Chapter 1

Introduction to Damping Models and Analysis Methods

It is true that the grasping of truth is not possible without empirical basis. However, the deeper we penetrate and the more extensive and embracing our theories become, the less empirical knowledge is needed to determine those theories.

Albert Einstein, December 1952

Problems involving vibration occur in many areas of mechanical, civil and aerospace engineering: wave loading of offshore platforms, cabin noise in aircrafts, earthquake and wind loading of cable-stayed bridges and high-rise buildings, performance of machine tools – to pick only few random examples. Quite often vibration is not desirable and the interest lies in reducing it by dissipation of vibration energy or *damping*. Characterization of damping forces in a vibrating structure has long been an active area of research in structural dynamics. Since the publication of Lord Rayleigh's classic monograph *Theory of Sound* (1877), a large body of literature can now be found on damping. Although the topic of damping is an age old problem, the demands of modern engineering have led to a steady increase of interest in recent years. Studies of damping have a major role in vibration isolation in automobiles under random loading due to surface irregularities and buildings subjected to earthquake loadings. The developments in the fields of robotics and active structures have provided impetus toward developing procedures for dealing with general dissipative forces in the context of structural dynamics. Beside these, in the last few decades, the sophistication of modern design methods together with the development of improved composite structural materials instilled a trend toward lighter structures. At the same time, there is also a constant demand for larger structures, capable of carrying more loads at higher speeds with minimum noise and vibration level as the safety/workability and environmental criteria become more stringent. Examples include very large wind turbines, which are being used increasingly for superior energy generation. Unfortunately, these two demands are conflicting and the problem cannot be solved without proper understanding of energy dissipation or damping behavior. Recent advances in vibration energy harvesting [ERT 11, ELV 13] demand further understanding of damping [LES 04, ALI 13], as it is crucial for the quantification of harvested energy. It is the aim of this book to provide fundamental techniques for the analysis and identification of damped structural systems.

In spite of a large amount of research, understanding of damping mechanisms is quite basic compared to the other aspects of modeling. A major reason for this is that, in contrast to inertia and stiffness forces, it is not, in general, clear which *state variables* are relevant to determining the damping forces. Moreover, it seems that in a realistic situation it is often the structural joints [SEG 06] which are more responsible for energy dissipation than the (solid) material. There have been detailed studies on the material damping (see [BER 73]) and also on energy dissipation mechanisms in the joints [EAR 66, BEA 77]. But here, difficulty lies in representing all these tiny mechanisms in different parts of the structure in a unified manner. Even in many cases, these mechanisms turn out to be locally nonlinear, requiring an equivalent linearization technique for a global analysis [BAN 83]. A well-known method for getting rid of all these problems is to use the so-called "viscous damping". This approach was first introduced by Lord Rayleigh [RAY 77] via his famous "dissipation function", a quadratic expression for the energy dissipation rate with a symmetric matrix of coefficients, the "damping matrix". A further idealization, also pointed out by Rayleigh, is to assume the damping matrix to be a linear combination of the mass and stiffness matrices. Since its introduction, this model has been used extensively and is now usually known as "Rayleigh damping", "proportional damping" or "classical damping". With such a damping model, the *modal analysis* procedure, originally developed for undamped systems, can be used to analyze damped systems in a very similar manner.

In this chapter, we review some existing works on damping and set the scene for the book. This book focuses on the mathematical analysis and identification of damped linear dynamic systems. We also look mainly at discrete or discretized continuous systems. This can be done by employing conventional finite element approximation to the original boundary value problem. This aspect is not discussed here as there are many excellent books which the readers can refer to [BAT 76, DAW 84, ZIE 91, BAT 95, FRI 95b, PET 98, HUG 00, COO 01]. Therefore, for the purpose of this book, we consider that the mass and stiffness matrices are available so that we mainly focus on the damping aspects.

Different mathematical models of damping used in literature are discussed in section 1.1. Damping models used for single-degree-of-freedom (SDOF), multiple-degrees-of-freedom (MDOF) and continuous systems have been included. The concepts of viscous and non-viscous damping are introduced. A brief review of modal analysis method for viscously damped systems is given in section 1.2. The state-space method and approximate methods based on the configuration space are reviewed. Analysis methods for non-viscously damped systems are discussed in section 1.3. State-space-based methods, time-domain-based methods and approximate methods in the configuration space are discussed. After the review of different analysis methods, we move to review various damping identification methods.

The methods for identification of viscous damping is discussed in section 1.4. Both SDOF and MDOF systems are considered. In section 1.5, the identification methods for non-viscous damping model in linear dynamic systems have been reviewed. For successful modeling and model updating of a dynamic system, it is necessary to know how much the eigenvalues and eigenvectors are sensitive to the parameters [MOT 93, FRI 95b, FRI 01]. Therefore, different methods for computing parametric sensitivity of eigenvalues and eigenvectors are reviewed in section 1.6. Sensitivity of undraped, viscously damped and non-viscously damped systems is discussed. Based on the review of existing works, the motivation behind this book is explained in section 1.7. Finally, in section 1.8, the scope of the book is outlined. Here, a summary of the topics that are covered in the following chapters is given.

1.1. Models of damping

Damping is the dissipation of energy from a vibrating structure. In this context, the term "dissipate" is used to mean the transformation of energy from one form into another and, therefore, a removal of energy from the vibrating system. The type of energy into which the mechanical energy is transformed is dependent on the system and the physical mechanisms that cause the dissipation. For most vibrating systems, a significant part of the energy is converted into heat.

The specific ways in which energy is dissipated in vibration are dependent on the physical mechanisms active in the structure. These physical mechanisms are complicated physical processes that are not totally understood. The types of damping that are present in the structure will depend on which mechanisms predominate in the given situation. Thus, any mathematical representation of the physical damping mechanisms in the equation of motion of a vibrating system will have to be a generalization and approximation of the true physical situation. As Scanlan [SCA 70] has observed, any mathematical damping model is really only a crutch that does not give a detailed explanation of the underlying mechanisms. The majority of vibration books, for example [MEI 67, MEI 80, PAZ 80, NEW 89, BAT 95, MEI 97, PET 98, GÉR 97, INM 03, FRI 10b], consider the classical viscous damping model. However, there are some excellent books and papers which specifically focus on vibration damping in engineering structures. The books by Bland [BLA 60] and Lazan [LAZ 68] give a detailed account of earlier works on viscoelastic damping and damped systems. The book by Nashif *et al.* [NAS 85] covers various material damping models and their applications in the design and analysis of dynamic systems. A valuable reference on dynamics analysis of damped structures is the book by Sun and Lu [SUN 95]. The book by Beards [BEA 96] takes a pedagogical approach toward structural vibration of damped systems. The handbook by Jones [JON 01] focuses on viscoelastic damping and analysis of structures with such damping models. The important role of damping in the context of earthquake engineering has been analyzed in the book by Liang *et al.* [LIA 11]. The recent book by Veselic [VES 11] focuses on mathematical aspects of dynamics of MDOF damped systems. The paper by Gaul [GAU 99] gives a comprehensive overview of viscoelastically damped systems. The review paper by Mead [MEA 02] gives an overview of damping modeling structural dynamics. The two linked review papers by Vasques *et al.* [VAS 10a, VAS 10b] and the article by Vasques and Cardoso [VAS 11] discuss both mathematical aspects and experimental identification of linear dynamic systems with viscoelastic damping.

For our mathematical convenience, we divide the elements that dissipate energy into three classes: (1) damping in SDOF systems, (2) damping in continuous systems and (3) damping in MDOF systems. Elements such as dampers of a vehicle-suspension fall into the first class. Dissipation within a solid body, on the other hand, falls into the second class and demands a representation that accounts for both its intrinsic properties and its spatial distribution. Damping models for MDOF systems can be obtained by discretization of the equation of motion. There have been attempts to mathematically describe the damping in SDOF, continuous and MDOF systems.

1.1.1. *Single-degree-of-freedom systems*

Free oscillation of an undamped SDOF system never dies out and the simplest approach to introduce dissipation is to incorporate an ideal viscous dashpot in the model. The damping force (F_d) is assumed to be proportional to the instantaneous velocity, that is

$$F_d(t) = c\,\dot{q}(t) \qquad [1.1]$$

and the coefficient of proportionality c is known as the dashpot constant or viscous damping constant. The loss factor, which is the energy dissipation per radian to the peak potential energy in the cycle, is widely accepted as a basic measure of the damping. For an SDOF system, this loss factor can be given by

$$\eta = \frac{c|\omega|}{k} \tag{1.2}$$

where k is the stiffness. The expression similar to this equation has been discussed in Ungar and Kerwin [UNG 62] in the context of viscoelastic systems. Equation [1.2] shows a linear dependence of the loss factor on the driving frequency. This dependence has been discussed by Crandall [CRA 70] where it has been pointed out that the frequency dependence, observed in practice, is usually not of this form. In such cases, we often resort to an equivalent ideal dashpot. Theoretical objections to the approximately constant value of damping over a range of frequency, as observed in aeroelasticity problems, have been raised by Naylor [NAY 70]. On the lines of equation [1.2], we are tempted to define the frequency-dependent dashpot as

$$c(\omega) = \frac{k\eta(\omega)}{|\omega|}. \tag{1.3}$$

This representation, however, has some serious physical limitations. In references [CRA 70, CRA 91, NEW 89, SCA 70], it has been pointed out that such a representation violates causality, a principle which asserts that the states of a system at a given point of time can be affected only by the events in the past and not by those of the future.

Now for the SDOF system, the frequency-domain description of the equation of motion can be given by

$$\left[-m\omega^2 + \mathrm{i}\omega c(\omega) + k\right] \bar{q}(\mathrm{i}\omega) = \bar{f}(\mathrm{i}\omega) \tag{1.4}$$

where $\bar{q}(\mathrm{i}\omega)$ and $\bar{f}(\mathrm{i}\omega)$ are the response and excitation, respectively, represented in the frequency domain. Note that the dashpot is now allowed to have frequency dependence. Inserting equation [1.3] into [1.4], we obtain

$$\left[-m\omega^2 + k\left\{1 + \mathrm{i}\eta(\omega)\mathrm{sgn}(\omega)\right\}\right] \bar{q}(\mathrm{i}\omega) = \bar{f}(\mathrm{i}\omega) \tag{1.5}$$

where $\mathrm{sgn}(\bullet)$ represents the sign function. The "time-domain" representations of equations [1.4] and [1.5] are often taken as

$$m\ddot{q} + c(\omega)\dot{q} + kq = f \tag{1.6}$$

and

$$m\ddot{q} + kq\left\{1 + \mathrm{i}\eta(\omega)\mathrm{sgn}(\omega)\right\} = f \qquad [1.7]$$

respectively. It has been pointed out by Crandall [CRA 70] that these are not the correct Fourier inverses of equations [1.4] and [1.5]. The reason is that the inertia, the stiffness and the forcing function are inverted properly, while the damping terms in equations [1.6] and [1.7] are obtained by mixing the frequency-domain and time-domain operations. Crandall [CRA 70] calls [1.6] and [1.7] the "non-equations" in the time domain. It has been pointed out by Newland [NEW 89] that only certain forms of frequency dependence for $\eta(\omega)$ are allowed in order to satisfy causality. Crandall [CRA 70] has shown that the impulse response function for the ideal hysteretic dashpot (η independent of frequency) is given by

$$h(t) = \frac{1}{\pi k \eta_0}.\frac{1}{t}, \quad -\infty < t < \infty. \qquad [1.8]$$

This response function is clearly non-causal because it states that the system responds before the excitation (or the cause) takes place. This non-physical behavior of the hysteretic damping model is a flaw, and further attempts have been made to address this problem. Bishop and Price [BIS 86] introduced the band-limited hysteretic damper and suggested that it might satisfy the causality requirement. However, Crandall [CRA 91] has further shown that the band-limited hysteretic dashpot is also non-causal. In view of this discussion, it can be said that most of the hysteretic damping models fail to satisfy the casualty condition. Based on the analyticity of the transfer function, Makris [MAK 99] has shown that for causal hysteretic damping the real and imaginary parts of the dynamic stiffness matrix must form a Hilbert transform pair. The Hilbert transform relation is also known as the Kramers–Kronig result. He has shown that the causal hysteretic damping model is the limiting case of a linear viscoelastic model with nearly frequency-independent dissipation that was proposed by Biot [BIO 58]. It was also shown that there is a continuous transition from the linear viscoelastic model to the ideally hysteretic damping model. More recently, Chen and Zhang [CHE 08] showed that ideal linear hysteretic damper possesses a non-causal impulse response precursor.

