1

Summary of Acoustic Equations

This chapter first discusses the acoustic equations and also focuses on the particle velocity and particle displacement. Conventional methods for estimating the particle velocity are briefly discussed with an emphasis on their advantages and disadvantages. This chapter also explains the advantages of a direct measurement of the velocity and the importance of using laser-based non-intrusive methods for nonlinear acoustics.

1.1. Basic equations

1.1.1. *Fluid- and thermodynamics*

Acoustic motion is governed by the laws of fluid mechanics and thermodynamics [KUN 90]. The quantities involved in acoustics are the pressure p in Pascal (Pa = N·m^{-1}), the particle velocity $\vec{v}$ (m·s^{-1}), the density of the fluid ρ (kg·m^{-3}) and two thermodynamic variables as the temperature (K) T and the entropy S (J/K). As the velocity is a three-component vector and the other variables are scalar, seven equations are needed to solve an acoustic problem. All variables generally have spatial and time dependences, which are omitted in order to simplify the presentation of the equations.

The first equation (vectorial) is the momentum equation derived from Newton's second principle applied to the fluid particle

$$\rho \frac{\partial}{\partial t}\vec{v} + \rho\left(\vec{v}.gr\vec{a}d\right)\vec{v} = \rho\vec{f} - gr\vec{a}d \quad p + div\left[\sigma\right], \qquad [1.1]$$

where $[\sigma]$ is the stress tensor including the effects of viscosity (see the following note) and $\rho\vec{f}$ is the body forces by unit volume (usually gravity height in fluid mechanics). The term

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\vec{v}.gr\vec{a}d \qquad [1.2]$$

is the convective derivative.

The second equation derived from the mass conservation law [BRU 06] is given as

$$\frac{\partial\rho}{\partial t}+div(\rho\vec{v})=\rho q\,, \qquad [1.3]$$

where ρq is the source by injection of mass expressed per unit of volume and per unit of time (dimension of q is s^{-1}). Without sources, the equation becomes the continuity equation of the fluid

$$\frac{\partial\rho}{\partial t}+div(\rho\vec{v})=0\,, \qquad [1.4]$$

which could be rewritten, using the property $div(\rho\vec{v})=\rho div(\vec{v})+\vec{v}.gr\vec{a}d\rho$

$$\frac{1}{\rho}\frac{d\rho}{dt}=-div(\vec{v})\,. \qquad [1.5]$$

Equation [1.5] expresses the relationship between compressibility and velocity and plays an important role in compressible flows.

To consider heat transfer in flows, the first law of thermodynamics (energy conservation) may be applied and leads to the third necessary equation [KUN 90, RIE 03]. If e is the internal energy (potential energy) per unit mass, the energy conservation could be written as

$$\frac{d}{dt}\left(\rho e+\rho\frac{v^2}{2}\right)=-div\left(\vec{J}\right)-div\left(p\vec{v}\right)+\rho\vec{f}.\vec{v}+div\left(\left[\sigma\right].\vec{v}\right), \qquad [1.6]$$

where $\vec{J}$ is the heat flux vector per unit mass.

As the velocities are quite a lot smaller than the sound celerity, by neglecting the viscosity and with $e = C_V T$ for a perfect gas with C_V, the specific heat at constant volume, equation [1.6] becomes

$$\rho C_p \frac{dT}{dt} = -div(\vec{J}) , \qquad [1.7]$$

where C_P is the specific heat at constant pressure [KUN 90]. Considering the heat transfer equation (Fourier's law) written here for an isotropic and homogeneous fluid of the first order

$$\vec{J} = -k\ gr\vec{a}d\left(T\right) , \qquad [1.8]$$

where k is the heat conductivity. Introducing equation [1.7] into [1.8] leads to

$$\frac{dT}{dt} = \kappa \Delta T , \qquad [1.9]$$

where $\kappa = \dfrac{k}{\rho C_P}$ is the coefficient of thermal diffusivity.

The second law of thermodynamics, entropy production, which is generally considered as a measure of disorder in a system, can be written as

$$T\frac{dS}{dt} = \frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} , \qquad [1.10]$$

where S is the entropy. An insulated system tends to a constant and maximum entropy.

To summarize, the analysis of a complete acoustic process must take into account seven scalar variables, pressure, three-dimensional (3D) motion variable (displacement, velocity or acceleration), density (mass flow), temperature and entropy. Equations [1.1], [1.3], [1.6] and [1.10] are sufficient to close the system for a perfect gas.

According to given problems, the hypotheses may considerably simplify the equations and in the following we will give some classical examples. Nevertheless, as we will see later, the use of lasers for measuring particle velocity is very useful in the case of complex experiments for which temperature gradient or viscosity, for instance, could not be neglected.

NOTE.– Stress tensor for acoustics.

