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Uncertainty

In a large number of optimization problems, simulation software, coupled with an appropriate mathematical optimization algorithm, is used for problems such as finance [LID 04], transport [CAP 03], manufacturing [CHE 08] and biochemical engineering and design engineering, [GAN 02]. This approach has proven to be much more effective than standard trial and error procedures. This is, for the most part, due to the development of faster digital computers, more sophisticated calculation techniques and the combination of simulation software based on the finite element method and mathematical optimization techniques [MAK 98]. In the case of deterministic optimization, the design variable can be accurately controlled and has a specific value. The input and output of the optimization procedure will be determinist in this case.

1.1. Introduction

Integrating uncertainty into the design process is a practice commonly used by engineers. This concerns the design of systems for critical values, the use of safety factors, and the more advanced techniques from the calculation of reliability. The aim is to design a system with statistically better performance that may often change according to uncertainty. For example, we may want to obtain a level of performance which is minimally sensitive to uncertainty. We may also not want to surpass a minimal performance threshold with a given probability. In addition, design problems are still constrained optimization problems. Where there is uncertainty, we want to identify it with a high degree of probability.

This uncertainty is a naturally inherent characteristic that cannot be avoided. We can, for example, cite exterior load point clouds and environmental conditions, such as variations in temperature and material properties. Alongside controllable design variables, processes are influenced by noise or stochastic variables. This type of variable cannot be precisely ordered and has either an unknown or known distribution. In the latter case, the variable may be commonly expressed by an average value and a corresponding standard deviation [JAN 08]. The input variation then translates the responding quantity that attaches a distribution rather than a deterministic value.

With continual demands on manufacturers to improve quality, quality control plays an increasingly important role in industrial procedures. One method that uses statistical techniques to monitor and control product quality is termed statistic process control. The different demands generally include three main tasks in the following order:

– a control procedure;

– process diagnostics;

– taking corrective measures.

What does it take to adjust the uncertain process so that it is evaluated on the basis of knowledge of the system and the experience [WER 07]?

With movement toward an integrated computer manufacturing environment, the need to develop applications that allow the implementation of various statistical process control tasks must be automatic. The ability to predict the response of a forming metal process to change in the number of input parameters is crucial. This is because, very often, an obtained deterministic optimum is found at the crossover between one and several constraints. The natural variation in materials, due to lubrication and process parameters, which could lead to an increased number of constraint violations, results in a higher amount of scrap iron [STR 10]. To avoid this unwanted situation, uncertainty must be specifically accounted for in the optimization strategy to avoid faults in the product such as wrinkling and material fractures and faults in form.

An initial approach of accounting for uncertainty in optimization problems has been carried out by considering security factors. The factor must compensate for the variation in yield caused by uncertainty in the system. The greatest security factors correlate with the highest levels of uncertainty. In the majority of cases, these factors are derived on the basis of past experience, but this does not absolutely guarantee security or a satisfying level of performance [BEN 02].

In recent years, several approaches have been developed to explicitly account for uncertainty [BEY 07, PAR 06]. This is examined in a special issue of the *Review of Computing Methods in Applied Mechanics* [HUG 05] and several research projects [PAD 03]. In addition, several modules (or courses) have already combined statistical control techniques for processes with simulations of finite elements to quantify robustness, such as, for example, Autoform-Sigma and LS-Opt [CLE 10]. However, these packets are mainly concentrated around quantifying reliability and robustness in a given solution, rather than optimization under uncertainty. The deterministic optimization strategy is extended to take design variables into consideration. This approach allows the quantification and optimization of a process or design performance.

