

Chapter 1

Uncertainty Representation Based on Set Theory

Real systems are often complex due to several factors: the system's nature (e.g. mechanical, electrical and chemical systems), interactions between its different components (e.g. multivariable systems), and its different behavior in a dynamic environment (e.g. influence of disturbances, noises and uncertainties). All these aspects have to be considered when modeling a given system, sometimes leading to a complicated model. In the context of control systems, a mathematical model is frequently used to describe the system behavior. On the one hand, the accuracy of the mathematical model is important to analyze and design control strategies for the considered system; on the other hand, in the context of industrial applications, it is suitable to use unsophisticated controllers designed using a simple model. In this context, a trade-off must be found: the system model should be simple but precise enough to characterize the dynamical behavior of the original system. Thus, the simple/simplified mathematical model cannot represent the real system exactly

due to a lack of knowledge of, or unreliable information about, the system. To validate this model, some uncertainties can be added to the mathematical model. Frequently, perturbations influencing the real system have to be taken into account in the mathematical model in order to ensure a similar behavior of the real system and the mathematical model. The importance of uncertainties in system design has been discussed by many authors (the interested reader can refer to [MAY 79, AUG 06, AYY 06] and the references therein). In the literature, there are two ways to represent uncertainties: the statistical (or stochastic) approach and the deterministic approach. An overview of these two directions is provided in the following.

In the *stochastic approach*, the uncertainty is modeled by a random process with a known statistical property. This technique is widely used in various domains (e.g. economics [BAT 08], biology [ULL 11] and engineering [MAY 79]), especially when estimates of the probability distribution of the uncertain parameters are available. But in many applications, there are situations when the probability distribution of the uncertain parameters is not known and only bounds of the uncertain domain can be fixed. In this case, the probabilistic assumptions on the uncertainty are no longer valid, making this method unsuitable for modeling the uncertainties.

In the *deterministic approach*, the uncertainty is assumed to belong to a set: a classical (crisp) set¹ or a fuzzy set². In the literature, different families of classical sets are used depending on their properties. Usually, the accuracy and the

¹ A classical set is a set wherein the degree of membership of any object to the set is either 0 or 1.

² A fuzzy set is a set wherein the degree of membership of any object to the set is between 0 and 1.

complexity of the uncertainties' representation are inversely proportional, depending on the particular problem related to the choice of a suitable geometric form. In the following sections, some popular families of sets are presented with their advantages and weaknesses. Note that in this book only convex classical sets are considered due to the important role of convexity in the optimization theory [BER 03].

1.1. Basic set definitions: advantages and weaknesses

Before presenting the most well-known families of sets, some basic set definitions and operations used in this book are introduced.

DEFINITION 1.1.— *A set $S \subset \mathbb{R}^n$ is called a convex set if for any $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in S$ and any $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}^+$ such that $\sum_{i=1}^k \alpha_i = 1$, then the element $\sum_{i=1}^k \alpha_i \mathbf{x}_i$ is in S .*

DEFINITION 1.2.— *A convex hull of a given set S , denoted $\text{conv}(S)$, is the smallest convex set containing S .*

DEFINITION 1.3.— *A set $S \subset \mathbb{R}^n$ is called a C-set if S is compact, convex and contains the origin. This is a proper set if its interior is not empty.*

DEFINITION 1.4.— *The inclusion operator between two sets is defined by $\mathcal{X} \subseteq \mathcal{Y}$, if and only if $\forall x \in \mathcal{X}$, then $x \in \mathcal{Y}$. This means that \mathcal{X} is a subset of \mathcal{Y} .*

DEFINITION 1.5.— *The intersection operator of two sets \mathcal{X} and \mathcal{Y} is defined by $\mathcal{X} \cap \mathcal{Y} = \{x : x \in \mathcal{X} \text{ and } x \in \mathcal{Y}\}$.*

DEFINITION 1.6.— *The image of a set S under a map (projection) \mathcal{M} is the set $\mathcal{M}(S) = \{y : y = \mathcal{M}(x), x \in S\}$.*

DEFINITION 1.7.— *The Minkowski sum of two sets \mathcal{X} and \mathcal{Y} is defined by $\mathcal{X} \oplus \mathcal{Y} = \{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\}$.*

DEFINITION 1.8.— *The Pontryagin difference of two sets \mathcal{X} and \mathcal{Y} is defined by $\mathcal{X} \ominus \mathcal{Y} = \{z : z + y \in \mathcal{X}, \forall y \in \mathcal{Y}\}$.*

DEFINITION 1.9.— *Let \mathcal{X} and \mathcal{Y} be two non-empty sets. The distance between these two sets \mathcal{X} and \mathcal{Y} is defined by $d(\mathcal{X}, \mathcal{Y}) = \min\{d(x, y) : x \in \mathcal{X}, y \in \mathcal{Y}\}$.*

DEFINITION 1.10.— *Let \mathcal{X} and \mathcal{Y} be two non-empty sets. The Hausdorff distance of these two sets \mathcal{X} and \mathcal{Y} is defined by $d_H(\mathcal{X}, \mathcal{Y}) = \max\{\bar{d}_H(\mathcal{X}, \mathcal{Y}), \bar{d}_H(\mathcal{Y}, \mathcal{X})\}$, with $\bar{d}_H(\mathcal{X}, \mathcal{Y}) = \max_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} d(x, y)$.*

The Hausdorff distance allows characterizing the quality of the approximation of \mathcal{X} by \mathcal{Y} [HUN 93]. If \mathcal{X} and \mathcal{Y} have the same closure³, then the Hausdorff distance is equal to 0. Figure 1.1 illustrates the difference between the “normal” distance (Definition 1.9) which is equal to 0 and the Hausdorff distance $d_H(\mathcal{X}, \mathcal{Y})$ between the two sets \mathcal{X} and \mathcal{Y} .

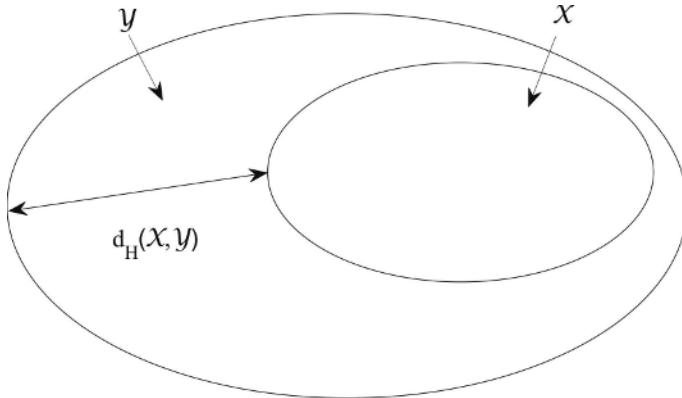


Figure 1.1. *Illustration of the Hausdorff distance between two sets*

³ The closure of a set S is defined by the union of S and its boundary ∂S .