The momentum equation in the context of acoustics is generally written as

$$\rho \frac{d\vec{v}}{dt} = \rho \frac{\partial}{\partial t}\vec{v} + \rho(\vec{v}.gr\vec{a}d)\vec{v} = \rho\vec{f} - gr\vec{a}d\ p + \mu\Delta\vec{v} + \left(\eta + \frac{\mu}{3}\right)gr\vec{a}d\ div\ \vec{v}$$

where μ is the coefficient of shear viscosity and η is the bulk viscosity coefficient. As we do not emphasize thermoviscous losses, the above equation will not be used later.

1.1.2. *Hypothesis of linear acoustics without losses*

In this first case, the fluid is homogeneous and without temperature gradient or fluctuation. The other acoustic variables may be divided into a mean value and fluctuating quantities assumed to be related to acoustic variations; then

$$p = p_0 + p' \qquad [1.11]$$

$$\rho = \rho_0 + \rho' \qquad [1.12]$$

$$\vec{v} = \vec{v}_0 + \vec{v}' \qquad [1.13]$$

with $\frac{p'}{p_0} << 1$, $\frac{\rho'}{\rho_0} << 1$, T = T_0 and $\vec{v}_0 \approx \vec{0}$.

The above hypotheses are still available for very few mean flows, the nonlinear terms of the second-order Navier–Stokes equation (advection term) remain much lower than the unstationary term (equation [1.1]) and low velocities do not impact the compressibility (equation [1.3]). Neglecting the weight forces and the shear and bulk viscosities, equation [1.1] becomes

$$\rho_0 \frac{\partial \vec{v}'}{\partial t} + gr\vec{a}d \quad p' = \rho_0 \vec{f}\ , \qquad [1.14]$$

where $\vec{f}$ is the force that an acoustic source may impose on the fluid. If in a given volume (source) a fluctuating mass flow is injected, $\rho_0 q(t)$, then equation [1.3] becomes

$$\frac{\partial}{\partial t}\rho' + div(\rho'\vec{v}') = \rho_0 q \,. \tag{1.15}$$

Without heat source and viscous dissipation, the equation of entropy [1.10] for a perfect gas ($e = C_V T$) becomes

$$\frac{dS}{dt} = \frac{C_v}{T}\frac{dT}{dt} - \frac{p}{T\rho^2}\frac{d\rho}{dt} \,. \tag{1.16}$$

As it is usual to write dS in relation to dp and $d\rho$, dT is replaced using the log-differentiation of the state equation

$$\frac{dT}{T} = \frac{dp}{p} + \frac{d\rho}{\rho} \tag{1.17}$$

into [1.16]. As the average value of T, p and ρ is much larger than their fluctuations and using the perfect gas equation, $p/\rho = rT$, with $r = 287$ J·kg^{-1}·K^{-1}, it gives

$$\frac{dS}{dt} = \frac{C_v}{p_0}\frac{dp}{dt} + \frac{(Cv - r)}{\rho_0}\frac{d\rho}{dt} = \frac{C_v}{p_0}\frac{dp}{dt} + \frac{C_p}{\rho_0}\frac{d\rho}{dt} = C_v\left(\frac{1}{p_0}\frac{dp}{dt} + \frac{\gamma}{\rho_0}\frac{d\rho}{dt}\right), \tag{1.18}$$

where $\gamma = \dfrac{C_p}{C_v} = 1.4$ for the air. If the fluid is considered as isentropic ($dS = 0$), i.e. with no temperature source, we obtain by integration from [1.18]

$$p' = \frac{\gamma p_0}{\rho_0}\rho' = \frac{\gamma}{\rho_0 \chi_T}\rho' = c^2 \rho' \,, \tag{1.19}$$

where p' and ρ' are zero at the origin. In the above equation, c_0 is the acoustic celerity for a homogeneous and quiet media and χ_T is the coefficient of isothermal compressibility (for a perfect gas $\chi_T = 1/p_0$).

1.2. Acoustic equations

1.2.1. *Linear acoustic equations with sources*

In taking the divergence of equation [1.14] and the gradient of equation [1.15] and combining both with equation [1.19], the equation for the acoustic pressure is obtained; then

$$\Delta p' - \frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} = \rho_0\left(div\vec{f} - \frac{\partial q}{\partial t}\right). \qquad [1.20]$$

If a fluctuating source of heat h is inserted into the volume (pulsed laser, flame, etc.), then the entropy becomes

$$TdS = hdt\ . \qquad [1.21]$$

By neglecting spatial variation on the source domain, the state equation [1.18] could be rewritten as

$$\frac{\partial \rho}{\partial t} = \frac{1}{c_0^2}\frac{\partial p}{\partial t} - \frac{\alpha}{C_p} h\ , \qquad [1.22]$$

where $\alpha = \beta\chi_T p_0$ is the thermal expansion coefficient and β is the thermal pressure variation. Whatever is inside the source, the density fluctuations are now dependent not only on the pressure variation but also on heat fluctuations.