1.2. The optimization problem

The basic idea of optimization consists of minimizing an objective function, f, by finding the optimum value of one or several design variables x. In addition, several types of restrictions or constraints can be present as equality constraints h, inequality constraints g or a box of constraints. This last type of constraint is often defined as boundaries that identify the area in which design variables are authorized to vary according to an upper and lower limit. These boundaries are respectively indicated by a and b. In general, an optimization problem must be mathematically described as follows:

$$\begin{cases} \min\limits_{x} & f(x) \\ \text{s.c.} & h(x) = 0 \\ & g(x) \leq 0 \\ & a \leq x \leq b \end{cases} \qquad [1.1]$$

The solution to an optimization problem entails finding one of the optimum design values or the variables that minimize the objective functions subject to different constraints. This process requires an optimization algorithm for specific problem. The inequality and equality constraints can be divided into linear or nonlinear and explicit or implicit constraints. Explicit constraints depend directly on the design variables while implicit constraints depend indirectly on the design variables. In the latter case, an evaluation of the constraint function is required to evaluate whether or not the constraint is satisfactory.

The aim of an optimization procedure is to find an optimal design with a high degree of precision. Some critical factors concerning optimization procedures are as follows:

– The optimization procedure is carried out using models and/or approximations of reality. In the majority of cases, we do not know what the error in the model is. If this is the case, we cannot be certain that the optimum model is the true optimum. This indicates that the optimum solution, even if calculated very accurately, may be difficult to replicate in a real world application. Verification of the optimum model against the physical process is therefore highly recommended.

– If the aim of optimization is to improve product quality, it is also helpful to verify the economic feasibility of improving products. There is, in general, a compromise between a potentially more complex and costly manufacturing procedure and an increase in the new design's performance. It is therefore recommended to favor an optimum economic product or tend rather towards the qualitative optimum.

– The following optimum design for an optimum procedure is a static optimum. However, it is, in reality, dynamic. For example, the process or environmental parameters may change over time and therefore the static optimum is only valid for a limited duration.

– Both the objective function and implicit constraints require an evaluation of the response since they inherently depend on the design variables. Each evaluation of the response is carried out by calculating the nonlinear finite elements, which is therefore costly in terms of calculation time.

– The evaluation of the response relating to the obtained optimum, a series of finite element simulations, must be carried out. As a result, an effective algorithm must reduce the number of these simulations.

– In particular, optimization procedures that incorporate uncertainty are significantly less time effective in relation to their deterministic counterparts. This is because the evaluation of the objective functions and constraints are more costly in uncertain conditions. It is therefore necessary to use an appropriate and effective technique to manage uncertainty in the optimization procedure.

– To limit the computational load when there is a combination of optimization strategies with finite element simulations, only a limited number of parameters may be studied. Other procedures which make the design problem less costly in terms of calculation include the use of approximation and parallel calculation [SCH 08].

– A final, crucial remark regarding digital simulation in the optimization process is that the latter may introduce new sources of point clouds, also known as digital noise [DES 08].

1.3. Sources of uncertainty

Optimization under uncertainty requires information regarding the uncertainty influencing the system. There are different sources of variation. Each type of uncertainty requires a different approach for use in the optimization procedure.

There are different scenarios where the designer has to deal with uncertainty. A metal forming the procedure or the product has an output or a response, f, that depends on the input. The input can be divided into design variables x and design parameters p. The design parameters are governed by its environment such as temperature and humidity. The behavior being displayed by the system can be controlled by design variables such as, for example, process parameters and geometric tools. Uncertainty is the input that the designer cannot control in an industrial context, such as with a procedure forming metal while it creates the variation in the response. Different types of uncertainty can be present:

– *Uncertainty in the design parameter:* this type of uncertainty, denoted by z_p, is caused by a change in the environmental and operational conditions. The environmental input can be classed as energy, information and material. Changes in temperature, etc., are examples of variations in the design parameters. Note that material parameters can also show variations. Having chosen a material, its parameters can no longer be controlled. Variation in the parameters must then be taken into account as a noisy variable.

– *Uncertainty in the design variable:* this type of uncertainty is caused by the limited degree of precision with which a design variable can be controlled. Examples of this include variations in the material's thickness, geometric tolerances, inaccuracy in the actuator, variations in the strength of the implementation process, etc. Uncertainty in the design variable is often involved in the process where there is z_x disturbance in the design variables x. This is conveyed by:

$$f = f(x + z_x, p) \qquad [1.2]$$

Uncertainty in the design variable may also depend on x via a coupling such as, for example, $z_x = \epsilon x$. There are some design parameters and variables that are interchangeable. For example, if the process group is used as a variable to influence the response, it will be a design variable. If this parameter has a constant value (with or without a specific variant), it will be a design parameter.