The physical mechanisms of damping, including various types of external friction, fluid viscosity and internal material friction, have been studied rather extensively in some detail and are complicated physical phenomena. However, a certain simplified mathematical formulation of damping forces and energy dissipation can be associated with a class of physical phenomena. Coulomb damping, for example, is used to represent dry friction present in sliding surfaces, such as structural joints. For this kind of damping, the force resisting the motion is assumed to be proportional to the normal force between the sliding surfaces and independent of the velocity except for the sign. The damping force is thus

$$F_d = \frac{\dot{q}}{|\dot{q}|} F_r = \mathrm{sgn}(\dot{q}) F_r \qquad [1.9]$$

where F_r is the frictional force. In the context of finding an equivalent viscous damping, Bandstra [BAN 83] has reported several mathematical models of physical damping mechanisms in SDOF systems. For example, velocity-squared damping, which is present when a mass vibrates in a fluid or when fluid is forced rapidly through an orifice. The damping force in this case is

$$F_d = \mathrm{sgn}(\dot{q}) a \dot{q}^2; \quad \text{or, more generally} \quad F_d = c\dot{q}|\dot{q}|^{n-1} \qquad [1.10]$$

where c is the damping proportionality constant. Viscous damping is a special case of this type of damping. If the fluid flow is relatively slow, i.e. laminar, then by letting $n = 1$ the above equation reduces to the case of viscous damping [1.1].

In the context of viscoelastically damped SDOF systems, there are several studies which analyze the dynamics in detail. Free and forced vibration of viscoelastic systems were considered in [MUR 98b, MUR 98a]. Muller [MUL 05] and Adhikari [ADH 05] considered the conditions of oscillatory motion for a viscoelastically damped SDOF system. Sieber *et al.* [SIE 08] considered exponential non-viscous damping with weak nonlinearities in a Duffing oscillator. Equation of motion of such a system can be given by

$$m\frac{\mathrm{d}^2 q}{\mathrm{d}\hat{t}^2} + c\int_{\hat{\tau}=0}^{\hat{\tau}=\hat{t}} \mu \mathrm{e}^{-\mu(\hat{t}-\hat{\tau})} \frac{\mathrm{d}q}{\mathrm{d}\hat{\tau}} \mathrm{d}\hat{\tau} + kq + \alpha k q^3 = A\cos(\Omega \hat{t}\,). \qquad [1.11]$$

Both hardening and softening type of nonlinearities were considered and the stability of the system was discussed. In [ADH 08], the dynamic response characteristics of a non-viscously damped oscillator was discussed in detail. Genta and Amati [GEN 10] considered dynamics of non-viscously damped SDOF system and proposed a general state-space approach. In [ADH 09a], some approximate methods (non-state-space approach) for the calculation of eigenvalues of non-viscously damped SDOF system were proposed. Palmeri and Muscolino [PAL 11b] proposed a Laguerre's polynomial approximation (LPA) technique for time-domain analysis of an oscillator with the generalized Maxwell's model. Some of the techniques can be extended to continuous and MDOF systems as discussed next.

1.1.2. *Continuous systems*

Construction of damping models becomes more difficult for continuous systems. Inman [INM 89] applied the GHM (Golla, Hughes and McTavish) approach to simple beams and used the separation of variables approach in conjunction with modal analysis. Banks and Inman [BAN 91] have considered four different damping models for a composite beam. These models of damping are:

1) *Viscous air damping:* For this model, the damping operator in the Euler–Bernoulli equation for beam vibration becomes

$$L_1 = \gamma \frac{\partial}{\partial t} \tag{1.12}$$

where γ is the viscous damping constant.

2) *Kelvin–Voigt damping:* For this model, the damping operator becomes

$$L_1 = c_d I \frac{\partial^5}{\partial x^4 \partial t} \tag{1.13}$$

where I is the moment of inertia and c_d is the strain-rate dependent damping coefficient. A similar damping model was also used in [MAN 98, ADH 99c] in the context of randomly parametered Euler–Bernoulli beams.

3) *Time hysteresis damping:* For this model, the damping operator is assumed as

$$L_1 = \int_{-\infty}^{t} g(\tau) q_{xx}(x, t+\tau) \mathrm{d}\tau \quad \text{where } g(\tau) = \frac{\alpha}{\sqrt{-\tau}} \exp(\beta\tau) \tag{1.14}$$

where α and β are constants. Later, this model will be discussed in detail.

4) *Spatial hysteresis damping:*

$$L_1 = \frac{\partial}{\partial x} \left[\int_0^L h(x,\xi) \{ q_{xx}(x,t) - q_{xt}(\xi,t) \} \mathrm{d}\xi \right] \tag{1.15}$$

The kernel function $h(x,\xi)$ is defined as

$$h(x,\xi) = \frac{a}{b\sqrt{\pi}} \exp[-(x-\xi)^2/2b^2] \tag{1.16}$$

where b is some constant.

They observed that the spatial hysteresis model combined with a viscous air damping model results in the best quantitative agreement with the experimental time histories. Again, in the context of Euler–Bernoulli beams, Bandstra [BAN 83] has considered two damping models where the damping term is assumed to be of the forms $\{\operatorname{sgn} q_t(x,t)\}\, b_1 q^2(x,t)$ and $\{\operatorname{sgn} q_t(x,t)\}\, b_2 |q(x,t)|$.

Lesieutre [LES 92] considered the dynamics of uniaxial rods with frequency-dependent material properties. Lei *et al.* [LEI 06] proposed a Galerkin method for dynamics of beam with distributed non-viscous damping. Friswell *et al.* [FRI 07a, FRI 07b] considered dynamics of Euler–Bernoulli beams with non-local and non-viscous damping. They considered the following integro-partial-differential equation as the equation of motion for the beam

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 q(x,t)}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 q(x,t)}{\partial t^2} + Q_N(x,t) = f\,(x,t) \qquad [1.17]$$

with the damping force $Q_N\,(x,t)$ given by

$$Q_N\,(x,t) = \frac{\partial^2}{\partial x^2}\left(\int_0^L \int_{-\infty}^t C\,(x,\xi,t-\tau)\,\frac{\partial^2 \dot{q}(\xi,\tau)}{\partial \xi^2} d\tau d\xi\right) \qquad [1.18]$$

Dynamic analysis of beams with general non-local and non-viscous damping [1.18] has been considered by several authors [ADH 07b, DI 13, FAI 13, GON 12, DI 11, CHE 11, TSA 09, CHI 09, PAN 13, POT 13, ABU 12]. Yuksel and Dalli [YUK 05] considered longitudinally vibrating elastic rods with locally and non-locally reacting viscous dampers. Cortes and Elejabarrieta [COR 06b] considered longitudinal vibration of a rod with exponential non-viscous damping model. They obtained expressions for complex natural frequencies and mode shapes. Damped vibration of spatially curved one-dimensional structures was considered by Otrin and Boltezar [OTR 07b, OTR 07a, OTR 09a]. Xue-Chuan *et al.* [XUE 08] studied axial vibration of non-local viscoelastic Kelvin bars in tension. Cortes *et al.* [COR 08] proposed a frequency-domain approach for the axial vibration problem of a uniform elastic rod with a viscoelastic end damper. They derived an analytical solution for the frequency response functions (FRFs). Calim [CAL 09] analyzed the dynamic behavior of Timoshenko beams on Pasternak-type viscoelastic foundation subjected to time-dependent loads. A Galerkin-type state-space approach for transverse vibrations of slender double-beam systems with viscoelastic inner layer was proposed by Palmeri and Adhikari [PAL 11a]. Calim and Akkurt [CAL 11] considered free vibration analysis of straight and circular Timoshenko beams on elastic foundation. Garcia-Barruetabena *et al.* [GAR 12] proposed both an analytical solution and finite element approach for axial vibration of rods with exponential non-viscous damping. A rotating Timoshenko beam with Maxwell–Weichert viscoelastic damping model is used in [SKA 12]. Lei *et al.* [LEI 13b, LEI 13a] studied free vibration of non-local Euler–Bernoulli and Timoshenko beams with non-viscous damping. A generalized one-dimensional elastoplastic model based on fractional calculus is presented in [MEN 12a]. Recently, Wang and Inman [WAN 13] considered GHM and anelastic displacement field (ADF) models of viscoelastic damping for the dynamics of constrained layer sandwich beams. A finite element approach and an experimental validation have been reported by the authors.

1.1.3. *Multiple-degrees-of-freedom systems*

The most popular approach to model damping in the context of MDOF systems is to assume viscous damping. This approach was first introduced by Lord Rayleigh [RAY 77]. By analogy with the potential energy and the kinetic energy, Rayleigh assumed the *dissipation function*, given by

$$\mathcal{F}(\mathbf{q}) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{jk}\dot{q}_j\dot{q}_k = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{C}\dot{\mathbf{q}}. \qquad [1.19]$$

In the above expression, $\mathbf{C} \in \mathbb{R}^{N\times N}$ is a non-negative definite symmetric matrix known as the viscous damping matrix. It should be noted that not all forms of the viscous damping matrix can be handled within the scope of classical modal analysis. Based on the solution method, viscous damping matrices can be further divided into classical and non-classical dampings. Further discussions on viscous damping will follow in section 1.2.

It is important to avoid the widespread misconception that viscous damping is the *only* linear model of vibration damping in the context of MDOF systems. Any causal model which makes the energy dissipation functional non-negative is a possible candidate for a damping model. There have been several efforts to incorporate non-viscous damping models in MDOF systems. References [BAG 83, TOR 87, GAU 91, MAI 98] considered damping modeling in terms of fractional derivatives of the displacements. Following Maia *et al.* [MAI 98], the damping force using such models can be expressed by

$$\mathbf{F}_d = \sum_{j=1}^{l} \mathbf{g}_j D^{\nu_j}[\mathbf{q}(t)]. \qquad [1.20]$$

Here, $\mathbf{g}_j$ are complex constant matrices and the fractional derivative operator

$$D^{\nu_j}[\mathbf{q}(t)] = \frac{\mathrm{d}^{\nu_j}\mathbf{q}(t)}{\mathrm{d}t^{\nu_j}} = \frac{1}{\Gamma(1-\nu_j)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\mathbf{q}(t)}{(t-\tau)^{\nu_j}}\mathrm{d}\tau \qquad [1.21]$$

where ν_j is a fraction and $\Gamma(\bullet)$ is the gamma function. The familiar viscous damping appears as a special case when $\nu_j = 1$. We refer the readers to the review papers

[SLA 93, ROS 97, GAU 99] for further discussions on this topic. The physical justification for such models, however, may not always be very clear for engineering problems.

