

Chapter 1

Introduction to Computational Contact

From a mechanical point of view, at macroscopic scale, contact is the notion for the interaction between bodies coming in to touch and exchanging load and energy (heat and electric charge). The physics of contact – rich and complex – is hard to study because of its multiscale and multiphysical nature and also because of the contact zone’s inaccessibility for direct observations¹. Both experimental and numerical investigations of the contact experience difficulties. Tribology is an experimental science that describes and characterizes the contact, adhesion, friction, wear and lubrication as well as the involved mechanical, physical and chemical effects at different scales. Mathematics formalizes these descriptions by some measurable quantities (the coefficient of friction, the real contact area, the heat transfer coefficient, etc.). On the basis of observations, we suggest some models for the evolution of these quantities and then integrate them in complete numerical models to study particular systems at a common basis. The simulations thus imply strong simplifications² that may be crucial for their validity. On the other hand, even the simplest models appear complicated for numerical simulations both from mathematical and programming points of view. All together, the oversimplification of the phenomenon and the imperfect implementation of established models may easily lead to incorrect results. The aim of this book is to resolve some of these problems, provide a consistent basis for the numerical treatment of contact problems at all stages, avoid unnecessary simplifications and enhance existing numerical models. The book focuses on the mechanics of contact and its numerical treatment by the

¹ The contact zone is visible only if one of the contact bodies is transparent.

² Sometimes these simplifications result from the limited computer resources or programming skills, so may be partly avoided in many cases.

finite element method (FEM). The underlying physics and mathematics are covered only partly and superficially³.

We suppose that the reader is unfamiliar with this field, so from the very beginning we provide the reader with the vocabulary and all the basic notions of computational contact mechanics. Many notions are very technical and complex, so we explain them in many ways throughout the book⁴ and often enhance these explanation with figures and examples. The material covers three subjects: geometry, detection and resolution. However, these parts are hard to comprehend independently. To ensure the readability of the book, this chapter introduces the subject, provides the global context of the numerical treatment and briefly discusses every component.

Contact problems in mechanics of deformable solids can be singled out into a particular class. First, they imply a discontinuity: the contact occurs at the interface between two *separate* continuous bodies. Second, the contact constraints at this interface cannot be replaced by ordinary boundary conditions imposed on both the contacting surfaces. Third, the contact interface itself cannot be simply considered as an internal surface. In an idealized case, the contact interface is a zero thickness layer, which sustains only compressive stress in the direction orthogonal to the contact interface (Figure 1.1(a)); any stretching leads to the vanishing of the contact interface (Figure 1.1(b)). In the case of frictionless contact, the contact interface contrary to an ordinary internal surface, does not sustain any tangential efforts, which allows two solids to slide relative to each other (Figure 1.1(a)). In the case of frictional contact, things become more complex. For example, in case of the classic Coulomb's friction law, in stick state, the contact interface under pressure is similar to an internal interface – no separation, no sliding – locally both surfaces remain glued to each other (Figure 1.1(c)). However, if locally we reach a critical shear stress, the surfaces start to slip relatively to each other (Figure 1.1(d)). It follows from this simple representation that the contribution of the contact interface to the energy of the system is always zero except in the case of frictional slip.

Many mechanical problems can be formulated as boundary value problems, where governing differential equations should be fulfilled within the domain Ω under ordinary boundary conditions imposed on its closure $\partial\Omega$. This formulation is called the *strong form*. By writing the balance of virtual works, we obtain a *weak (integral) form* of this boundary value problem. This weak form presents a basis on which the structural FEM is constructed. Contact constraints are formulated as sets of

³ We refer the reader interested in the physics and mechanics of contact to [RAB 65, BOW 50]; and those who prefer to gain a solid mathematical basis to [KIK 88].

⁴ It implies some repetitions and reformulations of all the notions, which hopefully will help the reader to better visualize them.

inequalities, which are not trivial to incorporate in the weak form. The rigorous construction leads to a *variational inequality* instead of a classical weak form formulated as a *variational equality*. This new mathematical structure requires new solution approaches. The problem becomes even more complex when we introduce a friction. Coulomb's friction law states that tangential resistance depends upon the normal contact pressure that is known only if the solution is known. Roughly speaking, the boundary conditions depend on the solution, which naturally leads to difficulties in the formulation of the frictional contact problem. Moreover, the Coulomb's friction law yields a non-smooth energy functional making the problem even more difficult from a numerical point of view. In summary:

“Frictional contact problem between continuous deformable solids involves formidable mathematical difficulties.”