– *Uncertainty in the model:* when using numerical techniques to describe the real physical process, the designer must tackle uncertainty in the model, such as digital noise. The response can be made, for example, by automatic adaptations in size in the simulation stage or adaptive remeshing. The uncertainty of the resulting model depends on the system's input.

– *Uncertainty in the constraint:* there are two kinds of constraint uncertainty. The first kind entails variations in the design space or constraints since they often rely on design and/or parameter variables and the accompanying uncertainty. The second type of uncertainty in the constraint will be specific to the application considered.

We can also use a different classification schema for uncertainty by differentiating between uncertainty in non-cognitive and cognitive sources. The previous source of uncertainty, also known as random uncertainty, is physical in nature. The random nature inherent in physical observations is a statistical uncertainty due to a lack of precise information regarding variation, etc. The last source of uncertainty, also known as epistemic uncertainty, reflects the designer's lack of knowledge about the problem being examined.

Another classification system proposed in [KIM 10] describes different kinds of uncertainty according to the stage of process or lifecycle of the product displaying variation. For example, in the design phase, uncertainty may be caused by errors in the model as well as incomplete knowledge about the system. During the manufacturing stage, the fabrication tolerances of the material introduce uncertainty. Changes in temperature and fluctuations in load may be recognized as sources of variation in a product or procedure's use. Finally, during aging, deterioration in the materials' properties can lead to variability in performance.

1.4. Dealing with uncertainty

This section will describe the main approaches developed for accounting for uncertainty. The majority of these methods have been developed to be applied to technically complex problems. We will not attempt to provide an exhaustive overview of these different approaches and their applications. However, we will provide a general overview to give the reader an idea of the models available when dealing with uncertainty.

A random stochastic reliability-based description is used in the optimization stage and the robust optimization stage where uncertainty is managed probabilistically.

In these practical engineering problems, uncertain random parameters are often modeled as a set of discretized random variables. Let us suppose that X is a random variable and there are n observations of X and the specific occurrence of a random variable where the samples of X are given by x or $x_1, x_2, \ldots, x_n$. The statistical description of a random variable X can be fully described by a cumulative distribution function (CDF) or a probability density function (PDF), denoted by $P_X(x)$ and $p_X(x)$, respectively. To calculate the probability Pr[] of X having a value between x_1 and x_2, the area in the PDF between these two points must be calculated. This can be expressed as:

$$\text{Pr}[x_1 \leq X \leq x_2] = \int_{x_1}^{x_2} p_X(x)dx = P_X(x_2) - P_X(x_1) \qquad [1.3]$$

The PDF is the first derivative of the CDF, such that:

$$p_X(x) = \frac{dP_X(x)}{dx} \qquad [1.4]$$

The effect of PDFs and CDFs on a Gaussian distribution with $\mu_X = 0$ and $\sigma_X = 1$ type deviation is shown in Figure 1.1.

