Chapter 1

Positive Systems: Discretization with Positivity and Constraints

In this chapter, we discuss the problem of preservation of two properties pertaining continuous-time systems under discretization, namely the properties of positivity and sparsity. In the first part of the chapter, the action of diagonal Padé transformations is studied together with the preservation of copositive quadratic and copositive linear Lyapunov functions. A variation of the scaling and squaring method is then introduced and shown to be able to preserve such Lyapunov functions and positivity for small sampling times. In the second part, besides positivity, the problem of preservation of the structure (sparseness) of the continuous-time system under discretization is analyzed. The action of the standard forward Euler discretization method is discussed and a new approximation method – *mixed Euler* – ZOH (mE-ZOH) is introduced that preserves copositive Lyapunov functions, the sparseness structure and the positivity property for all sampling times.

1.1. Introduction and statement of the problem

This chapter is devoted to the study of the effects of discretization in the preservation of two properties pertaining linear systems, namely (1) positivity and (2) structure. The first property characterizes systems whose inputs, state

---

Chapter written by Patrizio COLANERI, Marcello FARINA, Stephen KIRKLAND, Riccardo SCATTOLINI and Robert SHORTEN.

and outputs take non-negative values in forward time. As part of the more general class of *monotonic* systems [ANG 03], such systems characterize the dynamic behavior of processes frequently encountered in engineering and in social, economic and biological sciences. A few monographs are now available where both the mathematical properties and the application interest of such systems are underlined [BER 94, FAR 00].

The important problem of obtaining reliable discrete-time approximations to a given continuous-time system arises in many circumstances: in simulation issues, in control system design, in certain optimization problems and in model order reduction [ANT 05, FAL 08]. While a complete understanding of this problem exists for linear time-invariant (LTI) systems [WES 01], and some results are available for switched linear systems [ROS 09, SAJ 11], the analogous problems for positive systems are more challenging since discretization methods must preserve not only the stability properties of the original continuous-time system, but also physical properties, such as state positivity. To the best of our knowledge, this is a relatively new problem in the literature, with only a few recent works on this topic [BAU 10]. In particular, we stress the importance of this issue in the framework of switched positive systems, a research field still in its infancy, but with growing importance in telecommunications, biological networks and cloud computing (see [SHO 07, SHO 06, BAR 89, HAR 02]). Generally speaking, we are interested in the evolution of the system:

$$\dot{x}_c(t) = A_{\sigma_c(t)} x_c(t),\ \sigma_c(t) \in \{1, \dots, m\},\ x_c(0) = x_0, \qquad [1.1]$$

where $A_\sigma \in \mathbb{R}^n$ are Hurwitz stable Metzler matrices, $x_c(t) \in \mathbb{R}^{n \times 1}$ and $m \geq 1$. We are interested in obtaining from this continuous-time positive system, a discrete-time representation:

$$x_d(k+1) = F_{\sigma_d(k)}(h) x_d(k),\ \sigma_d(k) \in \{1, \dots, m\},\ x_d(0) = x_0, \qquad [1.2]$$

where $h > 0$ is the sampling interval. The first objective of this chapter is to study diagonal Padé approximations to the matrix exponential. Such a study is well motivated, as diagonal Padé approximations are methods used by control engineers. Following [ZAP 12], we deal with two fundamental questions. First, under what conditions are certain types of stability of the original positive switched system inherited by the discrete-time approximation? Second, we also ask if and when positivity itself is inherited by the discrete-time system. We give sufficient conditions under which the Padé approximation is positivity preserving, and identify a new approximation method that is guaranteed to preserve both stability and positivity.

The second objective of this chapter arises from the need of discretizing large-scale systems. In this context, we are often interested in discretization methods that preserve the structure of a dynamic system. We aim to find efficient discretization methods which preserve, for the elements of $F_{\sigma_c}(h)$, the same zero/non-zero pattern of A_{σ_c}. The attention here is focused on positive switched systems only, along the lines traced in [COL 12]. First, we analyze the properties of the forward Euler transformation, which intrinsically preserve the zero pattern of the off-diagonal entries of the dynamic matrix. However, it is well known that the forward Euler transformation can easily lead to a loss of stability even for short sampling times. We then propose a novel *mE-ZOH* discretization method that preserves the structure independently of the sampling time, with improved performance in terms of stability preservation.

The chapter is organized as follows: in section 1.2, we study Padé transformations and their properties, while in section 1.3 we propose the new *mE-ZOH* transformation and we analyze some of its properties. Section 1.4 concludes the chapter.

NOTATION. In this chapter, the following notations are used: capital letters denote matrices and small letters denote vectors. For matrices or vectors, $(')$ indicates transpose and $(^*)$ the complex conjugate transpose. For matrices X or vectors x, the notation X or $x > 0$ (≥ 0) indicates that X, or x, has all positive (non-negative) entries and it will be called a positive (non-negative) matrix or vector. The notation $X \succ 0$ ($X \prec 0$) or $X \succeq 0$ ($X \preceq 0$) indicates that the matrix X is positive (negative) definite or positive (negative) semi-definite. The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively, while $\mathbb{R}_+$ denotes the set of non-negative real numbers. A square matrix A_c is said to be Hurwitz stable if all its eigenvalues lie in the open left-half of the complex plane. A square matrix A_d is said to be Schur stable if all its eigenvalues lie inside the unit disc. A matrix A is said to be Metzler (or essentially non-negative) if all its off-diagonal elements are non-negative; moreover, we say that the diagonal entries are non-positive, with at least one negative diagonal entry. A matrix B is an M-matrix if $B = -A$, where A is both Metzler and Hurwitz; if an M-matrix is invertible, then its inverse is non-negative [BER 94]. The matrix I will be the identity matrix of appropriate dimensions. Finally, we denote with $\mathcal{M}_c$ the set of Hurwitz stable Metzler matrices, and with $\mathcal{M}_d$ the set of Schur stable non-negative matrices.