Possibly, the most general way to model damping within the linear range is to consider non-viscous damping models which depend on the past history of motion via convolution integrals over some kernel functions. A *modified dissipation function* for such damping model can be defined as

$$\mathcal{F}(\mathbf{q}) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\dot{q}_k \int_0^t \mathcal{G}_{jk}(t-\tau)\dot{q}_j(\tau)\mathrm{d}\tau = \frac{1}{2}\dot{\mathbf{q}}^T\int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{q}}(\tau)\mathrm{d}\tau. \quad [1.22]$$

Here, $\boldsymbol{\mathcal{G}}(t) \in \mathbb{R}^{N\times N}$ is a symmetric matrix of the damping kernel functions, $\mathcal{G}_{jk}(t)$. The kernel functions, or others closely related to them, are described under many different names in the literature of different subjects: for example, retardation functions, heredity functions, after-effect functions and relaxation functions. In the special case when $\boldsymbol{\mathcal{G}}(t-\tau) = \mathbf{C}\,\delta(t-\tau)$, where $\delta(t)$ is the Dirac-delta function, equation [1.22] reduces to the case of viscous damping as in equation [1.19]. The damping model of this kind is a further generalization of the familiar viscous damping. By choosing suitable kernel functions, it can also be shown that the fractional derivative model discussed before is also a special case of this damping model. Thus, as pointed by Woodhouse [WOO 98], this damping model is the most general damping model within the scope of a linear analysis.

Golla and Hughes [GOL 85] and McTavish and Hughes [MCT 93] have effectively used damping model of the form [1.22] in the context of viscoelastic structures. The damping kernel functions are commonly defined in the frequency/Laplace domain. Conditions which $\mathbf{G}(s)$, the Laplace transform of $\boldsymbol{\mathcal{G}}(t)$, must satisfy in order to produce dissipative motion were given by Golla and Hughes [GOL 85]. The approach pioneered by Lesieutre [LES 90, LES 92, LES 95, LES 96b, LES 96a] used a first-order state-space method called the anelastic displacement fields (ADF) method. A selection of different damping models proposed in literature is summarized in Table 1.1. Adhikari and Woodhouse [ADH 03b] proposed four indices to quantify non-viscous damping when the kernel function can have any form as given in Table 1.1.

1.1.4. *Other studies*

Another major source of damping in a vibrating structure is the structural joints, see [TAN 97] for a recent review. Here, a major part of the energy loss takes place through air-pumping. The air-pumping phenomenon is associated with damping when air is entrapped in pockets in the vicinity of a vibrating surface. In these situations, the entrapped air is "squeezed out" and "sucked in" through any available hole. Dissipation of energy takes place in the process of air flow and Coulomb-friction dominates around the joints. This damping behavior has been studied by many authors in some practical situations, for example by Cremer and Heckl [CRE 73]. Earls [EAR 66] has obtained the energy dissipation in a lap joint over a cycle under different clamping pressure. Beards and Williams [BEA 77] have noted that significant damping can be obtained by suitably choosing the fastening pressure at the interfacial slip in joints.

Model number	Damping functions	Author and reference
1	$G(s) = \sum_{k=1}^{n} \frac{a_k s}{s+b_k}$	Biot [BIO 55], [BIO 58]
2	$G(s) = as \int_0^\infty \frac{\gamma(\rho)}{s+\rho} \mathrm{d}\rho$ $\gamma(\rho) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha \leq \gamma \leq \beta \\ 0 & \text{otherwise} \end{cases}$	Buhariwala [BUH 82]
3	$G(s) = \frac{E_1 s^\alpha - E_0 b s^\beta}{1+bs^\beta}$ $0 < \alpha < 1, \quad 0 < \beta < 1$	Bagley and Torvik [BAG 83]
4	$sG(s) = G^\infty \left[1 + \sum_k \alpha_k \frac{s^2+2\xi_k \omega_k s}{s^2+2\xi_k \omega_k s + \omega_k^2}\right]$	Golla and Hughes [GOL 85] and McTavish and Hughes [MCT 93]
5	$G(s) = 1 + \sum_{k=1}^{n} \frac{\Delta_k s}{s+\beta_k}$	Lesieutre and Mingori [LES 90]
6	$G(s) = c \frac{1-e^{-st_0}}{st_0}$	Adhikari [ADH 98]
7	$G(s) = c \frac{1+2(st_0/\pi)^2 - e^{-st_0}}{1+2(st_0/\pi)^2}$	Adhikari [ADH 98]
8	$G(s) = c\, e^{s^2/4\mu} \left[1 - \mathrm{erf}\left(\frac{s}{2\sqrt{\mu}}\right)\right]$	Adhikari and Woodhouse [ADH 01c]

Table 1.1. *Summary of damping functions in the Laplace domain*

Energy dissipation within the material is attributed to a variety of mechanisms, such as thermoelasticity, grain-boundary viscosity and point-defect relaxation, see [LAZ 59, LAZ 68, BER 73]. Such effects are in general called material damping. In an imperfect elastic material, the stress-strain curve forms a closed hysteresis loop rather than a single line upon a cyclic loading. Much effort has been made by numerous investigators to develop models of hysteretic restoring forces and techniques to identify such systems. For a recent review on this literature, we refer the readers to [CHA 98]. Most of these studies are motivated by the observed fact that the energy dissipation from materials is only a weak function of frequency and almost directly proportional to q^n. The exponent on displacement for the energy dissipation of material damping ranges from 2 to 3, for example 2.3 for mild steel [BAN 83]. In this context, another large body of literature can be found on composite materials where many researchers have evaluated a material's specific damping capacity (SDC). Baburaj and Matsukai [BAB 94] and the references therein give an account of research that has been conducted in this area.

1.2. Modal analysis of viscously damped systems

Equation of motion of a viscously damped system can be obtained from the Lagrange's equation (see, for example, [MEI 67, GÉR 97, MEI 97] for further details). Using the Rayleigh's dissipation function given by [1.19], the damped forces can be obtained as

$$Q_{\mathrm{nc}_k} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_k}, \quad k = 1, \cdots, N \qquad [1.23]$$

and, consequently, the equation of motion can expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t). \qquad [1.24]$$

The aim is to solve this equation, together with the initial conditions, by modal analysis (which is described in detail in section 2.3.1). Using the modal transformation in [2.69], premultiplying equation [1.24] by the transpose of the modal matrix $\mathbf{X}^T$ and using the mode orthogonality relationships in [2.65] and [2.66], equation of motion of a damped system in the modal coordinates may be obtained as

$$\ddot{\mathbf{y}}(t) + \mathbf{X}^T\mathbf{C}\mathbf{X}\dot{\mathbf{y}}(t) + \mathbf{\Omega}^2\mathbf{y}(t) = \tilde{\mathbf{f}}(t). \qquad [1.25]$$

Clearly, unless $\mathbf{X}^T\mathbf{C}\mathbf{X}$ is a diagonal matrix, no advantage can be gained by employing modal analysis because the equations of motion will still be coupled. To solve this problem, it is common to assume *proportional damping*, that is $\mathbf{C}$ is simultaneously diagonalizable with $\mathbf{M}$ and $\mathbf{K}$. Such damping models allows us to analyze damped systems in very much the same manner as undamped systems. Later, Caughey and O'Kelly [CAU 65] have derived the condition which the system matrices must satisfy so that viscously damped linear systems possess classical normal modes. Adhikari [ADH 06a] introduced the concept of generalized proportional damping by which the damping matrix can be expressed as matrix functions of mass and stiffness matrices. This can significantly help in identifying the damping matrix from measured damping factors for multiple modes [ADH 09b, PAP 12]. Several authors have used a proportional damping modeling approach in wide-ranging applications [BIL 06, SUL 13, CAR 11, SUL 10, OTR 09b, LIN 09]. Phani [PHA 03] discussed the necessary and sufficient conditions for the existence of classical normal modes in damped linear dynamic systems. Recently, Chang [CHA 13] investigated the performance of proportional damping in the context of nonlinear MDOF systems. In Chapter 2, the concept of proportional damping or classical damping will be analyzed in more detail.

Modes of proportionally damped systems preserve the simplicity of the real normal modes as in the undamped case. Unfortunately, there is no physical reason why a general system should behave like this. In fact, practical experience in modal testing shows that most real-life structures do not do so, as they possess complex modes instead of real normal modes. This implies that, in general, linear systems are non-classically damped. When the system is non-classically damped, some or all of the N differential equations in [1.25] are coupled through the $\mathbf{X}^T\mathbf{C}\mathbf{X}$ term and cannot be reduced to N second-order uncoupled equations. This coupling brings several complication in the system dynamics – the eigenvalues and the eigenvectors no longer remain real and also the eigenvectors do not satisfy the classical orthogonality relationships. The methods for solving this kind of problem follow mainly two routes: the state-space method and the configuration space method or configuration space. A brief discussion of these two approaches is taken up in the following sections.

1.2.1. *The state-space method*

The state-space method is based on transforming the N second-order coupled equations into a set of $2N$ first-order coupled equations by augmenting the displacement response vectors with the velocities of the corresponding coordinates, see [NEW 89]. Equation [1.24] can be recast as

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{r}(t) \tag{1.26}$$

where $\mathbf{A} \in \mathbb{R}^{2N\times 2N}$ is the system matrix, $\mathbf{r}(t) \in \mathbb{R}^{2N}$ is the force vector and $\mathbf{u}(t) \in \mathbb{R}^{2N}$ is the response vector in the state-space given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_N & \mathbf{I}_N \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \quad \mathbf{u}(t) = \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix}, \quad \text{and} \quad \mathbf{r}(t) = \begin{Bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{f}(t). \end{Bmatrix} \tag{1.27}$$

In the above equation, $\mathbf{O}_N$ is the $N \times N$ null matrix and $\mathbf{I}_N$ is the $N \times N$ identity matrix. The eigenvalue problem associated with the above equation is an asymmetric matrix now. Uncoupling of equations in the state-space is again possible and has been considered by many authors, for example [MEI 80, NEW 89, VEL 86]. This analysis was further generalized by Newland [NEW 87] for the case of systems involving singular matrices. In the formulation of equation [1.26], the matrix $\mathbf{A}$ is no longer symmetric, and so eigenvectors are no longer orthogonal with respect to it. In fact, in this case, instead of an orthogonality relationship, we obtain a biorthogonality relationship, after solving the adjoint eigenvalue problem. The complete procedure for uncoupling the equations now involves solving two eigenvalue problems, each of which is double the size of an eigenvalue problem in the modal space. The details of the relevant algebra can be found in [MEI 80, MEI 97]. It should be noted that these solution procedures are exact in nature. One disadvantage of such an exact method is that it requires significant numerical effort to determine the eigensolutions. The effort required is evidently intensified by the fact that the eigensolutions of a non-classically damped system are complex. From the analyst's viewpoint, another disadvantage is the lack of physical insight afforded by this method which is intrinsically numerical in nature.

Another variation of the state-space method available in the literature is through the use of "Duncan form". This approach was introduced in [FOS 58], and later several authors, for example [BÉL 77, NEL 79, VIG 86, SUA 87, SUA 89, SES 94, REN 97, ELB 09], have used this approach to solve a wide range of interesting problems. The advantage of this approach is that the system matrices in the state-space retain symmetry as in the configuration space.

