Chapter 1

Mathematical Examination of Dielectrics

1.1. Introduction to dielectrics

The ideal insulator is a substance with infinite resistivity. In the real world, insulators have resistivity values which are very high, but finite. Table 1.1 gives an indication of the resistivity of a number of insulators, expressed using the MKSA (Meter, Kilogram, Second, Ampere) system.

PolyPropylene	**PolyImide**	**Epoxy**	**Phlogopite Mica**	**Silica Glass**
10^{15}	10^{14}	10^{13}	10^{12}	10^{11}

Table 1.1. *Resistivity of a number of dehydrated insulators expressed in [Ω.m] at 20°C*

For many applications, the value which best characterizes a material's insulating capacity is its relaxation time – the time constant of a condenser of any form using that material. This time is given by the following equation [1.1]:

$$\tau = R.C = \frac{\rho.l}{S}.\frac{\varepsilon.S}{l} = \varepsilon.\rho \qquad [1.1]$$

$$\underline{AN}: \tau \approx 8.85\times10^{-2}[s] \text{ with } \begin{cases} \varepsilon = 8.85\times10^{-12}[m^{-3}.kg^{-1}.s^{4}.A^{2}] \\ \rho = 10^{10}[\Omega.m] \end{cases}$$

where ε is the permittivity and ρ the resistivity.

If we take the value of ε to be that in a vacuum – ε_0 – we can see that for $\rho = 10^{10}$ [$\Omega.m$], the relaxation time of charges through the insulator is close to a fraction of [*s*]; hence, this is not a good insulator. The accepted materials in electrotechnics have relaxation times ranging from around a [*s*] to several [*min*] or more; in exceptional conditions, we have even seen relaxation times of around a [*year*].

The relaxation time is directly involved in the value of the electrical loss angle δ with alternating current (AC). Indeed, if conduction is the only cause of loss, and if the time constant τ is around a second, the loss angle will be too great for long-term operation at industrial frequencies without the risk of accident [1.2]:

$$\tan(\delta) = \frac{1}{\omega.\tau} \text{ where } \delta = \pi/2 + \arg(\underline{V}/\underline{I}) = \pi/2 + \varphi \qquad [1.2]$$

$$\underline{AN}: \tan(\delta) = \frac{1}{2.\pi.50.1} \approx 3.18\times10^{-3} \text{ where } \begin{cases} f = 50[Hz] \\ \tau = 1[s] \end{cases}$$

Whilst solid insulators, both organic and mineral-based, can easily deliver sufficiently high relaxation times, liquids are usually far too conductive to be usable. Only a very few aromatic hydrocarbons, including the infamous PolyChloroBiphenyl (PCB or pyralene) or the Mono-Dibenzyl-Toluene (used by Jarylec in Isère) have taken their place in industry, alongside mineral-based oils; other liquids, such as a1cohol, acetone and nitrobenzene, rarely reach values above 10^4 or 10^5 [$\Omega.m$] due to their sensitivity to the slightest electrolytic contamination.

Alongside resistivity *per se*, which depends on conduction in the volume, in practice we often find surface resistivity, which characterizes surface conduction. Many insulators which have very high volume resistivity actually conduct current quite easily along their surface. The most common example is that of a sheet of glass, covered with a layer of water condensation, and therefore no more insulating than wood when not hot-air dried. Other substances, such as paraffin and ebonite, do not present this problem. Yet it is meaningless to express this surface resistivity in numerical terms without drawing the connection between it and external causes such as humidity, temperature, etc. The surface resistance R_s of an "insulator" is expressed in [Ω_m], and is calculated by using equation [1.3].

$$R_S = R.\frac{l'}{l} \qquad [1.3]$$

where R is the measured resistance between two electrodes (placed at the surface of the insulator) of length l (transversal to the field) and separated by a distance l' (longitudinal to the field). The measured resistance R therefore relates to a rectangular "insulating" surface with long side l and short side l', and the surface resistance R_s is measured in a square "insulating" surface.

1.1.1. *Polarization*

Like any material, a dielectric material contains the "two electricities" in equal and considerable quantities, but unlike with conductors, these electricities cannot circulate within the materials under the influence of the field.

If we look at a molecule of dielectric, it contains positive and negative charges. These charges are not free: they are connected by an elastic force which is something like a spring. If we subject the molecule to an electrical field, the charges cannot move about within the insulator under its influence; the positive charge pulls on the spring in the direction of the field, the negative charge pulls in the opposite direction, and the result is that the spring becomes tenser and the two charges move apart from one another slightly. This separation is practically proportional to the field. Under its influence, the molecule is therefore transformed into a system with two equal positive and negative charges, a small distance apart. This is what is known as a doublet or dipole, and the appearance of dipoles constitutes the polarization of the dielectric. If the field is removed, the springs bring the charges back into contact and the polarization disappears (we shall see in Chapter 2 that there are other mechanisms which lead to macroscopic polarization of the material).