1.1.1. Interval set

A very simple way to define uncertainties is by using the *interval* notion. This is based on the idea of enclosing numerical errors into an interval. In many cases, obtaining the probability of occurrence of different uncertainties is not possible. Therefore, it can be easier and thus suitable to bound the uncertainties by intervals. Moreover, the interval analysis allows us to simplify most of the standard operations [MOO 66, HAN 65, JAU 01]. This approach is developed in many domains (e.g. identification, diagnosis and estimation), especially when a short computation time is required.

DEFINITION 1.11.— *An interval $\mathbf{I} = [a, b] \subset \mathbb{R}$ is defined by the set $\{x \in \mathbb{R} : a \leq x \leq b\}$.*

DEFINITION 1.12.— *The center and the radius of an interval $\mathbf{I} = [a, b]$ are, respectively, defined by $\text{mid}(\mathbf{I}) = \frac{a+b}{2}$ and $\text{rad}(\mathbf{I}) = \frac{b-a}{2}$.*

DEFINITION 1.13.— *The unitary interval is denoted by $\mathbf{B} = [-1, 1]$.*

DEFINITION 1.14.— *The set of real compact intervals $[a, b]$, where $a, b \in \mathbb{R}$ and $a \leq b$, is denoted by \mathbb{I} .*

DEFINITION 1.15.— *A box $([a_1, b_1], \dots, [a_n, b_n])^T$ is an interval vector.*

DEFINITION 1.16.— *A unitary box in \mathbb{R}^n , denoted by \mathbf{B}^n , is a box composed of n unitary intervals.*

Consider the intervals $[\underline{x}, \bar{x}]$ and $[\underline{y}, \bar{y}]$. An operation \circ between these two intervals can be formalized as:

$$[\underline{x}, \bar{x}] \circ [\underline{y}, \bar{y}] = \{x \circ y : x \in [\underline{x}, \bar{x}], y \in [\underline{y}, \bar{y}]\} \quad [1.1]$$

The four basic operations [MOO 66] with intervals are the following:

- 1) $[\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] = [\underline{x} + \underline{y}, \bar{x} + \bar{y}]$
- 2) $[\underline{x}, \bar{x}] - [\underline{y}, \bar{y}] = [\underline{x} - \bar{y}, \bar{x} - \underline{y}]$
- 3) $[\underline{x}, \bar{x}] * [\underline{y}, \bar{y}] = [\min(\underline{x} \cdot \underline{y}, \underline{x} \cdot \bar{y}, \bar{x} \cdot \underline{y}, \bar{x} \cdot \bar{y}), \max(\underline{x} \cdot \underline{y}, \underline{x} \cdot \bar{y}, \bar{x} \cdot \underline{y}, \bar{x} \cdot \bar{y})]$
- 4) $[\underline{x}, \bar{x}] / [\underline{y}, \bar{y}] = [\underline{x}, \bar{x}] * [1/\bar{y}, 1/\underline{y}]$, if $0 \notin [\underline{y}, \bar{y}]$

Despite the simplicity of interval analysis, a drawback of this approach is that the computation results can sometimes be conservative due to the *dependency effect*⁴ and the *wrapping effect*⁵ [MOO 66, KÜH 98b, JAU 01]. These two effects are further analyzed through two examples.

EXAMPLE 1.1.— DEPENDENCY EFFECT – Consider two functions $f_1(x, y) = x - y$ and $f_2(x) = x - x$, with the variables $x, y \in [-1, 1]$. Using the interval analysis, the value domain of f_1 and f_2 is the same $[-1, 1] - [-1, 1] = [-2, 2]$, even if the real value domain of f_2 is 0. This problem, called the “dependency effect”, relies on the fact that the occurrence of the same variable x in the function f_2 is independently considered and can lead to an important over-approximation of the result.

EXAMPLE 1.2.— WRAPPING EFFECT – Consider a function $f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$, with $x, y \in [-1, 1]$ and $A = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}$. Using the interval analysis leads to $f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} -0.5y \\ x + y \end{bmatrix}$, with the value domains $f_1 \in [-0.5, 0.5]$ and $f_2 \in [-2, 2]$, represented in dark gray in

⁴ The dependency effect appears when a variable occurs more than one time in a function and each occurrence is considered independently, increasing the resulting interval.

⁵ The wrapping effect appears when the domain representation grows due to the over-estimation at each sampling time.

Figure 1.2. The vertices of the exact solution of f (see Figure 1.2) $\left\{ \begin{bmatrix} -0.5 \\ 2 \end{bmatrix}, \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -2 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \right\}$ are obtained by computing the product of the matrix A with each vertex of the unitary box containing x and y (e.g. $[-0.5, 0]^T = A \cdot [-1, 1]^T$). Comparing these solutions, an important over-approximation of the interval analysis solution can be observed. If this operation is repeated several times, the difference between the exact solution and the solution of the interval analysis is more and more important. This problem is called the “wrapping effect”.

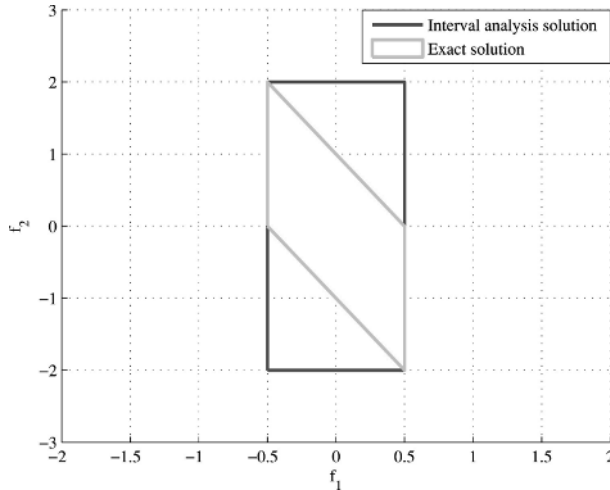


Figure 1.2. Illustration of the wrapping effect in interval analysis

1.1.2. Ellipsoidal set

Another way to represent bounded uncertainties is based on the use of bounded ellipsoids. Moreover, due to its low complexity, the *ellipsoidal* set is widely used in a large class of applications in automatic control [SCH 68, WIT 68,

BER 71, CHE 81, MIL 96, KUR 96, DUR 01, POL 04]. Some basic properties of ellipsoids are discussed in the following.

