Chapter 1

Origins and Topicality of a Concept

Limit state design is, to some extent, a familiar terminology within the syllabuses of civil engineers' education, as it appears explicitly in the stability analyses of various types of structures or is present "anonymously" in the methods used for such analyses. Nevertheless, the variety of the corresponding approaches often makes it difficult to recognize that they proceed from the same fundamental principles, which are now the basis of the ultimate limit state design (ULSD) approach to the safety analysis of structures. As an introduction to the theory, this chapter will both present some famous historical milestones and the topicality of the subject referring to the principles of ULSD.

1.1. Historical milestones

1.1.1. *Dialogs concerning two new sciences*

The fundamental concept to be acknowledged first is that of *yield strength* as introduced by Galileo in his *Discorsi* [GAL 38a] on the simple experiment of a specimen in pure tension (Figure 1.1).

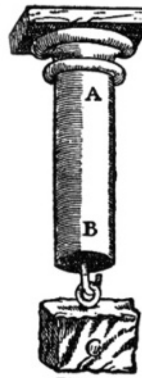

Figure 1.1. *Longitudinal pull test (Galileo, Discorsi, 1st day [GAL 38a])*

Galileo uses this first characterization of the tenacity and coherence (*tenacità e coerenza*) of the material to explain the difficulty he finds in breaking a rod or a beam in tension while it is far easier to break it in bending: "*A prism or solid cylinder of glass, steel, wood or other breakable material which is capable of sustaining a very heavy weight when applied longitudinally is, as previously remarked, easily broken by the transverse application of a weight which may be much smaller in proportion as the length of the cylinder exceeds its thickness*". Considering a cantilever beam (Figure 1.2) built in a wall (section *AB*) and subjected to a weight applied at the other extremity (section *CD*), he first defines the "*absolute resistance to fracture as that offered to a longitudinal pull*". Then, he assumes that this resistance to tension will be localized in the section of the beam where it is fastened to the wall and that "*this resistance opposes the separation of the part BD lying outside the wall, from that portion lying inside*". The reasoning follows "*it is clear that if the cylinder breaks, fracture will occur at the point B where the edge of the mortise acts as a fulcrum for the lever BC*" [GAL 38a]. Introducing the second fundamental concept of the yield design approach, namely *equilibrium*, by writing the balance equation for the lever about *B*, Galileo finally relates the "*absolute resistance of the prism BD*" to its "*absolute resistance to fracture*" through the ratio of the short lever arm *BA/*2 to the long lever arm *BC*.

Figure 1.2. *Prism subjected to the transverse application of a weight (Galileo, Discorsi, 2nd day)*

Galileo's reasoning has been criticized, as shown in Figure 1.3, on the basis that the equilibrium of the cross-section *BA* is not satisfied.

* The one fundamental error which is implicitly introduced into this proposition and which is carried through the entire discussion of the Second Day consists in a failure to see that, in such a beam, there must be equilibrium between the forces of tension and compression over any cross-section. The correct point of view seems first to have been found by E. Mariotte in 1680 and by A. Parent in 1713. Fortunately this error does not vitiate the conclusions of the subsequent propositions which deal only with proportions—not actual strength—of beams. Following K. Pearson (Todhunter's *History of Elasticity*) one might say that Galileo's mistake lay in supposing the fibres of the strained beam to be inextensible. Or, confessing the anachronism, one might say that the error consisted in taking the lowest fibre of the beam as the neutral axis. [*Trans.*]

Figure 1.3. *Translator note [GAL 54, p. 115]*

Staying within the framework of the yield design approach for the beam, the criticism amounts to pointing out that the global equilibrium equation for the horizontal resultant force has not been taken into consideration. As a matter of fact, by focusing his attention only on the moment equation for the global equilibrium of the beam, Galileo obtains a necessary condition for the beam to sustain the load in a model where the constituent material is considered at the mesoscale of the section, with its resistance determined through the longitudinal pull test, and not at a more local level such as the longitudinal fibers as the criticism in Figure 1.3 would require: this is consistent with the fact that resistance to compression is never referred.

1.1.2. *Note on an application of the rules of maximum and minimum to some statical problems, relevant to architecture*

The appearance of soil mechanics as an engineering science is often associated with Coulomb's memoir [COU 73] presented to the French Academy of sciences in 1773 after Coulomb returned from his eight year period in Martinique as a lieutenant in the French military corps of engineers. This *Essay* was devoted to various problems that he had encountered when building the "Fort Bourbon": stability of pillars, arches and vaults, calculation of earth pressure on retaining walls, etc. (Figure 1.4).