1.2.2. *Methods in the configuration space*

It has been pointed out that the state-space approach toward the solution of equation of motion in the context of linear structural dynamics is not only computationally expensive but also fails to provide the physical insight which modal analysis in configuration space or configuration space offers. The eigenvalue problem associated with equation [1.24] can be represented by the λ-matrix problem [LAN 66]

$$s_j^2\mathbf{M}\mathbf{z}_j + s_j\mathbf{C}\mathbf{z}_j + \mathbf{K}\mathbf{z}_j = \mathbf{0} \qquad [1.28]$$

where $s_j \in \mathbb{C}$ is the jth latent root (eigenvalue) and $\mathbf{z}_j \in \mathbb{C}^N$ is the jth latent vector (eigenvector). The eigenvalues s_j are the roots of the characteristic polynomial

$$\det\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right] = 0. \qquad [1.29]$$

The order of the polynomial is $2N$ and the roots appear in complex conjugate pairs. Several authors have studied non-classically damped linear systems using approximate methods. In this section, we briefly review the existing methods for this kind of analysis.

1.2.2.1. *Approximate decoupling method*

Consider the equation of motion of a general viscously damped system in the modal coordinates given by [1.25]. Earlier, it was mentioned that due to the non-classical nature of damping, this set of N differential equations is coupled through the $\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X}$ term. A usual approach in this case is simply to ignore the off-diagonal terms of the modal damping matrix $\mathbf{C}'$ which couple the equation of motion. This approach is termed the "decoupling approximation". For large-scale systems, the computational effort in adopting the decoupling approximation is an order of magnitude smaller than the methods of complex modes. The solution of the decoupled equation would be close to the exact solution of the coupled equations if the non-classical damping terms are sufficiently small. Analysis of this question goes back to Rayleigh [RAY 77]. A preliminary discussion on this topic can be found in [MEI 67, MEI 97]. Thomson *et al.* [THO 74] have studied the effect of neglecting off-diagonal entries of the modal damping matrix through numerical experiments and have proposed a method for improved accuracy. Warburton and Soni [WAR 77] have suggested a criterion for such a diagonalization so that the computed response is acceptable. Using the frequency-domain approach, Hasselsman [HAS 76] proposed a criterion for determining whether the equations of motion might be considered practically decoupled if non-classical damping exists. The criterion suggested by him was to have adequate frequency separation between the natural modes.

Using matrix norms, Shahruz and Ma [SHA 88] have tried to find an optimal diagonal matrix $\mathbf{C}_d$ in place of $\mathbf{C}'$. An important conclusion emerging from their study is that if $\mathbf{C}'$ is diagonally dominant, then among all approximating diagonal matrices $\mathbf{C}_d$, the one that minimizes the error bound is simply the diagonal matrix obtained by omitting the off-diagonal elements of $\mathbf{C}'$. Using a time-domain analysis, Shahruz [SHA 90] has rigorously proved that if $\mathbf{C}_d$ is obtained from $\mathbf{C}'$ by neglecting the off-diagonal elements of $\mathbf{C}'$, then the error in the solution of the approximately decoupled system will be small as long as the off-diagonal elements of $\mathbf{C}'$ are not too large. Udwadia [UDW 09] proved that, for systems with non-repeated eigenvalues, the best approximation of a diagonal modal damping matrix is simply to consider the diagonal of the $\mathbf{C}'$ matrix. Mentrasti [MEN 12b] considered complex modal analysis for a proportionally damped structure equipped with linear non-proportionally damped viscous elements.

Ibrahimbegovic and Wilson [IBR 89] have developed a procedure for analyzing non-proportionally damped systems using a subspace with a vector basis generated from the mass and stiffness matrices. Their approach avoids the use of complex eigensolutions. An iterative approach for solving the coupled equations is developed in [UDW 90] based on updating the forcing term appropriately. Felszeghy [FEL 93] presented a method that searches for another coordinate system in the neighborhood of the normal coordinate system so that in the new coordinate system removal of coupling terms in the equations of motion produces a minimum bound on the relative error introduced in the approximate solution. Hwang and Ma [HWA 93] have shown that the error due to the decoupling approximation can be decomposed into an infinite series and can be summed exactly in the Laplace domain. They also concluded that by solving a small number of additional coupled equations in an iterative manner, the accuracy of the approximate solution can be greatly improved. Felszeghy [FEL 94] developed a formulation based on biorthonormal eigenvectors for modal analysis of non-classically damped discrete systems. The analytical procedure takes advantage of the simplification that arises when the modal analysis of the motion separated into a classical and non-classical modal vector expansion.

From the above-mentioned studies, it has often been believed that either frequency separation between the normal modes [HAS 76], often known as "Hasselsman's criteria", or some form of diagonal dominance [SHA 88], in the modal damping matrix $\mathbf{C}'$, is sufficient for neglecting modal coupling. In contrast to these widely accepted beliefs, references [PAR 92a, PAR 92b, PAR 94] have shown, using Laplace transform methods, that within the practical range of engineering applications neither the diagonal dominance of the modal damping matrix nor the frequency separation between the normal modes would be sufficient for neglecting modal coupling. They have also given examples when the effect of modal coupling may even increase following the previous criterion. Phani and Adhikari [PHA 08] proposed three Rayleigh quotients for non-proportionally damped systems based on approximate complex modes. It was shown that the stationarity can only be obtained when the modal damping matrix is diagonally dominant.

In the context of approximate decoupling, Shahruz and Srimatsya [SHA 97] considered error vectors in modal and physical coordinates, say denoted by $\mathbf{e_N}(\bullet)$ and $\mathbf{e_P}(\bullet)$, respectively. They have shown that based on the norm (denoted here by $\| (\bullet) \|$) of these error vectors, three cases may arise:

1) $\| \mathbf{e_N}(\bullet) \|$ is small (respectively, large) and $\| \mathbf{e_P}(\bullet) \|$ is small (respectively, large);

2) $\| \mathbf{e_N}(\bullet) \|$ is large but $\| \mathbf{e_P}(\bullet) \|$ is small;

3) $\| \mathbf{e_N}(\bullet) \|$ is small but $\| \mathbf{e_P}(\bullet) \|$ is large.

From this study, especially in view of case (3), it is clear that the error norms based on the modal coordinates are not reliable to use in the actual physical coordinates. However, they have given conditions when $\| \mathbf{e_N}(\bullet) \|$ will lead to a reliable estimate of $\| \mathbf{e_P}(\bullet) \|$. For a flexible structure with light damping, it was shown [GAW 97] that neglecting off-diagonal terms of the modal damping matrix in most practical cases imposes negligible errors in the system dynamics. They also concluded that the requirement of diagonal dominance of the damping matrix is not necessary in the case of small damping, which relaxes the criterion earlier given by [SHA 88].

In order to quantify the extent of non-proportionality, several authors have proposed "non-proportionality indices". References [PAR 86, NAI 86] have developed several indices based on modal phase difference, modal polygon areas, relative magnitude of coupling terms in the modal damping matrix, system response and Nyquist plot. Based on the idea related to the modal polygon area, Bhaskar [BHA 99] has proposed two more indices of non-proportionality. Another index based on driving frequency and elements of the modal damping matrix is given in [BEL 90]. Bhaskar [BHA 95] has proposed a non-proportionality index based on the error introduced by ignoring the coupling terms in the modal damping matrix. An analytical index for the quantification of non-proportionality for discrete vibrating systems was developed in [TON 92, TON 94]. It has been shown that the fundamental nature of non-proportionality lies in finer decompositions of the damping matrix. Shahruz [SHA 95] has shown that the analytical index given by [TON 94] solely based on the damping matrix may lead to erroneous results when the driving frequency lies close to a system's natural frequency. He has suggested that a suitable index for non-proportionality should include the damping matrix and natural frequencies as well as the excitation vector. Prells and Friswell [PRE 00] have shown that the (complex) modal matrix of a non-proportionally damped system depends on an orthonormal matrix, which represents the phase between different degrees-of-freedom of the system. For proportionally damped systems, this matrix becomes an identity matrix and, consequently, they have used this orthonormal matrix as an indicator of non-proportionality. Three indices to measure the damping non-proportionality were proposed in [LIU 00]. The first index measures the correlation between the real and imaginary parts of the complex modes; the second index measures the magnitude of the imaginary parts of the complex modes; and the third index quantifies the degree of modal coupling. These indices are based on the fact that the complex modal matrix can be decomposed into a product of a real and complex matrix. Adhikari [ADH 04a] proposed the optimal normalization of complex modes and suggested a mode-by-mode non-proportionality index. Koruk and Sanliturk [KOR 13] quantified mode shape complexity based on conservation of energy principle when a structure is vibrating at a specific mode during a period of vibration.

In an another line of work, some researchers aimed at diagonalizing a linear dynamic system *exactly* using real transformations even when it is non-proportionally damped. Works by Garvey *et al.* [GAR 02b, GAR 02a, GAR 04, ABU 09, PRE 09, TIS 11] proposed the structure preserving transformation and for viscously damped systems and extended the idea to more general linear dynamical systems. In a series of papers, Ma *et al.* [KAW 11, MOR 11a, MOR 11b, MA 10, MA 09, MOR 09, MOR 08b, MOR 08a, MA 04] considered the possibility of decoupling the equation of motion using real modes. They showed that it is possible to diagonalize the **M**, **C**, **K** system using a real transformation even when these matrices are general in nature (i.e. not proportionally damped). These works have the potential to rethink the concept of proportional damping in linear dynamic systems in a new light.

1.2.2.2. *Complex modal analysis*

Other than the approximate decoupling methods, another approach toward the analysis of non-proportionally damped linear systems is to use complex modes. Since the original contribution of Caughey and O'Kelly [CAU 65], many papers have been written on complex modes. Several authors, for example [MIT 90, IMR 95, LAL 95], have given reviews on this subject. Placidi *et al.* [PLA 91] have used a series expansion of complex eigenvectors into the subspace of real modes, in order to identify normal modes from complex eigensolutions. In the context of modal analysis, Liang *et al.* [LIA 92] have proposed and analyzed the question of whether the existence of complex modes is an indicator of non-proportional damping and how a mode is influenced by damping. Analyzing the errors in the use of modal coordinates, references [SES 94, IBR 95] have concluded that the complex mode shapes are not necessarily the result of high damping. The complexity of the mode shapes is the result of particular damping distributions in the system and depends on the proximity of the mode shapes. Liu and Sneckenberger [LIU 94] have developed a complex mode theory for a linear vibrating deficient system based on the assumption that it has a complete set of eigenvectors. Complex mode superposition methods have been used by [OLI 96] in the context of soil structure interaction problems. Balmès [BAL 97] has proposed a method to find normal modes and the associated non-proportional damping matrix from the complex modes. He has also shown that a set of complex modes is complete if it verifies a defined properness condition which is used to find complete approximations of identified complex modes. Garvey *et al.* [GAR 95] have given a relationship between real and imaginary parts of complex modes for general systems whose mass, stiffness and damping can be expressed by real symmetric matrices. They have also observed that the relationship becomes most simple when all roots are complex and the real parts of all the roots have the same sign. Bhaskar [BHA 99] has analyzed complex modes in detail and addressed the problem of visualizing the deformed modes shapes when the motion is not synchronous.