DEFINITION 1.17.— Given a vector $\mathbf{c} \in \mathbb{R}^n$ and a symmetric positive definite matrix $P = P^T \succ 0$, an ellipsoid $\mathcal{E}(\mathbf{c}, P)$ is defined as follows:

$$\mathcal{E}(\mathbf{c}, P) = \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x} - \mathbf{c})^T P^{-1} (\mathbf{x} - \mathbf{c}) \leq 1\} \quad [1.2]$$

The vector \mathbf{c} is called the *center* of the ellipsoid $\mathcal{E}(\mathbf{c}, P)$ and the matrix P is called the *shape matrix* of the ellipsoid $\mathcal{E}(\mathbf{c}, P)$.

EXAMPLE 1.3.— An ellipsoid $\mathcal{E}(\mathbf{c}, P)$ with $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ is illustrated in Figure 1.3.

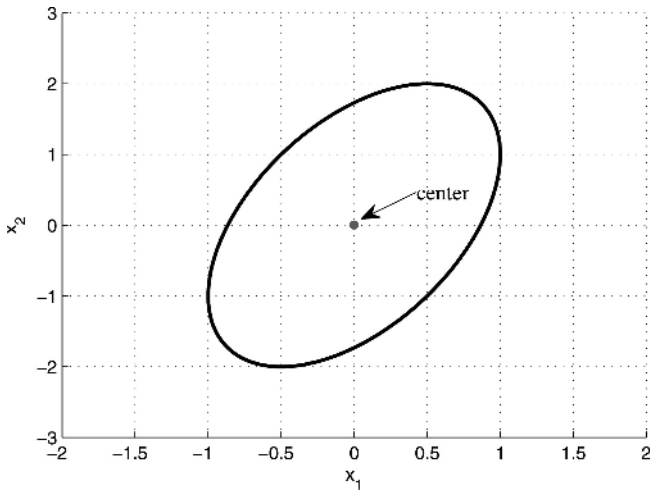


Figure 1.3. Illustration of a bounded ellipsoid in \mathbb{R}^2

PROPERTY 1.1.— Consider an ellipsoid $\mathcal{E}(\mathbf{c}, P) \subseteq \mathbb{R}^n$, a matrix $A \in \mathbb{R}^{n \times n}$ and a vector $\mathbf{b} \in \mathbb{R}^n$. An *affine transformation* of this ellipsoid is also an ellipsoid $A\mathcal{E}(\mathbf{c}, P) + \mathbf{b} = \mathcal{E}(A\mathbf{c} + \mathbf{b}, APA^T)$.

The interested reader can find concrete studies on ellipsoids and their operations in [BOY 94] and [KUR 96] and can easily use the Ellipsoidal Toolbox [KUR 06] to implement different operations of ellipsoidal calculus within a MATLAB[®] environment.

Despite the simple representation of ellipsoids, there are still some drawbacks that can be a source of conservatism. The main inconvenience is that ellipsoids are not closed under some operations⁶. Furthermore, ellipsoids allow representing bounded uncertainties which are necessarily coupled [LAL 08] due to the low flexibility of the ellipsoidal shape compared to polyhedral sets which are presented in the following section.

1.1.3. *Polyhedral set*

The *polyhedral* set is one of the most popular geometrical forms used in control systems and optimization. A polyhedral set in a finite-dimensional Euclidean space is the intersection of a finite amount of closed half-spaces [ZIE 95]. A bounded polyhedral is denoted as a *polytope*. Because of its flexibility, polytopes offer a good approximation of any convex set [LAY 82, BRO 08, SCI 11]. In addition, polytopes are closed under the basic operations⁷ unlike ellipsoids. Moreover, its dual representations (half-space representation and vertex representation) allow us to choose the appropriate form for a particular problem. The main weakness of polytopes is due to

⁶ For example, the Minkowski sum or the intersection of two ellipsoids is generally not an ellipsoid. In order to preserve the simplicity offered by ellipsoids, ellipsoidal outer approximations of the obtained sets can be made, leading to a trade-off between accuracy of the representation and simplicity of the computation.

⁷ For example, the Minkowski sum or the intersection of two convex polytopes is generally a polytope.

their complexity related to the number of vertices, but not fixed by the space dimension. Therefore, even if a polytope can approximate any convex set well, the complexity can quickly increase with the number of vertices even in a low space dimension. In order to formalize the notations, the main definitions of polytopes are summarized below.

DEFINITION 1.18.— **HALF-SPACE REPRESENTATION** – A polyhedral set $\mathcal{P} \in \mathbb{R}^n$ can be defined as the intersection of a finite number of half-spaces: $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n : H\mathbf{x} \leq \mathbf{k}\}$, with $H \in \mathbb{R}^{m \times n}$, $\mathbf{k} \in \mathbb{R}^m$. If \mathcal{P} is bounded, then \mathcal{P} is a polytope.

EXAMPLE 1.4.— A polytope \mathcal{P} in the half-space representation with:

$$H = \begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -4 \end{bmatrix}^T, \mathbf{k} = [1 \ 1 \ 1 \ 4]^T$$

is shown in Figure 1.4. H_i and k_i denote the i th row of matrix H and the i th element of vector \mathbf{k} , respectively.

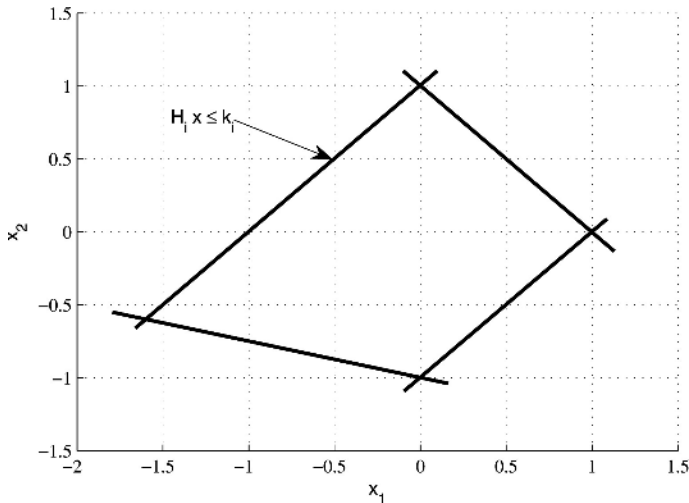


Figure 1.4. *H-representation of a polytope*

DEFINITION 1.19.– **VERTEX REPRESENTATION** – *For a finite set of points $\mathcal{V} \subset \mathbb{R}^n$, with $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$, a polytope \mathcal{P} can be defined as the convex hull of the set \mathcal{V} :*

$$\mathcal{P} = \text{conv}(\mathcal{V}) = \left\{ \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_m \mathbf{v}_m : \alpha_i \in \mathbb{R}^+, \sum_{i=1}^m \alpha_i = 1, \mathbf{v}_i \in \mathbb{R}^n \right\}$$

EXAMPLE 1.5.– Consider the set of vertices $\mathcal{V} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1.6 \\ -0.6 \end{bmatrix} \right\}$. Figure 1.5 shows the V -representation of a polytope based on the set \mathcal{V} .