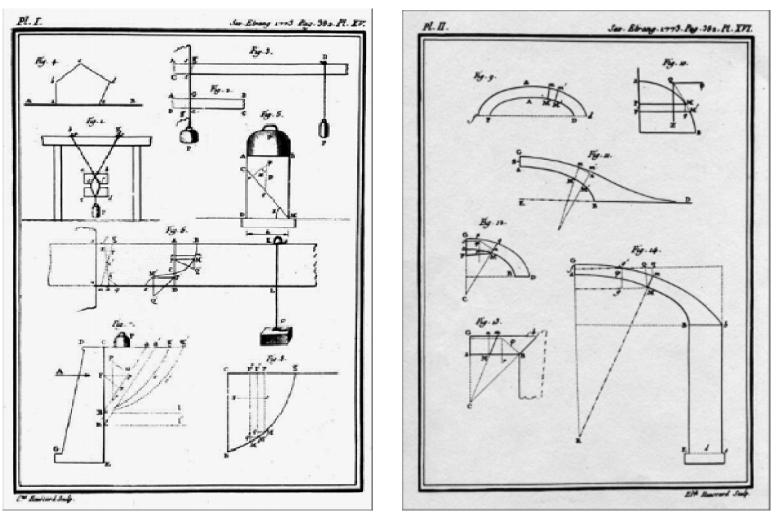

Figure 1.4. *Figure plates in Coulomb's Essay [COU 73]*

The first guiding idea of Coulomb's rationale in tackling these problems is making a clear distinction between the *active forces*, which are the prescribed loads acting on the structure under consideration, and the characteristics of *resistance* of the material, which set the bounds to the "*coherence*" forces that can be mobilized (Figure 1.5).

I V.

Du Frottement.

Le frottement & la cohéſion ne ſont point des forces actives comme la gravité, qui exerce toujours ſon effet en entier, mais ſeulement des forces coërcitives; l'on eſtime ces deux forces par les limites de leur réſiſtance. Lorſqu'on dit, par exemple, que dans certains bois polis, le frottement ſur un plan horizontal d'un corps peſant neuf livres, eſt trois livres; c'eſt dire que toute force au-deſſous de trois livres ne troublera point ſon état de repos.

Je ſuppoſerai ici que la réſiſtance dûe au frottement eſt proportionnelle à la preſſion, comme l'a trouvé M. Amontons; quoique dans les groſſes maſſes le frottement ne ſuive pas exactement cette loi. D'après cette ſuppoſition, l'on trouve dans les briques le frottement, les trois quarts de la preſſion. Il ſera bon de faire des épreuves ſur les matériaux que l'on voudra employer. Il eſt impoſſible de fixer ici le frottement

Xx ij

5

Figure 1.5. *Defining friction and cohesion in Coulomb's Essay [COU 73]*

The second guiding idea is that the resistance forces are exerted locally along an assumed failure surface, anticipating, to a certain extent, the concept of the stress vector to be introduced some 50 years later. In the simple case of a stone column under a compressive load (Figure 1.4), Coulomb explains the principles of the analysis: the active force on the assumed fracture surface must be balanced by the "coherence" force; the fracture surface will be determined through a minimization process.

On the basis of the same principle, Coulomb's stability analysis of a retaining wall is a fundamental landmark for the theory of yield design. Coulomb starts with the celebrated "Coulomb's wedge" reasoning (Figures 1.4, 1.6), where he assumes the failure surface to be plane and states a condition for stability that the active forces on the assumed fracture surface *Ba* must be balanced by the "coherence" forces, from which he derives, through minimization and maximization processes, two bounds for the horizontal force that can be applied to *CB* so that the wall be stable. Because of its simplicity, this reasoning is often presented as the Coulomb analysis of the stability of a retaining wall. In fact, Coulomb, after showing how the friction along the wall could be taken into account, states that, to be complete, the analysis should look for the curve that produces the highest pressure on *CB* and sketches the process for this determination.

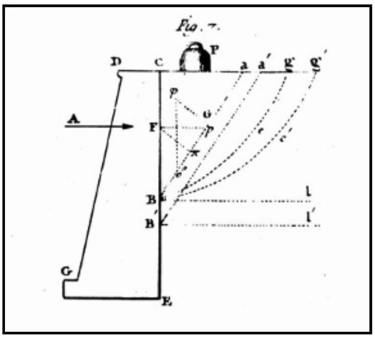

Figure 1.6. *Coulomb's wedge [COU 73]*

1.1.3. *Compatibility between equilibrium and resistance*

It is not difficult to point out the common features of the analyses that have been briefly presented here.

– First, the concept of *resistance* is introduced as a mechanical characteristic of the constituent material. After having been determined through a given simple experiment, it is used in any other circumstances and sets the *limits* to the resisting forces that can be actually mobilized.

– Then, the idea that the resistance of a given structure – a result at the *global* level – can be derived from the knowledge of the resistance of its constituent material(s), which is a property at the *local* level.

– For this determination, the rationale is based upon the statement that equilibrium equations of the structure must be satisfied while complying with the limits imposed by the resistance of the constituent material(s). In other words, *equilibrium and resistance must be mathematically compatible*.