Using the same transformation, the equation with thermal contribution may be written as

$$\Delta p' - \frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} = \rho_0\left(div\vec{f} - \frac{\partial q}{\partial t} - \frac{\alpha}{C_p}\frac{\partial h}{\partial t}\right). \qquad [1.23]$$

1.2.2. *Some remarks on acoustic sources*

The concept of acoustic sources is non-trivial. First, equation [1.23] is only valid in the source domain. Second, the analysis of the sources practically requires an other discipline such as vibration or thermal science. Figure 1.1 shows the principle of a model of the acoustic problem with sources. In the presence of acoustic sources (musical instrument, rotating machinery, burner, etc.), the analysis allows us to estimate $\vec{f}$, q or h, three types of canonical sources but possibly present at the same time in a physical process responsible for the generation of the sound. These three sources are, respectively, called:

– force source because it derives from the momentum equation;

– debit source because it derives from the mass equation;

– heat source.

The actual sources are often a combination of these three sources located more or less in the same area of production. A good example is an airplane turboreactor.

Further study will show that the source of force leads to a bidirectional directivity (acoustic dipole) due to the term $div\vec{f}$ in the right-hand side of equation [1.23]; the other terms are omnidirectional directivity (monopole).

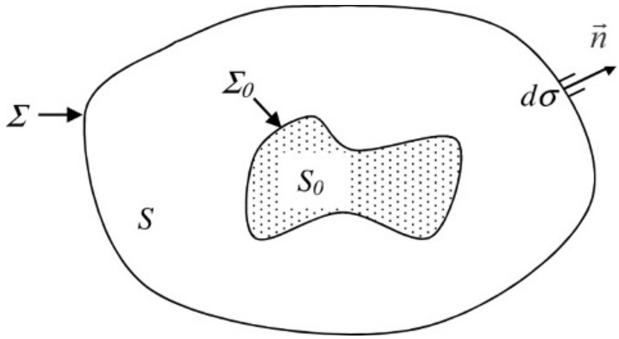

Figure 1.1. *Principle scheme of a model of a resolution of an acoustic problem by taking into account sources: S is the study domain with a border Σ and S_0 is the source domain with a border Σ_0*

1.2.3. *Without sources*

In the (S-S_0) domain (Figure 1.1), the source activity has no more direct effect and the sound propagates until final attenuation. Then, the model for this part of the space needs only the left-hand side of equation [1.23], equation [1.14] without source and equation [1.19]. The set of equations

$$\begin{cases} \Delta p' - \dfrac{1}{c_0^2}\dfrac{\partial^2 p'}{\partial t^2} = 0 \\ \rho_0 \dfrac{\partial \vec{v}'}{\partial t} = -grad\left(p'\right), \\ p' = c_0^2 \rho' \end{cases} \qquad [1.24]$$

Make it possible solve many acoustic problems, if sources are physically described and border and time conditions are known. The first equation is the well-known D'Alembert equation (called the propagation equation in the following), the second equation is the linear Euler equation for small motions (simply called the Euler equation in the following).

Remembering the Fourier transform of the signal, $x(t)$ is written as

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \qquad [1.25]$$

and one of its properties is

$$TF\left\{\frac{dx(t)}{dt}\right\} = j2\pi fX(f) . \qquad [1.26]$$

The previous set of equations [1.24] could be rewritten as

$$\begin{cases} \Delta p' + k^2 p' = 0 \\ \vec{v}' = \dfrac{j}{\rho_0 \omega} grad(p'), \\ p' = c_0^2 \rho' \end{cases} \qquad [1.27]$$

where the first equation is called the Helmholtz equation and p' is now dependent on frequency. The wave number is

$$k = \frac{\omega}{c_0} = \frac{2\pi f}{c_0} . \qquad [1.28]$$

and the wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{c_0}{f} . \qquad [1.29]$$

This above characterictic has much impact on practical acoustics because the solid objects considerably modify the acoustic field (diffraction to ultrasounds for instance). The measurement method have to take into account the solid environment of an acoustic field and must avoid to disturb the field with too large probes. The use of laser probes is one answer of this problem.

1.2.4. *Acoustic intensity and source power*

Far from sources, inside (S–S_0), assuming that sound propagates in a homogeneous and quiet media and neglecting the dissipation, equation [1.6] directly becomes

$$\frac{\partial e_t}{\partial t} = -div\left(p'\vec{v}'\right), \qquad [1.30]$$

where e_t is the total mechanical energy (kinetic and potential) per unit of volume. In the absence of source in the domain, this equation vanishes. The product, $\vec{P} = p'\vec{v}'$, called the acoustic energy flux, contains the information on source activities whatever their physical origin. An actual estimation of the flux is given by $\vec{P} = \Re(p')\Re(\vec{v}')$ where $\Re(\;)$ is the real value of the variables. By definition, the acoustic intensity is

$$\vec{I} = <\vec{P}> = <\Re\left(p'\right)\Re\left(\vec{v}'\right)>, \qquad [1.31]$$

where <.> is the temporal means (see note below).

The above equation shows that knowledge of the velocity is absolutely essential for knowledge of the source intensity and direction.

