the concrete, the equivalence coefficient being classically denoted by α_e. The acting solicitation is evaluated at point O and is characterized by two components, the acting moment $\boldsymbol{M}_{act} = M_{act}\boldsymbol{k}$ and the normal force resultant $\boldsymbol{N}_{act} = N_{act}.\boldsymbol{i}$. The unknowns of the problem are, respectively, the tensile reinforcement area A_{s1} and the relative height of the neutral axis α. It is assumed that the neutral axis is located inside the web, that is $h_0 \leq \alpha d \leq h$.

To simplify the theoretical study of the T-cross-section analysis, the summation of the forces resistant screw is induced by a decomposition of the total cross-section within piecewise rectangular cross-sections, without forgetting the steel reinforcements, both in tension and compression.

1.11.2. *Decomposition of the cross-section*

The cross-section S is decomposed into three rectangular parts: two rectangular parts, S_1 that can be added, and S_2 that can be subtracted, plus the additional steel reinforcement S_3 that has to be added to the total cross-section (see Figure 1.44). These subsections have the same neutral axis and the same curvature for compatibility reasons.

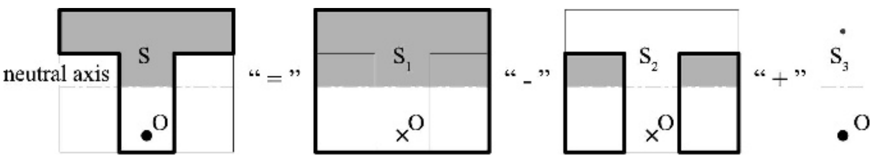


Figure 1.44. *Decomposition of the T-cross-section*

The characteristics of each subdomain are given in Table 1.1.

	S	S_1	S_2	S_3
Effective depth	d	d	$d\text{-}h_0$	/
Relative depth of the neutral axis	α	α	$\alpha' = (\alpha d\text{-}h_0)/(d\text{-}h_0)$	/
Force	/	$\alpha.\psi.b.d.f_{cs}$	$\alpha'.\psi'.(b\text{-}b_w).(d\text{-}h_0).f_{cs}$	$A_{s1}.\sigma_{s1}+A_{s2}.\sigma_{s2}$
Moment with respect to O	/	$-\ \alpha.\psi.(1\text{-}\alpha.\delta_g).b.d^2.f_{cs}$	$-\ \alpha'.\psi'.(1\text{-}\alpha'.\delta'_g).(b\text{-}b_w).(d\text{-}h_0)^2.f_{cs}$	$-A_{s2}.\sigma_{s2}.(d\text{-}d')$

Table 1.1. *Characterization of each subdomain of the T-cross-section*

The center of pressure of the compression block in concrete is at a distance $\alpha d/3$ of the most compressed fiber of the cross-section, that is the upper concrete fiber, and then $\delta g = 1/3$. As for the rectangular cross-section, the shape filling coefficient ψ has been introduced in Table 1.1. The normal force in the compressed part of concrete can be expressed through this dimensionless coefficient:

$$N_c = \psi f_{cs} \alpha bd \qquad [1.123]$$

In Table 1.1, the dimensionless shape filling coefficient ψ depends on the considered pivot, Pivot A or Pivot B.

$$\psi = \frac{-\alpha}{2\alpha_e(1-\alpha)} \cdot \frac{f_{ss}}{f_{cs}} \text{ in Pivot A, and } \psi = 1/2 \text{ in Pivot B} \qquad [1.124]$$

In Table 1.1, the dimensionless parameter ψ' denotes another shape filling coefficient that also depends on the considered pivot, Pivot A or Pivot B:

$$\psi' = -(\alpha - h_0/d) f_{ss} / [2\alpha_e(1-\alpha) f_{cs}] \text{ in Pivot A,}$$
$$\text{and } \psi' = (\alpha - h_0/d)/(2\alpha_e \alpha) \text{ in Pivot B} \qquad [1.125]$$

1.11.3. *Case of pivot A for a T-cross-section*

1.11.3.1. *Equations formulation*

By setting $M = M_{ser}$, the bending moment equilibrium consideration leads to a nonlinear equation of the dimensionless relative depth α, similar to what has been found for the rectangular cross-section in section 1.8.4 as:

$$M = \frac{A_{s2}(d-d')(\alpha d-d')f_{ss}}{d(1-\alpha)} + \frac{\alpha^2(3-\alpha)}{6\,\alpha_e(1-\alpha)} bd^2 f_{ss} -$$
$$\frac{(\alpha - h_0/d)^2(3-2\,h_0/d-\alpha)}{6\,\alpha_e(1-\alpha)}(b-b_w)d^2 f_{ss} \qquad [1.126]$$

or, equivalently, by using the dimensionless coefficients:

$$M = [\alpha_e (1-\delta')(\alpha-\delta') dA_{s2} + \frac{\alpha^2(3-\alpha)}{6} bd^2 -$$

$$\frac{(\alpha - h_0/d)^2(3 - 2\, h_0/d - \alpha)}{6}(b-b_w)d^2]\frac{f_{ss}}{\alpha_e(1-\alpha)} \quad [1.127]$$