N. Kikuchi and J.T. Oden [KIK 88]

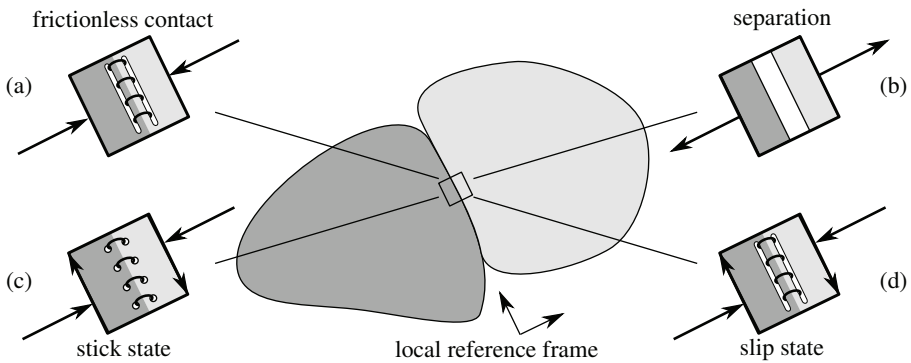


Figure 1.1. Analogy between contact and internal interfaces: (a) frictionless contact sustains compressive stress in the local reference frame, (b) any stretching leads to the vanishing of contact interface, (c) frictional contact interface can transfer shear stress, (d) in Coulomb's friction law, in stick state there is no relative sliding up to reaching a critical shear stress

Another mathematical difficulty in contact mechanics arises from a rigorous description of continuous interacting surfaces. First, contacting bodies may penetrate each other or be separated: in both cases, a one-to-one correspondence between points of the contacting surfaces does not always exist. Second, the finite element discretization renders only piecewise-smooth surfaces, which leads to mathematical and numerical difficulties. Third, a considerable effort has to be undertaken to find variations of the weak form, which requires second-order variations of the normal

gap and the tangential sliding, which is not an easy task. We need a basic knowledge of the differential geometry to obtain the relevant quantities.

The discretization of the contact interface⁵ is the third challenge in computational contact mechanics. A simple and stable *node-to-node* (NTN) discretization for conforming meshes⁶ can be established only in the case of small deformations and infinitely small relative sliding. A less simple but multipurpose *node-to-segment* (NTS) *discretization* implies the creation of contact pairs consisting of a node belonging to one surface and a corresponding segment of another surface. However, this discretization does not fulfill the so called Babuška–Brezzi conditions and leads to an unstable discretization. It means, for example, that a uniform contact pressure cannot be obtained at the interface between non-matching meshes. Recently, new techniques based on *segment-to-segment* (STS) *discretizations* – *Nitsche* and *mortar methods* – have been successfully introduced in computational contact mechanics; however, the computer implementation of these methods for a general case presents a challenge both from algorithmic and technical points of view. Seeking for a stable and relatively simple contact discretization is still in progress.

All aforementioned difficulties are related to the resolution phase of the contact algorithm. This phase follows the detection that determines the contacting pairs on discretized surfaces. At first glance, the contact detection phase is a standalone task, but in reality it appears to be strongly connected with the contact discretization, the definition of the gap and with the type of contact (e.g. simple contact or self-contact). The detection may present a bottleneck for the numerical treatment of contact problems both in terms of rapidity and robustness. The contact detection becomes one of the crucial points for an efficient parallelization of the whole resolution scheme. Elaboration and implementation of an efficient contact detection algorithm is an absolute necessity for a robust and fast finite element analysis (FEA) of large contact problems.

First, we make a short historical remark on contact and computational contact mechanics. In the coming sections, we present the general concept of contact treatment in the framework of the FEM and implicit integration. To provide the reader with a general understanding of the numerical treatment, we expose, here, in a first approximation, the contact detection, contact discretization and contact resolution.

⁵ By contact discretization or discretization of the contact interface we mean the establishment of contact elements over which the transferred forces are transmitted. Even though two contacting surfaces are already discretized, there are different ways to discretize the interface between them.

⁶ By *conforming* or *matching* meshes we understand a situation where each node on one contacting surface matches a corresponding node on another surface.