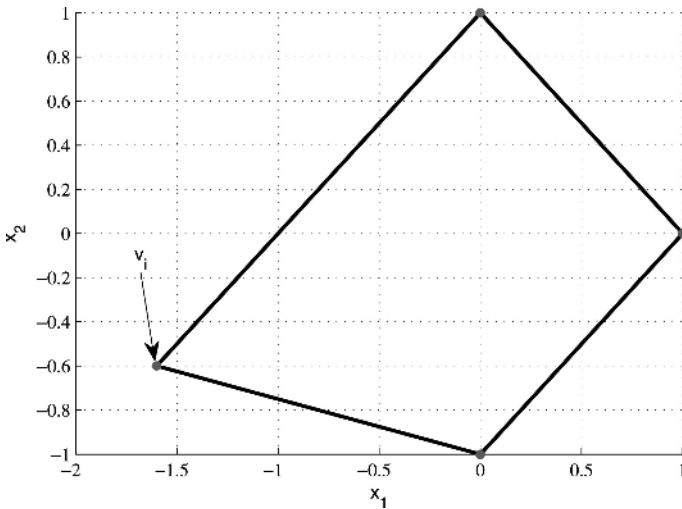


Figure 1.5. V -representation of a polytope. For a color version of this figure, see www.iste.co.uk/stoica/zonotopes.zip

Theorem 1.1 [ZIE 95] shows the equivalence of these two definitions of a polytope (i.e. H -polytope and V -polytope) allowing us to choose a suitable representation for a

particular problem. For example, the proof that the polytopic set is closed under the Minkowski sum is trivial via the V -representation but can become complicated via the H -representation.

THEOREM 1.1.– EQUIVALENCE OF H -REPRESENTATION AND V -REPRESENTATION [ZIE 95] – A set $\mathcal{P} \subset \mathbb{R}^n$ is the convex hull of a finite point set (a V -polytope) if and only if it is a bounded intersection of half-spaces (a H -polytope).

This theorem shows that the H -polytope can be transformed into the V -polytope and vice versa. In the literature, this problem is well known as the *vertex enumeration problem* for the transformation of a V -polytope to a H -polytope and the *facet enumeration problem* for the transformation of a H -polytope to its equivalent V -polytope. There exist algorithms to solve these transformation problems, but they are time consuming (e.g. [DAN 72] and [FUK 99]). The interested reader can find more details on polytopes in [ZIE 95] and [BLA 07] and the references therein. An example of the same polytope defined by H -representation and V -representation is illustrated in Figures 1.4 and 1.5.

Even if polytopes can approximate any convex set well, their complexity increases with its number of vertices. This can lead to some limitations for the direct application of the methods based on polytopes. In the following section, a geometrical form which offers a good trade-off between complexity and flexibility is presented.

1.1.4. *Zonotopic set*

In recent years, zonotopes are used more and more to represent uncertainties due to the flexibility, the reduced

complexity and specially the efficient computation of linear transformations and Minkowski sums. Zonotopes are a special class of convex symmetric polytopes. Thus, the properties developed for convex polytopes can also be applied to zonotopes. In this way, zonotopes can be represented as H -polytopes or V -polytopes. There also exist different representations specific to zonotopes, which are presented below.

DEFINITION 1.20.— **GENERATOR REPRESENTATION** — *Given a vector $\mathbf{p} \in \mathbb{R}^n$ and a set of vectors $\mathcal{G} = \{\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_m\} \subset \mathbb{R}^n$, $m \geq n$, a zonotope \mathcal{Z} of order m (also called a m -zonotope) is defined as follows:*

$$\begin{aligned} \mathcal{Z} &= (\mathbf{p}; \mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_m) \\ &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{p} + \sum_{i=1}^m \alpha_i \mathbf{g}_i; -1 \leq \alpha_i \leq 1\} \end{aligned} \quad [1.3]$$

The vector \mathbf{p} is called the *center* of the zonotope \mathcal{Z} . The vectors $\mathbf{g}_1, \dots, \mathbf{g}_m$ are called the *generators* of \mathcal{Z} . The *order* of a zonotope is defined by the number of its generators (m in this case). A m -zonotope $\mathcal{Z} \subset \mathbb{R}^n$ with m generators and $m < n$ is called a *degenerated m -zonotope*.

This definition is equivalent to the definition of zonotopes obtained by the Minkowski sum of a finite number of line segments defined by $\mathbf{g}_i \mathbf{B}^1$:

$$\mathcal{Z} = (\mathbf{p}; \mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_m) = \mathbf{p} \oplus \mathbf{g}_1 \mathbf{B}^1 \oplus \dots \oplus \mathbf{g}_m \mathbf{B}^1 \quad [1.4]$$

EXAMPLE 1.6.— Consider the following zonotope of third order in \mathbb{R}^2 with $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and its generators $\mathbf{g}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{g}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{g}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. This 3-zonotope is illustrated in Figure 1.6.

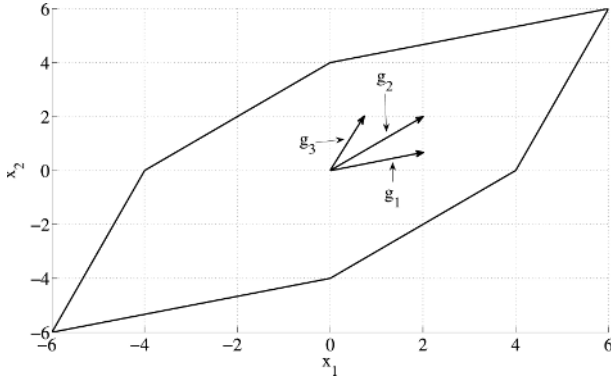


Figure 1.6. 3-zonotope and its generators in \mathbb{R}^2

EXAMPLE 1.7.— Figure 1.7 represents a sixth-order *centered zonotope*⁸ in \mathbb{R}^3 with:

$$\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{g}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{g}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{g}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{g}_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$\mathbf{g}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{g}_6 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

From these two examples, it can be observed that the complexity of zonotopes depends on the number of generators and the dimension of the space. The complexity grows quickly with the number of generators: the number of vertices of the zonotope is 6 in Figure 1.6 and 24 in Figure 1.7. When the number of generators is increased, the space dimension is also increased in Figure 1.7 compared to Figure 1.6.