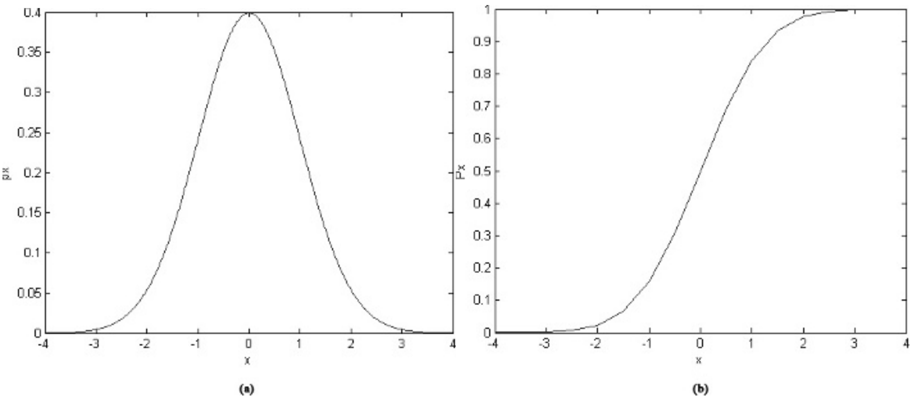

Figure 1.1. *a) Probability density function (PDF) and b) cumulative distribution function (CDF) of a Gaussian distribution with mean* μ_X *= 0 and standard deviation* σ_X *= 1*

Now, taking a general expression to evaluate the mathematical mean value $E(X)$, the variance var(X) and the asymmetry of the random variable are given by equations [1.5], [1.6] and [1.7], respectively. When these values are known, we can identify other parameters such as the standard deviation σ_X and μ_X:

$$E(X) = \mu_X = \int_{-\infty}^{+\infty} x p_X(x) dx \quad [1.5]$$

$$Var(X) = \sigma^2(X) = \int_{-\infty}^{+\infty} (x - \mu_X)^2 p_X(x) dx \quad [1.6]$$

$$m_3 = \int_{-\infty}^{+\infty} (x - \mu_X)^3 p_X(x) dx \quad [1.7]$$

In general, it is necessary to account for several random variables to formulate the problem. These variables can be modeled separately but it is better to model uncertainty jointly. For example, we can cite correlations between specific parameters in the material. Modeling joint uncertainties from two random variables will be examined later on. However, this can also easily be extended to more than just two random variables.

Let us suppose that X and Y are two random variables with the joint PDF denoted by $p_{X,Y}(x,y)$. The CDF is given by:

$$P_{X,Y}(x,y) = Pr[X \leq x,\ Y \leq y] = \int_{-\infty}^{x} \int_{-\infty}^{y} p_{X,Y}(u,v)dudv \qquad [1.8]$$

For two random variables, the PDF can be described according to three dimensions. If the random variables are statistically dependent on the values of another random variable, it is necessary to calculate the separate conditional PDFs:

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \qquad [1.9]$$

or

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)} \qquad [1.10]$$

If X and Y are statistically independent, it can be expressed as:

$$p_{X,Y}(x,y) = p_X(x)p_Y(y) \qquad [1.11]$$

A measure of the dependency or independence between two random variables is given by covariance. Covariance cov(X,Y) indicates the degree of linear relation between these two variables. For statistically independent variables, cov $(X,Y) = 0$. In the opposite case, it can be positive or negative.

EXAMPLE 1.1.– We are studying an elastic pendulum (coefficient with a radius of k, mass m) whose period is: $T = 2\pi\sqrt{m/k}$. Its mass m is 100 g and is known at 1%. We measure 15 times during the period T. We find the average $T = 1.253$ s, with an estimated standard deviation 0.3 s. The stopwatch is graduated after 10 s. The direct calculation of k gives:

$$k = \frac{4\pi^2 m}{T^2} = 2.52453 \text{ kg·s}^{-2}$$

and the variance σ_k^2

$$\sigma_k^2 = \sigma_m^2 \left(\frac{4\pi^2}{T^2}\right)^2 + \sigma_T^2 \left(\frac{8\pi^2 m}{T^3}\right)^2$$

The variance of m (rectangular law with a width of 2 g) is given by:

$$\sigma_m^2 = \frac{1}{3}g^2$$

Type A variance for the period T is given by:

$$\sigma_A^2 = \frac{9 \times 10^{-2}}{15} = 6 \times 10^{-3}\, s^2$$

$$\sigma_B^2 = \frac{0.05^2}{3} = 8.33 \times 10^{-4}\, s^2$$

The overall variance T is written as: $\sigma_T^2 = 6.833 \times 10^{-3}\, s^2$, where

$$\begin{aligned}\sigma_k^2 &= 0.3333 \times 10^{-6} \times 632.3 + 6.833 \times 10^{-3} \times 16.1092\\ &= 2.107 \times 10^{-4} + 0.11 = 0.11\end{aligned}$$

We therefore have: $\sigma_k = 0.33$ s, which can be expressed as:

– $k = 2.51(0.33)$ kg·s^{-2} with the enlargement coefficient 1.