1.1.2. *Ionization*

When we pull too hard on a spring, it will eventually break. Thus, we may imagine that with a critical value of the electrical field, the charges that the springs were holding would suddenly become free, with the insulator becoming a good conductor. In practice, this phenomenon of ionization of the molecules does not occur homogeneously throughout the volume. We shall see later on (in Chapter 2) that ionization phenomena can lead to the breakdown of the material at values of the macroscopic electrical field that are far less than the critical value mentioned above.

The value of the breakdown field strength is one of the most important characteristics of the insulator. It goes without saying that in practice, we allow ourselves a significant safety margin. The maximum field is generally expressed in [$MV.m^{-1}$] (Table 1.2).

Air	Silica glass	Jarylec® (liquid)	Phlogopite mica	PolyEthylene
3	20	35	60	300

Table 1.2. *Dielectric strength under direct current (DC) of a number of insulators, expressed in [MV.m⁻¹], with a thickness of around a [mm] at 25°C*

The values encountered in practice are highly variable depending on the impurities: transformer oil contaminated with a little water vapor has a strength (or "rigidity") of 15 [$MV.m^{-1}$]; when perfectly free of water, this value can reach up to 500 [$MV.m^{-1}$]. The "vacuum" itself is not a perfect insulator: with a field strength of around 100 [$MV.m^{-1}$], the negative charges are torn away from metals – a phenomenon which is greatly facilitated by an increase in temperature (thermo-electronic effect) and by light (photo-electric effect).

In addition, ionization may be caused by friction (triboelectric effect), X-rays and particle/molecule collisions.

The positive and negative charges into which the insulator molecule is split are called "ions", which is a Greek word meaning "to walk". In effect they are charges moving (or "walking") under the influence of the field, once the spring holding them has been broken.

1.1.3. *Polarized dielectrics*

Polarization is the characteristic property of dielectrics. Here, we propose looking at the potential produced by dipoles.

Calculation of the potential produced by a dipole

Let us consider two electrical charges $+q$ and $-q$, separated by a length L, and calculate the potential produced at O at a distance r which is far greater than L (see Figure 1.1):

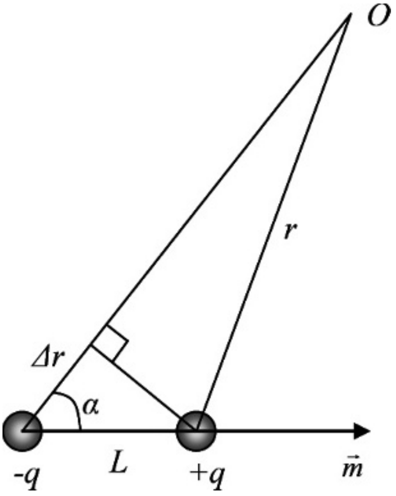

Figure 1.1. *Potential and dipole moment*

$$\begin{aligned}V &= \frac{q}{4.\pi.\varepsilon_0.r} - \frac{q}{4.\pi.\varepsilon_0.(r+\Delta r)}\\ &= \frac{q.\Delta r}{4.\pi.\varepsilon_0.(r^2 + r.\Delta r)}\\ &\approx \frac{q.\Delta r}{4.\pi.\varepsilon_0.r^2} \text{ where } r >> \Delta r\\ &\approx \frac{q.L.\cos(\alpha)}{4.\pi.\varepsilon_0.r^2} \text{ where } \Delta r = L.\cos(\alpha)\\ &= \frac{m.\cos(\alpha)}{4.\pi.\varepsilon_0.r^2} \text{ where } m = q.L\text{:electric dipole moment}\end{aligned}$$

This equation can also be written in vector form:

$$V \approx \frac{1}{4.\pi.\varepsilon_0}.\vec{m}.grad\left(\frac{1}{r}\right) \text{ where } \begin{cases} grad(1/r)\text{: vector from } -q \rightarrow O \\ \vec{m}\text{: vector from } -q \rightarrow +q \end{cases}$$

If we have multiple dipoles in the vicinity of a point in space, the expression of V becomes:

$$V \approx \frac{1}{4.\pi.\varepsilon_0}.\left[\vec{m}_1.grad\left(\frac{1}{r}\right)+\vec{m}_1.grad\left(\frac{1}{r}\right)+\cdots+\vec{m}_n.grad\left(\frac{1}{r}\right)\right]$$
$$\approx \frac{1}{4.\pi.\varepsilon_0}.\left[\vec{m}_1+\vec{m}_1+\cdots+\vec{m}_n\right].grad\left(\frac{1}{r}\right)$$