Another definition of zonotopes that is more convenient for the approach considered in this book is based on the fact that

⁸ A centered zonotope is a zonotope whose center is the origin (i.e. $\mathbf{p} = \mathbf{0}_n$ in equation [1.3]).

a m -zonotope in \mathbb{R}^n can be defined as the affine image of a m -dimensional hypercube in \mathbb{R}^n .

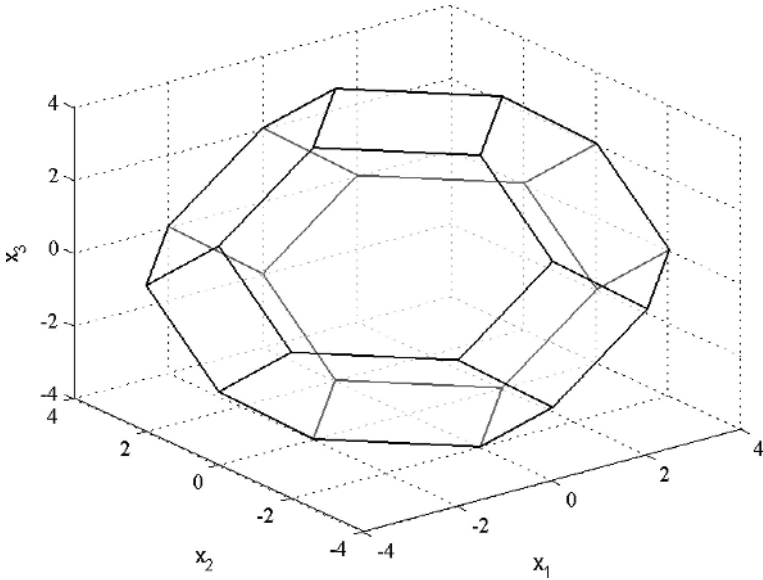


Figure 1.7. 6-zonotope in \mathbb{R}^3

DEFINITION 1.21.— **HYPERCUBE AFFINE PROJECTION** – A m -zonotope in \mathbb{R}^n ($m \geq n$) is the translation by the center $\mathbf{p} \in \mathbb{R}^n$ of the image of a unitary hypercube of dimension m in \mathbb{R}^n under a linear transformation. Given a matrix $H \in \mathbb{R}^{n \times m}$ representing the linear transformation, the zonotope \mathcal{Z} is defined by:

$$\mathcal{Z} = (\mathbf{p}; H) = \mathbf{p} \oplus HB^m \quad [1.5]$$

Considering the matrix $H = [\mathbf{g}_1 \ \mathbf{g}_2 \ \dots \ \mathbf{g}_m]$ proves the equivalence of the two proposed definitions. From now on, to simplify the chapter, the zonotope \mathcal{Z} is described by $\mathcal{Z}(\mathbf{p}; H)$.

EXAMPLE 1.8.— Consider the zonotope $\mathcal{Z}(\mathbf{p}; H)$ with $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Using the hypercube affine projection, this zonotope can be constructed as the image of a hypercube in \mathbb{R}^3 (with its eight vertices $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$) under the projection H in \mathbb{R}^2 (see Figure 1.8).

The reader can note that the same zonotope is obtained via the hypercube affine projection (Figure 1.8) and via the generator representation (Figure 1.6).

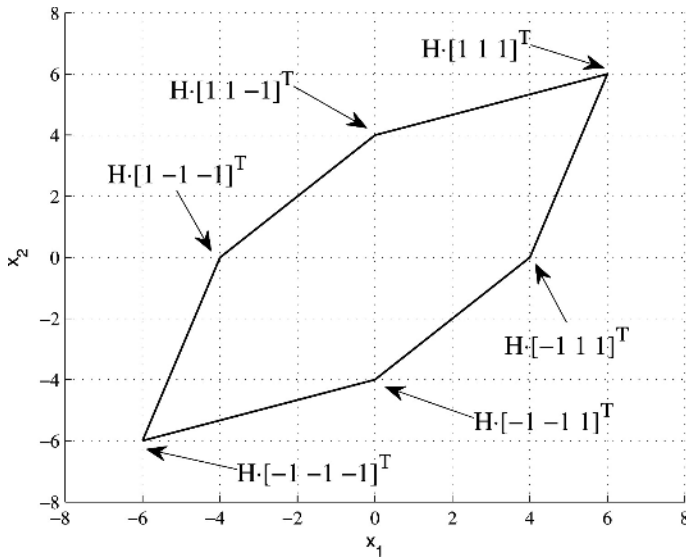


Figure 1.8. 3-zonotope and its vertices in \mathbb{R}^2

The conversion between these particular representations of zonotopes is studied by many authors

[GRI 93, SEY 94, FUK 04, SCH 05b, ALT 10]. As zonotopes are special cases of polytopes, the generator representation and the hypercube affine projection of a zonotope can be converted to the V -representation and also to the H -representation. These conversions are related to the Minkowski sum of two polytopes because the generator representation is equivalent to the Minkowski sum of a finite number of line segments, which is a polytope. In addition, the generator representation illustrates a significant advantage of zonotopes: a complex geometrical form represented via a simple matrix and a vector (that can be zero for zonotopes centered to the origin). For example, the centered zonotope from Figure 1.7 with 24 vertices in \mathbb{R}^3 is represented by a matrix $H \in \mathbb{R}^{3 \times 6}$. This leads us to simplify the mentioned basic set operations by simple matrix computation as presented in the following section.

1.2. Main properties of zonotopes

This section proposes an overview of the main properties of zonotopes that will be used throughout this book. Some examples are also considered in order to make the book easier to comprehend.

PROPERTY 1.2.– PARTICULAR ZONOTOPES – *Given a centered zonotope $\mathcal{Z} = HB^m \in \mathbb{R}^n$. Because of the properties of the matrix H , some particular forms of zonotopes can be obtained. If H is the identity matrix, then \mathcal{Z} is the *unit box*. If H is diagonal, orthogonal or invertible, then \mathcal{Z} is a *box*, a *hypercube* or a *parallelotope*⁹, respectively.*

⁹ A parallelotope is a special zonotope whose number of generators is equal to the dimension of the space.

PROPERTY 1.3.– GENERATOR PERMUTATION – The permutation of the matrix columns in the generator representation of a zonotope does not modify the zonotope.

PROOF – This property results from the commutativity of the Minkowski sum of several line segments. \square

PROPERTY 1.4.– SUM OF TWO ZONOTOPES – Given two zonotopes $\mathcal{Z}_1 = \mathbf{p}_1 \oplus H_1 \mathbf{B}^{m_1} \in \mathbb{R}^n$ and $\mathcal{Z}_2 = \mathbf{p}_2 \oplus H_2 \mathbf{B}^{m_2} \in \mathbb{R}^n$, the Minkowski sum of two zonotopes is also a zonotope defined by $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2 = (\mathbf{p}_1 + \mathbf{p}_2) \oplus [H_1 \ H_2] \mathbf{B}^{m_1+m_2}$.