1.1. Historical remark

Modern contact mechanics is about 130 years old. It started in 1882 with the publication of Hertz's famous paper *On the Contact of Elastic Solids* [HER 82], which gives the solution for frictionless contact between two ellipsoidal bodies. This problem came from the optical interference between glass lenses. Further developments in contact theory appear only at the beginning of 20th Century in application to railways, reduction gears and rolling contact bearings. Progress in contact mechanics was associated with removing the restrictions of the Hertz theory: linear elasticity, frictionless contact and small deformations. The Russian school of mechanicians, starting from Galin [GAL 53, GAL 76] and Muskhelishvili [MUS 66], made a significant contribution to this development. A synthesis of analytical solutions and approaches for contact problems can be found, for example, in monographs by [LUR 70, ALE 83, JOH 94, GOR 98, GOR 01, VOR 01].

Since the analytical solution is achievable only for a few simple geometries, boundary conditions and mostly for linear materials, it can only provide rough approximations for complex industrial contact problems, which include complex geometries, friction, wear, adhesion, large deformations and nonlinear materials. More powerful tools are needed to fulfill the industrial demand for a fast and accurate solution of these problems.

Since 1965 (NASTRAN), the FEM becomes one of the most usable and efficient tools for the treatment of problems in structural mechanics. To answer the requirements from industry, the scientific community worked out a rigorous mathematical framework valid for incorporation of the contact in the FEM, which required formidable efforts from mathematicians and mechanicians. First, the frictionless Signorini's problem (unilateral contact between a deformable body and a rigid foundation) has been treated, further the developed approaches were extended to the case of unilateral frictional contact in small and large deformations and finally to bilateral⁷ or multibody contact. At the same time, engineering practice tested the solution schemes and proposed new challenging tasks. The work on an accurate and reliable method to treat large-sliding frictional contact is still in progress.

The history of computational contact began in 1933 with the works of Signorini. He was the first who formulated the general problem of the equilibrium of a linearly elastic body in frictionless contact with a rigid foundation [SIG 33, SIG 59]. The works of Fichera represent the first treatment of the questions of existence and uniqueness of the solution of *variational inequalities* arising from the minimization of functionals on convex subsets of Banach spaces, which yields from his rigorous

⁷ *Bilateral* contact – contact between two or more deformable solids, in contrast to *unilateral* contact – contact between a deformable and a rigid solid.

analysis of a class of Signorini's problems [FIC 63, FIC 64, FIC 72]. Variational inequality is a new structure in the field of optimization theory; new approaches are required to use such a formulation for practical problems in physics and mechanics. "Inequalities in mechanics and physics" by Duvaut and Lions (first published in French and translated in English [DUV 76]) was a scientific breakthrough in this direction, the authors investigated the solution of frictional contact problems and large deformation contact. Among the early relevant contributions, we can enumerate Cocu [COC 84], Panagiotopoulos [PAN 85] and Rabier *et al.* [RAB 86]. A consistent description of the variational inequality approach to contact problems is given in the book by Kikuchi and Oden [KIK 88], where among other important results the existence and uniqueness of the solution of Signorini's problem is proven. The questions of stability of contact problem's solution were discussed by Klarbring [KLA 88]; the examples of non-uniqueness or non-existence were demonstrated by Klarbring [KLA 90] and by Martins *et al.* [MAR 94]. The existence and uniqueness results for dynamic contact problems can be found in Martins and Oden [MAR 87], Jarusek and Eck [JAR 99] and others.

The *frictionless* contact problem formulated as a variational inequality presents a special type of minimization problem with inequality constraints, which can be efficiently treated in a standard manner (penalty method (PM), Lagrange multiplier method (LMM), augmented Lagrangian method (ALM)). Unfortunately, there is no associated minimization principle for the *frictional* contact problem [KIK 88, MIJ 00]. Such a problem is complicated and unusual for optimization theory as the energy of the system (objective function in optimization) depends on the frictional status, which in turn depends on the normal contact pressure, which itself depends on the displacements, finally the energy depends on the solution of the problem, which is unknown. As there is no smooth energy functional associated with the frictional contact problem, its formulation and resolution present real challenges.