This system of doublets is equivalent to a single doublet whose electric moment would be the geometric sum of the electric moments. If we now consider an element with volume *dv* containing doublets with many directions: the point *O* is practically at the same distance from each of them, and *grad(1/r)* is the same for all of them, so the moment of *dv* is the resultant *dm* of the moments in the volume *dv*. Thus, we are able to define the polarization vector:

$\vec{P} = d\vec{m}/dv$: electric moment per unit volume

Calculation of the potential produced by a polarized dielectric

Let us now consider a finite volume for which we know the vector *P* at all points.

$$dV \approx \frac{1}{4.\pi.\varepsilon_0}.d\vec{m}.grad\left(\frac{1}{r}\right)$$
$$= \frac{1}{4.\pi.\varepsilon_0}.\vec{P}.dv.grad\left(\frac{1}{r}\right) \text{ where } \vec{P} = d\vec{m}/dv$$

$$V = \frac{1}{4.\pi.\varepsilon_0}.\iiint_{volume} \vec{P}.grad\left(\frac{1}{r}\right).dv$$

however $div\left(\frac{1}{r}.\vec{P}\right) = \frac{1}{r}.div(\vec{P}) + grad\left(\frac{1}{r}\right).\vec{P}$

$$V=\frac{1}{4.\pi.\varepsilon_0}.\left[\underbrace{\iiint_{\text{volume}} div\left(\frac{1}{r}.\vec{P}\right).dv}_{1^{st}\ \text{term}}-\underbrace{\iiint_{\text{volume}} \frac{1}{r}.div(\vec{P}).dv}_{2^{nd}\ \text{term}}\right]$$

With regard to the second term, note that the potential V created by a charge with volume density ρ is:

$$V=\frac{1}{4.\pi.\varepsilon_0}.\iiint_{\text{volume}} \frac{\rho}{r}.dv$$

By identification, we can say that *div(P)* has the dimension of a volume density of charge, and thus write that $\rho'=-div(P)$.

With regard to the first term:

– if we apply Green/Ostrogradsky's theorem:

$$\iiint_{\text{volume}} div\left(\frac{1}{r}.\vec{P}\right).dv=\iint_{\text{surface}} \frac{\vec{n}.\vec{P}}{r}.ds$$

– finally, remember that a surface density of charge σ gives us a potential V:

$$V=\frac{1}{4.\pi.\varepsilon_0}.\iint_{\text{surface}} \frac{\sigma}{r}.ds$$

By identification, we can say that $n.P$ has the dimension of a surface density of charge and thus write that $\sigma'=n.P$.

Hence, the polarized dielectric produces the same potential as charges with a volume density $\rho'=-div(P)$ and a surface density $\sigma'=n.P$. We then speak of an equivalent fictitious charge density. The charges involved in the dipolar bonds that cause the phenomenon of polarization of the dielectric are generally referred to as bound charges.

Let us try to give an idea of what this means.

Figure 1.2 represents the polarized dielectric and the different dipoles which make it up. These dipoles form chains; they touch, and the effect of a + pole is compensated by the – pole of the next dipole. Hence, only the last poles in the chain, at the surface of the dielectric, are involved. This electricity has a charge density of σ'. The direction of the chain at each point is P, and σ' will be greater the more perpendicular the chain is to the surface – i.e. $\sigma'=n.P$.

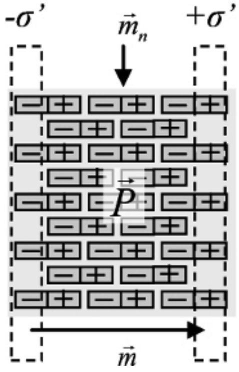

Figure 1.2. *Representation of a uniformly-polarized dielectric and the dipoles which make it up*

In a dielectric, there may be incomplete chains which do not reach the surface at both ends (see Figure 1.3). In this case, we have uncompensated charges within the material. If the rows are parallel, there is no ρ'; for a line to begin, there must be a separation. We then say that the vector P diverges and $\rho'=-div(P)$.