While the above-mentioned works concentrate on the properties of the complex modes, several authors have considered the problem of determination of complex modes in the configuration space. Cronin [CRO 76] has obtained an approximate solution for a non-classically damped system under harmonic excitation by perturbation techniques. Clough and Mojtahedi [CLO 76] considered several methods of treating generally damped systems, and concluded that the proportional damping approximation may give unreliable results for many cases. Similarly, it was shown [DUN 79] that significant errors can be incurred when dynamic analysis of a non-proportionally damped system is based on a truncated set of modes, as is commonly done in modeling continuous systems. Meirovitch and Ryland [MEI 85] have used a perturbation approach to obtain the left and right eigenvectors of damped gyroscopic systems. Chung and Lee [CHU 86] applied perturbation techniques to obtain the eigensolutions of damped systems with weakly non-classical damping. Cronin [CRO 90] has developed an efficient perturbation-based series method to solve the eigenproblem for dynamic systems having a non-proportional damping matrix. To illustrate the general applicability of this method, Peres-Da-Silva *et al.* [PER 95] have applied it to determine the eigenvalues and eigenvectors of a damped gyroscopic system. In the context of non-proportionally damped gyroscopic systems, Malone *et al.* [MAL 97] have developed a perturbation method that uses an undamped gyroscopic system as the unperturbed system. Based on a small damping assumption, Woodhouse [WOO 98] has given the expression for complex natural frequencies and mode shapes of non-proportionally damped linear discrete systems with viscous and non-viscous damping.

Adhikari [ADH 99a] derived approximate expressions of complex modes using a Neumann expansion for each mode. A general expression of the FRF when the system matrices are asymmetric was derived. This is particularly useful when it is not possible to transform an asymmetric system [INM 83, AHM 87, SHA 89, AHM 84a, ADH 00c, LIU 05], into a symmetric system. Gallina [GAL 03] discussed the effect of damping on asymmetric system. Liu and Zheng [LIU 10] proposed a synthesis method for transient response of non-proportionally damped structures. An iterative approach to obtain complex modes for non-proportionally damped systems was proposed in [ADH 11a].

1.2.2.3. *Response bounds and frequency response*

Previously, we have mainly discussed the calculation of the eigensolutions of non-classically damped systems. Here, we briefly consider the problem of obtaining dynamic response of such systems. Nicholson [NIC 87b] and Nicholson and Baojiu [NIC 96] have reviewed the literature on stable response of non-classically damped mechanical systems. Nicholson [NIC 87a] gave upper bounds for the response of non-classically damped systems under impulsive loads and step loads. Yae and Inman [YAE 87] have obtained bound on the displacement response of non-proportionally damped discrete systems in terms of physical parameters of the system and input. They have also observed that the larger the deviation from proportional damping, the less accurate their results become.

Bellos and Inman [BEL 90] have given a procedure for computing the transfer functions of a non-proportionally damped discrete system. Their method was based on Laplace transformation of the equation of motion in modal coordinates. A fairly detailed survey of the previous research is made in [BEL 90]. Yang [YAN 93] has developed an iterative procedure for calculation of the transfer functions of non-proportionally damped systems. Bhaskar [BHA 95] has analyzed the behavior of errors in calculating FRF when the off-diagonal terms of modal damping matrix are neglected. It has been shown that the exact response can be expressed by an infinite Taylor series and the approximation of ignoring the off-diagonal terms of modal damping matrix is equivalent to retaining one term of the series.

Finally, it should be noted that frequency responses of viscously damped systems with non-proportional damping can be obtained *exactly* in terms of the complex frequencies and complex modes in the configuration space, see, for example, [LAN 66] (section 7.5) and [GÉR 97, pp. 126–128]. Similar expressions are also derived in [FAW 76, VIG 86, WOO 98]. This in turn requires determination of complex modes in the configuration space. This problem will be discussed in detail later in this book.

1.3. Analysis of non-viscously damped systems

In section 1.1.3, it was pointed out that the most general way to model (non-viscous) damping within the scope of linear theory is through the use of the modified dissipation function given by equation [1.22]. Equations of motion of such non-viscously damped systems can be obtained from Lagrange's equation (see, for example, [MEI 67, GÉR 97, MEI 97]). The damping forces can be obtained as

$$Q_{\mathrm{nc}_k} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_k} = -\sum_{j=1}^{N} \int_0^t \mathcal{G}_{jk}(t-\tau)\dot{q}_j(\tau)\mathrm{d}\tau, \quad k = 1, \cdots, N \qquad [1.30]$$

and, consequently, the equation of motion can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{q}}(\tau)\mathrm{d}\tau + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t). \qquad [1.31]$$

This is a set of coupled second-order integro-differential equations. The presence of the "integral" term in the equations of motion complicates the analysis. Unlike the viscously damped systems, the concept of "proportional damping" cannot easily be formulated for such systems. The question of the existence of classical normal modes in such systems, i.e. if proportional damping can occur in such systems, will be discussed in Chapter 5.

Equations similar to [1.31] occur in many different subjects. Bishop and Price [BIS 79] have considered equations of motion similar to [1.31] in the context of ship dynamics. The convolution term appeared in order to represent the fluid forces and moments. They have discussed the eigenvalue problem associated with equation [1.31] and presented an orthogonality relationship for the right and left eigenvectors. They have also given an expression for the system response due to sinusoidal excitation. Their results were not very efficient because the orthogonality relationship of the eigenvectors was not utilized due to the difficulty associated with the form of the orthogonality equation, which itself became frequency dependent. Here, we briefly discuss different numerical methods proposed for linear dynamic systems with non-viscous damping.

1.3.1. *State-space-based methods*

Equation of motion like [1.31] arises in the dynamics of *viscoelastic structures*. A method to obtain such equations using a time-domain finite element formulation was proposed in [GOL 85, MCT 93]. Their approach (the GHM method), which introduces additional dissipation coordinates corresponding to the internal dampers, increases the size of the problem. Dynamic responses of the system were obtained by using the eigensolutions of the augmented problem in the state-space. A method to obtain the time- and frequency-domain descriptions of the response by introducing additional coordinates like the GHM method was proposed in [MUR 97a, MUR 98a]. To reduce the order of the problem, references [FRI 97, PAR 99, FRI 99] have proposed a state-space approach that employs a modal truncation and uses an iterative approach to obtain the eigensolutions. A state-space model reduction approach was considered by Yiu [YIU 93, YIU 94] using sub-structuring techniques for linear systems with exponential viscoelastic damping model. Trindade *et al.* [TRI 00] considered frequency-dependent viscoelastic material models for active–passive vibration damping and compared two widely used models. Adhikari [ADH 01a] derived the conditions for existence of proportional damping for non-viscously damped systems. Under such conditions, undamped normal modes can diagonalize the dynamic system. Palmeri [PAL 03] considered a state-space approach linear dynamic systems with memory. Time-domain approaches for viscoelastically damped systems were considered in [MUS 05, PAL 04]. Trindade [TRI 06] proposed a reduced order approach for viscoelastically damped beams through projection of the dissipative modes onto the structural modes. Wagner and Adhikari [WAG 03] proposed a symmetric extended state-space approach for exponentially damped systems using internal dissipation coordinates. Adhikari and Wagner [ADH 03a] considered general asymmetric non-viscously damped systems and explained the structures of the left and right eigenvectors. Zhang and Zheng [ZHA 07a] proposed a state-space approach for general linear dynamic systems with Biot viscoelastic model. Vasques *et al.* [VAS 06] considered finite element modeling and experimental validation of a beam with frequency-dependent viscoelastic damping. Vasques *et al.* [VAS 10a] discussed computational methods for viscoelastically damped systems using reduced approaches in the extended state-space. Genta and Amati [GEN 10] considered dynamics of non-viscously damped MDOF systems in the context of rotor dynamics using a state-space approach. Friswell *et al.* [FRI 10a] used internal variables for the time-domain analysis of rotors with frequency-dependent damping. The papers by de Lima *et al.* [DE 10, DE 09] proposed a novel component mode synthesis approach for general viscoelastic linear dynamic systems. Wang and Inman [WAN 13] used a symmetric state-space formulation linear system with GHM damping.

1.3.2. *Time-domain-based methods*

While the above methods often aimed at determining the eigensolutions of the system, few authors have considered the calculation of the dynamic response in the time domain. Adhikari and Wagner [ADH 04b] proposed a direct time-domain approach for exponentially damped systems which avoids the use of dissipation coordinates. Shen and Duan [SHE 09] proposed a Guass integration approach in conjunction with state-space representation of the equation of motion for linear MDOF systems with exponential damping. An efficient time-domain approach for linear dynamic systems with fractional damping was proposed by Trinks and Ruge [TRI 02]. Cortes and Elejabarrieta [COR 07a] proposed a time-domain integration approach for linear systems with fractional derivative damping model. Later, Cortes *et al.* [COR 09] proposed a direct integration formulation for linear dynamics systems with exponential non-viscous damping model. Pan and Wang [PAN 13] proposed a frequency as well as a time-domain approach using a discrete Fourier transform (DFT) method in combination with the fast Fourier transform (FFT) for exponentially damped systems.

1.3.3. *Approximate methods in the configuration space*

Computational cost for non-viscously damped systems can be prohibitive for large dimensional problems. To address this, several authors have proposed reduced approximate methods in the configuration space. Using a first-order perturbation approach, Woodhouse [WOO 98] has obtained expressions for the eigensolutions and transfer functions of system [1.31]. His method, although it avoids the state-space representations and additional dissipation coordinates, is valid for small damping terms only. Adhikari [ADH 02b] proposed an approximate method based on Neumann expansion for the eigenvectors of linear systems with general non-viscous damping. Several mathematical properties of the eigensolutions of such systems were derived in [ADH 01b]. Cortes and Elejabarrieta [COR 06a, COR 06c] proposed a new approximate method for the complex eigensolutions of a non-viscously damped system. The key idea proposed used the solution of the undamped system and approximated the complex eigensolutions by finite increments using the eigenvector derivatives and the Rayleigh quotient. Garcia-Barruetabena [GAR 11] demonstrated that non-viscous modes only contribute to the transient response in of a linear system with exponential non-viscous damping. Some approximate methods to obtain the eigensolutions of non-viscously damped systems using the eigensolutions of the underlying undamped systems were proposed in references [ADH 09a, ADH 10, ADH 11b]. Lázaro *et al.* [LAZ 12, LAZ 13a, LAZ 13b] proposed an approach for the computation of eigensolutions and dynamic response of MDOF system with exponential damping. The motivation was to approximate the response of the original viscoelastic system using the eigensolutions of the underlying undamped or proportionally damped system. Li *et al.* [LI 13a] approximated the FRF matrix without using the dissipation modes of the linear MDOF systems with viscoelastic hereditary terms. Pawlak and Lewandowski [PAW 13] proposed a reduced computational approach for nonlinear eigenvalue problems arising in non-viscously damped systems.

1.4. Identification of viscous damping

In section 1.2, we have discussed several methods for *analysis* of viscously damped linear dynamic systems. In this section, we focus our attention on the methodologies available for identification of viscous damping parameters from experimental measurements.

1.4.1. *Single-degree-of-freedom systems*

Several methods are available for identifying the viscous damping parameters for SDOF systems for linear and nonlinear damping models, see [NAS 85]. For linear damping models, these methods can be broadly described as:

1) *Methods based on transient response of the system:* This is also known as logarithmic decrement method: if q_i and q_{i+i} are heights of two subsequent peaks, then the damping ratio ζ can be obtained as

$$\delta = \log_e \left(\frac{q_i}{q_{i+i}} \right) \approx 2\pi\zeta \qquad [1.32]$$

For applicability of this method, the decay must be exponential.

2) *Methods based on harmonic response of the system:* These methods are based on calculating the half-power points and bandwidth from the frequency response curve. It can be shown that the damping factor ζ can be related to a peak of the normalized frequency response curve by

$$|H|_{\max} \approx \frac{1}{2\zeta} \qquad [1.33]$$

3) *Methods based on energy dissipation:* Consider the force-deflection behavior of a spring-mass-damper (equivalent to a block of material) under sinusoidal loading at some particular frequency. In steady state, considering conservation of energy, energy

loss per cycle (Δq_{cyc}) can be calculated by equating it with the input power. Here, it can be shown that the damping factor ζ can be related as

$$2\zeta = \frac{\Delta q_{\mathrm{cyc}}}{2\pi U_{\mathrm{max}}} \qquad [1.34]$$

where U_{max} is maximum energy of the system.