– The practical implementation of this statement is made through the choice or the assumption of some particularly crucial zone in the structure (cross-section in the first case and failure surface in the second case), where it is anticipated that compatibility between equilibrium and resistance should be checked.

As it is shown in Figure 1.3 in the case of Galileo's analysis, it may be objected that such approaches do not take into account the behavior of the material, that is the fact that the material deforms under the forces it is subjected to. But it must be recalled that although the concept of linear elasticity was first introduced by Hooke in the 1660s, it was only in 1807 that Young's recognized shear as an elastic deformation; three-dimensional linear elasticity itself was only really formalized in the 1820s (Navier, Cauchy and others) at the same time as the concept of the stress tensor. As noted before, the yield design approach implicitly embodies an anticipation of the concept of internal forces. This is not surprising since the intuition of internal forces is primarily linked to that of rupture being localized on surfaces or lines as observed on full-, reduced- or small-scale experiments (Figures 1.7 and 1.8).

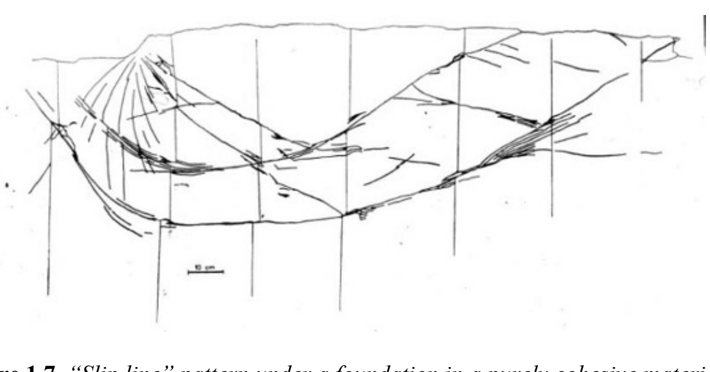

Figure 1.7. *"Slip line" pattern under a foundation in a purely cohesive material (medium-scale experiment) [HAB 84]*

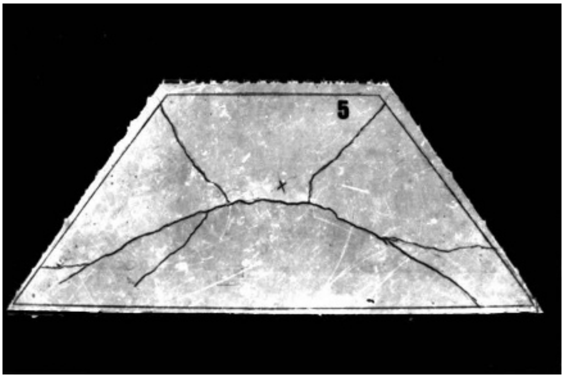

Figure 1.8. *Bending of a reinforced plaster slab: evidence of hinge curves (M. Milicevic)*

1.2. Topicality of the yield design approach

1.2.1. *The Coulomb's Essay legacy*

Coulomb's *Memoir* was at the origin of many methods used by engineers for the stability analyses of various types of structures. In the case of masonry vaults, the works by Méry [MER 40] and Durand-Claye [DUR 67, DUR 80] have been extensively studied by Heyman [HEY 66, HEY 69, HEY 72, HEY 80, HEY 82, HEY 98] and Delbecq [DEL 81, DEL 82]: it is interesting to note that they often combined Coulomb's original reasoning with elastic arguments, thus losing its original theoretical meaning without any damage from the practical point of view.

Soil mechanics, which is sometimes considered as having found its very origin in Coulomb's *Memoir*, exhibits numerous methods clearly related to it for the stability analysis of slopes, retaining walls, fills and earth dams or for the calculation of the bearing capacity of the surface foundations [BER 52, BIS 54, BØN 77, BRI 53, BU 93, CHA 07, CHE 69a, CHE 69b, CHE 70a, CHE 70b, CHE 73a, CHE 75a, CHE 75b, COU 79, JOS 80, DRU 52, GRE 49, HIL 50, HOU 82, KÖT 03, KÖT 09, LAU 11, MAN 72, MAR 05, MAR 09, MAS 99, MAT 79, MEY 51, MEY 53, MEY 63, MIC 98, MIC 09, PRA 55, REN 35, SAL 74, SAL 76, SAL 82, SAL 85, SAL 95a, SAL 95b, SAL 06, SOK 55, SOK 60, SOK 65, SAR 91, TAY 37a, TAY 37b, UKR 98], including the limit equilibrium methods and the slip line methods, which were also applied to solving metal forming problems. Finite element methods have also been developed and used extensively within this framework for applications to soil mechanics and to some related problems [AND 72, DEL 77, FRÉ 73, KAM 10, KRA 03, KRA 05, LYA 02a, LYA 02b, LYS 70, MAK 06, MAK 07, MAK 08, MAR 11, PAS 09].