The assumption of a known *a priori* contact interface on the current computational step results in a reformulation of the variational inequality into a variational equality problem with a special contact term; the form of this term depends on the method chosen to enforce the contact constraints. Among the well-known and widely used methods, there are barrier and penalty methods, LMMs and their combinations. Another branch of methods makes use of different techniques from mathematical programming: application of the simplex method to contact problems can be found in [CHA 76], parametric quadratic programming method is used in [KLA 86, ZHO 88]. Separately from these two branches, there is a group of direct methods, which treats the contact problem independently from the structural problem. The *flexibility method* proposed by Francavilla and Zienkiewicz [FRA 75] and modified and improved by Jean [JEA 95] is rarely mentioned in the scientific literature on computational contact. In practice [WRO 94], this method demonstrates a higher robustness and rapidity in

comparison to ordinary methods if the number of nodes in contact remains moderate. But it is not applicable for large contact problems and the parallelization of the method is hardly possible. A complete list of methods used for the numerical treatment of contact problems can be found in [WRI 06, LAU 02].

1.1.1. *The augmented Lagrangian method*

As mentioned in the previous section, the assumption of a known *a priori* contact surface allows us to replace the variational inequality by a variational equality with an additional contact term. The form of this contact term depends upon the choice of the optimization method; the most usable in contact mechanics are the LMM, the linear PM and an ALM, the two latter methods are implemented in leading modern FEA softwares: ANSYS [BHA 02, OAT 07, ANS 05], ABAQUS [ABA 07], COMSOL [COM 10] and others. In this chapter, all aforementioned methods are considered, but particular attention is paid to the ALM, possessing several advantages in comparison to other methods.

Within the framework of classical LMM, contact conditions are exactly satisfied by the introduction of extra degrees of freedom (dof) called Lagrange multipliers. The constrained minimization problem converts into an unconstrained saddle point problem often called the min–max problem. Because of inequality constraints, this formulation has to be considered in combination with an active set strategy [LUE 03, MUR 88], that is a check and update of active and passive constraints should be integrated in the convergence loop. Moreover, the additional dof of the LMM introduce supplementary computational efforts. PM is simple to implement and interpret from the physical point of view, but, on the other hand, the contact conditions are fulfilled exactly only in case of the infinite penalty parameter that results in ill-conditioning of the numerical problem. The ALM is a sort of Lagrange multiplier formulation regularized by penalty functions. It yields a smooth energy functional and fully unconstrained problem, resulting in exact fulfillment of contact constraints with a finite value of the penalty parameter. In this section, a few historical remarks concerning the ALM are given. For a more detailed background, the reader is referred to the articles and books cited below.

The ALM has been proposed in the first raw approximation by Arrow and Solow in 1958 [ARR 58b]. Further a more elaborated version of the ALM method for optimization problems subjected to equality constraints has been independently proposed by Hestenes [HES 69] and Powell [POW 69] in 1969. As mentioned by Pietrzak [PIE 97], it was proposed “rather in an intuitive way” and a lot of questions have not been considered. The way to apply the ALM method to optimization problems with inequality constraints has been developed by Rockafellar [ROC 70, ROC 73b] and Wierzbick [WIE 71].

Using the ALM as well as the LMM leads to the saddle point problem, that is the objective function is to be minimized by “ordinary” primal variables (e.g. displacement dof in the displacement-based FEM) and is to be maximized by dual variables – Lagrange multipliers that represent contact stresses. All aforementioned authors approach this min–max (saddle point) problem by an independent consecutive updating of the primal and dual dof. An algebraic formula is used to update the Lagrange multipliers at each iteration step and consequently a standard minimization procedure is used to update the primal dof. This idea has been worked out by Powell [POW 69]. Nowadays such an approach is used under the name of Uzawa’s algorithm and the full method is referred to as a nested augmented Lagrangian algorithm. Another approach has been developed by Fletcher [FLE 70]. It consists of a continuous minimization of the resulting saddle problem with a simultaneous update of both primal and dual variables.

One of the first applications of the ALM to frictionless contact problem can be found in Glowinski and Le Tallec [GLO 89] and Wriggers, Simo and Taylor [MID 85]. The first application of the ALM with Uzawa’s algorithm to frictional problems has been reported by Simo and Laursen [SIM 92]. The first successful attempt to apply the coupled ALM to frictional contact problems has been undertaken by Alart [ALA 88], and Alart and Curnier [ALA 91]. The augmented Lagrangian approach has been elaborated by developing the perturbation approach to convex minimization as proposed in [ROC 70] and first applied by Fortin [FOR 76] to visco-plastic flow problems (rather similar to frictional contact problems).