– $k = 2.51(0.66)$ kg·s^{-2} with the enlargement coefficient 2.

EXAMPLE 1.2.– We are measuring the period of a hanging pendant $T = 2\pi\sqrt{l/g}$ to deduce the value of g. The period T has a value of 1.82 s with a standard deviation of 0.15 s. The length l has a value of 0.83 m with a standard deviation of 0.03 m. We therefore obtain:

$$g = 4\pi^2 \frac{l}{T^2} = 9.89\, \mathrm{ms}^{-2}$$

its variance can therefore be written as:

$$\sigma_g^2 = \sigma_l^2 \left(\frac{4\pi^2}{T^2}\right)^2 + \sigma_T^2 \left(\frac{8\pi^2 l}{T^3}\right)^2 = 1.7\, \mathrm{m}^2\mathrm{s}^{-4}$$

where the standard deviation is expressed as:

$$\sigma_g = 1.7\, \mathrm{ms}^{-2}$$

where g is between 11.6 ms^{-2} and 8.2 ms^{-2}.

EXAMPLE 1.3.– In this example, we are interested in the problem of identifying a Young's modulus E in a section of a rectangular beam. This example was introduced by [PER 08] into the identification framework of uncertain systems. Identification is carried out by measuring the curve in the beam being bent on the two supports in Figure 1.2.

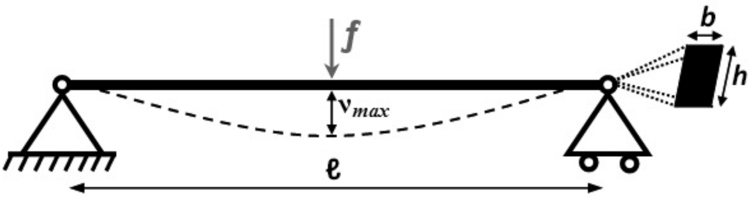

Figure 1.2. *Beam being bent*

The beam's geometry is considered to be deterministic with $L = 2$ m, $l = h = 20$ cm. A sudden force is applied to the center of the beam. In the linear elasticity model, the material resistance theory predicts that the curve v_{max} is equal to:

$$v_{max} = \frac{fL^3}{4Elh^3} \tag{1.12}$$

Theoretically, a measure of maximum movement may identify the elasticity module E. However, the experimental conditions could cause errors in the load f and the curve's measure. If we consider that the measure of the beam's dimensions and the applied force is less precise then we model the parameters l, h and f using known random law variables. The description of their probabilistic model is given in Table 1.1. In addition, all the variables are presumed to be independent.

Parameter	Probability law	Average	Covariance (%)
Width L	Determinist	2 m	–
Width l	Normal	0.1 m	3
Height h	Normal	0.1 m	3
Force f	Lognormal	10 kN	5

Table 1.1. *The different parameters and their successive laws*

If, for example, the experimental device measures the curve perfectly, we will have a sample of measures for the curve. Let us suppose that the Young's modulus E follows a log normal law with an average of $\mu_E = 10,000$ MPa and a variation coefficient $CV_E = 25\%$. A data set of the input parameters $X = \{L, l, h, f, E\}$ with a size of $Q = 50$ is simulated and we calculate the corresponding curves. The calculation of the four first statistical points is presented in Table 1.2.

Statistical points	Identified	Sample	Relative error (%)
Average	$9,966.6$	$9,972.8$	0.1
Standard deviation	$2,348.7$	$2,434.4$	3.5
Asymmetry	0.43	0.54	20.2
Kurtosis	2.79	2.94	4.8

Table 1.2. *Results calculated, taking uncertainties into account*

We can see that the values of the first four statistical points differ from the theoretical values used in sampling ($\mu_E = 10,000$ MPa, $\sigma_E = 2,500$ MPa, $\delta_E = 0.766$ and $\kappa_E = 4.06$). This is due to statistical uncertainty caused by the small size of sample used ($Q = 50$).