The source power is written as

$$P\left(S_0\right) = \iint_{\Sigma} \vec{I}.\vec{n}d\sigma, \qquad [1.32]$$

where Σ is the integrating surface, including the S domain larger than S_0 by definition. The surface surrounding the source domain may be chosen arbitrarily. Nevertheless, depending on the applications, the standards of measurement recommend respecting a certain distance from the assumed location of the sources to the measuring devices (typically a few wavelengths).

The above equation is the basis of standard measurement for noise engineering.

NOTE.– Averages of time-varying variables.

The notation <.> usually means the expected value or statistical averages. In the case of an ergodic process, the statistical means are assumed to be equal to time averages. This concept will be detailed in Chapter 2. In this chapter, the hypothesis of ergodicity is assumed; then

$$< x >= \lim_{T \to +\infty} \frac{1}{T} \int_{(T)} x(t) dt.$$

If $x'(t) = x(t)\text{-}m_x$ where $m_x = <x>$, mean of the signal, the root mean square (RMS) value of $x(t)$ is defined as

$$e_x = \sqrt{< x'^2 >} = \sqrt{\lim_{T \to +\infty} \frac{1}{T} \int_{(T)} (x(t) - m_x)^2 dt},$$

if the process is ergodic. The standard deviation (STD) and the RMS value (e_x) are equivalent. This enables us to quantify the characteristic variations of a signal.

1.2.5. *Acoustic impedance and border conditions*

If an incident wave is reflected on a surface (Figure 1.2), it is assumed that the surface as a local reaction can be modeled by the surface impedance

$$Z_n(\omega) = \frac{p'}{v_n'}, \qquad [1.33]$$

where v_n is the surface normal velocity in the direction of the external normal of the element of the surface [DOW 83]. We should note that the impedance depends on frequency and is generally measured with adapted setup (Kundt's tube for example).

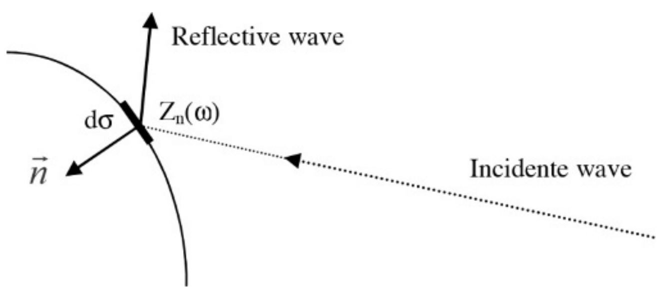

Figure 1.2. *On a border (wall, media changing, etc.), the wave is assumed to have a local reaction due to the impedance $Z_n(\omega)$ of the element of surface $d\sigma$*

From Euler equation [1.27] in the frequency domain, the normal velocity could be written as

$$v_n' = \frac{j}{\rho_0 \omega} \frac{\partial p'}{\partial n}. \qquad [1.34]$$

This means that the knowledge of the pressure gradient is theoretically sufficient to solve a border problem. Practically, this is not so simple and this pressure gradient has to be estimated using a finite number of pressure captions, as we will see later [ALL 09].

In the particular case where the normal velocity is close to zero, $v_n' \approx 0$, the pressure gradient is given by

$$\frac{\partial p'}{\partial n} = 0 \qquad [1.35]$$

which is called Neumann's condition. This condition is nearly followed when the border is highly reflective (solid body).

Conversely, if $Z_n \approx 0$, the border condition could be written as

$$p' = 0 \qquad [1.36]$$

which is called Diriclet's condition. This condition is much more difficult to respect in actual experiments because the pressure rarely vanishes in a singularity. Nevertheless, in a first approximation, this is the case of an open end of an acoustic pipe.

The last case is when the wall is vibrating with a velocity V_0; by continuity, the impedance could be rewritten as

$$Z_n(\omega) = \frac{p'}{V_0 - v_n}. \qquad [1.37]$$

Not only should the acoustic velocity be known, but also the vibration of the wall. This condition is very important in coupling between vibrating surfaces and an acoustic domain (loudspeaker, radiation of mechanical engine, wall transparency, etc.).

The acoustic impedance, which is the ratio between a scalar variable on the projection of a vectorial variable, has to be used very carefully and its measurement must be carried out in a sensitive manner. For one-dimensional propagation, typically for wind instrument analysis [DAL 01], or when wavelengths are much larger than the object dimensions, in electroacoustics [ROS 86], the concept of impedance is very useful and it enables us to solve numerous problems at the first order.

1.3. Constants, units and magnitude orders of linear acoustics

The celerity in the air is mainly dependent on temperature T

$$c_0^{\ 2} = \gamma \frac{p_0}{\rho_0} = \gamma \frac{RT}{M} \qquad [1.38]$$

for a perfect gas where R is the perfect gas constant and M is the molar mass of the gas[1]. A good linear approximation about $T_0 = 15°C$ is given by

$$c_0 = (331.4 + 0.607 \ T) \ \text{m/s}, \qquad [1.39]$$

where T is in degrees Celsius.