With the Maxwell-Gauss equation (or Poisson's equation), we see that generally in a polarized dielectric, the electrical field E is not conservative (equation [1.4]):

$$div\left(\vec{E}\right) = \frac{\rho}{\varepsilon_0} : \text{Maxwell-Gauss equation} \qquad [1.4]$$

however $\rho' = -div(\vec{P}) \neq 0$: generally the case with a polarized dielectric

hence $div(\vec{E}) = -\frac{1}{\varepsilon_0}.div(\vec{P}) \neq 0$

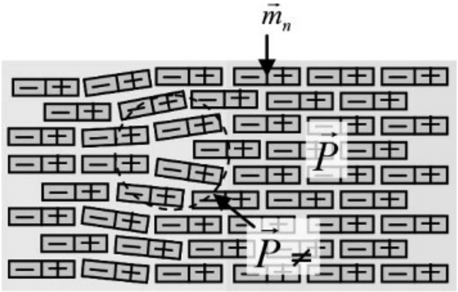

Figure 1.3. *Representation of a divergently-polarized dielectric and the dipoles which make it up, around a flaw causing an uncompensated charge*

Note finally that on passing through the surface of the dielectric, because of the density $\sigma'=n.P$, the normal component of the field experiences a jump in value (equation [1.5]):

$$\sigma' = \vec{n}.\vec{P} \neq 0 \qquad [1.5]$$

thus $\Delta(E_n) = \left|\frac{\sigma'}{\varepsilon_0}\right| \neq 0$

1.1.4. *Electrical induction*

Everything we have just seen leads us to define electrical induction. Consider the induction vector or electric displacement D [1.6]:

$$\vec{D} = \varepsilon_0.\vec{E} + \vec{P} \qquad [1.6]$$

where $div(\overrightarrow{D}) = 0$ in dielectrics with no free charges

If in the dielectric there are real charges other than those equivalent to polarization – which occurs, for instance, if we have conductors within the dielectric, or if the dielectric is electrified by friction, by depositing of ions on it, etc. – we have [1.7]:

$$div(\overrightarrow{D}) = \rho : \text{density of real charges} \qquad [1.7]$$

If we wish to apply Gauss's theorem to real charges not equivalent to polarization, we apply it to the induction rather than to the electrical field.

1.1.5. *Move from one dielectric to another*

The field and the induction usually exhibit a discontinuity on moving from one polarized medium to another.

We shall use the notation E_N and E_T to denote the normal and tangential components of the field in the first medium, and E'_N and E'_T for its components in the second. Consider (Figure 1.4) two nearby points located on either side of the limitrophe surface S.

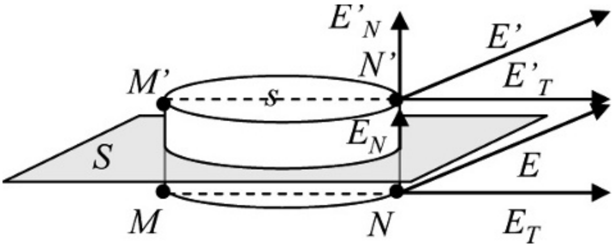

Figure 1.4. *Representation of the normal and tangential components of the field in two limitrophe media*

What is the value of the field at these two points?

If a + charge makes an infinitesimal move parallel to S, only the tangential component is involved. We can therefore envisage the closed cycle $MNN'M'$ where E and E' are involved alternately. As E derives from a potential, the work generated along this circuit is zero; we can also consider it to be negligible along the segments $[M,M']$ and $[N,N']$. We can also write $d(M,N)=d(M',N')$, and therefore $E_T.d(M,N)=E'_T.d(M',N')$, which gives us $E'_T=E_T$.

From the surface S, let us take a small surface s and take it to be the base of a very flat cylinder whose bases can be considered equal. The induction flow D across this closed surface is equal to the sum of the real charges not due to polarization. (Note: because of lack of density in the real volume ρ, it may happen that there is a real density σ on the separation surface). Across the lateral surface, the induction flow is negligible, as this surface is infinitely small. Thus, we can write that the total flow of D across the bases is: $s.D'_N-s.D_N=s.\sigma$, so that $D'_N-D_N=\sigma$. Thus, when the density σ of the real charges (not due to polarization) is null, $D'_N=D_N$.

The equations $E'_T=E_T$ and $D'_N-D_N=\sigma$ are valid in all unit systems and for all dielectrics.

1.1.6. *Move from a dielectric to a conductor*

In any conductor at equilibrium, the internal electrical field E and the induction field D are null. In the dielectric, we deduce from the transition equations that $E'_T=0$ and $D'_N=\sigma$. Thus, we show that the field is always normal to the conductor, in line with Coulomb’s theorem.

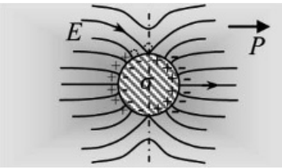

Figure 1.5. *Field in the vicinity of a conductive sphere placed in a polarized dielectric*

1.1.7. *Energy contained in a dielectric*

The density of energy stored in a dielectric by plunging it into an electrical field E satisfies equation [1.8]:

$$W / v = \frac{1}{2}.\varepsilon.E^2 \qquad [1.8]$$

The following calculation proves it.