A third-order polynomial equation is finally obtained for the determination of the neutral axis position, characterized by α, for a given cross-section:

$$a_0\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3 = 0 \text{ with}$$

$$\begin{cases} a_0 = \dfrac{b_w}{b} \\ a_1 = -\dfrac{3b_w}{b} \\ a_2 = -6\alpha_e\left(\dfrac{M}{bd^2 f_{ss}} + \dfrac{(1-\delta')A_{s2}}{bd}\right) - 3\left(\dfrac{h_0}{d}\right)\left(2-\dfrac{h_0}{d}\right)\left(1-\dfrac{b_w}{b}\right) \\ a_3 = +6\alpha_e\left(\dfrac{M}{bd^2 f_{ss}} + \dfrac{\delta'(1-\delta')A_{s2}}{bd}\right) + \left(\dfrac{h_0}{d}\right)^2\left(3-\dfrac{2h_0}{d}\right)\left(1-\dfrac{b_w}{b}\right) \end{cases} \quad [1.128]$$

where the dimensionless coefficient $\delta' = d'/d$ has been used.

1.11.3.2. *Resolution – Pivot A*

Cardano's method can be used for the resolution of this third-order polynomial equation (see Appendix 1). The canonical parameters are:

$$p = \frac{3a_0a_2 - a_1^2}{3a_0^2} \text{ and } q = \frac{27a_0^2a_3 + 2a_1^3 - 9a_0a_1a_2}{27a_0^3} \quad [1.129]$$

Typically, for the physical parameters associated with reinforced concrete design application, the third-order polynomial equation in α has three roots, which the one of interest in our study is:

$$\alpha = 2\sqrt{\frac{-p}{3}}\cos\left[\frac{Arc\cos\left(\frac{3q}{2p}\sqrt{\frac{3}{-p}}\right)+4\pi}{3}\right]-\frac{b}{3a} \qquad [1.130]$$

Once the neutral axis position α is numerically calculated, the normal force equilibrium equation gives the area of tensile steel reinforcement as found in section 1.8.5 for the rectangular cross-section:

$$A_{s1} = \frac{N_{act}}{f_{ss}} + \frac{\alpha - \delta'}{1 - \alpha}A_{s2} + \frac{\alpha^2 bd}{2\alpha_e(1-\alpha)} - \frac{(\alpha - h_0/d)^2 (b-b_w)d}{2\alpha_e(1-\alpha)} \qquad [1.131]$$

Different specific cases can be deduced from this general equation valid for T-cross-section. In the case of simple bending $N_{act} = 0$, the steel area equation is reduced to:

$$A_{s1} = \frac{\alpha - \delta'}{1 - \alpha}A_{s2} + \frac{\alpha^2 bd}{2\alpha_e(1-\alpha)} - \frac{(\alpha - h_0/d)^2 (b-b_w)d}{2\alpha_e(1-\alpha)} \qquad [1.132]$$

In the case of simple bending $N_{act} = 0$, and with only tensile steel reinforcement ($A_{s2} = 0$), this equation is simplified into:

$$A_{s1} = \frac{\alpha^2 bd}{2\alpha_e(1-\alpha)} - \frac{(\alpha - h_0/d)^2 (b-b_w)d}{2\alpha_e(1-\alpha)} \qquad [1.133]$$

Finally, in the case of simple bending $N_{act} = 0$, with only tensile steel reinforcement ($A_{s2} = 0$), and for rectangular cross-section ($b_w = b$), we find again what has been found in sections 1.8 and 1.9:

$$A_{s1} = \frac{\alpha^2 bd}{2\alpha_e(1-\alpha)} \qquad [1.134]$$

1.11.4. *Case of pivot B for a T-cross-section*

1.11.4.1. *Equations formulation*

By setting $M = M_{ser}$, the bending moment equilibrium consideration leads to a nonlinear equation of the dimensionless relative depth α, similar to what has been found for the rectangular cross-section in section 1.9.4 as:

$$M = -\frac{\alpha_e A_{s2}(d\text{-}d')(\alpha d\text{-}d')f_{cs}}{\alpha d} - \frac{\alpha(3 - \alpha)}{6} bd^2 f_{cs} +$$

$$\frac{(\alpha - h_0/d)^2(3 - 2\, h_0/d - \alpha)}{6\alpha}(b–b_w)d^2 f_{cs} \qquad [1.135]$$

or, equivalently, by using the dimensionless coefficients:

$$M = [\alpha_e\,(1 - \delta')(\alpha – \delta')dA_{s2} + \frac{\alpha^2(3 - \alpha)}{6} bd^2 -$$

$$\frac{(\alpha - h_0/d)^2(3 - 2\, h_0/d - \alpha)}{6}(b–b_w)d^2]\,\frac{-f_{cs}}{\alpha} \qquad [1.136]$$