1.2. Discretization of switched positive systems via Padé transformations

This section is a summary of the recent work described in [ZAP 12] and some other related papers. The interested reader is referred to [ZAP 12] for proofs and examples. Concerning the problem of obtaining a discrete-time approximation [1.2] to system [1.1], the Padé approximation can be used, where h is the sampling time. The $[L/M]$ order Padé approximation to the exponential function e^s is the rational function C_{LM} defined by:

$$C_{LM}(s) = Q_L(s)Q_M^{-1}(-s),$$

where

$$Q_L(s) = \sum_{k=0}^{L} l_k s^k, \; Q_M(s) = \sum_{k=0}^{M} m_k s^k,$$

$$l_k = \frac{L!(L+M-k)!}{(L+M)!k!(L-k)!} \text{ and } m_k = \frac{M!(L+M-k)!}{(L+M)!k!(M-k)!}.$$

Thus, given a matrix A, the diagonal Padé approximant to the matrix exponential e^{Ah} with sampling time h is given by taking $L = M = p$

$$C_p(Ah) = Q_p(Ah)Q_p^{-1}(-Ah),$$

where $Q_p(Ah) = \sum_{k=0}^{p} c_k(Ah)^k$ and $c_k = \frac{p!(2p-k)!}{(2p)!k!(p-k)!}$. It is known that diagonal Padé approximations map the open left-half of the complex plane to the interior of the unit disc, and hence are A-stable [BUT 02].

1.2.1. *Preservation of copositive Lyapunov functions*

Recently, it was shown in [SAJ 11] that quadratic Lyapunov functions are preserved for sets of matrices that arise in the study of systems of the form of equation [1.1]. We now ask whether copositive Lyapunov functions are preserved when discretizing an LTI positive system using Padé-like approximations. Since trajectories of positive systems are constrained to lie in the positive orthant, the stability of these systems is completely captured by Lyapunov functions whose derivative is decreasing for all such positive trajectories. Such functions are referred to as copositive Lyapunov functions. With this background in mind, we observe the following elementary result.

LEMMA 1.1.– Let $A \in \mathcal{M}_c$ and let α be a positive real number. Fix any sampling time $h > 0$ such that $F(h) = (\alpha(h)I + A)(\alpha(h)I - A)^{-1}$ is a non-negative matrix, where $\alpha(h) = \frac{\alpha}{h}$. Then the following statements hold.

1) If $v(x) = x'Px$, with $P = P' \succ 0$, is a quadratic Lyapunov function for A, that is:

$$x'(A'P + PA)x < 0, \; \forall \, x \geq 0, \; x \neq 0,$$

then $v(x)$ is a quadratic Lyapunov function for $F(h)$, that is:

$$x'(F'(h)PF(h) - P)x < 0, \; \forall \, x \geq 0, \; x \neq 0.$$

2) If $v(x) = w'x$, $w > 0$ is a linear copositive Lyapunov function for A, that is $w'A < 0$, then $v(x)$ is a linear copositive Lyapunov function for $F(h)$, namely $w'F(h) < w'$.

It is shown in [ZAP 12] that some Padé approximations may result in the loss of certain copositive Lyapunov functions. In such situations, the usual approach is to make the sampling rate h smaller to make the approximation more likely to inherit desired properties. We now summarize some results in this direction with the following lemma.

LEMMA 1.2.– Let $A \in \mathcal{M}_c$, and suppose that $\hat{\lambda}$ is a complex number with a positive real part. For all $h > 0$, let $\lambda(h) = \frac{\hat{\lambda}}{h}$, and consider the following matrices:

$$\begin{aligned} \Theta_1 &= (\lambda(h)I + A)(\lambda^*(h)I + A); \\ \Theta_2 &= (\lambda(h)I - A)(\lambda^*(h)I - A); \\ F(h) &= \Theta_1 \Theta_2^{-1}. \end{aligned}$$

Suppose that there exists $h_0 > 0$ such that for all $0 < h \leq h_0, \Theta_2$ is an M-matrix and $F(h)$ is a non-negative matrix. Then, the following statements hold.

1) If $v(x) = x'Px$, with $P = P' \succ 0$, is a copositive quadratic Lyapunov function for A, that is:

$$x'(A'P + PA)x < 0, \; \forall \, x \geq 0, \; x \neq 0,$$

then there exists $h_1 > 0$ such that for all $0 < h \leq h_1$, $v(x)$ is a quadratic Lyapunov function for $F(h)$, that is:

$$x'(F(h)'PF(h) - P)x < 0, \; \forall \; x \geq 0, \; x \neq 0.$$

2) If $v(x) = w'x$, $w > 0$, is a linear copositive Lyapunov function for A, that is $w'A < 0$, then for $0 < h \leq h_0$, $v(x)$ is a linear copositive Lyapunov function for $F(h)$, namely $w'F(h) < w'$.

We can now state the following result, which formalizes the intuition that Lyapunov stability is indeed preserved provided that h is chosen to be small enough, for diagonal Padé approximations. We then have the following result.

THEOREM 1.1.– Let $A \in \mathcal{M}_c$ and let $C_p\,(Ah)$ be the pth-order diagonal Padé approximation of e^{Ah}. Suppose also that there exists $h_0 > 0$ such that for all $0 < h \leq h_0$, the following conditions hold:

1) For each real pole α of $C_p(\cdot)$, the matrix $(\frac{\alpha}{h}I + A)(\frac{\alpha}{h}I - A)^{-1}$ is non-negative.

2) For each complex pole λ of $C_p(\cdot)$, the matrix $(\frac{\lambda}{h}I - A)(\frac{\lambda^*}{h}I - A)$ is an M-matrix.