Further developments of the ALM method to large deformations, large sliding and nonlinear materials can be found in [HEE 93, MIJ 04a, MIJ 04b], etc. A comprehensive investigation on the implementation of the ALM method in the framework of the FEM to large deformation frictional contact problems has been carried out by Pietrzak and Curnier [PIE 97, PIE 99]. The attempts to work out a technique for penalty parameter updating are worth mentioning, since it became a crucial factor for convergence of the ALM. A direction was proposed in early works [HES 69] and [POW 69]. The need was mentioned by Rockafellar [ROC 73b], discussed in [ALA 97] and an approach has been proposed by Mijar and Arora [MIJ 04a, MIJ 04b]; another phenomenological approach has been proposed in [BUS 09]. An early attempt to parallelize the ALM has been undertaken by Barboteu [BAR 99] for particular structures.

The ALM combines advantages of both methods LMM and PM and avoids their drawbacks, it converges precisely to the exact solution for a finite value of the penalty coefficient and if a nested update of dual variables is used, there are almost no additional computational efforts. Following Pietrzak, we would like to emphasize the smoothing effect of the ALM, which is not the only advantage over ordinary LMM. Even in the case of a smooth objective function, the ALM method shows its superiority. The ordinary LMM does not fully reduce the optimization problem with inequality constraints to an unconstrained problem, since the condition of positivity

of the Lagrange multipliers $\lambda \geq 0$ has to be satisfied. The ALM method does not have this restriction and therefore is better for practical use. An elaborated presentation of the method will be given in section 4.7.

1.2. Basics of the numerical treatment of contact problems

A finite element code – capable of treating contact problems – should include the following steps: detection of contact, construction and update of “contact elements”, their incorporation in associated residual vectors and tangential matrices and finally resolution of the resulting problem. Here, we give basic ideas and a general view of these steps, which will be presented in detail in the corresponding chapters.

*Contact elements*⁸ are a sort of “bridge elements” between locally separated but potentially interacting bodies. Each contact element contains components (nodes, edges, faces) of both surfaces; the composition of these components depends upon the *contact discretization* method. Besides structural dof contact element may have their proper dof. These unknowns and the structure of the residual vector and the tangential matrix are determined by the *resolution method*. For example, in addition to primal or structural dof (e.g. displacements) contact elements may contain dual dof (Lagrange multipliers) representing contact forces.

The *contact detection* is a step preceding all others. The aim of this step is to create contact elements containing the proximal components of both surfaces that *may* come into contact during the current solution step. Thus, the detection is an algorithmic task that relies on the search for the closest components. The criterion of proximity is either provided by the user or is chosen automatically based on boundary conditions and/or discretization of contacting surfaces. If we use an *implicit integration*, then to incorporate contact elements in the resolution cycle, they should be created before a contact occurs and if needed should be removed and recreated at each solution step. In the case of *explicit integration*, generally, the searching step should ensure only the detection of already occurred penetrations.

1.2.1. Contact detection

The development of numerical methods and the increasing demands on complexity (large deformation/sliding, self-contact, remeshing) and size of problems

⁸ Contact elements are similar to ordinary structural finite elements in the way that they are assembled with the latter in the global residual vector and the global tangential matrix. But their elemental residual vectors and the tangential matrices are very different from structural elements; normally, contact elements do not need shape functions and they are not visualized in the graphical user interface; also they are generally created and removed “on flight” during the solution.

in computational contact mechanics entail the progress in contact detection techniques. Construction of an efficient technique, which is based on the notion of proximity, presents a purely algorithmic problem and depends upon the contact discretization. For example, in the case of NTN discretization, the contact detection consists simply of establishing close pairs of nodes: nodes of one surface form pairs with their closest opponents from another surface. As the NTN discretization is limited to small deformations and slidings, once created contact pairs do not change during the solution steps. NTS discretization requires a more elaborated detection procedure: for nodes of one surface (slave), we have to find the closest point on another surface (master), the master segment possessing this point complemented by the slave node forms an NTS contact element.