Another field of application is the bearing capacity of metallic plates and reinforced concrete slabs through the yield hinges theory as developed by Johansen, Save, Massonnet and others [JOH 31, JOH 43, MAS 63, SAV 73, SAV 95, BRA 07].

Considerable attention has been devoted by Chen, Drucker and co-authors to applying the theorems of limit analysis to the determination of the bearing capacity of concrete blocks and fiber reinforced concrete [CHE 69c, CHE 70c, CHE 71, CHE 73b, CHE 74].

More recently, it has been applied to the determination of the resistance of long fiber composites from the knowledge of the resistances of the components through a homogenization process leading to the definition and determination of a homogenized yield criterion [BU 86a, BU 98, BU 89, BU 86b, BU 90, BU 91, SUQ 82, SUQ 83].

1.2.2. *Topicality*

Obviously, the yield design approach did play a highly important role in civil engineering and construction as a scientific approach before the theory of elasticity was elaborated and could be practically implemented for the design of structures. We may wonder now about its topicality, taking into account both the constant improvement of the formulation and determination of constitutive laws and the development of computational methods and tools that can be applied to determine the behavior of a structure along a given loading path. It must be understood that there is no inconsistency between the different approaches provided they are used within their proper domain of validity, depending on the available data, and with their results interpreted accordingly. Moreover, the yield design approach proves quite efficient for back calculations after the collapse of a structure without knowing the exact circumstances of its occurrence.

Recent construction codes such as the Eurocodes are based on the concept of limit state design that includes ULSD, the principle of which may be stated as follows [OVE 89]:

> *The design criterion is simply to design for equilibrium [under the design loads] in the design limit state of failure. The design criterion could be expressed in the following way:*
>
> $R_d \geq S_d$

which means that the *design load effect* S_d should be inferior to the *effect of the design resistances* R_d.

Three words are familiar to us in this statement, namely "*equilibrium*", "*loads*" and "*resistances*", as a follow up to Coulomb's *Memoir*. The word "*design*" needs to be explained and "*effect*" must be defined. As far as *design* is concerned, it means that the values that are considered for the design and the dimensioning of the structures are not the actual values of the loads or of the resistances but conventional values derived from them through properly chosen partial safety coefficients ("partial factors") and thus setting the "rules of the game". Regarding the *effect*, it must be quantified as a scalar in order to make the inequality practically meaningful.

Because of the theoretical basis of the ULSD approach to safety provided by the theory of yield design [SAL 94], it is possible[1] to make the necessary clear distinction between the active forces and the resisting forces, exactly in the same spirit as explained by Coulomb more than 200 years ago. Also, through a quantified definition of the

1 See Chapter 7.

effects, it provides at the same time, scientifically consistent and efficient methods for its implementation [ANT 91, SIM 09].

1.3. Bibliography

[AND 72] ANDERHEGGEN E., KNÖPFEL H., "Finite element limit analysis using linear programming", *International Journal of Solids and Structures*, vol. 8, pp. 1413–1431, 1972.

[ANT 91] ANTHOINE A., DE BUHAN P., DORMIEUX L., *et al.*, *STARS 2.00. Computer Code for the Stability analysis of Earth Structures*, Presses de l'École Nationale des Ponts et Chaussées, Paris, 1991.

[BER 52] BEREZANCEW B.G., *Axisymmetric Problem of the Theory of Limiting Equilibrium of a Granular Medium*, Gostekhizdat, Moscow 1952.

[BIS 54] BISHOP A.W., "The use of the slip circle in the stability analysis of slopes", *Géotechnique*, vol. 5, no. 1, pp. 7–17, 1955.

[BØN 77] BØNDING N., "Kinematical admissibility of the pure N_γ rupture figure", *Proceedings of the 9th International Conference on Soil Mechanics and Foundation Engineering*, vol. 1, pp. 415–418, 1977.

[BU 86a] DE BUHAN P., Approche fondamentale du calcul à la rupture des ouvrages en sols renforcés, PhD Thesis, Pierre and Marie Curie University, Paris, 1986.

[BU 86b] DE BUHAN P., SALENÇON J., SIAD L., "Critère de résistance pour le matériau terre armée", *C. R. Ac. Sc. Paris,* vol. 302, no. II, pp. 377–381, 1986.

[BU 89] DE BUHAN P., MANGIAVACCHI R., NOVA R., *et al.*, "Yield design of reinforced earth walls by homogenisation method", *Géotechnique*, vol. 39, no. 2, pp. 189–201, 1989.

[BU 90] DE BUHAN P., SALENÇON J., TALIERCIO A., "Lower and upper bound estimates for the macroscopic strength criteria of fiber composite materials", *Proceedings of IUTAM Symposium on Inelastic Deformation of Composite Materials*, 29 May–1 June, Springer Verlag, Troy, NY, pp. 563–580, 1990.