PROOF – From the definition of the Minkowski sum, it results in $\mathcal{Z}_1 \oplus \mathcal{Z}_2 = \{\mathbf{p}_1 + \mathbf{p}_2 + H_1 \mathbf{z}_1 + H_2 \mathbf{z}_2 : \mathbf{z}_1 \in \mathbf{B}^{m_1}, \mathbf{z}_2 \in \mathbf{B}^{m_2}\}$, which can be further rewritten in a matrix form as:

$$\mathcal{Z}_1 \oplus \mathcal{Z}_2 = \{\mathbf{p}_1 + \mathbf{p}_2 + [H_1 \ H_2] \cdot \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} : \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} \in \mathbf{B}^{m_1+m_2}\} =$$

$$(\mathbf{p}_1 + \mathbf{p}_2) \oplus [H_1 \ H_2] \mathbf{B}^{m_1+m_2} = \mathcal{Z}. \quad \square$$

PROPERTY 1.5.– LINEAR IMAGE OF A ZONOTOPE – *The image of a zonotope $\mathcal{Z} = \mathbf{p} \oplus H \mathbf{B}^m \in \mathbb{R}^n$ by a linear mapping K can be computed by a standard matrix product $K \cdot \mathcal{Z} = (K \cdot \mathbf{p}) \oplus (K \cdot H) \mathbf{B}^m$.*

PROOF – Using matrix multiplication leads to $K \mathcal{Z} = \{K(\mathbf{p} + H \mathbf{z}) : \mathbf{z} \in \mathbf{B}^m\} = \{K \mathbf{p} + K H \mathbf{z} : \mathbf{z} \in \mathbf{B}^m\} = (K \mathbf{p}) \oplus (K H) \mathbf{B}^m$. \square

The basic operations with zonotopes can increase the size of the obtained zonotopes. Thus, an interesting issue concerns the techniques that reduce the complexity of zonotopes, allowing us to limit the number of generators of a zonotope. For example, if the problem of *reachable set*¹⁰ is addressed using zonotopes, the complexity of this zonotope increases at each sample time due to the Minkowski sum operation. This

¹⁰ This is the problem of computing all states visited by trajectories of a system starting from any $x_0 \in X_0$.

part focuses on over-approximations of a high-order zonotope leading to compute a reduced order zonotope enclosing the initial zonotope.

PROPOSITION 1.1.— **COMPLEXITY REDUCTION OF ZONOTOPES VIA INTERVAL HULL METHOD** – Considering a zonotope $\mathcal{Z} = \mathbf{p} \oplus H\mathbf{B}^m \in \mathbb{R}^n$, the smallest box containing this zonotope is computed by:

$$\blacksquare \mathcal{Z} = \mathbf{p} \oplus rs(H)\mathbf{B}^n \quad [1.6]$$

with $rs(H)$ a diagonal matrix such that $rs(H)_{ii} = \sum_{j=1}^m |H_{ij}|$, $i = 1, \dots, n$.

PROOF.— As a box is an axis-aligned set, the over-approximation of a zonotope by a box can be made by considering its extreme points in each direction. The extreme point in the direction i can be computed by $p_i + \sum_{j=1}^m |H_{ij}|$. All extreme points in all n directions are similarly computed and the smallest box containing the zonotope \mathcal{Z} is obtained as $\blacksquare \mathcal{Z} = \mathbf{p} \oplus rs(H)\mathbf{B}^n$. \square

This proposition provides a simple and fast over-approximation of a zonotope by a box. The result has a minimal complexity which is given by the dimension of the space. However, the result obtained with this proposition is conservative because the form of the initial zonotope is lost.

An example is proposed in the following in order to better illustrate this proposition.

EXAMPLE 1.9.— Consider a centered zonotope $\mathcal{Z} = H\mathbf{B}^3 \in \mathbb{R}^2$, with $H = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Applying the interval hull approximation leads to a box (in Figure 1.9) containing the original zonotope (hexagon in Figure 1.9).

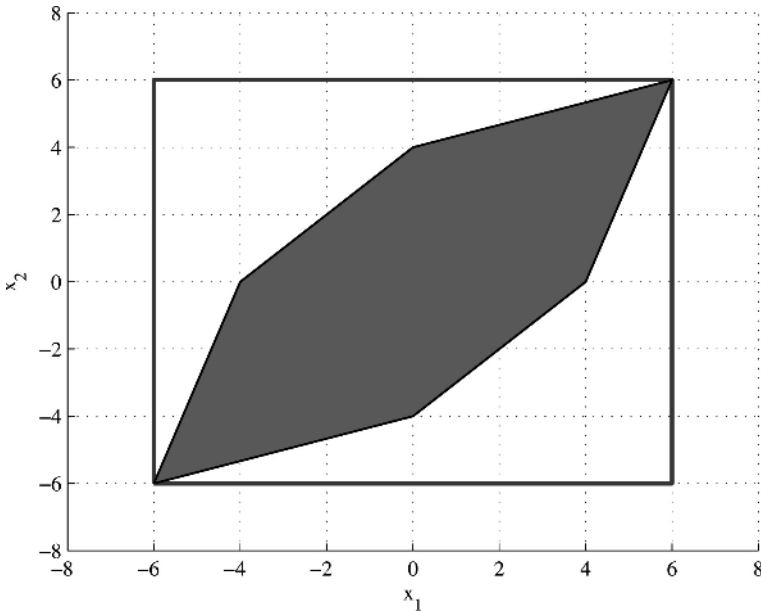


Figure 1.9. Interval hull of a zonotope. For a color version of this figure, see www.iste.co.uk/stoica/zonotopes.zip

In order to reduce the conservatism, an outer approximation based on the parallelotope hull is further presented.

PROPOSITION 1.2.— COMPLEXITY REDUCTION OF ZONOTOPES VIA PARALLELOTOPE HULL METHOD – Given a zonotope $\mathcal{Z} = \mathbf{p} \oplus H\mathbf{B}^m \in \mathbb{R}^n$ ($m > n$), an over-approximation of this zonotope by a parallelotope is computed as:

$$\blacklozenge \mathcal{Z} = \Gamma \cdot \blacksquare(\Gamma^{-1} \mathcal{Z}) \quad [1.7]$$

where $\Gamma \in \mathbb{R}^{n \times n}$ is an invertible matrix containing n columns taken from H .