1.4.1. *Reliability optimization*

The models predominantly used to treat uncertainty in structural engineering manage noisy variables probabilistically. This is also the case in a reliability-based design optimization approach (RBDO). It provides a means of identifying the optimum solution to a specific objective function while ensuring the predefined low probability missing in a product or procedure.

The probability of infringing a limited state or a predefined constraint is calculated using complete or partial information on PDFs for uncertain parameters. To reach a specific level of reliability, the PDF set of the result is scaled, as shown in Figure 1.3. We can see that this specifically and precisely identifies the zone in the distribution queue outside the specified limit. In general, reliability-based optimization is formulated as follows:

$$\begin{cases} \min\limits_{x} \ f(x) \\ s.c. \ \ Pr[g(x, z_x, z_p) \leq 0] \geq P_0 \\ \qquad a \leq x \leq b \end{cases} \qquad [1.13]$$

with Pr[] being the probability of satisfying the constraints. The limited state $g = 0$ separates the area of failure ($g > 0$) and success ($g < 0$) and is a function of design variables x and the uncertain variables z_x and z_p. P_0 is the level of reliability or performance expectation. Equality constraints h are generally eliminated before the optimization process. The inequality above can be expressed by a multidimensional integral that results in:

$$Pr[g(x, z_x, z_p) \leq 0] = \int_{g(x,z_x,z_p)\leq 0} p(z_x, z_p) dz_x dz_p \geq P_0 \qquad [1.14]$$

where $p(z_x, z_p)$ is the conjoint PDF of the probabilistic variables z_x and z_p. If the variables are statistically independent, the joint probability function can be replaced by the product of the individual PDFs in the integral, as illustrated by equation [1.11].

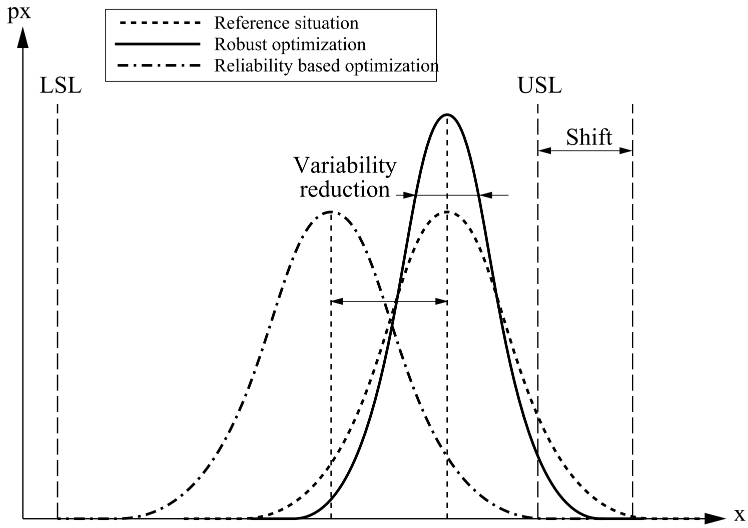

Figure 1.3. *Reliability-based optimization and robust optimization [BON 07]*

From a theoretical perspective, the RBDO is a well-established concept. However, the calculation of integrals in equation [1.14] appears to be a technically complicated problem, which is analytically feasible in very simple cases. Because it is a multidimensional integral equation where we do not know the joint PDF and/or the limited state function g in an explicit form. In practice, it is used for reliability analysis techniques using the Monte Carlo method, for example. Other well-known techniques are first- and second-order reliability methods, known as the first-order reliability method (FORM) and the second-order reliability method (SORM), respectively. These techniques and their applications found in the literature will be discussed further in Chapter 2.