[BU 91] DE BUHAN P., TALIERCIO A., "A homogenization approach to the yield strength of composite materials", *European Journal of Mechanics A/Solids*, vol. 10, no. 2, pp. 129–154, 1991.

[BU 93] DE BUHAN P., SALENÇON J., "A comprehensive stability analysis of soil nailed structures", *European Journal of Mechanics A/Solids*, vol. 12, no. 3, pp. 325–345, 1993.

[BU 98] DE BUHAN P., DORMIEUX L., SALENÇON J., "Modélisation multipolaire de la résistance d'un milieu renforcé par inclusions", *C. R. Ac. Sc. Paris*, vol. 326, no. IIb, pp. 163–170, 1998.

[BRA 07] BRAESTRUP M.W., "Yield line theory and concrete plasticity", *Proceedings of the Morley Symposium on Concrete Plasticity and its Application*, University of Cambridge, pp. 43–48, 23 July 2007.

[BRI 53] BRINCH-HANSEN J., *Earth Pressure Calculation*, Danish Technical Press, Copenhagen, 1953.

[CHA 07] CHATZIGOGOS C.T., PECKER A., SALENÇON J., "Seismic bearing capacity of a circular footing on a heterogeneous soil", *Soils and Foundations*, vol. 47, no. 4, pp. 783–797, 2007.

[CHE 69a] CHEN W.F., "Soil mechanics and theorems of limit analysis", *Journal of the Soil Mechanics and Foundations Division*, *ASCE*, vol. 95, no. SM2, pp. 493–518, 1969.

[CHE 69b] CHEN W.F., GIGER M.W., FANG H.Y., "On the limit analysis of stability of slopes", *Soils and Foundations*, vol. 9, no. 4, pp. 23–32, 1969.

[CHE 69c] CHEN W.F., DRUCKER D.C., "Bearing capacity of concrete blocks or rock", *J. Eng. Mech. Found. Div., ASCE*, vol. 95, no. EM4, pp. 955–978, 1969.

[CHE 70a] CHEN W.F., SCAWTHORN C.R., "Limit analysis and limit equilibrium solutions in soil mechanics", *Soils and Foundations*, vol. 10, no. 3, pp. 13–49, 1970.

[CHE 70b] CHEN W.F., "Discussion on circular and logarithmic spiral slip surfaces", *Journal of the Soil Mechanics and Foundations Division, ASCE,* vol. 96, no. SM1, pp. 324–326, 1970.

[CHE 70c] CHEN W.F., "Extensibility of concrete and theorems of limit analysis", *J. Eng. Mech. Found. Div.*, *ASCE*, vol. 96, no. EM3, pp. 341–352, 1970.

[CHE 71] CHEN W.F., COVARRUBIAS S., "Bearing capacity of concrete blocks", *J. Eng. Mech. Found. Div.*, *ASCE*, vol. 96, no. EM5, pp. 1413–1430, 1971.

[CHE 73a] CHEN W.F., "Bearing strength of concrete blocks", *J. Eng. Mech. Found. Div.*, *ASCE,* vol. 99, no. EM6, pp. 1314–1321, 1973.

[CHE 73b] CHEN W.F., ROSENFARB J.L., “Limit analysis solutions of earth pressure problems”, *Soils and Foundations*, vol. 13, no. 4, pp. 45–60, 1973.

[CHE 74] CHEN W.F., CARSON J.L., “Bearing capacity of fiber reinforced concrete”, *International Symposium on Fiber Reinforced Concrete,* ACI Special Publication, SP-44-12, pp. 209–220, 1974.

[CHE 75a] CHEN W.F., SNITBHAN N., FANG H.Y., “Stability of slope in anisotropic nonhomogeneous soils”, *Canadian Geotechnical Journal,* vol. 12, no. 1, pp. 146–152, 1975.

[CHE 75b] CHEN W.F., *Limit Analysis and Soil Plasticity*, Elsevier, 1975.

[COU 73] COULOMB C.-A., Essai sur une application des règles de Maximis et Minimis à quelques problèmes de statique relatifs à l’architecture. *Mémoires de Mathématiques et de Physique présentés à l’Académie Royale des Sciences*, vol. 7, pp. 343–382, 1776.

[COU 79] COUSSY O., SALENÇON J., “Analyse de la stabilité des ouvrages en terre par le calcul à la rupture”, *Annales des Ponts et Chaussées*, vol. 12, pp. 7–35, 1979.

[DEL 77] DELBECQ J.M., FREMOND M., PECKER A., *et al.*, “Éléments finis en plasticité et viscoplasticité”, *Journal de Mécanique Appliquée*, vol. 1, no. 3, pp. 267–304, 1977.