3) For each complex pole λ of $C_p(\cdot)$, the matrix $(\frac{\lambda}{h}I + A)(\frac{\lambda^*}{h}I + A)$ $(\frac{\lambda}{h}I - A)^{-1}(\frac{\lambda^*}{h}I - A)^{-1}$ is non-negative.

Finally, suppose there exists a linear copositive Lyapunov function for the continuous-time system, that is $w'A < 0$ with $w > 0$. Then, for all $0 < h \leq h_0$, the discretized system, with $F(h) = C_p(Ah)$, shares the same linear copositive Lyapunov function, that is $w'F(h) < w'$.

An analogous theorem holds for preservation of copositive quadratic functions, that are preserved, however, for $h \leq h_1$, where $h_1 \leq h_0$, see [ZAP 12]. To conclude this section, note that theorem 1.1 can be formulated so as to cope with preservation of a *common* copositive Lyapunov function $v(x) = w'x$ for the switched systems [1.1] and [1.2]. In this case, h_0 should be such that conditions $1 - 3$ of the theorem hold for any matrix A_i, $i = 1, 2, \cdots, m$ defining the switched system [1.1]. This number h_0, however, can be arbitrarily low and even zero in some cases. A notable exception is for 2×2 matrices (second-order systems) where the existence of $h_0 > 0$ is guaranteed.

1.2.2. *Non-negativity of the diagonal Padé approximation*

Our results in the previous section were concerned with the preservation of linear and quadratic copositive Lyapunov functions. In this section, we address the fundamental question of whether $C_p(Ah)$ is non-negative when A is Metzler and Hurwitz. Our approach will be first to analyze the situation for some simple, Padé-like, rational functions, then to decompose C_p into a suitable product of such functions. We begin with the following straightforward result, which has also been noted in [ALO 10] and [ZAP 10] in conjunction with preservation of quadratic Lyapunov functions and is a special case of the main result in [BOL 78].

LEMMA 1.3.– Let $A \in \mathcal{M}_c$ and denote with a_{ij} the (i,j)th entry of A, $i,j = 1\ldots,n$. Set $\alpha_0 > 0$, set $\alpha(h) = \frac{\alpha_0}{h}$ and define $F(h)$ by:

$$F(h) = (\alpha(h)I + A)\,(\alpha(h)I - A)^{-1}. \qquad [1.3]$$

If

$$h \leq \min_{i\,:\,a_{ii}\neq 0} \frac{\alpha_0}{|a_{ii}|}, \qquad [1.4]$$

then $F(h) \in \mathcal{M}_d$.

The first-order Padé transformation is obtained by letting $\alpha_0 = 2$ in [1.3]. As a result, see [1.4], if $h \leq \min_{i\,:\,a_{ii}\neq 0} \frac{2}{|a_{ii}|}$, then $C_1(Ah)$ is a non-negative and Schur stable matrix.

We now consider a Padé-like rational function where the numerator and denominator are both quadratic. Specifically, suppose that λ_0 is a complex number with $Re(\lambda_0) > 0$. Set $\lambda(h) = \frac{\lambda_0}{h}$, and define $F(h)$ via

$$F(h) = (\lambda(h)I + A)\,(\lambda^*(h)I + A)\,(\lambda(h)I - A)^{-1}\,(\lambda^*(h)I - A)^{-1}. \qquad [1.5]$$

Set

$$\Theta_1 = \left(|\lambda(h)|^2 I + 2Re(\lambda(h))A + A^2\right), \qquad [1.6]$$

$$\Theta_2 = \left(|\lambda(h)|^2 I - 2Re(\lambda(h))A + A^2\right), \qquad [1.7]$$

so that $F(h) = \Theta_1 \Theta_2^{-1}$. Furthermore we note that taking $\lambda_0 = 3 + \sqrt{3}i$, [1.5] yields $C_2(Ah)$. Define $A = \{a_{ij}\}$ and $A^2 = \{b_{ij}\}$, then let $\mathcal{P}$ be the set of indices i, j, $i \neq j$, such that $b_{ij} \neq 0$.

LEMMA 1.4.– Let $A = \{a_{ij}\} \in \mathcal{M}_c$ and let F be the matrix achieved through the transformation [1.5]. If

$$h \leq 2Re(\lambda_0) \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}, \qquad [1.8]$$

then Θ_1 of [1.6] is a non-negative matrix, Θ_2 of [1.7] is an M-matrix and $F(h) \in \mathcal{M}_d$.

Lemmas 1.3 and 1.4 will now yield the following result regarding the non-negativity of a pth-order diagonal Padé approximation.

THEOREM 1.2.– Let $A \in \mathcal{M}_c$ and $F(h) = C_p(Ah)$ be the pth-order diagonal Padé approximation to e^{Ah}. Let $\alpha_l, l = 1, \ldots, m$ denote the real poles of $C_p(x)$, and let $\lambda_k, \lambda_k^*, k = 1, \ldots, \frac{n}{2}$ denote the complex conjugate pairs of poles $C_p(x)$. If $m \geq 1$, we define $\hat{\alpha} = \min_{l=1,\ldots,m} \alpha_l$, and if $n \geq 2$, we define $\hat{\lambda} = \min_{k=1,\ldots,\frac{n}{2}} Re(\lambda_k)$. Then, $F(h) \in \mathcal{M}_d$ for every $h \leq h^*$, where

$$h^* = \min_{i\ ,:\, a_{ii} \neq 0} \frac{\hat{\alpha}}{|a_{ii}|}, \quad \text{if } n = 0, m \geq 1,$$

$$h^* = 2\hat{\lambda} \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}, \quad \text{if } m = 0, n \geq 2,$$

$$h^* = \min_{i\,:\, a_{ii} \neq 0} \frac{\hat{\alpha}}{|a_{ii}|}, 2\hat{\lambda} \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}, \quad \text{if m} \geq 1, n \geq 2,$$

where a_{ij} and b_{ij} denote the (i, j) element of A and A^2, respectively.