Calculation of the potential energy of polarization of 1[m³] of dielectric

Let us imagine that this dielectric serves as an insulator in a flat condenser, whose electrodes of surface area S are separated by a distance e. The voltage is therefore linked to the electrical field by the relation $V=E.e$, and the charge is linked to the induction by the relation $Q=S.\sigma=S.D_N$.

Where V is the voltage, if we apply a charge dQ to the casing, the energy injected into the condenser is $dW=V.dQ$, meaning that $dW=E.e.S.dD_N$.

By integration, the energy of the condenser is:

$$W = e.S.\int_0^D E.dD_N$$

and the density of energy in the dielectric is:

$$W / v = \int_0^D E.dD_N$$

This can also be written in vector form as follows:

$$W / v = \int_0^{\vec{D}} \vec{E}.d\vec{D}_N$$

The integral of $E.dD_N$ thus represents the energy needed to polarize $1[m^3]$ of dielectric.

This energy is not really a recoverable potential energy unless E is a unique function of D or, more strictly speaking, if there is no hysteresis. This condition is fulfilled in the following cases:

– perfect dielectrics:

$$\overrightarrow{D} = \varepsilon.\overrightarrow{E} \text{ where } \varepsilon = \varepsilon_0.\varepsilon_r$$

– anisotropic perfect dielectrics:

$$\vec{D}_i = \varepsilon_{ij}.\vec{E}_j$$

ε_{ij} being a constant tensor independent of E_j (which is the case with crystals);

– “ferro-electric” substances.

In certain temperature ranges, E is no longer a linear function of D, but there is no hysteresis.

In the first two cases, because of the linear relations between E and D, the energy density can be written without the integral:

$$W / v = \frac{1}{2}\vec{E}.\vec{D}$$

In the case of an isotropic perfect dielectric, we can also write:

$$W/v=\frac{1}{2}.\varepsilon.E^2$$

1.2. Perfect dielectrics

Experience tells us that in many dielectrics, polarization is proportional (approximately) to the electrical field (equation [1.9]):

$$\vec{P}=\chi.\varepsilon_0.\vec{E} \text{ where } \chi \text{ is the susceptibility } (>0) \qquad [1.9]$$

This observation may appear obvious, in that the polarization is the result of the application of the electrical field. The two vectors *P* and *E* are therefore parallel and proportional.

On the other hand [1.10]:

$$\vec{D}=\varepsilon_0.\vec{E}.(1+\chi)=\varepsilon_0.\varepsilon_r.\vec{E} \qquad [1.10]$$

where $\varepsilon_r>0$ is the dielectric power or the specific inductive power, also known as the dielectric constant. The expression “dielectric constant” is practically the only term in common usage today.

This leads to a significant simplification, because *div(D)*=0 leads to *div(E)*=0, which in turn leads to *div(P)*=0, which finally yields $\rho'=0$. Thus, a perfect dielectric has only a surface density of bound charges (equation [1.11]):

$$\sigma'=\vec{n}.\vec{P} \qquad [1.11]$$

and its energy per [m^3] is equal to [1.12]:

$$W/v=\frac{1}{2}.\varepsilon.E^2 \qquad [1.12]$$

Note that gases are perfect dielectrics.

The same is also true of liquids, provided the frequency is below a certain limit, which is generally at least several megahertz. However, with liquids, we have to take account of losses due to their conductivity. Liquids with a high dielectric constant (ε_r >10) cannot be used as insulators, because their conductivity is too high.

Heterogeneous solids are the most imperfect dielectrics. Conversely, certain homogeneous polymers (polyethylene: $-[CH_2\text{-}CH_2]_n-$, polystyrene: $-[CH_2\text{-}CH(Ph)]_n-$) are practically perfect, even at very high frequency. In general, with solids used as insulators, ε_r varies between 2 and 8, except in certain ceramics specially formulated for condensers where ε_r reaches 100.

To characterize a perfect dielectric, we only need to know its dielectric susceptibility χ; thus we know ε_r, which is the "specific inductive power" (Table 1.3).