[DEL 81] DELBECQ J.-M., “Analyse de la stabilité des voûtes en maçonnerie de Charles-Augustin Coulomb à nos jours”, *Ann. Pts et Ch., Journal de Mécanique Appliquée*, vol. 19, pp. 36–43, 1981.

[DEL 82] DELBECQ J.-M., “Analyse de la stabilité des voûtes en maçonnerie par le calcul à la rupture”, *J. Mec. Appl.*, vol. 1, no. 1, pp. 91–121, 1982.

[DRU 52] DRUCKER D.C., PRAGER W., “Soil mechanics and plastic analysis or limit design”, *Quarterly of Applied Mathematics*, vol. 10, no. 2, pp. 157–165, 1952.

[DUR 67] DURAND-CLAYE A.,“Stabilité des voûtes en maçonnerie”, *Ann. Pts et Ch.*, vol. I, pp. 63–96, 1867.

[DUR 80] DURAND-CLAYE A., “Stabilité des voûtes et des arcs”, *Ann. Pts et Ch.*, vol. I, pp. 416–440, 1880.

[FRÉ 73] FRÉMOND M., SALENÇON J., “Limit analysis by finite-element method”, in PALMER A.C. (ed.), *Proceedings of the Symposium on the Role of Plasticity in Soil Mechanics*, Cambridge, UK, pp. 297–308, September 1973.

[GAL 38a] GALILEI G., *Discorsi e dimostrazionimatematicheintorno à due nuovescienze*, Elsevier, Leiden, 1638.

[GAL 54] GALILEI G., *Dialogs Concerning Two New Sciences*, CREW H., SALVIO A., (transl. 1954), Dover Publ. Inc., New York, 1638.

[GRE 49] GREENBERG H.J., PRAGER W., "On limit design of beams and frames", Technical Report A 18, 1, 1, Brown University, *Trans. ASCE,* vol. 117, pp. 447–484, 1952.

[HAB 84] HABIB P., "Les surfaces de glissement en mécanique des sols", *Revue française de géotechnique*, vol. 27, pp. 7–21, 1984.

[HEY 66] HEYMAN J., "The stone skeleton", *International Journal of Solids and Structures*, vol. 2, no. 2, pp. 249–279, 1966.

[HEY 69] HEYMAN J., "The safety of masonry arches", *International Journal of Mechanical Sciences*, vol. 11, pp. 363–385, 1969.

[HEY 72] HEYMAN J., *Coulomb's Memoir on Statics. An Essay in the History of Civil Engineering*, Cambridge University Press, 1972.

[HEY 80] HEYMAN J., "The estimation of the strength of masonry arches", *Proceedings of the American Society of Civil Engineers*, vol. 69, no. 2, pp. 921–937, 1980.

[HEY 82] HEYMAN J., *The Masonry Arch*, Ellis Horwood Ltd., John Wiley, Chichester, UK, 1982.

[HEY 98] HEYMAN J., *Structural Analysis*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 1998.

[HIL 50] HILL R., *The Mathematical Theory of Plasticity*, Clarendon Press, Oxford, 1950.

[HOU 82] HOULSBY G.T., WROTH C.P., "Direct solution of plasticity problems in soils by the method of characteristics", *Proceedings of the 4th International Conference on Numerical Methods in Geomechanics,* Edmonton, vol. 3, pp. 1059–1071, 1982.

[JOH 31] JOHANSEN K.W., "Beregning af krydsarmerede jernbetonpladers brudmoment", *Bygningsstatiske Meddelelser*, vol. 3, no. 1, pp. 1–18, 1931.

[JOH 43] JOHANSEN K.W., *Brudlinieteorier*, Gjellerup, Copenhagen, pp. 189, 1943 (English translation: *Yield-Line Theory*, Cement and Concrete Association, London, 1962).

[JOS 80] DE JOSSELIN DE JONG G., "Application of the calculus of variations to the vertical cut off in cohesive frictionless soil", *Géotechnique*, vol. 30, no. 1, pp. 1–16, 1980.

[KAM 10] KAMMOUN Z., PASTOR F., SMAOUI H., *et al.*, “Large static problem in numerical analysis: a decomposition approach”, *International Journal for Numerical and Analytical Methods in Geomechanics*, vol. 34, no. 18, pp. 1960–1980, 2010.

[KRA 03] KRABBENHOFT K., DAMKILDE L., “A general optimization algorithm for lower bound limit analysis”, *International Journal for Numerical Methods in Engineering*, vol. 56, pp. 165–184, 2003.

[KRA 05] KRABBENHOFT K., LYAMIN A.V., HIJAJ M., *et al.*, “A new discontinuous upper bound analysis formulation”, *International Journal for Numerical Methods in Engineering*, vol. 63, pp. 1069–1088, 2005.