The proof of theorem 1.2 shows that for each complex pole λ of C_p, the matrix Θ_2 is an M-matrix whenever $h \leq h^*$. We thus find from theorem 1.1 that if $h_0 \leq h^*$, the linear copositive Lyapunov functions are preserved. For switched systems [1.1] and [1.2] this means that a *common* copositive Lyapunov function $v(x) = w'x$ is preserved if $h_0 \leq \min_i h_i^*$, where h_i^* are defined in theorem 1.2 for the generic matrix A_i of the switched system [1.1].

1.2.3. *An alternative approximation to the exponential matrix*

The results in the previous section are somewhat unsatisfactory from the point of view of stability and positivity. In this section, we present a Padé-like approximation that has the following properties: we can always find a sampling time such that positivity is preserved, and in addition, for any h, both linear and quadratic copositive Lyapunov functions are preserved. This approximation may have worse numerical properties, but these two basic qualitative properties are preserved. To do this, we introduce the following approximation to the exponential matrix e^{Ah}:

$$F(h) = \left[\left(I + \frac{Ah}{2p}\right)\left(I - \frac{Ah}{2p}\right)^{-1}\right]^p, \quad p \in \mathbb{N}. \qquad [1.9]$$

Writing $F(h)$ as $\left(I + \frac{Ah}{2p}\right)^p \left(I - \frac{Ah}{2p}\right)^{-p}$, and applying the binomial expansion to each of the two factors in that expression, we find readily that $F(h)$ converges to e^{Ah} as $p \to \infty$. Note that if p is chosen as a power of two, then [1.9] coincides exactly with the scaling and squaring method, where the Padé approximant computed is the first-order diagonal Padé approximant. The nice property of this transformation is that, given $A \in \mathcal{M}_c$, if $I + \frac{Ah}{2}$ is non-negative, then $F(h)$ is non-negative for each positive power p. Then lemmas 1.1 and 1.3 can be applied with

$$h < \min_i \frac{2}{|a_{ii}|} \leq \min_i \frac{2p}{|a_{ii}|}, \quad \forall p \geq 1.$$

We are now in the position to state the main result relative to the discretization of switched positive linear systems [1.1].

THEOREM 1.3.– Let $\{A_1, \ldots, A_m\}$ be such that $A_i \in \mathcal{M}_c$ for all $i = 1, \ldots, m$ and let $F_i(h) = C_{ap}(A_i h)$ be the pth-order approximation to the exponential matrix $e^{A_i h}$ defined in equation [1.9]. Then the following properties hold:

1) Fix an i between 1 and m, and suppose that

$$0 < h \leq h_i = \min_j \frac{2}{|a_{i,jj}|}, \qquad [1.10]$$

where $a_{i,jj}$ are the elements on the main diagonal of the matrix A_i. Then, $F_i(h) \in \mathcal{M}_d$.

2) Consider the continuous-time switching positive system [1.1] Suppose that [1.10] holds. Then, the discretized system [1.2] is positive. Moreover, if there exists a common quadratic or linear copositive Lyapunov function for system [1.1], then the origin $x = 0$ is globally uniformly exponentially stable for system [1.2].

1.3. Discretization of positive switched systems with sparsity constraints

In this section, the main objective is to discuss a novel discretization method, previously presented in [COL 12], which is denoted as as mE-ZOH discretization (*mE-ZOH*). The properties of this new method are manyfold: (1) it conserves positivity, (2) it conserves asymptotic stability in case of LTI positive systems for all sampling times and (3) it conserves the matrix sparsity, that is the zero/non-zero pattern of the elements of A, which can be extremely useful to unveil the so-called system's structural properties (in the sense defined in [SIL 91] and [LUN 92]). In particular, we will say that matrix F in [1.2] belongs to the same class $\mathcal{S}$ of A if its entries f_{ij} are $f_{ij} = 0$ if $a_{ij} = 0$ for all $j \neq i$.

Besides guaranteeing the preservation of the system's structural properties [SIL 91], the conservation of the sparsity of the system allows minimizing the number of communication links (corresponding to non-zero terms a_{ij}) needed for digital distributed control of large-scale interconnected continuous-time systems (see [VAD 03, FAR 12, SCA 09]).

1.3.1. *Forward Euler discretization*

Consider system [1.1] and a sampling time $h > 0$. Considering matrix $A \in R^{n \times n}$, the standard forward Euler approximation for the exponential matrix e^{Ah} is given by:

$$F(h) = I + hA. \qquad [1.11]$$

The first result below deals with:

1) the preservation of stability for positive systems when using the forward Euler approximation [1.11];

2) the proof that $F(h)$ in [1.11] and A share the same zero pattern of the off-diagonal entries, that is $A \in \mathcal{S} \to F \in \mathcal{S}$;

3) the preservation of Lyapunov functions when applying the forward Euler approximation [1.11].

THEOREM 1.4.– Let $A \in \mathcal{M}_c \cap \mathcal{S}$. Then, under [1.11], $F(h) \in \mathcal{M}_d \cap \mathcal{S}$ if and only if:

$$h < h^\star = \min_i \frac{1}{|a_{ii}|}. \qquad [1.12]$$

Furthermore, if $v(x) = \max_i \left(\frac{x_i}{w_i}\right)$, $w > 0$, is a copositive linear Lyapunov function for A, that is $Aw < 0$, then, for $h < h^\star$, $v(x)$ is a copositive linear Lyapunov function for $F(h)$, namely $F(h)w < w$.