Gases	Liquids	Solids
$\varepsilon_r \approx 1$	ε_r = 2 to 100	ε_r = 2 to 1000

Table 1.3. *Order of magnitude of the dielectric constant* ε_r *(static)*

1.2.1. *Refraction of force- and induction-lines*

Determination of the line of force refraction law

To the general relations: $E'_T = E_T$ and $D'_N - D_N = \sigma$, we must add $D=\varepsilon.E$ and $D'=\varepsilon'.E'$ where $\varepsilon= \varepsilon_0.\varepsilon_r$ (the lines of force and of induction are therefore identical). Consider (Figure 1.6) the angle φ' (or φ) formed by D' (or D) with the normal to the surface. We then have:

$$\tan\varphi' = \frac{E'_T}{E'_N} = \frac{D'_T}{D'_N} \text{ and } \tan\varphi = \frac{E_T}{E_N} = \frac{D_T}{D_N}$$

so:

$$\frac{\tan\varphi'}{\tan\varphi} = \frac{D'_T}{D_T}.\frac{D_N}{D'_N} = \frac{\varepsilon'.E'_T}{\varepsilon.E_T}.\frac{D_N}{D'_N} = \frac{\varepsilon'}{\varepsilon}.\frac{D_N}{D'_N}$$

When $\sigma=0$, $D'_N=D_N$ then we get the simplified force-line refraction law:

$$\frac{\tan\varphi'}{\tan\varphi} = \frac{\varepsilon'}{\varepsilon}$$

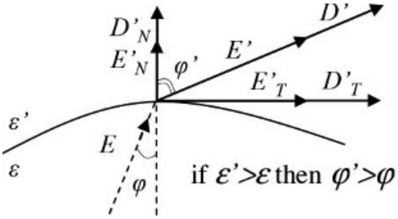

Figure 1.6. *Refraction of lines of force on moving from one medium to another*

The lines of force move closer to the normal when they pass from a dielectric into air. We can express this fact by saying that dielectrics attract the lines of force. If $\varepsilon'_r<\varepsilon_r$ then $\varphi'<\varphi$. If, for instance, $\varepsilon_r=100$ and if φ is not overly large, the lines come out of the dielectric into the air in an almost normal direction.

There is a similar formula in the case of magnetism: for unsaturated iron, μ_r is very large, so $tan\varphi\#0$; the lines of force in the air are practically normal to the surface of the iron, which is approximately a magnetic force equipotential surface.

All the above results on the refraction of lines of force are valid on condition that $D_N = D'_N$, meaning that the surface density σ is null at the separation surface between the two dielectrics. This point is generally accepted unquestioningly, which is a serious mistake.

Indeed, charges may reach the surface by any conductive phenomenon in dielectrics. Exactly as happens in the case of combination of condensers, we cannot be certain that the density σ will remain null unless the field is generated by alternating current and its frequency is sufficient. Otherwise, if it is continuous or if only its average value is non-zero, the inevitably conductive phenomena will cause a density σ and the force-line refraction will therefore be reversed, unless the resistivities ρ (or ρ') and dielectric constants ε (or ε') of the different media satisfy the relation $\rho.\varepsilon = \rho'.\varepsilon' = \rho''.\varepsilon''$, which expresses that the time constants $\tau = \rho.\varepsilon$ of these media are the same.

There is a particularly striking example which shows the enormous errors which can occur with the conventional refraction law if we do not take account of the density σ. Such is the case with solid dielectrics immersed in an ionized gas and subjected to a DC electrical field. If these solids have high resistivity, like most plastic materials used in electrotechnics, and if the air is sufficiently ionized, as is the case in the vicinity of a high-voltage piece of machinery, we may consider that the amount of charge which reaches the dielectric through the air cannot practically be evacuated by conduction through it. In other words, the gas is much more conductive than the solid. In these conditions, ions will accumulate on the surface of the solid, forming a more or less dense layer. If a state of equilibrium is reached (which is not always the case, with the layer sometimes being swept by a disruptive discharge when it reaches a certain density), this means that the electrical field can no longer deposit ions on the surface, so its normal component in the air is null. The final shape of the lines of force is therefore the same as if the solid's dielectric constant, instead of being higher than that of air, had become zero. The lines of force in the air are tangents to the surface instead of being nearly normal to it.

1.2.2. *Field in the vicinity of a charged conductor*

We have already established that in the vicinity of a conductor immersed in any dielectric, $E_T=0$, meaning that the field is normal to the conductor (the surface is an equipotential surface) and $D_N=\sigma$.

In a perfect dielectric placed at the surface of a conductor, $E'=\sigma/\varepsilon$ (Figure 1.7, left). The involvement of ε is explicable by the appearance of an electrical density with fictitious charge σ' which is opposite to the density σ of the conductor, and has the opposite sign.