1.4.2. *Robust optimization*

This is intrinsically linked to Taguchi who first initiated a design philosophy of robust optimization that has since been highly influential. Taguchi, the pioneer of robust design, stated that "robustness is the state where technology is a product or process whose performance is minimally sensitive to factors causing variability (whether in the manufacturing or user environments) at the lowest possible cost". Similarly to the RBDO approach, uncertainty is managed probabilistically. Robust optimization focuses on design centered optimization that is relatively minimally sensitive with regard to uncertainty.

This indicates that variation in the response f is minimized by changing the average of the stochastic variable. The selection of the design variable replacing x_1 with x_2 provides a more precise result and, therefore, a more robust design. This approach is evidently different to the RBDO approach that emphasizes the point in the distribution queue, which is outside of the specified limit, as shown in Figure 1.3.

The probabilistic measure of robustness is generally expressed by an anticipated value and variation in the objective function, given, respectively, by equations [1.5] and [1.6]. Fundamentally, the mathematical formulation of robust optimization is given by:

$$\begin{cases} \min\limits_{x} \sigma_f(x) \\ s.c. \;\; Pr[g(x, z_x, z_p) \leq 0] \geq P_0 \\ \qquad a \leq x \leq b \end{cases} \qquad [1.15]$$

In this case, the width of the response distribution is minimized by reducing the standard deviation as far as possible. To ensure the required level of reliability, an additional constraint is added to the result. The respect for uncertain constraints is governed by their guarantee probabilistically that is using equation [1.14]. The robust design optimization problems which also incorporate this type of formulation of constraint are also called robust design based reliability optimization problems. For some applications, robustness and the reliability-based approach are combined and may not even be distinguishable from one another.

1.4.3. *Multi-object optimization*

It is possible to apply a multi-objective formulation. This formulation is implemented to study the minimization of the average performance and variance of the response at the same time. It is composed of the average and the standard deviation of the objective function:

$$\begin{cases} \min\limits_{x} \alpha\frac{\mu_f}{\mu_f^*} + (1-\alpha)\frac{\sigma_f}{\sigma_f^*} \\ \text{s.c.} \;\; \mu_g + k\sigma_g \leq L \\ \qquad a \leq x \leq b \end{cases} \qquad [1.16]$$

where μ_f^* and σ_f^* are the values of the optimum function. We can see that the point corresponding to the formulation is used to manage the constraints. The value of the weighting factor α is determined according to the importance of the reduction in the average performance or variance. More precisely, it is evident that when using this measure of robustness, seeking an optimum design seems to require several decision criteria [LEE 01a, SHI 09]. Seeking a compromise solution is called robust multi-objective optimization when a set of Pareto-optimal solutions can be considered as solutions to possible compromises.

1.4.4. *Stochastic optimization*

Stochastic optimization can be seen as an extension of robust optimization [WET 89]. This is not within the context of reducing the variance in which it was developed. These techniques are focused around linear problems requiring a decision taken without knowing a number of factors. One of the most commonly examined examples is that of programming with recourse [KLE 01]. This involves maximizing rentability in a production line without knowing future sales costs. Sahinidis [SAH 04] states that the original formulation of the problem was developed to treat more general problems such as constraints in probability and nonlinearity. Two main categories of algorithm are used to solve this kind of problem. On the one hand, approximation algorithms based on a statistical estimation of the quasi-gradient are used. On the other hand, there are simulation methods that solve a range of problems. These problems have been found in a number of applications in a variety of fields such as production planning [PER 91], itinerary research [LIU 04] and finance.

1.4.5. *Worst-case scenario based optimization*

Precise information on the probabilistic distribution of uncertainty is rare or even non-existent. This may be caused by a lack of experimental data or knowledge about the product or procedure. If this is the case, a worst-case scenario optimization strategy can be applied to combat uncertainty. In this approach, the aim is to optimize toward a point located as far as possible from the rupture constraints. This is based on the notion of minmax, that is maximizing the minimum distance between the optimum point and the failure constraints. Note, however, that uncertainty cannot be explicitly recorded. Uncertain parameters are modeled using a deterministic set instead of a PDF, for example. This approach has simplified the incorporation of variation in a deterministic problem and is conveyed as a reduced admissible region where we want an optimum solution. As a result, quantitatively we know nothing about the variation in the result [PAR 06, LEE 01a]. Worst-case scenario optimization based applications are also found in [BEN 02].