A third-order polynomial equation is finally obtained for the determination of the neutral axis position, characterized by α, for a given cross-section:

$$a_0\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3 = 0 \text{ with}$$

$$\begin{cases} a_0 = \dfrac{b_w}{b} \\ a_1 = -\dfrac{3b_w}{b} \\ a_2 = -6\left(\dfrac{M}{bd^2 f_{cs}} + \dfrac{\alpha_e(1-\delta')A_{s2}}{bd}\right) - 3\left(\dfrac{h_0}{d}\right)\left(2 - \dfrac{h_0}{d}\right)\left(1 - \dfrac{b_w}{b}\right) \\ a_3 = +\dfrac{6\alpha_e\delta'(1-\delta')A_{s2}}{bd} + \left(\dfrac{h_0}{d}\right)^2\left(3 - \dfrac{2h_0}{d}\right)\left(1 - \dfrac{b_w}{b}\right) \end{cases} \qquad [1.137]$$

whose solution is also given by Cardano's method (see section 1.11.3).

1.11.4.2. *Resolution*

Once the neutral axis position α is numerically calculated, the normal force equilibrium equation gives the area of tensile steel reinforcement as found in section 1.9.5 for the rectangular cross-section:

$$A_{s1} = \frac{-\alpha N_{act}}{\alpha f_{cs}(1\text{-}\alpha)} + \frac{\alpha - \delta'}{1 - \alpha} A_{s2} + \frac{\alpha^2\, bd}{2\,\alpha_e(1\text{-}\alpha)} - \frac{(\alpha - h_0/d)^2\,(b\text{-}b_w)d}{2\,\alpha_e(1\text{-}\alpha)} \qquad [1.138]$$

The stresses in the tensile and in the compression steel reinforcements are obtained from the equivalent stress diagram:

$$\sigma_{s1} = -\frac{\alpha_e(1-\alpha)}{\alpha} f_{cs} \text{ and } \sigma_{s2} = \frac{\alpha_e(\alpha - \delta')}{\alpha} f_{cs} \qquad [1.139]$$

It is worth mentioning that a third-order polynomial equation is generally obtained for the determination of the position of the neutral axis at Pivot B, as in the case of Pivot A (even if the third-order equations in both pivots are not the same). However, in the specific case of a rectangular cross-section ($b_w = b$) without compression steel reinforcement ($A_{s2} = 0$), the last term a_3 vanishes in the third-order equation and the third-order equation degenerates mathematically into a second-order equation:

$$a_3 = 0 \quad \Rightarrow \quad \alpha^3 - 3\alpha^2 - \frac{6M}{bd^2 f_{cs}} \alpha = 0 \qquad [1.140]$$

We recognize the second-order equation for the determination of the position of the neutral axis at Pivot B in the case of a rectangular cross-section without compression steel reinforcement. However, in the case of a T-cross-sectional, with $b \neq b_w$, a third-order equation is obtained for the determination of the position of neutral axis α, even in a case without compression steel reinforcement, which is a notable difference with the case of a rectangular cross-section.

Finally, in the case of simple bending $N_{act} = 0$, with only tensile steel reinforcement ($A_{s2}=0$), and for a rectangular cross-section ($b_w=b$), we find again, as in the case of Pivot A:

$$A_{s1} = \frac{\alpha^2 bd}{2\alpha_e(1-\alpha)} \qquad [1.141]$$

1.11.5. *Example – design of reinforced concrete beams composed of T-cross-section*

1.11.5.1. *Data of the design problem*

The geometrical characteristics of the T-cross-section are given in Figure 1.45 and detailed below:

$$b = 800 \text{ mm}; b_w = 300 \text{ mm}; h_0 = 200 \text{ mm};$$
$$d = 920 \text{ mm}; d' = 40 \text{ mm}; A_{s2} = 0 \text{ mm}^2. \qquad [1.142]$$

The reinforced concrete section is composed of steel bars with f_{yk} = 300 MPa and *C25/30* type of concrete. Hence, we calculate the limit stresses at the SLS as:

$$f_{ss} = 0.8 f_{yk} = 0.8 \times 300 = 240 \text{ MPa}$$
$$\text{and } f_{cs} = -0.6 f_{ck} = -0.6 \times 25 = -15 \text{ MPa} \qquad [1.143]$$

The equivalence coefficient is given for this problem as: α_e = 15. The steel Young's modulus is equal to *Es* = 200,000 MPa.