PROOF.– First note that the (i, j)th entry $f_{ij}(h)$ of $F(h)$ can be written as:

$$f_{ij}(h) = ha_{ij}, \quad i \neq j,$$
$$f_{ii}(h) = 1 + ha_{ii}.$$

If $a_{ij} = 0, i \neq j$, then $f_{ij}(h) = 0$. This means that $F(h) \in \mathcal{S}$ for all $h > 0$. Moreover, since A is Metzler, the entries $a_{ij}, i \neq j$ are non-negative and hence $f_{ij}(h)$ are non-negative for all $h > 0$. In order $f_{ii}(h)$ to be non-negative, it is necessary and sufficient that $f_{ii}(h)$ are non-negative. Recalling that $a_{ii} < 0$ for all $i = 1, \ldots, n$ (due to stability), it means $h < h^\star$. Let now w be a strictly positive vector such $Aw < 0$ (it exists due to stability). Then, for it follows that $F(h)w < w$, this means that if $F(h)$ is non-negative, then it is also Schur stable, and this concludes the proof.

1.3.2. *The mixed Euler-ZOH discretization*

Consider system [1.1] and a sampling time $h > 0$. Considering matrix $A \in R^{n\times n}$ with entries a_{ij}, we define the *mE-ZOH* approximation $F(h)$ of e^{Ah} as follows:

$$F(h) = I + hD(h)A, \qquad [1.13]$$

and

$$hD(h) = \begin{bmatrix} \int_0^h e^{a_{11}t}dt & 0 & \cdots & 0 \\ 0 & \int_0^h e^{a_{22}t}dt & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \int_0^h e^{a_{mm}t}dt \end{bmatrix}.$$

REMARK 1.1.– Note that the continuous-time system [1.1] can be decomposed into n differential subsystems:

$$\dot{x}_i = a_{ii}x_i + \sum_{j \neq i} a_{ij}x_j. \qquad [1.14]$$

Transformation [1.13] comes from integrating each differential equation [1.14], by considering $x_j(t)$, $j \neq i$, as constant in the sampling interval, that is $x_j(t) = x_j(kh)$ for $t \in [kh, (k+1)h)$. This is a very natural approach to discretization in a distributed context; where there is access to local states, and where the "interconnected" states are communicated to each subsystem.

The following result deals with

1) the preservation of stability using [1.13], under suitable assumptions;

2) the proof that $A \in \mathcal{S} \to F(h) \in \mathcal{S}$ so that A and $F(h)$ in [1.13] share the same zero pattern of the off-diagonal entries;

3) the preservation of Lyapunov functions when using the *mE-ZOH* transformation [1.13].

Finally, note that $\lim_{h \to 0} D(h) = I$, so that the standard forward Euler discretization formula [1.11] can be recovered from [1.13].

THEOREM 1.5.– Let $A \in \mathcal{M}_c \cap \mathcal{S}$. Then, under [1.13], $F(h) \in \mathcal{M}_d \cap \mathcal{S}$, $\forall h > 0$. Furthermore, if $v(x) = \max_i \left(\frac{x_i}{w_i}\right)$, $w > 0$, is a copositive linear Lyapunov function for A, that is $Aw < 0$, then, for all h, $v(x)$ is a copositive linear Lyapunov function for $F(h)$, namely $F(h)w < w$.

PROOF.– First note that the (i, j) entry $f_{ij}(h)$ of $F(h)$ can be written as:

$$f_{ij}(h) = \left(\int_0^h e^{a_{ii}\tau} d\tau \right) a_{ij}, \quad i \neq j,$$

$$f_{ii}(h) = e^{a_{ii}h}.$$

If $a_{ij} = 0$, $i \neq j$, then $f_{ij}(h) = 0$. This means that $F(h) \in \mathcal{S}$ for all $h > 0$. Moreover, since A is Metzler, (1) the diagonal entries $e^{a_{ii}t} \geq 0, t \geq 0$; (2) the non-diagonal entries $f_{ij}(h)$ are non-negative for all $i, j = 1, \ldots, n$ and hence $F(h)$ is a non-negative matrix. Finally, since A is Hurwitz, there exists a strictly positive vector w such that $Aw < 0$. Hence, by noticing that $D(h)$ is a non-negative matrix, $F(h)w < w$, implying that $F(h)$ is a Schur stable matrix, for any $h > 0$.

REMARK 1.2.– Concerning the case where A is not necessarily a Metzler matrix, note first that the forward Euler transformation maps the eigenvalue λ_i of A to those of $F(h)$ in a very simple way, inherited by formula [1.11]. Indeed a simple computation shows that if A is Hurwitz stable, then $F(h)$ in [1.11] is Schur stable if and only if:

$$h < \hat{h} = \min_i \frac{-2Re(\lambda_i)}{|\lambda_i|^2}.$$

The map of the eigenvalues of the *mE-ZOH* transformation is much more complicated and cannot be given an explicit formula. In the particular case when the diagonal elements a_{ii} are all equal, say to a, then this formula can be easily found [COL 12]. We obtain, that, if A is Hurwitz stable, then $F(h)$ in [1.13] is Schur stable if and only if:

$$h < \tilde{h} = \begin{cases} \frac{1}{a} ln(1 + a\hat{h}), & |a| < \frac{1}{\hat{h}} \\ \infty & \text{otherwise} \end{cases}.$$

Note that, since $a < 0$ due to the stability assumption, $\tilde{h} \geq \hat{h}$ so that, even for non-positive systems, at least in the simple case of equal diagonal entries, it can be shown that the mE-ZOH transformation outperforms the forward Euler transformation in terms of stability preservation.