To obtain estimations of the robustness of the deterministic optimum, we carry out a Monte Carlo analysis. The robustness of an optimum can be calculated by averaging a specific number of samples while keeping the design point x constant.

The noisy variables are randomly varied according to a distribution and we can therefore calculate the average and the variance of the optimum design. As a result, if the specification's limits are known, the level of robustness can be calculated.

1.4.6. *Non-probabilistic optimization*

Non-probabilistic methods have been developed in recent years to solve optimization problems where the probabilistic distribution of uncertain variables is unknown. These methods, also known as probabilistic approaches [BEY 07, DEM 08], do not require *a priori* hypotheses about the PDF to describe uncertain variables.

1.4.7. *Interval modeling*

The first model of non-probabilistic uncertainty is the interval model. An interval can be described as:

$$X = [x_{min}, x_{max}] = \{x \in \mathbb{R} |\ x_{min} \leq x \leq x_{max}\} \qquad [1.17]$$

In this case, only a range of values between the clean limits x_{min} and x_{max} is known for its variation. The principal objective of the interval model is the simplest means of calculating the upper and lower values or the limits [MÖL 08] of the response (and constraints) for the given range of uncertainty. The limits of the result's interval are known and can then be used in a general nonlinear optimization technique to find a reliable optimum design by reducing the objective function as far as possible.

1.4.8. *Fuzzy sets*

Fuzzy sets are a generalization and an improvement on the interval model. In the interval approach, uncertainty is characterized by clean sets resulting in a design that may or may not be possible. In a fuzzy approach, the interval approach is extended using a progressive allocation component. The values in the interval $x \in [x_{min}, x_{max}]$ are considered using adhesion values $\mu_\Omega(x)$ in the interval $[0, 1]$, describing the degree of adhesion to the possible set Ω.

The attribution of intermediary adhesion values allows us to model uncertain values that are too "rich" in content to be adequately reflected in interval modeling. We can examine records for more information on the values more or less supposed to be in the interval or the integration of expertise.

1.5. Analyzing sensitivity

We have so far highlighted the link between variability in the input of the problem and the model's response. Nevertheless, the complete modelization of a complex phenomenon often involves using a large number of random parameters. A study of the relative influence of each parameter on the solution can therefore reduce the number of variables to be taken into account while only retaining the most important. This is often called sensitivity analysis. This sensitivity can be estimated at either a local or global level.

1.5.1. *Local sensitivity analysis*

Analyzing local sensitivity has an impact on the response of the model in terms of parameter variability around a nominal value. The majority of these methods are based on calculating gradients of observations in relation to uncertain parameters [HOM 96]. In [SUD 08], it was noted that these quantities can often be directly deduced from the results of a preliminary study of uncertainty propagation. Ditlevsen, for example, has proposed a direct measure of sensitivity after reliability analysis [DIT 05]. Madsen has also deduced a sensitivity factor by expressing the error by replacing a variable parameter with a fixed value [MAD 88].

1.5.2. *Global sensitivity analysis*

In contrast, global methods seek to quantify the influence of input parameters by taking into account the whole of their area of variability. The more parameter dependent the model is, the more uncertainty in the response generated by input variability is measured. Two categories of techniques are often presented [SAL 04], the first being regression principle based methods. These entail constructing a linear model of the response by regressing on uncertain parameters [HOM 96]. The model's Pearson coefficients measure correlations. Nevertheless, this method does not effectively account for nonlinearities in the model M. To combat this problem, variance analysis based techniques provide an attractive alternative. These methods seek to calculate the variance of the response expressed by each variable at input. This ratio is often called the first-order Sobol index [SOB 01]. First-order Sobol indexes are obtained by examining the interactions between variables. These indexes are very good indicators but their evaluation, most often using Monte Carlo methods, is costly.