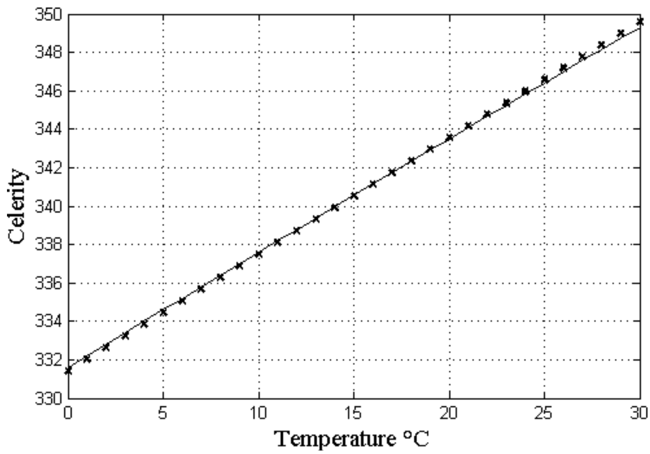

Figure 1.3. *Acoustic celerity versus temperature in degrees Celsius. The solid line represents the values in equation [1.38] and the dotted line represents the linear approximation given in equation [1.39]*

1 M = 28.92 g/mol and R = 8.314472 $J \cdot mol^{-1} \cdot K^{-1}$.

To simplify the exposition of the order of magnitude of acoustic variables, we will only consider one-dimensional wave propagation. Whatever the functions f and g, the solutions of system [1.14] are

$$\begin{cases} p'(x,t) = f\left(t - \dfrac{x}{c_0}\right) + g\left(t + \dfrac{x}{c_0}\right) \\ v'(x,t) = \dfrac{1}{\rho_0 c_0} f\left(t - \dfrac{x}{c_0}\right) - \dfrac{1}{\rho_0 c_0} g\left(t + \dfrac{x}{c_0}\right) \end{cases}. \qquad [1.40]$$

The function $f\left(t - \dfrac{t}{c_0}\right)$ is the progressive wave when x increases and $g\left(t + \dfrac{t}{c_0}\right)$ is the progressive wave when x decreases. In actual problems, both kinds of waves occur when an obstacle is present in the field. In the case of a harmonic wave, the solution could be

$$\begin{aligned} p'(x,t) &= Ae^{j\omega t}e^{-jkx} + Be^{j\omega t}e^{+jkx} \\ &= Ae^{j\omega\left(t - \frac{x}{c_0}\right)} + Be^{j\omega\left(t + \frac{x}{c_0}\right)}, \end{aligned} \qquad [1.41]$$

where A and B are arbitrary complex constants. Assuming that only one progressive wave is propagating,

$$\begin{cases} p'(x,t) = f\left(t - \dfrac{x}{c_0}\right) \\ v'(x,t) = \dfrac{1}{\rho_0 c} f\left(t - \dfrac{x}{c_0}\right) \end{cases} \qquad [1.42]$$

and the ratio of pressure and velocity is

$$\rho_0 c_0 = \frac{p'(x,t)}{v'(x,t)}, \qquad [1.43]$$

called the characteristic impedance because it contains the essential elements of the thermodynamics of a homogeneous and quiet media. At temperature T_0 = 15°C, $\rho_0 \approx 1.23$ kg/m^3 and $\rho_0 c_0 \approx 417$ kg m^{-2} s^{-1} for the air.

For a pure tone of 1,000 Hz, the RMS value of the minimum pressure perceived by human being (on average) is

$$p_r = 2 \times 10^{-5} \text{ Pa} \qquad [1.44]$$

which is a very small value of magnitude (called the reference acoustic pressure in the following). The highest level before the theshold of painful hearing is

$$p_h = 20 \text{ Pa} . \qquad [1.45]$$

In terms of pressure, audition is being able to perceive about six orders of magnitude without difficulties. It has seemed necessary to build a scale adapted to the audition range avoiding manipulating such a large scale of numbers. A logarithmic scale seemed to be a good solution because of the relative resolution of magnitude perception[2]. Thus, by definition, the decibel sound pressure level (dB_{SPL}) is defined as

$$p_{dB} = 20 \log \frac{p_{rms}}{2 \times 10^{-5}} = 20 \log \frac{p_{rms}}{p_r} \text{ dB}_{\text{SPL.}} \qquad [1.46]$$

The threshold of audition at 1,000 Hz (equation [1.44]) is $p_{dB} = 0$ dB and the maximum (equation [1.45]) is $p_{dB} = 120$ dB.