The above-mentioned methods, although developed for SDOF systems, can be used for separate modes of MDOF systems, for example a cantilever beam vibrating in the first mode. Chassiakos *et al.* [CHA 98] proposed an online parameter identification technique for an SDOF hysteretic system. Some authors [KHA 09, KHA 10, SRA 11] have proposed methods to identify damping parameters in nonlinear systems.

1.4.2. *Multiple-degrees-of-freedom systems*

For MDOF systems, most of the common methods for experimental determination of the damping parameters use the proportional damping assumption. A typical procedure can be described as follows, see [EWI 84] for details:

1) Measure a set of transfer functions $H_{ij}(\omega)$ at a set of grid points on the structure.

2) Obtain the natural frequencies ω_k by a pole-fitting method.

3) Evaluate the modal half-power bandwidth $\Delta\omega_k$ from the FRFs, then the Q-factor $Q_k = \frac{\omega_k}{\Delta\omega_k}$ and the modal damping factor $\zeta_k = \frac{1}{2Q_k}$.

4) Determine the modal amplitude factors a_k to obtain the mode shapes, $\mathbf{x}_k$.

5) Finally, reconstruct some transfer functions to verify the accuracy of the evaluated parameters.

Such a procedure does not provide reliable information about the nature or spatial distribution of the damping, though the reconstructed transfer functions may match the measured ones well.

The next stage, followed by many researchers, is to attempt to obtain the full viscous damping matrix from the experimental measurements. Pilkey and Inman [PIL 98] have given a recent survey on methods of viscous damping identification. These methods can be divided into two basic categories [FAB 88]: (1) damping identification from modal testing and analysis and (2) direct damping identification from the forced response measurements.

The modal testing and analysis method seeks to determine the modal parameters, such as natural frequencies, damping ratio and mode shapes, from the measured transfer functions, and then fit a damping matrix to these data. In one of the earliest works, Lancaster [LAN 61] gave an expression from which the damping matrix can be constructed from complex modes and frequencies. Unfortunately, this expression relies on having all the modes, which is almost impossible in practice. For this reason, several authors have proposed identification methods by considering the modal data to be incomplete or noisy. Hasselsman [HAS 72] proposed a perturbation method to identify a non-proportional viscous damping matrix from complex modes and frequencies. Béliveau [BÉL 76] proposed a method that uses eigensolutions, phase angles and damping ratios to identify the parameters of viscous damping matrix. His method utilizes a Bayesian framework based on eigensolution perturbation and a Newton–Raphson scheme. Ibrahim [IBR 83b] uses the higher order analytical modes together with the experimental set of complex modes to compute improved mass, stiffness and damping matrices. Minas and Inman [MIN 91] proposed a method for viscous damping identification in which it is assumed that the mass and stiffness are *a priori* known and modal data, obtained from experiment, allowed to be incomplete. Starek and Inman [STA 97] proposed an inverse vibration problem approach in which it is assumed that the damping matrix has an *a priori* known structure. Their method yields a positive-definite damping matrix but requires the full set of complex modes. Pilkey and Inman [PIL 97] developed an iterative method for damping matrix identification by using Lancaster's [LAN 61] algorithm. This method requires experimentally identified complex eigensolutions and the mass matrix. Alvin *et al.* [ALV 97] proposed a method in which a correction was applied to the proportionally damped matrix by means of an error minimization approach. Halevi and Kenigsbuch [HAL 99] proposed a method for updating the damping matrix by using the reference basis approach in which error and incompleteness of the measured modal data were taken into account. As an intermediate step, their method corrects the imaginary parts of the measured complex modes which are more inaccurate than their corresponding real parts.

Direct damping identification methods attempt to fit the equations of motion to the measured forced response data at several time/frequency points. Caravani and Thomson [CAR 74] proposed a least-square error minimization approach to obtain the viscous damping matrix. Their method uses a measured frequency response at a set of chosen frequency points and utilizes an iterative method to successively improve the identified parameters. Fritzen [FRI 86] used the instrumental variable method for identification of the mass, damping and stiffness matrices. It was observed that the identified values are less sensitive to noise compared to what was obtained from the least-square approach. Fabunmi *et al.* [FAB 88] presented a damping matrix identification scheme that uses forced response data in the frequency domain and assumes that the mass and stiffness matrices are known. Mottershead [MOT 90] used the inverse of the FRFs to modify the system matrices so that the modified model varies minimally from an initial finite element model. Using a different approach, Roemer and Mook [ROE 92] developed methods in the time domain for simultaneous identification of the mass, damping and stiffness matrices. It was observed that the identified damping matrix has a larger relative error than that of the mass and stiffness matrices. Chen *et al.* [CHE 96a] proposed a frequency-domain technique for identification of the system matrices in which the damping matrix was determined independently. It was shown that separate identification of the damping matrix improves the result as relative magnitude of the damping matrix is less than those of the mass and stiffness matrices. Later, Baruch [BAR 97] proposed a similar approach in which the damping matrix was identified separately from the mass and stiffness matrices.

Adhikari and Woodhouse [ADH 01c, ADH 02e] proposed a complex mode based approach for the identification of viscous damping matrix. Later, a method to identify symmetric damping matrices [ADH 02d] was proposed. Li [LI 05] used modulations of the responses to identify damping. Damping identification in pneumatic tyres was discussed by Geng *et al.* [GEN 07] using complex modes. A pattern recognition approach was used to identify damping in sucker-rod pumping system [LIU 07]. Erlicher and Argoul [ERL 07] proposed a wavelet transform based method for damping identification. Lin and Zhu [LIN 06] and Phani and Woodhouse [PHA 07, PHA 09] discussed methods for damping matrix identification from measured FRFs. Khalil *et al.* [KHA 07] proposed a proper orthogonal decomposition approach for the identification of the damping matrix along with the mass and stiffness matrices. Arora *et al.* [ARO 10, ARO 09a, ARO 09d, ARO 09b, ARO 09c] considered updating the damping matrix using analytical and experimental approaches. Prandina *et al.* [PRA 09] discussed the philosophy and performance of different damping identification approaches and investigated the role of the first-order perturbation methods and modal truncation on damping identification. A system identification algorithm based on the free vibration response of structures was proposed to identify damping in references [ROY 09, CHA 10]. Cavacece *et al.* [CAV 09] used a Moore–Penrose pseudo-inverse for the identification of damping. Pradhan and Modak [PRA 12a, PRA 12b] considered the determination of damping matrices from the FRF data. They developed an updating formulation that seeks to separate updating of the damping matrix from the updating of the stiffness and the mass matrix. Holland *et al.* [HOL 12a, HOL 12b] considered identification of damping in bladed disks. In a series of works, Liu *et al.* [LIU 08c, LIU 08d, LIU 08b, LIU 08a, LIU 09] pioneered the Lie-group estimation method for the inverse problem and damping identification in linear structural dynamics. Li and Law [LI 09] proposed a time-domain approach for damping identification using the sensitivity of the acceleration response of the analytical model along with a model updating technique. Algorithms for the mass normalization of the mode shapes in the context of experimental modal analysis were proposed by Yang *et al.* [YAN 12]. A common-base proper orthogonal decomposition approach was used by Andrianne and Dimitriadis [AND 12]. A pattern recognition approach was proposed to identify damping in a sucker-rod pumping system [LIU 07]. Cheonhong *et al.* [MIN 12] discussed a direct method for the identification of a non-proportional damping matrix using modal parameters. A two-step model updating algorithm for parameter identification of linear elastic damped structures was proposed by García-Palencia and Santini-Bell [GAR 13]. More recently, Holland and Epureanu [HOL 13] suggested a technique to identify the overall damping matrix utilizing identified (simple) damping matrices from different components. They demonstrated the approach for a mistuned blisk with varying levels of measurement noise.

1.5. Identification of non-viscous damping

Unlike viscous damping, there is little available in the literature which discusses generic methodologies for identification of non-viscous damping. Most of the methods proposed in the literature are system specific. Banks and Inman [BAN 91] have considered the problem of estimating damping parameters in a non-proportionally damped beam. They have taken four different models of damping: viscous air damping, Kelvin–Voigt damping, time hysteresis damping and spatial hysteresis damping, and used a spline inverse procedure to form a least-square fit to the experimental data. A procedure for obtaining hysteretic damping parameters in free-hanging pipe systems is given by Fang and Lyons [FAN 94]. Assuming material damping is the only source of damping, they have given a theoretical expression for the loss factor of the n-th mode. Their theory predicts higher modal damping ratios in higher modes. Maia *et al.* [MAI 97b] have emphasized the need for development of identification methodologies of general damping models and indicated several difficulties that might arise. Dalenbring [DAL 99] has proposed a method for identification of (exponentially decaying) damping functions from the measured FRFs and finite element displacement modes. A limitation of this method is that it neglects the effect of modal coupling, that is the identified non-viscous damping model is effectively proportional.

Adhikari and Woodhouse [ADH 01d] proposed a complex mode based approach for the identification of exponential non-viscous damping model. Zhang and Zheng [ZHA 07a] considered the Biot model in the context of MDOF systems and experimentally identified the model parameters. Vasques *et al.* [VAS 10b] discussed experimental identification and model validation of viscoelastically damped systems. They have compared several viscoelastic models in their study. Adhikari [ADH 02c] used a modified Lancaster's method to identify viscous and non-viscous damping matrices from the FRF matrix. Cortes and Elejabarrieta [COR 07b] characterized the viscoelastic damping properties of cantilever beams using the seismic response. Parameter identification of dynamical systems with fractional derivative damping models using methods based on inverse sensitivity analysis of damped eigensolutions and FRFs was proposed by Sivaprasad *et al.* [SIV 09]. Ding and Law [DIN 11] proposed an iterative regularization method for the identification of structural damping. Wang and Inman [WAN 13] proposed methods to identify parameters of GHM and ADF models from vibration experiments.

1.6. Parametric sensitivity of eigenvalues and eigenvectors

As seen so far, the characterization of eigenvalues and eigenvectors constitutes a central role in the design, analysis and identification of damped dynamic systems. As a result, the study of the variation of the eigenvalues and eigenvectors due to variations in the system parameters, or more precisely the sensitivity of eigensolutions, has emerged as an important area of research. For physically representative damping modeling and model updating of a dynamic system, it is necessary to know how much the eigenvalues and eigenvectors might change due to the changes in the parameters [MOT 93, FRI 95b, FRI 01]. For generally damped systems, this tantamounts to computing sensitivity of complex eigenvalues and eigenvectors in general. Sensitivity of eigenvalues and eigenvectors with respect to some system parameters may be represented by their derivatives with respect to those parameters. We briefly review some of the existing works on sensitivity of eigensolutions of undamped and damped systems.