This simple detection procedure contains several difficulties. First, the detection of the closest point on the master segments may be ambiguous if the slave node is not sufficiently close to the master surface or if the latter is not smooth – the case of finite element discretization of the surface. The numerical scheme of the closest point detection is based on the looking for a minimum of the distance function, but on the one hand this minimum does not always exist, and on the other hand there may be several or infinitely many minimum points. Second, the detection has to be organized in a smart way. Large contact problems imply a large number of contacting nodes on both surfaces, that is why a simple detection technique, based on a comparison of distances from each slave node to all components of the master surface, leads to an excessive time consumption, especially if contact elements must be frequently updated.

STS discretization requires totally different detection algorithms based on surface topologies. In this book, we confine ourselves to consideration of the NTS contact discretization, so we omit the detection for other discretizations. In Chapter 2, we discuss the geometrical aspects of the closest point search and in Chapter 3, the detection algorithms.

1.2.2. *Contact discretization*

The contact discretization predetermines the structure of contact elements that transfer efforts from one contacting surface to another. We distinguish three types of discretization:

- Node-to-node (NTN);
- Node-to-segment, (NTS);
- Segment-to-segment, (STS).

The simplest and the oldest NTN discretization [FRA 75] introduces restrictions on mesh generation and does not permit any finite sliding or large deformations (see

Figure 1.2(a)). But it passes the contact patch test [TAY 91] – contact pressure is transferred correctly through the conforming contact interface. The NTN discretization is applicable for linear and quadratic elements in the two-dimensional case and to linear elements in the three-dimensional case. The NTN technique smoothes the asymmetry between contacting surfaces. However, the normal vector for each pair of nodes is usually determined⁹ according to one of the surfaces.

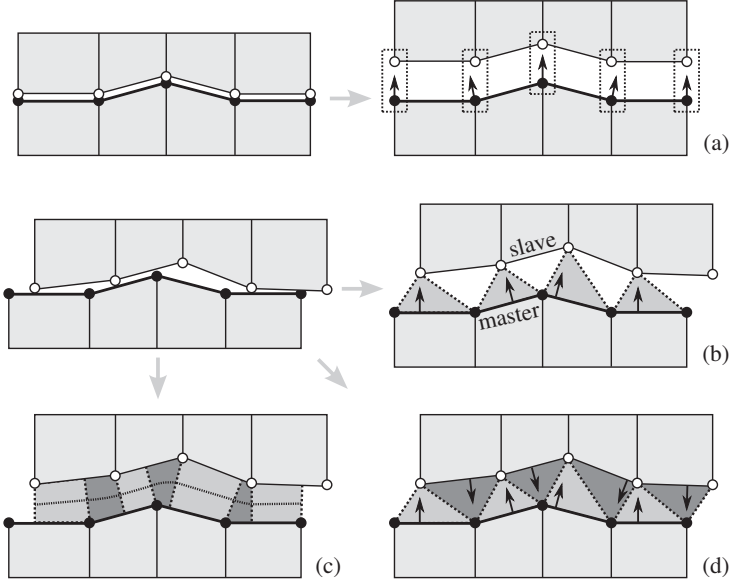


Figure 1.2. Graphical representation of different contact discretizations: for small deformations/sliding (a) node-to-node, associated pairs of nodes and normals; for large deformations/sliding (b) node-to-segment, slave nodes and associated master segments; (c) segment-to-segment, contact elements and intermediate integration line; (d) contact domain discretization, contact elements

NTS (Figure 1.2(b)) is a multipurpose discretization technique [HUG 77], valid for non-conforming meshes and large deformations/sliding. But this discretization is not stable and does not pass the contact patch test [TAY 91] for non-conforming meshes – a uniform contact pressure cannot be transferred correctly through the contact interface. This drawback can be omitted in the method of Lagrange multipliers using a “two-pass” technique [HAL 85], which means that at each solution step both surfaces serve as slave and master to create two layers of contact

⁹ Different possibilities of normal definition are presented in remark 3.2 in section 3.4.

elements. However, it may lead to locking problems [KIK 88]. A comprehensive discussion of the contact patch test for NTS discretizations can be found in [CRI 00c], where a new approach combining linear and quadratic shape functions is also suggested. Recently, a modification of the NTS discretization has been proposed for the PM [ZAV 09a], which also passes the patch test. Besides these drawbacks, the NTS discretization is simple and robust. Contact detection and resolution techniques that we present in this book are suitable for this discretization.