PROOF [ALT 10].— This approach first transforms the coordinates of \mathcal{Z} by the linear mapping Γ^{-1} (leading to

$\Gamma^{-1}\mathcal{Z} = (\Gamma^{-1}\mathbf{p}) \oplus (\Gamma^{-1}H)\mathbf{B}^m$, where the new coordinate axes are the column vectors of Γ . In these new coordinates, the zonotope is over-approximated by a box using the interval hull. This box is transformed back to the original coordinate system, resulting in a parallelotope. The over-approximation is guaranteed by the fact that the parallelotope is over-approximated in the transformed coordinate system by the interval hull operator, such that it is also over-approximated after the transformation to the original coordinate system. \square

EXAMPLE 1.10.– The same zonotope as in Example 1.9 is taken in order to compare the two approximation methods. In Figure 1.10, this zonotope \mathcal{Z} is over-approximated by three different parallelotopes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ due to the different choice of the matrix Γ ($\Gamma_1 = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $\Gamma_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $\Gamma_3 = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$). Comparing the two examples (Figures 1.9 and 1.10), the over-approximation by the parallelotope hull is less conservative than the over-approximation by interval hull, but with a higher complexity because n generators must be chosen among m generators to have the best approximation (the \mathcal{P}_2 parallelotope). Some criteria for selecting the suitable generators are given in [MOO 66] and [ALT 10].

PROPOSITION 1.3.– **CASCADE REDUCTION OF ZONOTOPES VIA THE GENERATOR SELECTION METHOD** – Given a zonotope $\mathcal{Z} = \mathbf{p} \oplus H\mathbf{B}^{m-n} \in \mathbb{R}^n$ ($m \geq n$), with H a m -block matrix of $n \times n$ matrices ($H = [H_1 \dots H_m]$), let $D(l) = [H_1 \dots H_l]$ be the matrix obtained by choosing l blocks of H . Choosing the biggest l ($2 \leq l \leq m$) for which $\|D(l-1)\|_\infty > \|H_l\|_\infty$ or $l = 1$ if such an integer does not exist, this norm criterion is called the *fullness criterion* and imposes that the small parallelotope will be over-approximated more frequently than the big parallelotope. Then an over-approximation of \mathcal{Z} is defined by:

$$\mathcal{Z} \subseteq \mathbf{p} \oplus [rs(D(l)) H_{l+1} \dots H_m] \quad [1.8]$$

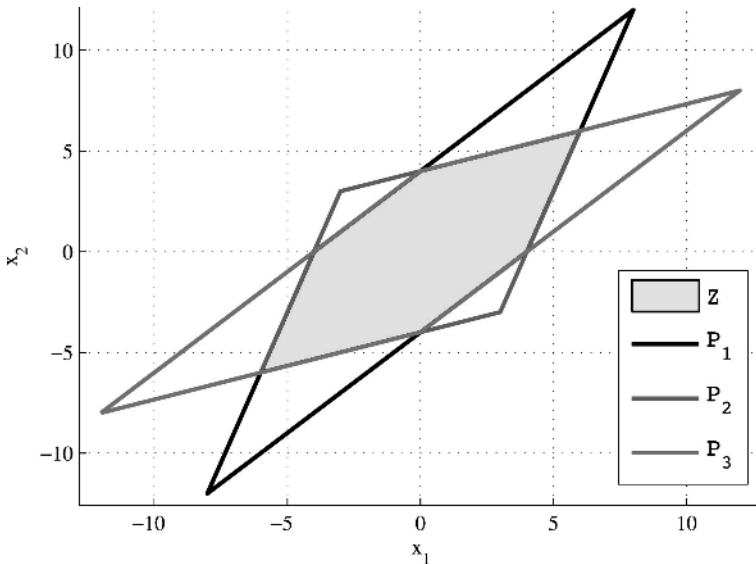


Figure 1.10. *Parallelotope hull of a zonotope. For a color version of this figure, see www.iste.co.uk/stoica/zonotopes.zip*

This proposition is based on the representation of a zonotope by the Minkowski sum of parallelotopes. The interested reader can find more details on this proposition and its performance in [KÚH 98a].

PROPOSITION 1.4.— **CRITERION-BASED REDUCTION OF A ZONOTOPE** – Given the zonotope $\mathcal{Z} = \mathbf{p} \oplus H\mathbf{B}^m \in \mathbb{R}^n$ and the integer s , with $n < s < m$, denote \hat{H} the matrix resulting from the reordering of the columns of the matrix H by a criterion which is detailed in the following ($\hat{H} = [\hat{h}_1 \dots \hat{h}_i \dots \hat{h}_m]$). The zonotope is rewritten as: $\mathcal{Z} = \mathbf{p} \oplus \hat{H}_1\mathbf{B}^{s-n} \oplus \hat{H}_2\mathbf{B}^{m-s+n}$, where \hat{H}_1 is obtained from the first $s - n$ columns of matrix \hat{H} and \hat{H}_2 is the remainder of \hat{H} . Then the initial zonotope is over-approximated by a zonotope of reduced order s as follows: $\mathcal{Z} \subseteq \mathbf{p} \oplus \hat{H}_1\mathbf{B}^{s-n} \oplus Q\mathbf{B}^n$, where $Q\mathbf{B}^n$ is the over-approximation of the zonotope $\hat{H}_2\mathbf{B}^{m-s-n}$.

This over-approximation can be a box using Proposition 1.1 or a parallelotope using Proposition 1.2.

PROOF.— [ALA 05] – Since a column of the matrix H represents a segment of the zonotope \mathcal{Z} , then a column permutation in the matrix H does not modify the zonotope \mathcal{Z} . It means that $\mathcal{Z} = \mathbf{p} \oplus H\mathbf{B}^m = p \oplus \hat{H}\mathbf{B}^m$. From the definition of matrix \hat{H} and applying Property 1.4, it results in:

$$\begin{aligned} \mathcal{Z} &= \mathbf{p} \oplus [\hat{H}_1 \hat{H}_2] \mathbf{B}^m = \\ &= \mathbf{p} \oplus \hat{H}_1 \mathbf{B}^{s-n} \oplus \hat{H}_2 \mathbf{B}^{m-s+n} = \\ &= \mathbf{p} \oplus \hat{H}_1 \mathbf{B}^{s-n} \oplus [\hat{h}_{s-n+1} \dots \hat{h}_m] \mathbf{B}^{m-s+n} \end{aligned} \quad [1.9]$$

Propositions 1.1 and 1.2 show that the centered zonotope $[\hat{h}_{s-n+1} \dots \hat{h}_m] \mathbf{B}^{m-s+n}$ can be approximated by $Q\mathbf{B}^n$, where Q is a diagonal matrix (if a box-based approximation is used) or a full matrix (if a parallelotope-based approximation is used). Therefore, the following expression is true: $\mathcal{Z} \subseteq \mathbf{p} \oplus \hat{H}_1 \mathbf{B}^{s-n} \oplus Q\mathbf{B}^n = \mathbf{p} \oplus [\hat{H}_1 Q] \mathbf{B}^s$. \square

Using Proposition 1.4, the quality of the approximation depends on the value of s which limits the complexity, the criterion used to split the zonotope \mathcal{Z} and the approximation method (box or parallelotope) used for the zonotope $\hat{H}_2 \mathbf{B}^{m-s-n}$. A large value of s means high accuracy of the approximation and high complexity of the computation. Two methods to split the zonotope \mathcal{Z} are described below.