EXAMPLE 1.1.– Let

$$A = \begin{bmatrix} -1 & 0.5 & 1 \\ 0 & -1 & 0.5 \\ 0.5 & 0 & -1 \end{bmatrix}.$$

Figure 1.1 depicts the maximum modulus eigenvalue and the minimum diagonal entry of $F(h)$ as a function of h for three cases; (1) when using the forward Euler transformation (fE), (2) when using the mE-ZOH transformation and (3) when using the ZOH transformation.

The eigenvalues of A are $-0.19, -1.5$ and -1.309. Recalling remark 1.2, $F(h)$ given by the forward Euler transformation becomes unstable for $h \geq 4/3$, but, accordingly to [1.12], fails to be a non-negative matrix for $h \geq h^\star = 1$. On the other hand, the mE-ZOH transformation is always Schur stable and non-negative for all $h > 0$.

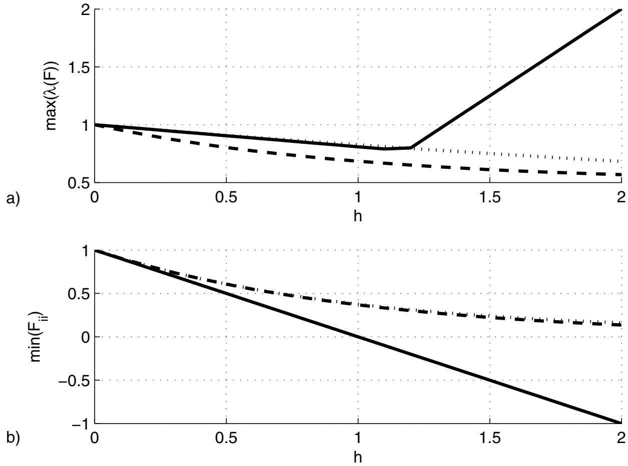

Figure 1.1. *a) Maximum modulus eigenvalue of $F(h)$; b) minimal diagonal entry of $F(h)$; solid line: fE, dotted line: mE-ZOH and dashed line: ZOH*

1.3.3. *The mixed Euler-ZOH discretization for switched systems*

Consider the switching continuous-time system [1.1], where the switching signal $\sigma_c(t)$ allows the system matrix to jump in the finite set $A_1, A_2, \cdots, A_m$. The matrices A_i are assumed to be Metzler, Hurwitz stable and sparse, that is $A_i \in \mathcal{M}_c \cap \mathcal{S}_i$, $i = 1, 2, \ldots, m$, where $\mathcal{S}_i$ indicates the sparsity structure of A_i. The (i, j) entry of matrix A_p is denoted by $a_{p,ij}$. The discrete-time switched system [1.2] is obtained using the *mE-ZOH* transformation in [1.13], where, associated with A_p, matrix $D(h)$ is replaced by the diagonal matrix $D_p(h)$, whose diagonal entries are $h^{-1} \int_0^h e^{a_{p,ii}t} dt$.

For systems [1.1] and [1.2] the following results hold (see [GUR 07]).

LEMMA 1.5.– System [1.1] with $A_p \in \mathcal{M}_c$, $\forall p$, is stable under arbitrary switching if there exists a vector $w > 0$ such that $A_p w < 0, \forall p$.

System [1.2] with $F_p \in \mathcal{M}_d$, $\forall p$, is stable under arbitrary switching if there exists a vector $w > 0$ such that $F_p w < w, \forall \sigma$.

In both the cases, $v(x) = \max_i \left(\frac{x_i}{w_i}\right)$ is a common copositive Lyapunov function.

We now provide conditions for stability under arbitrary switching for the discrete-time system [1.2]. Recall theorem 1.5, that is $A_p \in \mathcal{M}_c \cap \mathcal{S}_p \rightarrow F_p(h) \in \mathcal{M}_d \cap \mathcal{S}_p$. The following result can be easily proven.

THEOREM 1.6.– Let $A_p \in \mathcal{M}_c \cap \mathcal{S}_p$ for all $p = 1, \ldots, m$ and let w be a strictly positive vector. If $v(x) = \max_i \left(\frac{x_i}{w_i}\right)$ is a common copositive Lyapunov function for the continuous-time switched system [1.1], then it is also a common copositive Lyapunov function for the discrete-time switched system [1.2], and therefore system [1.2] is stable under arbitrary switching.

EXAMPLE 1.2.– Recalling example 1.1, let

$$A_1 = \begin{bmatrix} -1 & 0.5 & 1 \\ 0 & -1 & 0.5 \\ 0.5 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 & 0.5 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}.$$

Note that $A_p w < 0$, $p = 1, 2$, with $w' = \begin{bmatrix} 2 & 1 & 1.4 \end{bmatrix}$ so that the switched continuous-time system is stable under arbitrary switching with a copositive linear Lyapunov function $V(x) = \max_i \left(\frac{x_i}{w_i}\right)$. Let us consider the discrete-time system obtained by the *mE-ZOH* transformation, that is:

$$F_1(h) = \begin{bmatrix} e^{-h} & 0.5(1-e^{-h}) & 1-e^{-h} \\ 0 & e^{-h} & 0.5(1-e^{-h}) \\ 0.5(1-e^{-h}) & 0 & e^{-h} \end{bmatrix},$$

$$F_2(h) = \begin{bmatrix} e^{-2h} & 0.5(1-e^{-2h}) & 0.25(1-e^{-2h}) \\ 0 & e^{-2h} & 0.5(1-e^{-2h}) \\ 0.5(1-e^{-2h}) & 0 & e^{-2h} \end{bmatrix}.$$

It is possible to verify that $F_p(h)w < w$, $p = 1, 2$, so that the discrete-time system is stable under arbitrary switching with the same Lyapunov function.

The following example considers a switched system, taken from [FAI 09], that is not stable under arbitrary switching even though any convex combination of the two matrices is Hurwitz stable.