[KÖT 03] KÖTTER W.T., “Die Bestimmung des Druckes an den gekrümmten Gleitflächen, eine Aufgabe aus der Lehre vom Erddruck”, *Berlin Akademie Bericht,* pp. 229–233, 1903.

[KÖT 09] KÖTTER W.T., “Über den Druck von Sand gegen Öffnungsverschlüsse im horizontalen Boden kastenförmiger Gefäße”, *Berlin Akademie Bericht*, pp. 493–510, 1909.

[LAU 11] LAU C.K., BOLTON M.D., “The bearing capacity of footings on granular soils. I: numerical analysis”, *Géotechnique*, vol. 61, no. 8, pp. 627–638, 2011.

[LYA 02a] LYAMIN A.V., SLOAN M.W., “Lower bound limit analysis using nonlinear programming”, *International Journal for Numerical Methods in Engineering*, vol. 55, pp. 573–611, 2002.

[LYA 02b] LYAMIN A.V., SLOAN M.W., “Upper bound limit analysis using linear finite elements and non-linear programming”, *International Journal for Numerical and Analytical Methods in Geomechanics*, vol. 26, pp. 181–216, 2002.

[LYS 70] LYSMER J., “Limit analysis of plane problems in soil mechanics”, *Journal of the Soil Mechanics and Foundations Division*, *ASCE*, vol. 96, pp. 1311–1334, 1970.

[MAK 06] MAKRODIMOPOULOS A., MARTIN C.M., “Lower bound limit analysis of cohesive-frictional materials using second-order cone programming”, *International Journal for Numerical Methods in Engineering*, vol. 66, no. 4, pp. 604–634, 2006.

[MAK 07] MAKRODIMOPOULOS A., MARTIN C.M., “Upper bound limit using simplex strain element and second-order cone programming”, *International Journal for Numerical and Analytical Methods in Geomechanics*, vol. 31, pp. 835–865, 2007.

[MAK 08] MAKRODIMOPOULOS A., MARTIN C.M., "Upper bound limit using discontinuous quadratic displacement fields", *Communications in Numerical Methods in Engineering*, vol. 24, no. 11, pp. 911–927, 2008.

[MAN 72] MANDEL J., SALENÇON J., "Force portante d'un sol sur une assise rigide (étude théorique)", *Géotechnique*, vol. 22, no. 1, pp. 79–93, 1972.

[MAR 05] MARTIN C.M., "Exact bearing capacity calculations using the method of characteristics", *Proceedings of the 11th International Conference on Computer Methods and Advances in Geomechanics*, vol. 4, pp. 441–450, 2005.

[MAR 09] MARTIN C.M., "Undrained collapse of a shallow plane-strain trapdoor", *Géotechnique*, vol. 59, no. 10, pp. 855–863, 2009.

[MAR 11] MARTIN C.M., "The use of adaptive finite-element limit analysis to reveal slip-line fields", *Géotechnique Letters,* vol. 1, no. 2, pp. 23–29, 2011.

MAS 99] MASSAU J., Mémoire sur l'intégration graphique des équations aux dérivées partielles ; chap. IV: équilibre des terres sans cohésion. *Annales de l'Association des Ingénieurs de I'École de Gand*, 1899; Édition du centenaire, Comité national de mécanique, Brussels, Mons, 1952, 1899.

[MAS 63] MASSONNET C.E., SAVE M., *Calcul Plastique Des Constructions, II, Structures Spatiales*, CBLIA, Brussels, 1963.

[MAT 79] MATAR M., SALENÇON J., "Capacité portante des semelles filantes", *Revue Française de Géotechnique*, vol. 9, pp. 51–76, 1979.

[MER 40] MERY E., "Équilibre des voutes en berceau", *Annales des Ponts et Chaussées*, vol. I, pp. 50–70, 1840.

[MEY 51] MEYERHOF G.G., "The ultimate bearing capacity of foundations", *Géotechnique*, vol. 2, no. 4, pp. 301–332, 1951.

[MEY 53] MEYERHOF G.G., "The bearing capacity of foundations under eccentric and inclined loads", *Proceedings of the 3rd International Conference on Soil Mechanics and Foundations Engineering,* Zurich, vol. 1, pp. 440–445, 1953.

[MEY 63] MEYERHOF G.G., "Some recent research on the bearing capacity of foundations", *Canadian Geotechnical Journal*, vol. 1, no. 1, pp. 16–21, 1963.

[MIC 98] MICHALOWSKI R.L., YOU L., "Non-symmetrical limit loads on strip footings", *Soils and Foundations*, vol. 38, no. 4, pp. 195–203, 1998.

[MIC 09] MICHALOWSKI R.L., DRESCHER A., "Three-dimensional stability analysis of slopes and excavations", *Géotechnique*, vol. 59, no. 10, pp. 839–850, 2009.