The acoustic energy flux has been previously defined as *P=R(p)R(v)* where the pressure and the velocity are obtained by Fourier transform [1.25]. As *R(x)=x+x*/2*, it becomes

$$\vec{P} = \frac{1}{4}\left(p'\vec{v}' + p'^{*}\vec{v}' + p'\vec{v}'^{*} + p'^{*}\vec{v}'^{*} \right), \qquad [1.47]$$

if the acoustic wave is harmonic such as

$$\begin{aligned} p' &= e^{j(\omega t + \Phi)} \\ \vec{v}' &= e^{j(\omega t + \Phi')}\vec{n} \end{aligned} . \qquad [1.48]$$

As the terms $p'\vec{v}'$ and $p'^{*}\vec{v}'^{*}$ are time dependent and zero mean by definition, the intensity becomes

2 The audible gap from 0.01 to 0.1 Pa is nearly the same from 0.1 to 1 Pa.

$$\vec{I} = <\vec{P}> = \frac{1}{4}\left(<p'\vec{v}'^*> + <p'^*\vec{v}'>\right)\vec{n} = \frac{1}{2}\Re\left(p'\vec{v}'^*\right)\vec{n}\ . \quad [1.49]$$

Note that the intensity is a vector that indicates locally the sense of the acoustic energy flux. For a plane wave (equation [1.42]), the acoustic intensity is

$$I = \frac{p_{rms}^2}{\rho_0 c_0}\ , \quad [1.50]$$

where $p_{rms} = \sqrt{<p'^2>}$. Then, the decibel unit defined on intensity

$$I_{dB} = 10\ \log\frac{I}{10^{-12}}\ , \quad [1.51]$$

where the reference intensity I_r = 10^{-12} W comes from p_r = 2×10^{-5} Pa and $\rho_0 c_0 = 400$ Pa·s/m, an approximate value of the characteristic impedance[3]. These last equations are the basis of sound intensity measurements in noise engineering.

Now, assuming that the real part of the velocity of equation [1.48] (from a measurement for instance) is written

$$v'_x = V_a \cos\left(2\pi f t + \phi'\right) = V_a \cos\left(\omega t + \phi'\right) \quad [1.52]$$

the particle displacement becomes, if the equilibrium position of the particle is assumed to be zero,

$$d_x = \frac{V_a}{2\pi f}\sin\left(2\pi f t + \phi'\right) = \frac{V_a}{\omega}\sin\left(\omega t + \phi'\right). \quad [1.53]$$

Then, the values of the RMS value of velocity and displacement for the reference pressure $p_r = 2\ \times\ 10^{-5}$ Pa (0 dB at 1,000 Hz) are

$$V_r = \frac{p_r}{\rho_0 c} = 5\ \times\ 10^{-8}\ \ \text{m/s}\ ,$$

3 If not, the small error occurs, approximately 0.17 dB for measurement at 15°C.

$$d_r = \frac{V_r}{2\pi f} = 10^{-11} \text{ m}.$$

At the limits of comfortable audition, $p_h = 20$ Pa (120 dB at 1,000 Hz), it becomes

$$V_h = \frac{p_h}{\rho_0 c} = 5 \times 10^{-2} \text{ m/s}$$

$$d_h = \frac{V_h}{2\pi f} = 10^{-5} \text{ m}.$$

These calculations justify *a posteriori* the linearization of the basic equations, particularly the order of two advective terms of equation [1.1] (second term on the left side), which is low in comparison to the others, in a large range of magnitude. Furthermore, they also show that the sensitivity of human audition is considerably high; ears are sensitive to an acoustic displacement of about 10^{-11} m, that is the order of the molecular dimension.

1.4. Acoustic velocity measurement and applications

As we have seen before, the velocity is an indispensable variable for a complete knowledge of the velocity field, as well as for impedance estimation or intensity. Moreover, in noise engineering or control, it is often insufficient to deduce the intensity from equation [1.50] because of the high complexity of the acoustic field. The velocities have to be deduced from the gradient of pressure under the assumption of linearity. This measurement needs at least two or more microphones and the velocity is estimated using finite difference schemes.

1.4.1. *Velocity estimation from pressure gradient*

Considering the Euler equation in the frequency domain (equation [1.27]) projected on the x component to simplify the presentation

$$v_x' = \frac{j}{\rho_0 \omega} \frac{\partial p'}{\partial x}. \qquad [1.54]$$

Figure 1.4 shows the principle of the estimation of the pressure gradient using two microphones distant from L. Using a finite difference scheme for the gradient, the estimated velocity becomes

$$\tilde{v}_x{}' = \frac{j}{\rho_0 \omega} \frac{(p_2 - p_1)}{L} = \frac{j(p_2 - p_1)}{2\pi\rho_0 c_0} \frac{\lambda}{L}. \qquad [1.55]$$

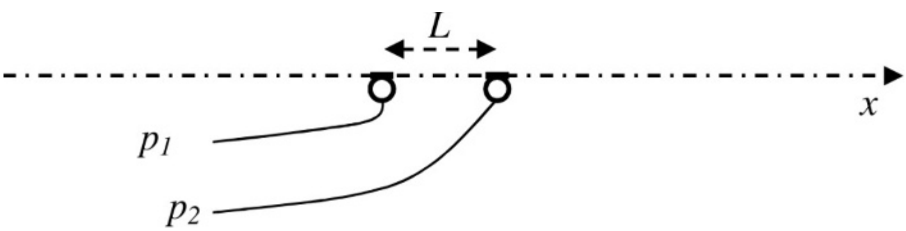

Figure 1.4. *Principle of velocity measurement by means of the microphone doublet*

It clearly appears in this simple example that a dimensional parameter λ/L carries all the difficulties of this kind of measurement:

– If $\lambda/L >> 1$, then $p_1 \approx p_2$; then the difference p_2-p_1 is close to zero and if an actual signal is noisy, the relative error could dramatically increase; at given L, this often occurs at low frequency.