EXAMPLE 1.3.– Let

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 0 & -10 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -10 & 0 & 10 \\ 0 & -10 & 0 \\ 0 & 10 & -1 \end{bmatrix}.$$

This continuous-time system is not stable under arbitrary switching since, for example, $e^{A_1}e^{A_2}$ is not a Schur stable matrix. It can be shown that the switched system is stable for any switched systems with dwell time $\tau \geq 1.44$. This means that it is stable for all switching sequences with switching instants satisfying $t_{k+1} - t_k \geq 1.44$.

Using the mE-ZOH transformation, we get a switched system with

$$F_1(h) = \begin{bmatrix} e^{-h} & 0 & 0 \\ 10(1-e^{-h}) & e^{-h} & 0 \\ 0 & 0 & e^{-10h} \end{bmatrix},$$

$$F_2(h) = \begin{bmatrix} e^{-10h} & 0 & 1-e^{-10h} \\ 0 & e^{-10h} & 0 \\ 0 & 10(1-e^{-h}) & e^{-h} \end{bmatrix}.$$

In Figure 1.2, we show the plot of the dwell time $T_{\text{dwell}}(h)$ for the discretized systems [1.2] obtained with *mE-ZOH* and with ZOH as a function of the sampling interval h. As expected, if $h \geq 1.44$, the dwell time for the discrete-time system in case of ZOH transformation is one step. On the other hand, for the *mE-ZOH* transformation $T_{\text{dwell}}(h) \geq 2$ for all $h > 0$, which implies that the switched system is not stable under arbitrarily switching laws for any h.

Furthermore, in Figure 1.3, we depict the plot of the maximum modulus eigenvalue of $F_1(h)$, $F_2(h)$ and of $F_1^{T_1}(h)F_2^{T_2}(h)$, with T_1 and T_2 spanning from 1 to $T_{\text{dwell}}(h)$, for the cases of fE, *mE-ZOH* and ZOH transformations. As expected, the discrete-time switching system obtained with the ZOH transformation, that is with $F_1(h) = e^{A_1 h}$ and $F_2(h) = e^{A_2 h}$, with $h \geq 1.44$, is stable under arbitrary switching. Furthermore, using the forward Euler transformation with $h = 1.44$, neither $F_1(h)$ nor $F_2(h)$ are Schur stable. Finally, in case of *mE-ZOH* matrices, $F_1(h)$ and $F_2(h)$ are Schur stable for any $h > 0$ (indeed it is easy to see that their eigenvalues correspond to those of $e^{A_1 h}$ and $e^{A_2 h}$), while $F_1^{T_1}(h)F_2^{T_2}(h)$ is unstable for suitable choices of T_1 and T_2. Note that however such a system admits a dwell time equal to 2 for $h \geq 5.2$.

As a final remark, note that theorem 1.6 ensures the preservation of copositive Lyapunov function only. There exist switched systems that are stable under arbitrary switching but do not admit a common copositive Lyapunov function. For such systems, the properties of the *mE-ZOH* transformation should be further explored.

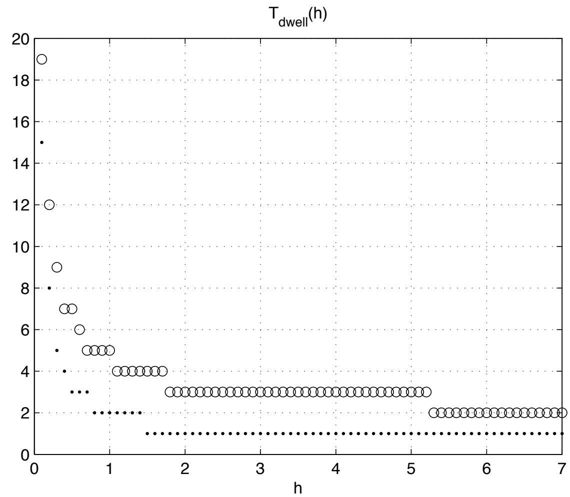

Figure 1.2. *Dwell time $T_{dwell}(h)$ for the discrete-time systems* [1.2] *obtained with mE-ZOH (circles) and with ZOH (dots) as a function of the sampling interval h*

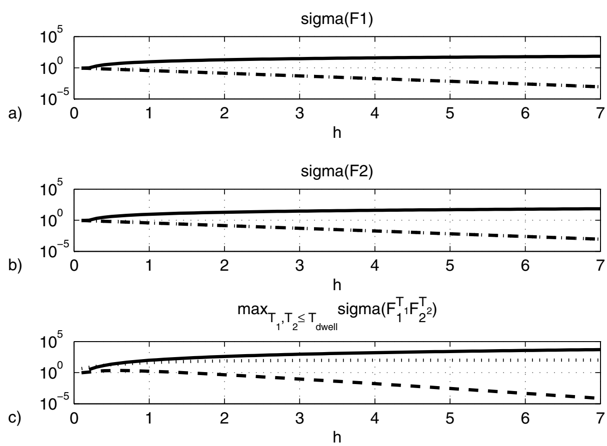

Figure 1.3. *a) Maximum modulus eigenvalue of $F_1(h)$; b) maximum modulus eigenvalue of $F_2(h)$; c) maximum modulus eigenvalue of $F_1^{T_1}(h)F_2^{T_2}(h)$ for $T_1, T_2 = 1, \ldots, T_{dwell}$; solid line: fE, dotted line: mE-ZOH and dashed line: ZOH*

1.4. Conclusions

This chapter addressed the problem of analyzing different discretization methods.

First, the suitability of diagonal Padé transformations for positive systems has been examined. Unfortunately, the results of this investigation are uniformly bad. In particular, a number of problems with this transformation have been noted, and an alternative method has been presented that avoids these pitfalls.