A first approach consists of sorting the generators of the zonotope in decreasing order of the Euclidean norm

[COM 03, ALA 05], which is equivalent to arranging the segments of zonotope from the longest to the shortest. Then the longest segments which have a more important role in the shape of the zonotope are kept and the contribution of the shortest segments is over-approximated by a box or a parallelotope in order to limit the complexity.

A second approach consists of reordering the columns of matrix H in decreasing order of the term $\|h_i\|_1 - \|h_i\|_\infty$ [GIR 05]. The chosen generators (whose contribution is approximated) are close to vectors with only one non-zero component and are therefore well approximated by an interval hull.

An example is proposed in order to better illustrate the quality of these methods of complexity reduction.

EXAMPLE 1.11.— Consider a centered zonotope $\mathcal{Z} = HB^8 \in \mathbb{R}^2$, with $m = 8$, $n = 2$ and $H = \begin{bmatrix} 0.9169 & 0.8936 & 0.3529 & 0.0099 & 0.2028 & 0.6038 & 0.1988 & 0.7468 \\ 0.4103 & 0.0579 & 0.8132 & 0.1389 & 0.1987 & 0.2722 & 0.0153 & 0.4451 \end{bmatrix}$.

This zonotope is approximated using Propositions 1.3 and 1.4¹¹. Using the cascade reduction, the value of $l = 3$ is obtained. Figure 1.11 shows the approximation of the initial zonotope using the cascade reduction. Figure 1.12 shows the over-approximation (Z_{initial}) of the zonotope \mathcal{Z} obtained using, as the criterion, the Euclidean norm with different values of s (with $s = 4$, $s = 5$). This example confirms that a bigger value of s leads to a better approximation.

¹¹ In Proposition 1.4, the over-approximation by a box is used.

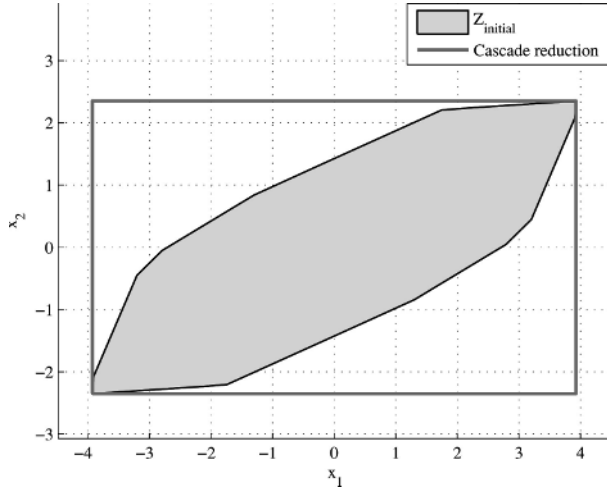


Figure 1.11. Complexity reduction of a zonotope using the cascade reduction. For a color version of this figure, see www.iste.co.uk/stoica/zonotopes.zip

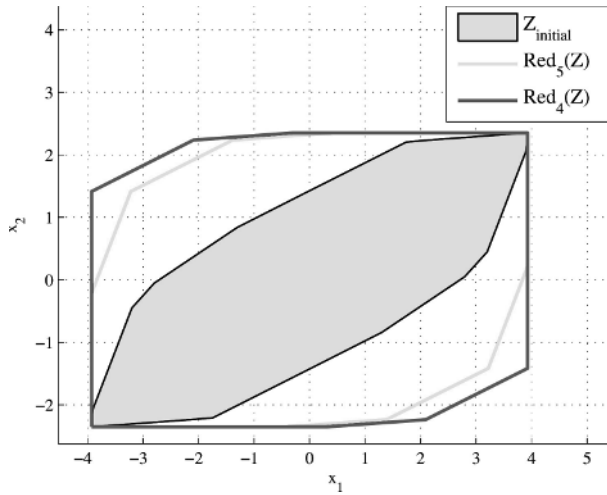


Figure 1.12. Complexity reduction of a zonotope using the Euclidean norm-based criterion. For a color version of this figure, see www.iste.co.uk/stoica/zonotopes.zip

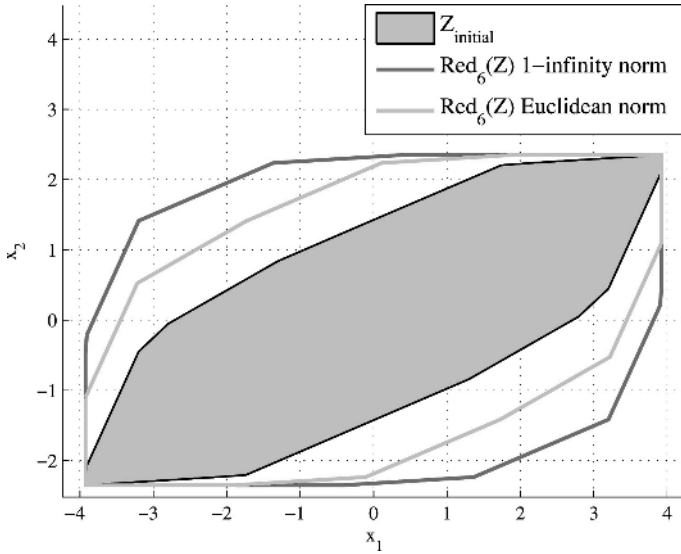


Figure 1.13. Complexity reduction of a zonotope: comparing two criteria. For a color version of this figure, see www.iste.co.uk/stoica/zonotopes.zip

Figure 1.13 compares the performance of the over-approximation of the same zonotope Z based on the two criteria of the generators' reordering and selection: the Euclidean norm and the difference between the H_1 norm and the H_∞ norm. The same value $s = 6$ is chosen for both cases. In this example, the best approximation is obtained using the Euclidean norm-based criterion. Therefore, in the further examples proposed in this book, the over-approximation based on the Euclidean norm criterion is used.