Thus, we can replace this dielectric with the plane of fictitious charge $\sigma'=n.P$ equivalent to its polarization (Figure 1.7, right). The field at point M' therefore has the expression $E'=(\sigma'+\sigma)/\varepsilon_0$. Thus, we derive the relation [1.13] between σ' and σ.

$$E'=\sigma/\varepsilon=(\sigma+\sigma')/\varepsilon_0 \Rightarrow \sigma'=-\sigma\left(1-\frac{1}{\varepsilon_r}\right) \qquad [1.13]$$

If ε increases, with the value of σ remaining constant, E' decreases, and the removal of the field (from the dielectric) is almost total with a high value of ε (though this is only true if the charges in the conductors are constant); the same is true in magnetism for substances with high permeability.

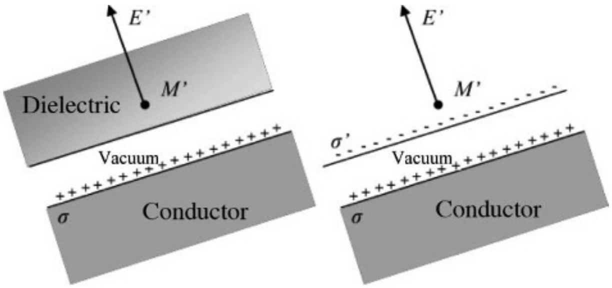

Figure 1.7. *Dielectric in the vicinity of a charged conductor: calculating the field*

1.2.3. *Electrostatic pressure*

Calculation of the ordinary electrostatic pressure with the energy

Remember that the variation in energy $d\xi$ is equal to the work of the forces:

$$d\xi = p.dv \text{ because } F = p.dS \text{ and } d\xi = F.dl\,[J]$$

Remember the expression for the volume electrostatic energy:

$$W/v = \frac{1}{2}.\varepsilon_0.E^2\ [J/m^3]$$

From this, we deduce the following equality and the expression of the pressure:

$$\frac{1}{2}.\varepsilon_0.E^2.dv = p.dv \Rightarrow p = \frac{1}{2}.\varepsilon_0.E^2\ [Pa]$$

Thus, when the volume of the conductor increases, the domain of *E* decreases, and with it the energy that it holds.

The same reasoning is valid if the conductor is immersed in a dielectric, but here the electrostatic pressure is the resultant of the forces exerted on the conductor and the dielectric:

$$p=\frac{1}{2}.\varepsilon.E^{2}[Pa] \text{ or indeed } p=\frac{1}{2}.\frac{\sigma^{2}}{\varepsilon}[Pa] \text{ where } E=\sigma/\varepsilon$$

Thus, electrostatic pressure tends to cause an increase in the volume of the conductor.

1.2.4. *Corresponding elements*

Consider two conductors C_1 and C_2 and an induction tube which runs from the first to the second, and then close this surface with Σ_1 and Σ_2 (see Figure 1.8). According to Gauss's theorem, the flux Φ of induction D across this closed surface is null. Thus, we are able to show the electrical charges carried by two corresponding elements which are perfectly opposite [1.14]:

$$\Phi=0=q+q'\Rightarrow q=-q' \qquad [1.14]$$

When the elements are in total influence, i.e. when the set of field lines emanating from one conductor leads to the other conductor, then the first conductor carries a charge which is totally opposite to the second.

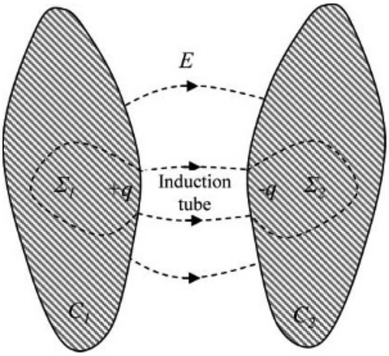

Figure 1.8. *Illustration of corresponding elements*

1.2.5. *Equilibrium in a system of conductors*

The equilibrium of a system of conductors placed in a vacuum is determined if we know the charge Q or the potential V for each of the conductors. The same is true when the conductors are surrounded by a perfect dielectric.

In the vacuum, we base our reasoning on the following equality ([1.15]):

$$W = \frac{1}{2}\sum_k Q_k.V_k = \frac{1}{2}.\varepsilon_0.\iiint E^2 dv \qquad [1.15]$$

If V=0 or Q=0 for each of the conductors, W=0, then the triple integral of $E^2.dv$ is a sum of positive quantities which can only be null if E is null everywhere.

The same thought process can be employed for a system of conductors immersed in a perfect dielectric. The quantity $\varepsilon_r.E^2$ which replaces E^2 in the energy expression is always positive: hence, its integral can only be null of it is null everywhere, meaning that E=0.

The state of a system of conductors immersed in a perfect dielectric is therefore determined completely by knowing the total charge or the potential for each conductor.