1.6.1. *Undamped systems*

In one of the earliest works, Fox and Kapoor [FOX 68] gave exact expressions for the first derivative of eigenvalues and eigenvectors with respect to any design variable. Their results were obtained in terms of changes in the system property matrices and the eigensolutions of the structure, and have been used extensively in a wide range of application areas of structural dynamics. The expressions derived in [FOX 68] are valid for symmetric undamped systems. In many problems in dynamics, the inertia, stiffness and damping properties of the system cannot be represented by symmetric matrices or self-adjoint differential operators. These kinds of problems typically arise in the dynamics of actively controlled structures and in many general damped dynamic systems, for example moving vehicles on roads, missiles following trajectories, ship motion in sea water or the study of aircraft flutter. The asymmetry of damping and stiffness terms is often addressed in the context of gyroscopic and follower forces. Many authors [ROG 70, PLA 73, GAR 73, RUD 74] have extended Fox and Kapoor's [FOX 68] approach to determine eigensolution derivatives for more general asymmetric *conservative* systems. For these kinds of systems, Nelson [NEL 76] proposed an efficient method to calculate the first-order derivative of eigenvectors which requires only the eigenvalue and eigenvector under consideration. Murthy and Haftka [MUR 88] have written an excellent review on calculating the derivatives of eigenvalues and eigenvectors associated with general (non-Hermitian) matrices. Eigensensitivity analysis of a defective matrix with zero first-order eigenvalue derivatives was considered in [ZHA 04]. A method for modal reanalysis due to topological modifications (which changes the degree-of-freedom of the system) of structures was discussed by Zhi *et al.* [ZHI 06]. A new eigensolution reanalysis method was developed by Chen *et al.* [CHE 06] based on the Neumann series expansion and epsilon-algorithm. A general approach for incorporating the eigenvector normalization condition in the computation of eigenvector design sensitivities was proposed in [SMI 06]. Cha and Sabater [CHA 11] considered eigenvalue sensitivities of a linear structure carrying lumped attachments.

First-order derivatives are useful for practical problems as long as the perturbations of the system parameters remain "small". To consider a wide range of changes in the design parameters, the linear approximation intrinsic to the first-order derivatives may not be sufficient. Apart from large perturbations of system parameters, Brandon [BRA 84] has shown that the second-order eigensolution derivatives are not negligible compared to the first-order derivatives when the system has closely spaced natural frequencies. Second-order eigensolution derivatives are also required in design optimization to calculate the so-called "Hessian Matrix". For these reasons, there has been considerable interest in obtaining the second-order derivatives of the eigensolutions. Plaut and Huseyin [PLA 73] gave an expression for the second derivative of the eigenvalues for asymmetric systems. Rudisill [RUD 74] suggested a similar expression for the second derivative of the eigenvalues and went on to derive the second derivative of the eigenvectors. Brandon [BRA 91] derived the second derivative of the eigenvalues and eigenvectors for the case when the system matrices are linear functions of the design variables. Chen *et al.* [CHE 94a, CHE 94b] derived the second-order derivative of eigenvectors in terms of a series in the eigenvectors. Friswell [FRI 95a] proposed a method, similar to [NEL 76], to obtain the second-order derivative of the eigenvectors which employs only the eigensolutions of interest. Most of the methods discussed so far do not explicitly consider damped systems. In order to apply these results to obtain the second derivatives of the eigensolutions of general (non-proportionally) damped systems, the state-space formalism is required.

1.6.2. *Damped systems*

The work discussed so far does not explicitly consider the damping present in the system. In order to apply these results to systems with general non-proportional damping, it is required to convert the equations of motion into state-space form (see [ZEN 95], for example). Although exact in nature, the state-space methods require significant numerical effort as the size of the problem doubles. Moreover, these methods also lack some of the intuitive simplicity of the analysis based on configuration space. For these reasons, the determination of the derivatives of eigenvalues and eigenvectors in the configuration space for damped systems is very desirable. Unlike undamped systems, in damped systems the eigenvalues and eigenvectors, and consequently their derivatives, become complex in general. Some authors have considered the problem of the calculation of first-order derivatives of eigensolutions of viscously damped symmetric systems. Lee *et al.* [LEE 99a, LEE 99b] have proposed first-order formalism to determine natural frequency and mode shape sensitivities of damped systems. Adhikari [ADH 99b] derived an exact expression for the first-order derivative of complex eigenvalues and eigenvectors. The results were expressed in terms of the complex eigenvalues and eigenvectors of the second-order system and the first-order representation of the equation of motion was avoided. Later, Adhikari [ADH 00a] suggested an approximate method for calculating the first derivative of complex modes using a modal series involving only classical normal modes. An expression for the derivatives of eigenvalues and eigenvectors of non-conservative systems is presented by Choi *et al.* [CHO 04] in the configuration space. Moon *et al.* [MOO 04] proposed modified modal methods for calculating eigenpair sensitivity of an asymmetric damped system. They have used few lowest sets of modes to reduce the computational time. Guedria *et al.* [GUE 06] presented a new approach for simultaneously calculating the derivatives of the eigenvalues and their associated derivatives of the left and right eigenvectors for asymmetric damped systems. Friswell and Adhikari [FRI 00] extended Nelson's method to symmetric and asymmetric viscously damped systems. Guedria *et al.* [GUE 07] considered the computation of the second-order derivatives of the eigenvalues and eigenvectors of symmetric and asymmetric damped systems using Nelson's method. A modal approach for efficient calculation of complex eigenvector derivatives was proposed by Zhang-Ping and Jin-Wu [ZHA 07b]. Derivatives of repeated complex eigenvalues and corresponding eigenvectors of non-proportionally damped systems were considered in [HUI 07]. Chouchane *et al.* [CHO 07] proposed an algebraic approach for the calculation of eigensensitivity of asymmetric damped systems. Calculation of derivatives of multiple eigenvalues and eigenvectors of general unsymmetrical quadratic eigenvalue problems was considered in [XIE 08]. Abuazoum and Garvey [ABU 09] used structure-preserving equivalences to obtain eigenvalue and eigenvector derivatives of general second-order systems. Burchett [BUR 09] proposed a QZ-based algorithm for calculating derivatives of the system pole, transmission zero and residues. For the case when the system matrices are defective, efficient approaches for calculating the sensitivity of the eigensolutions were proposed in [XU 10, ZHA 11]. In the context of bridge deck flutter problems, Omenzetter [OME 12] considered sensitivity analysis of the eigenvalues for general dynamic systems. Some iterative methods for the derivatives of eigenvectors of quadratic eigenvalue problems arising in damped systems were suggested by Xie [XIE 12, XIE 13]. Li *et al.* [LI 12b, LI 13e, LI 13b] proposed efficient computational methods for the problem of eigensensitivity analysis of damped systems using both distinct and repeated eigenvalues.

Most of the above studies consider viscously damped systems. Adhikari [ADH 02a] proposed a modal approach for the eigensensitivity of linear systems with general non-viscous and non-proportional damping. It was shown that the eigenvector derivative can be expressed as a linear combination of other eigenvectors even when they do not satisfy any simple orthogonality relationships. Later, Adhikari and Friswell [ADH 06b] used Nelson's method to calculate the eigenvector derivatives of general non-viscously damped systems. Li *et al.* [LI 12a] proposed an algebraic method to compute the eigensolution derivatives for non-viscously damped systems. Later, they extended the formulation to asymmetric non-viscous systems [LI 13d]. More recently, Li *et al.* [LI 13c] discussed sensitivity analysis for general nonlinear eigenproblems arising in non-proportional and non-viscously damped systems.

1.7. Motivation behind this book

From the discussions so far in this chapter, it emerges that significant developments in the analysis of damped systems have taken place in the past two decades. This is fueled by the emergence of new materials such as composite and nanocomposite materials and the need to predict the system response ever more accurately in an efficient way. Based on the existing literature, it is clear that there are some pressing questions of general interest. These questions include, but are not limited to:

1) What damping model has to be used for a given structure, i.e. viscous or non-viscous, and if non-viscous then what kind of model should it be?

2) How can conventional modal analysis be extended to systems with non-viscous damping?

3) Can we physically understand the role of non-viscous damping in structural dynamics, as we do for viscous damping?

4) How is it possible to determine the damping parameters by conventional modal testing if a system is non-viscous?

5) How can we efficiently calculate the dynamic response of a large complex system in an efficient manner if the damping is non-proportional and non-viscous?

6) How sensitive is the dynamics of a system to the damping parameters? Does it matter if we get errors in some damping parameters?

7) How can we quantify damping in a system? What measures and tools can we use when the damping is, in general, non-proportional and non-viscous?

This book is motivated by these types of questions. We do not necessarily provide precise answers to these questions. The aim is to develop mathematical tools so that we can at least appreciate and investigate these types of questions for practically relevant engineering problems.

The first question is a major issue, and in the context of general vibration analysis, has been "settled" by assuming viscous damping, although it has been pointed out in the literature that, in general, it will not be the correct model. The next three questions are related to each other in the sense that for the identification of non-viscous damping parameters, a reliable method of modal analysis is also required. The fifth question on computational efficiency is becoming an issue as structural dynamic finite element models are getting larger. The consideration of parametric sensitivity of dynamic systems in the sixth question is important due to the recent drive toward model validation and uncertainty quantification of computational models. Finally, the last question regarding the quantification of damping is related to conceptual and intuitive feeling about how much damping there is in a system provided by certain parametric models.

Most of the techniques for detecting damping in a structure either consider the structure to be viscously damped or *a priori* assume some particular non-viscous model of damping and try to fit its parameters with regard to some specific structure. This *a priori* selection of damping no doubt hides the mechanisms of the system and there has not been any indication in the literature on how to find a damping model by doing conventional vibration testing. However, another relevant question in this context is whether this *a priori* selection of a damping model matters from an engineering point of view: it may be possible that a pre-assumed damping model with a "correct" set of parameters may represent the system response quite well, although the actual physical mechanism behind the damping may be different. These issues will be discussed in this book. Next, the scope of the book is discussed together with a brief overview of the chapters.

1.8. Scope of the book

Motivated by the pressing questions identified in the last section, a systematic study on the *analysis* and *identification* of damped discrete linear dynamic systems has been carried out. This book deals with analysis of linear systems with general damping models. The related book [ADH 14b] deals with identification and quantification of damping. The focus of these books is toward theoretical and computational aspects. However, some limited experimental results are given to support the theoretical developments. In section 1.1, it has been brought forward that the convolution integral model is the most general damping model for MDOF linear systems. Attention is specifically focused on this kind of general damping model. However, for comparing and establishing the relationship with current practice, viscously damped systems are also discussed. The book is divided into 10 chapters and an Appendix.

In Chapter 2, we begin by reviewing the theory of dynamics of SDOF undamped systems. The concept of resonance frequency is explained and methods to calculate dynamic response with initial conditions are discussed. Next, viscously damped SDOF systems are considered. Fundamental ideas such as damped natural frequency, damping ratio, FRF and impulse response function are discussed. A general expression of the forced dynamic response with non-zero initial condition is derived. Undamped vibration of MDOF system is discussed next. Classical concepts of eigenfrequencies, eigenmodes and mode orthogonality are introduced. Expressions of the dynamic response in the frequency domain and time domain are derived using the eigensolutions. The discussions are then extended to viscously damped MDOF systems. The idea of classical damping or proportional damping is critically reviewed and generalized proportional damping is introduced. Expressions of the dynamic response of proportionally damped systems are derived in terms of the classical normal modes and modal damping factors. General non-proportionally damped MDOF systems are discussed within the scope of the state-space method. Expressions of the dynamic response in the frequency and time domain due to general forcing and initial conditions are derived. The idea of Rayleigh quotient for damped systems is discussed. Stationarity properties for systems with proportional damping and non-proportional damping are derived. Numerical examples are provided to illustrate the theoretical developments.