– If $\lambda/L << 1$, then the velocity is underestimated or does not have any sense; at given L, this occurs at high frequencies.

These kinds of measurements need a good choice of microphone positions and L should be adapted to the analyzed frequency domain. Above all, a precise estimation in magnitude and phase should be achieved; the calibration procedure is quite delicate. It is also possible to estimate velocity using more than two microphones; the principles are similar.

1.4.2. *Intensity estimation*

The intensity defined in equation [1.31] may be estimated from the experimental setup displayed in Figure 1.4. The pressure is estimated by

$$\tilde{p}' = \frac{p_1 + p_2}{2} \qquad [1.56]$$

and from equation [1.49], the estimation of acoustic intensity becomes

$$\tilde{I}_x = \frac{\lambda}{8\pi L \rho_0 c_0} \Re\left(-j\left(p_1 + p_2\right)\left(p_2 - p_1\right)^*\right) = \frac{\lambda \Im\left(p_1 p_2^*\right)}{4\pi L \rho_0 c_0}, \\ = \frac{\Im\left(p_1 p_2^*\right)}{2\omega L \rho_0} \qquad [1.57]$$

where $\Im(\;)$ is the imaginary part. Equation [1.57] shows that the estimation of intensity necessitates only the interpectrum calculation of the pressures given by both microphones (see chapter 2). In practice, some difficulties may occur such as for example: the calibration in magnitude and phase must be achieved carefully, the distance between microphones should be set in relation to the frequency range to be measured. However, this technique is extensively used in standard measurements.

1.4.3. *Application to impedance estimation*

With the same experimental setup, the impedance between both microphones becomes

$$\tilde{Z}_x = \frac{\tilde{p}}{\tilde{v}_x} = \rho_0 c_0 \frac{2\pi L}{j\lambda} \frac{\left(p_1 + p_2\right)}{\left(p_2 - p_1\right)} = \rho_0 c_0 \frac{kL}{j} \frac{\left(p_1 + p_2\right)}{\left(p_2 - p_1\right)}. \qquad [1.58]$$

The exact formulation of estimation impedance using two microphones in a wave guide would be obtained by replacing $\tan(kL)$ instead of (kL), which is a first-order approximation at large wavelength [DAL 01]. As discussed above, there are critical bands for which the precision of the estimation decreases, particularly around $p_1 \approx p_2$.

1.5. Beyond linear equations

The understanding of the previous sections 1.1–1.4 is sufficient to explain the measuring methods based on lasers and their capabilities. The present section briefly presents some cases for which laser techniques have been used successfully and when the classical theory presented here is insufficient. The development of each application would require further developments, which are far beyond the focus of this book.

1.5.1. *Acoustic equations with mean flow*

The velocity could be decomposed into a mean and fluctuating velocities as $\vec{v} = \vec{v}_0 + \vec{v}'$ (equation [1.13]) with a mean flow non-negligible and much higher than the fluctuating velocity; the convective derivative (equation [1.1]) becomes

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}_0.gr\vec{a}d \qquad [1.59]$$

and without losses and sources, and with the same method discussed in section 1.1.3, the linearization of equations [1.1], [1.5] and [1.19] gives

$$\rho_0 \left(\frac{\partial}{\partial t} + \vec{v}_0.gr\vec{a}d \right) \vec{v}' + gr\vec{a}d \quad p' = 0, \qquad [1.60]$$

$$\left(\frac{\partial}{\partial t} + \vec{v}_0.gr\vec{a}d \right) \rho' + \rho_0 div(\vec{v}') = 0 \qquad [1.61]$$

$$p' = c_0 \rho'. \qquad [1.62]$$

By using simular calculations as in section 1.2.1, an equation for pressure is easily derived

$$\Delta p' - \frac{1}{c_0^2} \left(\frac{\partial}{\partial t} + \vec{v}_0.gr\vec{a}d \right)^2 p' = 0. \qquad [1.63]$$

In a special case where the moving flow is in the same direction of propagation, the projection (hypothesis of the flat profile) of equation [1.63] gives

$$\Delta p' - \frac{1}{c_0^2} \left(\frac{\partial^2}{\partial t^2} + 2\left\|\vec{v}_0\right\| \frac{\partial^2}{\partial x \partial t} + \left\|\vec{v}_0\right\|^2 \frac{\partial^2}{\partial x^2} \right)^2 p' = 0 \qquad [1.64]$$

or introducing the Mach number

$$M = \frac{\left\|\vec{v}_0\right\|}{c_0} \qquad [1.65]$$

$$\Delta p' - M \frac{\partial^2 p'}{\partial x^2} - 2 \frac{M}{c_0} \frac{\partial^2 p'}{\partial x \partial t} + \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} = 0 \qquad [1.66]$$

and the frequential version, in the Fourier domain,

$$\Delta p' - M\frac{\partial^2 p'}{\partial x^2} - 2jMK\frac{\partial p'}{\partial x \partial t} + K\frac{\partial^2 p'}{\partial t^2} = 0 \qquad [1.67]$$

where $K = \frac{\omega}{c_0} = \frac{2\pi f}{c_0}$ [1.68]