Second, the *mE-ZOH* transformation has been introduced and studied. The main merit of this transformation is to outperform stability preservation (with respect to the traditional Euler transformation) besides preserving the matrix positivity and sparseness properties. This is in particular appealing in the analysis of positive systems since it has been shown that stability is preserved independent of the sampling period. The case of switched systems is also addressed. Future works include the use of channel-distributed sampling times in many applications and a closer analysis of the eigenvalues of the discrete-time systems in the more general case of non-positive systems.

1.5. Bibliography

[ALO 10] ALONSO H., ROCHA P., "A general stability test for switched positive systems based on a multidimensional system analysis", *IEEE Transactions on Automatic Control*, vol. 55, pp. 2660–2666, 2010.

[ANG 03] ANGELI D., SONTAG E.D., "Monotone control systems", *IEEE Transactions on Automatic Control*, vol. 48, pp. 1684–1698, 2003.

[ANT 05] ANTOULAS A., *Approximation of Large-Scale Dynamical Systems*, SIAM Advances in Design and Control, 2005.

[BAR 89] BARNSLEY M.F., DEMKO S.G., ELTON J.H., *et al.*, "Erratum Invariant measures for Markov processes arising from iterated function systems with place-dependent probabilities", *Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques*, vol. 25, no. 4, pp. 589–590, 1989.

[BAU 10] ANN-KRISTIN BAUM, "Positivity preserving simulation of differential-algebraic equations", *Proc. Numerical Solution of Differential-Algebraic Equations (NUMDIFF-13)*, Martin-Luther Halle-Wittemberg (Germany), 10.14 September 2012.

[BER 94] BERMAN A., PLEMMONS R.J., *Nonnegative Matrices in the Mathematical Sciences*, SIAM, 1994.

[BOL 78] BOLLEY C., CROUZEIX M., “Conservation de la positivite lors de la discrétisation des problèmes d'évolution paraboliques”, *R.A.I.R.O. Analyse Numerique*, vol. 12, no. 3, pp. 81–88, 1978.

[BUT 02] BUTCHER J.C., “The A-Stability of methods with Padé and generalized Padé stability functions”, *Numerical Algorithms*, vol. 31, pp. 47–58, 2002.

[COL 12] COLANERI P., FARINA M., SCATTOIINI R., *et al.*, “A note on discretization of sparse linear systems”, *Proceedings of the IFAC Workshop ROCOND 2012*, Aalborg, Denmark, vol. 54, 2012.

[FAI 09] FAINSHIL L., MARGALIOT M., CHIGANSKY P., “On the stability of positive linear switched systems under arbitrary switching laws”, *IEEE Transactions on Automatic Control*, vol. 54, pp. 807–809, 2009.

[FAL 08] FALCONE P., TSENG H.E., BORRELLI F., *et al.*, “MPC-based yaw and lateral stabilisation via active front steering and braking”, *Vehicle System Dynamics*, vol. 46, pp. 611–628, 2008.

[FAR 00] FARINA L., RINALDI S., *Positivie Linear Systems*, John Wiley & Sons, New York, NY, 2000.

[FAR 12] FARINA M., SCATTOLINI R., “Distributed predictive control: a non-cooperative algorithm with neighbor-to-neighbor communication”, *Automatica*, vol. 48, pp. 1088–1096, 2012.

[GUR 07] GURVITS L., SHORTEN R., MASON O., “On the stability of switched positive linear systems”, *IEEE Transactions on Automatic Control*, vol. 52, no. 6, pp. 1099–1103, 2007.

[HAR 02] HARTFIEL D., *Nonhomogeneous Matrix Products*, World Scientific, 2002.

[LUN 92] LUNZE J., *Feedback Control of Large Scale Systems*, Systems and Control Engineering, Prentice Hall, Upper Saddle River, NJ, 1992.

[ROS 09] ROSSI F., COLANERI P., SHORTEN R., “Pade discretization for linear systems with polyhedral Lyapunov functions”, *IEEE Transactions on Automatic Control*, vol. 56, pp. 2717–2722, 2009.

[SAJ 11] SAJJA S., CORLESS M., SHORTEN R., *et al.*, “On Pade discretization and preservaitions of quadratic stability for switched linear systems”, *Systems and Control Letters*, vol. 60, pp. 683–689, 2011.

[SCA 09] SCATTOLINI R., “Architectures for distributed and hierarchical model predictive control – a review”, *Journal of Process Control*, vol. 19, no. 5, pp. 723–731, 2009.

[SHO 06] SHORTEN R., WIRTH F., LEITH D.J., “A positive systems model of TCP-like congestion control: asymptotic results”, *IEEE/ACM Transactions on Networking*, vol. 14, no. 3, pp. 616–629, 2006.

[SHO 07] SHORTEN R., MASON O., “On linear copositive Lyapunov functions and the stability of switched positive linear systems”, *IEEE Transactions on Automatic Control*, vol. 52, pp. 1346–1349, 2007.

[SIL 91] SILJAC D.D., *Decentralized Control of Complex Systems*, Academic Press, Inc., New York, NY, 1991.

[VAD 03] VADIGEPALLI R., DOYLE III F.J., "Structural analysis of large-scale systems for distributed state estimation and control applications", *Control Engineering Practice*, vol. 11, pp. 895–905, 2003.

[WES 01] WESTPHAL L.C., *Handbook of Control Systems Engineering*, Springer, 2001.

[ZAP 10] ZAPPAVIGNA A., Stability analysis and stabilization of switched positive linear systems, PhD Thesis, Politecnico di Milano, 2010.

[ZAP 12] ZAPPAVIGNA A., COLANERI P., KIRKLAND S., *et al.*, "Essentially negative news about positive systems", *Linear Algebra and its Applications*, vol. 436, pp. 3425–3442, 2012.