[OVE 89] OVESEN N.K., "General report, session 30: codes and standards", *Proceedings of the 12th International Conference on Soil Mechanics and Foundation Engineering*, Balkema, Rotterdam, pp. 2751–2764, 1989.

[PAS 09] PASTOR F., LOUTE E., PASTOR J., "Limit analysis and convex programming: a decomposition approach of the kinematic mixed method", *International Journal for Numerical Methods in Engineering*, vol. 78, pp. 254–274, 2009.

[PRA 55] PRAGER W., "Théorie générale des états d'équilibre limite", *Journal de Mathématiques Pures et Appliquées*, vol. 34, pp. 395–406, 1955.

[REN 35] RENDULIC L., "Ein Beitrag zur Bestimmung der Gleitsicherheit", *Der Bauingenieur*, vol. 19–20, no. 16, pp. 230–233, 1935.

[SAL 74] SALENÇON J., "Bearing capacity of a footing on a purely cohesive soil with linearly varying shear strength", *Géotechnique*, vol. 24, no. 3, pp. 443–446, 1974.

[SAL 76] SALENÇON J., FLORENTIN P., GABRIEL Y., "Capacité portante globale d'une fondation sur un sol non homogène", *Géotechnique*, vol. 26, no. 2, pp. 351–370, 1976.

[SAL 82] SALENÇON J., MATAR M., "Capacité portante des fondations superficielles circulaires", *Journal de Mécanique Théorique et Appliquée*, vol. 1, no. 2, pp. 237–267, 1982.

[SAL 85] SALENÇON J., "Yield-strength of anisotropicsoils", *Proceedings XVI International Conference of Theoretical and Applied Mechanics, Lyngby Denmark*, 1984, North Holland Publishing Company, pp. 369–386, 1985.

[SAL 94] SALENÇON J., "Approche théorique du calcul aux états limites ultimes", in HOROWITZ J., LIONS J.-L. (eds), *Les grands systèmes des sciences et des technologies*, Masson, Paris, pp. 701–722, 1994.

[SAL 95a] SALENÇON J., PECKER A., "Ultimate bearing capacity of shallow foundations under inclined and eccentric loads. Part I: purely cohesive soil", *European Journal of Mechanics A/Solids*, vol. 14, no. 3, pp. 349–375, 1995.

[SAL 95b] SALENÇON J., PECKER A., "Ultimate bearing capacity of shallow foundations under inclined and eccentric loads. Part II: purely cohesive soil without tensile strength", *European Journal of Mechanics A/Solids*, vol. 14, no. 3, pp. 377–396, 1995.

[SAL 06] SALENÇON J., CHATZIGOGOS C., PECKER A., "Yield design theory applied to the determination of the seismic bearing capacity of surface footings", *Proceedings of the International Conference on Nonlinear Analysis and Engineering Mechanics Today,* 11–14 December, Institute of Applied Mechanics, Ho Chi Minh City, Vietnam, 2006.

[SAV 73] SAVE M., MASSONNET C.E., *Calcul Plastique Des Constructions*, 2nd ed., CBLIA, Brussels, 1973.

[SAV 95] SAVE M., *Atlas of Limit Loads of Metal Plates, Shells and Disks*, Elsevier, Amsterdam, 1995.

[SIM 09] SIMON B., "Yield design calculation of earth retaining structures", *Ground Engineering*, vol. 42, no. 9, pp. 20–25, 2009.

[SOK 55] SOKOLOVSKI V.V., *Theorie der Plastizität*, VEB Verlag Technik, Berlin, 1955.

[SOK 60] SOKOLOVSKI V.V., *Statics of Soil Media*, Butterworths Scientific Publications, London, 1960.

[SOK 65] SOKOLOVSKI V.V., *Statics of Granular Media*, Pergamon Press, Oxford, 1965.

[SUQ 82] SUQUET P., Plasticité et homogénéisation, PhD Thesis, Pierre and Marie Curie University, Paris, 1982.

[SUQ 83] SUQUET P., "Analyse limite et homogénéisation", *C. R. Ac. Sc. Paris*, vol. 296, no. II, pp. 1355–1358, 1983.

[SAR 91] SARAN S., ARGAWAL R.K., "Bearing capacity of eccentrically obliquely loaded footing", *Journal of Geotechnical Engineering*, vol. 117, no. 11, pp. 1669–1690, 1991.

[TAY 37] TAYLOR D.W., "Stability of earth slopes", *Journal of the Boston Society of the Civil Engineers*, vol. 24, no. 3, pp. 337–386, 1937.

[TAY 37] TAYLOR D.W., *Fundamentals of Soil Mechanics*, John Wiley, 1937.

[UKR 98] UKRITCHON B., WHITTLE A., SLOAN W., "Undrained limit analyses for combined loading of strip footings on clay", *Journal of Geotechnical and Geoenvironmental Engineering*, vol. 124, no. 3, pp. 265–276, 1998.