Dynamics of SDOF non-viscously damped oscillators is considered in Chapter 3. It is assumed that the non-viscous damping force depends on the history of velocity via a convolution integral over an exponentially decaying kernel function. Classical qualitative dynamic properties known for viscously damped oscillators have been generalized to such non-viscously damped oscillators. The following questions of fundamental interest have been addressed: (1) under what conditions can a non-viscously damped oscillator sustain oscillatory motions? (2) How does the natural frequency of a non-viscously damped oscillator compare with that of an equivalent undamped oscillator? And (3) How does the decay rate compare with that of an equivalent viscously damped oscillator? Next, the characteristics of the FRF are discussed. The classical dynamic response properties known for viscously damped oscillators have been generalized to such non-viscously damped oscillators. The following questions of wide interest have been investigated: (1) under what conditions can the amplitude of the FRF reach a maximum value? (2) At what frequency will it occur? And (3) what will be the value of the maximum amplitude of the FRF? Introducing two non-dimensional factors, namely the viscous damping factor and the non-viscous damping factor, answers to these questions are provided. Wherever possible, attempts have been made to relate the new results with equivalent classical results for a viscously damped oscillator. It is shown that the classical concepts based on viscously damped systems can be extended to a non-viscously damped system only under certain conditions. Such conditions have been explicitly determined and illustrated numerically.

Chapter 4 extends the study in Chapter 3 to MDOF systems. Possible choices of non-viscous kernel functions are discussed. A general non-proportionally damped

MDOF system with exponential non-viscous damping is considered. The traditional state-space approach, well-known for viscously damped systems, is extended to such non-viscously damped systems using a set of internal variables. Two physically realistic cases: (1) when all the damping coefficient matrices are of full rank and (2) when the damping coefficient matrices have rank deficiency, are presented. For both cases, the equation of motion has been represented in terms of two symmetric matrices. The eigenvalues and the corresponding eigenvectors of the system are obtained by solving the state-space eigenvalue problem. It is shown that, unlike viscously damped systems, the number of eigensolutions is more than $2N$ and depends on the rank of the damping coefficient matrices. The ideas of elastic modes and non-viscous modes are introduced. The nature of these eigensolutions in the extended state-space has been explored. Some useful results relating the modal matrix in the extended state-space to the modal matrix in the original space are derived. Closed-form expressions of the dynamic response in the time domain and frequency domain due to arbitrary forcing and initial conditions are derived. It is shown that, even for general non-viscously damped systems, the response can be obtained using an approach similar to classical modal superposition method. A direct time-domain analysis of linear systems with exponentially decaying damping memory kernels is also considered. The method is based on the extended state-space representation of the equations of motion. Numerical examples are provided to illustrate the theoretical expressions.

Chapter 5 is aimed at extending classical modal analysis to treat lumped-parameter *general* non-viscously damped linear dynamic systems. This chapter extends the results of the last chapter where the *special case* of exponential damping was considered. The analytical approach adopted here is very different as the state-space approach has not been used. The nature of the eigenvalues and eigenvectors is discussed under certain simplified but physically realistic assumptions concerning the system matrices and the damping kernel functions. A numerical method based on the Neumann series expansion for the calculation of the eigenvectors is suggested. The transfer function matrix of the system is derived in terms of the eigenvectors of the second-order system. Exact closed-form expressions for the dynamic response due to general forces and initial conditions are derived. The mode-orthogonality relationships, known for undamped or viscously damped systems, have been generalized to such non-viscously damped systems. Some expressions are suggested for the normalization of the complex eigenvectors. A number of useful results which relate the system matrices with the eigensolutions are established. The approach taken in this chapter uses neither the state-space approach nor employs additional dissipation coordinates. The concept of the Rayleigh quotient for non-viscously damped systems is discussed. Three new Rayleigh quotients are proposed and their stationary properties are investigated. Suitable examples are given throughout the chapter to illustrate the derived analytical results.

Chapter 6 is devoted to reduced computational methods for damped dynamic systems. First, non-proportionally damped system with viscous model is considered. An iterative method to calculate complex modes from classical normal modes is proposed. A simple numerical algorithm is given to implement the iterative method. The calculation of eigenvalues of SDOF linear non-viscously damped systems with exponential model is considered next. An approximate non-state-space based approach is proposed for this type of problem. The proposed approximations are based on certain physical assumptions which simplify the underlying characteristic equation to be solved. Closed-form approximate expressions of the complex and real eigenvalues of the system are derived. These approximate expressions are obtained as functions of the undamped eigenvalues only. The methods are then extended to exponentially damped MDOF systems. This technique enables us to approximately calculate the eigenvalues of non-viscously damped systems by simple post-processing of the undamped eigenvalues. Beside these reduced modal methods, another model reduction approach based on an equivalent second-order form is discussed. This method is applicable to any general non-viscous model and *not* only the exponential model. The proposed approximation utilizes the idea of generalized proportional damping and expressions of approximate eigenvalues of the system. A closed-form expression of the equivalent second-order system has been derived. The new expression is obtained by elementary operations involving the mass, stiffness and the kernel function matrix only. This enables us to approximately calculate the dynamic response of general non-viscously damped systems using the standard tools for conventional second-order viscously damped systems. Representative numerical examples are given throughout to verify the accuracy of the derived expressions.

The ability to efficiently calculate the eigensolutions and dynamic response of general non-viscously damped MDOF systems is useful. However, it is also necessary to understand how different parameters in the model impact the dynamics of the system. [ADH 14b], Chapter 1 is devoted to calculating the parametric sensitivity of the eigensolutions of damped systems with respect to the system parameters. Sensitivity of eigenvalues and eigenvectors of MDOF undamped systems is considered first. Then, we consider MDOF systems with a non-proportional viscous damping model. Because of the non-proportional nature of the damping, the mode shapes and natural frequencies become complex, and as a consequence the sensitivities of eigenvalues and eigenvectors are also complex. The results are presented in terms of the complex modes and frequencies of the second-order system and avoid the use of rather undesirable state-space representation. The eigenvector derivatives are obtained using two approaches, namely via the superposition of complex modes and Nelson's method. Next, the parametric sensitivity of the eigensolutions of MDOF systems with general non-viscous damping is considered. Because of the general nature of the damping, eigensolutions are generally complex valued and eigenvectors do not satisfy any orthogonality relationship. It is shown that even under such general conditions, the derivative of the eigensolutions can still be expressed in a way similar to that of undamped or viscously damped systems. All the results derived in the chapter are explained by many numerical examples.

The above-mentioned studies give a firm basis for the modal analysis of non-viscously damped systems. Motivated by these results, in [ADH 14b], Chapter 2, studies on damping identification are considered. First, the identification of a viscous damping matrix from the modal damping factors using the generalized proportional damping model is discussed. The theoretical formulation is presented and a step-by-step procedure is given. The practical utility of the proposed identification scheme is illustrated on three representative structures: (1) a free-free beam in flexural vibration, (2) a quasi-periodic three-cantilever structure model by inserting slots in a plate, in out-of-plane flexural vibration and (3) a point-coupled-beam system. Then, the identification of a non-proportional viscous damping matrix under the circumstances when the actual damping model in the structure can be non-viscous is considered. A method is presented to obtain a full (non-proportional) viscous damping matrix from complex modes and complex natural frequencies. It is assumed that the damping is “small” so that a first-order perturbation method is applicable. The proposed method and several related issues are discussed by considering numerical examples based on a linear array of damped spring-mass oscillators. It is shown that the method can predict the spatial location of damping with good accuracy, and also gives some indication of the correct mechanism of damping. In some cases, the identified damping matrix becomes non-symmetric, which in a way is a non-physical result because the original system is reciprocal. Through an error analysis, how the identified damping matrix is influenced by errors in the identified modal quantities is discussed. Methods are also developed to identify damping models which preserve the symmetry of the system. This procedure is based on a constrained error minimization approach. Another approach based on the Lancaster method is presented. Here, the damping matrix is obtained directly from the poles and residues of the measured transfer functions. The proposed methods are supported by suitable numerical examples.

From the studies in [ADH 14b], Chapter 2, it is observed that when a system is non-viscously damped, an identified equivalent viscous damping model does not accurately represent the damping behavior. This demands new methodologies to identify non-viscous damping models. [ADH 14b], Chapter 3, takes a first step, by outlining a procedure for identifying a damping model involving an exponentially decaying relaxation function. The method uses experimentally identified complex modes and complex natural frequencies, together with the knowledge of the mass matrix of the system. The proposed method and several related issues are discussed by considering numerical examples of a linear array of damped spring-mass oscillators. It is shown that good estimates can be obtained for the exponential time constant and the spatial distribution of the damping. However, when the fitted model does not match with the “true” model, the identification method results in an asymmetric coefficient matrix. A symmetry preserving damping identification method is presented, which guarantees a symmetric coefficient matrix. This method is based on a matrix variate constrained optimization method. Finally, another approach for damping identification directly from the poles and residues of the measured transfer functions is presented. This approach bypasses the need to obtain complex modes and frequencies from experiments. Numerical examples with simulated data are given to illustrate the damping identification methods.

Chapter 4 of [ADH 14b] is inspired by a simple question – given a damping model, how much damping is there in a system? In the context of a proportionally damped system, the answer to this question is straightforward as the damping is completely quantified by the modal damping factors. However, for non-proportionally damped systems and non-viscously damped systems, the answer to this question is less obvious. Here, numerical tools are developed to measure damping. First, a simple method is proposed to normalize the complex modes of a non-proportionally damped system so that they are closest to their corresponding classical normal modes. Based on these "optimal complex modes", an index of damping non-proportionality is proposed. The methodologies developed here are applicable to both viscous and non-viscously damped systems. Later, indices to quantify the extent of any departures of a non-viscous model from a viscous model are developed. In other words, the amount of "non-viscosity" of damping in discrete linear systems is quantified. Four indices are proposed. Two of these indices are based on the non-viscous damping matrix of the system. A third index is based on the residue matrices of the system transfer functions and the fourth is based on the (measured) complex modes of the system. The performance of the proposed indices is examined by considering numerical examples.

Theory of dynamics of MDOF *symmetric* systems has been studied in this book. However, dynamical behavior of some systems encountered in practice can be asymmetric in nature. In the Appendix methods are proposed by which asymmetric dynamic systems can be transformed into symmetric systems. In this way, the methods proposed in the book can in turn be applied to asymmetric systems as well. The conditions for transforming multiple-degrees-of-freedom linear asymmetric dynamical systems to equivalent symmetric systems by non-singular linear transformations are discussed. An approach is proposed to transform asymmetric systems into symmetric systems by an equivalence transformation. The existing approach of symmetrization by similarity transformation is the "first kind" and the proposed approach by equivalence transformation is the "second kind". Because equivalence transformations are the most general non-singular linear transformations, conditions of symmetrizability obtained here are more "liberal" compared to the first kind and numerical calculations also become more straightforward. Several examples are provided to illustrate this approach.

The intended readers of this book are primarily senior undergraduate students, graduate students and practicing engineers working in the field of advanced vibration. Limited examples are provided to support of the theoretical developments. The book is written with the aim of being self-contained. However, a recommended prerequisite is an undergraduate-level vibration course. There are many excellent books which cover the fundamentals of the theory of vibration, for example references [MEI 67, MEI 80, PAZ 80, NEW 89, CLO 93, BAT 95, MEI 97, PET 98, GÉR 97, INM 03, RAO 11]. Readers will greatly benefit by familiarizing themselves with the basics of the theory of vibration.

In spite of the attempt at being exhaustive at the time of writing, clearly many relevant and possibly important bibliographic references are missed. This is inevitable as a huge amount of literature was published recently due to the significant rise in the interest in this topic. However, the author expects that the book covers the necessary background so that at least the readers will appreciate existing publications and future research works and developments in the field of damping. It is hoped that the readers will not only gain an understanding of the material presented in the book, but will also be able do their personal research and take this field forward.