The solicitation is a simple bending solicitation characterized at the center of gravity of the tensile steel reinforcement by M_{ser} = 0.49 MN.m and N_{ser} = 0 MN. In simple bending, the solicitation would have been the same in another point of the cross-section.

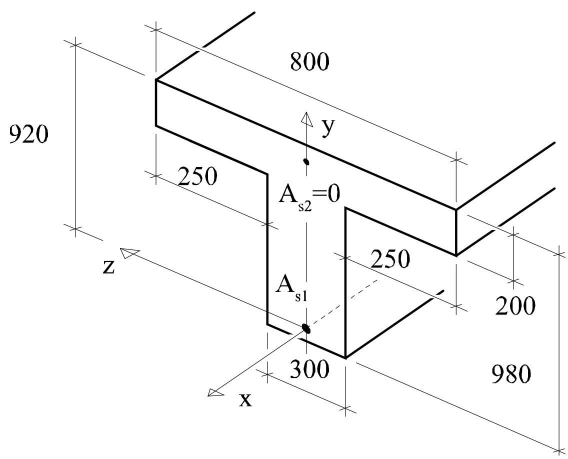


Figure 1.45. *Steel reinforcement design of a reinforced concrete T-cross-section at SLS; lengths given in meters*

It is assumed that there is no compression steel reinforcement (A_{s2} = 0 m^2). The exercise consists of designing the tensile reinforcement of this *T*-cross-section at the stress SLS.

1.11.5.2. *Resolution*

As presented in Chapter 5.5, the limit value of the neutral axis position at the boundary of the two pivots, Pivot A and Pivot B, is given from:

$$\alpha_{AB} = \frac{\alpha_e}{\alpha_e - \frac{f_{ss}}{f_{cs}}} = \frac{15}{15 + \frac{240}{15}} = 0.48387 \qquad [1.144]$$

The bending moment solicitation at the boundary of Pivot A and Pivot B can be calculated from:

$$M_{AB} = \frac{A_{s2}(d-d')(\alpha_{AB}d-d')f_{ss}}{d(1-\alpha_{AB})} + \frac{\alpha_{AB}^2(3-\alpha_{AB})}{6\,\alpha_e(1-\alpha_{AB})}bd^2f_{ss}$$

$$-\frac{(\alpha_{AB}-h_0/d)^2(3-2\,h_0/d-\alpha_{AB})}{6\,\alpha_e(1-\alpha_{AB})}(b-b_w)d^2f_{ss} = 1.737\ 78\ \text{MN.m}$$

$$>M_{ser,act} = 0.49\ \text{MN.m} \qquad [1.145]$$

Hence, the T-cross-section has to be calculated with the Pivot A rule. The dimensionless neutral axis position is the solution of a third-order equation given by equation [1.128] and written with the numerical application as:

$$0.375\ \alpha^3 - 1.125\ \alpha^2 - 0.997976725\ \alpha + 0.347138392 = 0 \qquad [1.146]$$

The three solutions can be computed from Cardano's method according to Appendix 1 – see equation [A1.23].

$$\alpha_1 = -0.930295125;\ \alpha_2 = 0.272002024;\ \alpha_3 = 3.658293101 \qquad [1.147]$$

Only $\alpha_2 = 0.2720$ is of interest for physical reasons. Furthermore, we can check that for this value of the neutral axis, the T-cross-section does not behave as a rectangular cross-section as:

$$\alpha d = 0.272 \times 0.92 = 0.25024\ \text{m} \geq h_0 = 0.2\,\text{m} \qquad [1.148]$$

In other words, it is confirmed that the neutral axis is located in the web of the T-cross-section, which justifies the calculation with the web part of the T-cross-section. In Pivot A, the stress in the tensile steel reinforcement

σ_{s1} is equal to the serviceability limit stress f_{ss} and the steel area A_{s1} can be calculated from equation [1.132]:

$$\sigma_{s1} = f_{ss} = 240 \text{ MPa} \quad \text{and}$$

$$A_{s1} = \frac{\alpha^2 \, bd}{2\alpha_e(1-\alpha)} - \frac{(\alpha - h_0/d)^2 \, (b-b_w)d}{2\alpha_e(1-\alpha)} = 24.304 \; 10^{-4} \text{ m}^2 \qquad [1.149]$$

We finally obtain for this cross-section designed at Pivot A, A_{s1} = 24.30 cm^2. We can take $3\phi 32$ (A_{s1} = 24.13cm^2) for the tensile steel reinforcement (see Appendix 2). But $5\phi 25$ (A_{s1} = 24.54cm^2) would probably be safer, even if the first solution based on $3\phi 32$ could also be used, if the anticipated value of d, the distance from the center of gravity of the tensile steel reinforcement to the upper fiber of the cross-section, is a little bit larger than 0.92 m.