Another technique *contact domain method* proposed in [OLI 09, HAR 09] is based on a symmetric NTS discretization and contact element shape functions. The discretization of the contact interface is based on a full triangulation of the zone between contacting surfaces based on surface nodes (Figure 1.2(c)). This formulation seems to be rather stable and passes the patch test, but its three-dimensional implementation reported in [OLI 10] is not applicable for arbitrary discretizations of the contacting surfaces.

STS discretization (Figure 1.2(d)) has been first proposed by Simo *et al.* [SIM 85] for the two-dimensional case (see also [ZAV 98]). Recently such a discretization has been efficiently applied to two- and three-dimensional problems coupled with the *mortar method* for non-conforming meshes, inspired by the domain decomposition methods [WOH 01]. This technique is stable and passes the patch test but its implementation for a general case presents a great challenge, “a nightmare”, according to Tod A. Laursen, one of the authors of the mortar method’s implementation for both two- and three-dimensional structural and contact problems [PUS 03, PUS 04, YAN 05, YAN 08b, MCD 00]. A standalone discretization technique is needed for the *Nitsche* method [BEC 03, WRI 08], the Gauss points of one surface play the role of slave nodes. The comparison of Nitsche and mortar techniques can be found in [FRI 04].

The basic idea of the mortar method appeared in the second half of the 1980s and in the beginning of the 1990s for domain decomposition techniques between non-conforming subdomains, see, for example, [BER 90]. In 1998, Belgacem *et al.* [BEL 98] adapted the mortar method for the multibody (bilateral) frictionless contact problem. Further at the beginning of 2000s, the rigorous formulation adapted to the frictional contact problem subjected to large deformations/slidings has been established by McDevitt and Laursen [MCD 00]. The mortar method consists of either introducing an intermediate contact surface, where contact pressure is defined, or using one of the contacting surfaces as mortar surface, for details see [WRI 06]. The mortar-based formulation leads to a consistent formulation of the frictional contact problem for large sliding and large deformations. It allows us to pass the contact patch test for non-conforming meshes and, contrary to the NTS, does not suffer from locking and spurious penetrations.

1.2.3. *Contact resolution*

The rigorous formulation of a variational principle for contact problems results in a variational inequality subjected to geometrical constraints [KIK 88]. These constraints can be brought as additional terms in the weak form by means of PM, LMM or other optimization methods [BER 84, BER 03, LUE 03, BON 06], etc. Such an operation converts the constrained optimization problem, where constraints are given as inequalities, into an unconstrained or partly unconstrained problem. If we suppose the active contact zone to be known, then the variational inequality can be replaced by a variational equality, which finally results in an unconstrained problem written in a standard form of variational equality [WRI 06]. This problem can be treated as a standard nonlinear minimization problem by the means available in a finite element code. A solver for systems of linear equations and a method for the treatment of nonlinear problems are needed. Note that since the contact constraints are given as inequalities, a special attention has to be paid to the definition of the active contact zone. For PM and ALM, this task is trivial. For the LMM, an active set strategy should be used (e.g. see, [LUE 03]).

The resulting unconstrained minimization problem is not sufficiently smooth, which may result in a slow convergence or even divergence of the resolution scheme. The stability of the numerical scheme depends upon the discretization and on the solution parameters. The frictional contact renders a non-symmetric tangent matrix, which presents a problem for several solvers (like conjugate gradient method) and for the parallel treatment of the problem: the Schwarz theory for non-symmetric problems is less satisfactory than for positive definite symmetric problems [TOS 05]. The way out has been proposed in [LAU 92, LAU 93] for ALM with Uzawa's algorithm – governing equations of Coulomb's friction have been linearized by the operator splitting technique, first recognized in [GLO 89], where the entire problem is recast in two subproblems, that are solved once at each solution step. ALM and LMM derive non-symmetric tangent matrices only for the slip state. This is due to the non-associativity of Coulomb's friction law, that is slip occurs in the plane of the constant contact pressure. The PM suffers from a non-symmetry both in stick and slip states. A solution has been proposed in [WRI 06], which consists of a similar treatment of normal and tangential deviations from the stick state. Another approach yielding a symmetric tangent matrix in stick state has been proposed in [KON 05], based on a rigorous covariant description of the contact geometry. The same author proposed a symmetrization of different friction models based on the ALM coupled with Uzawa's algorithm [KON 07b].