It is straightforward to prove that the progressive wave $p'(x,t) = Ae^{j\omega t}e^{-jk_x x}$ will propagate with a wave number when x increases

$$k_x = \frac{\omega}{c_0 + \|\vec{v}_0\|} = \frac{K}{1+M} \qquad [1.69]$$

and when x decreases

$$k_x = -\frac{\omega}{c_0 - \|\vec{v}_0\|} = -\frac{K}{1-M} = \frac{K}{M-1} \qquad [1.70]$$

As shown in section 1.3, the acoustic velocity still remains lower at 1m/s, even when the pressure reaches 120 dB. The mean velocities in actual applications of acoustic propagation with flow may range from 10 to 300 m/s, such as for cars, trains and aircraft (we do not consider supersonic flows). Thus, the measurement of acoustic velocity about 1% or less of a mean flow is challenging. Furthermore, the flow velocity profile could also be linear (roughly close to a wall), parabolic or with power laws. Also, turbulence occurs in a major part of these applications which adds more difficulties.

Nevertheless, a complete description of phenomena which occur in these applications has not been fulfilled up to now. Thus, advanced measurement techniques must be achieved by means of microphone arrays or laser techniques, in order to improve our comprehension of the physics involved.

1.5.2. *High acoustic displacement*

Even when the hypothesis of linear acoustics seems valid, in some cases, nonlinear behavior exists due to an abrupt change of section or discontinuities. Figure 1.5 shows the principle of what happens.

Figure 1.5. *a) The radius of curvature of an open end of a guide is larger than acoustic displacement (dashed arrow). b) The radius of curvature of an open end of a guide is smaller (sharp end) than the acoustic displacement. The fluid displacement (dashed arrow) does not follow the wall and vortex shedding occurs*

If the wave guide is excitated with a sine-wave, the particle movement at the end of the pipe may have two kinds of behavior as shown in Figure 1.5. If the particle displacement is the smaller of the radius of curvature of the end of the pipe (smooth end) the authors [MAR 08] note that when the acoustic displacement is higher than *R*

$$\frac{\delta_x}{R} = \frac{V_a}{\omega L} > 1 \qquad [1.71]$$

the movement generates a vortex at the end.

These kinds of phenomena occur in loudspeaker ports [ROO 98], thermoacoustic devices [BLA 03] or musical instruments [ATI 04]. Laser techniques are required for analyzing these phenomena.

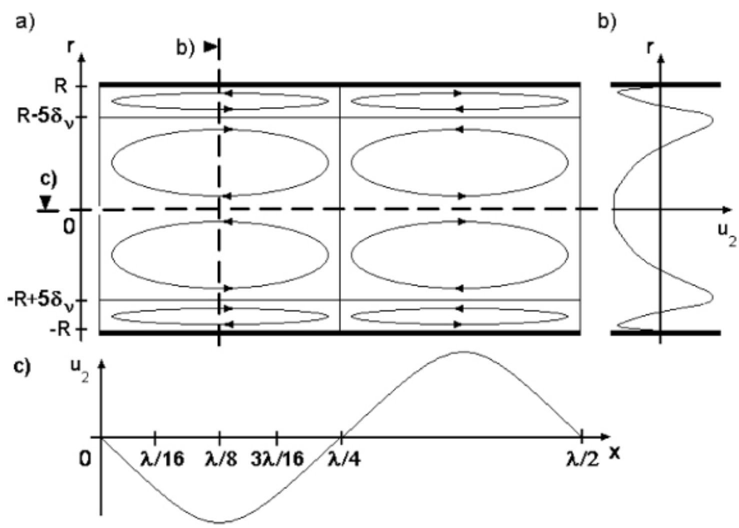

Figure 1.6. *Streaming velocity field; δ_v is the boundary layer depth, R is the radius of the wave guide and λ is the acoustic wavelength [MOR 07]*

1.5.3. *Acoustic streaming*

A last example of a phenomenon with in margin of classical acoustics is acoustic streaming, i.e. the flow induced by the order of two terms when the level increases. First described by Rayleigh (central vortices, called outer cells), a complete analytic description is now available, including the vortices close to the wall called the inner cells [BAI 01]. Figure 1.6 shows the theoretical behavior of acoustic streaming.

The understanding of this effect needs a complete measurement in the acoustic boundary layer (less than 1 mm) and a joint estimation of the acoustic and flow velocity, the last being about 1% of the acoustic velocity. This problem has been addressed by a large number of authors, as we will see in Chapter 4.

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