If, however, the dielectric is imperfect, the reasoning is no longer valid. We can perform a calculation which shows the following equality ([1.16]):

$$\sum_k Q_k.V_k = \iiint \vec{E}.\vec{D}.dv \qquad [1.16]$$

However, in general, this quantity is no longer equal to the energy, and we can no longer state definitively that it will be positive. The usual properties for systems of conductors disappear: thus, the casings on both sides of a condenser may be taken to the same potential, but this does not necessarily mean that their charge will be null. In addition, the system's energy is no longer really null either, although $½.\Sigma_k Q_k.V_k=0$ if we consider $V=0$, because residual charges may gradually appear.

1.2.6. *Capacities and influence coefficients*

Firstly, it should be remarked upon that the distribution of the potential does not change if we fill the whole space with a homogeneous dielectric, because the equation $div(E)=0$ remains valid. The volumetric density ρ' is null and the surface density of the bound charges σ' is such that $\sigma+\sigma'=\sigma/\varepsilon_r$. From the viewpoint of the field and the potential, it is as if σ had been divided by ε_r.

We conclude that the distribution of the electricity on the conductors and of the field in space is the same as if there were no dielectric, except that the field and the potential are ε_r times less than in the vacuum, while the charges of the conductors remain the same. This can also be stated as follows:

– where the potentials of the conductors are assumed to be constant, their charges are ε_r times greater than in the vacuum. The capacities and influence coefficients are therefore multiplied by ε_r.

Note that this result is true only if the whole space between the conductors is filled with a dielectric. Thus, the dielectric medium has to extend to infinity, or at least extend to fill the enclosed space containing the conductors.

With regard to the forces which are exerted on the conductors, the electrostatic pressure $½.\varepsilon.E^2$ can be written as $½.\sigma^2/\varepsilon$. Hence, if the charges remain constant, the forces are ε_r times lesser than in the vacuum. This tendency can also be seen is the expression of the force of interaction between two point charges (equation [1.17]).

$$F = \frac{1}{4\pi\varepsilon_0\varepsilon_r}.\frac{q.q'}{r^2} \qquad [1.17]$$

The above results apply to condensers. We can see that the capacity of a closed condenser, completely filled with a dielectric, is multiplied by ε_r. This is valid for modern models of condensers which are practically closed (although not rigorously, with the exception of the spherical condenser).

This can be seen by a direct reasoning process in the case of a flat condenser. Consider V_1 and V_2 to be the potentials of the casings. The field between them is uniform, and we have: $V_1\text{-}V_2=e.E$ (e being the distance between the casings). In addition, in a dielectric material: $E=\sigma/\varepsilon$, where σ is the density of the free charge carriers on the surface of the metal. However, the total charge is $Q=\sigma.S$, which gives us equations [1.18] and [1.19]:

$$V_1 - V_2 = \frac{\sigma.e}{\varepsilon} = \frac{Q.e}{\varepsilon.S} \qquad [1.18]$$

$$\Rightarrow C = \frac{Q}{V_1 - V_2} = \frac{\varepsilon.S}{e} \qquad [1.19]$$

Let us now examine what happens if only part of the gap between the casings is filled with dielectric, as illustrated by Figure 1.9.

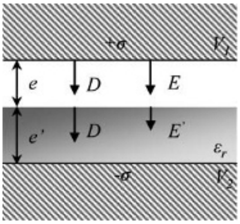

Figure 1.9. *Electrode / vacuum / dielectric / electrode condenser*

$$V_1 - V_2 = E'.e' + E.e \qquad [1.20]$$

however:

$$E = \frac{\sigma}{\varepsilon_0} \text{ and } E' = \frac{\sigma}{\varepsilon_0 . \varepsilon_r} \qquad [1.21]$$

hence:

$$V_1 - V_2 = \frac{\sigma . e}{\varepsilon_0} + \frac{\sigma . e'}{\varepsilon_0 . \varepsilon_r} = \frac{\sigma}{\varepsilon_0}\left(\frac{e'}{\varepsilon_r} + e\right) \qquad [1.22]$$

but:

$$C = \frac{Q}{V_1 - V_2} \text{ and } Q = \sigma . S \qquad [1.23]$$

so:

$$C = \frac{\sigma . S}{V_1 - V_2} = \frac{S . \varepsilon_0}{\left(\frac{e'}{\varepsilon_r} + e\right)} \qquad [1.24]$$

If e=0, we again see the well-known formula. Let us note in passing that the induction D retains the same value everywhere, as it is perpendicular to the surface between the dielectrics (equation [1.25]):

$$D = \varepsilon . E' = \varepsilon_0 . E \qquad [1.25]$$

1.2.7. *Calculation of the interstitial field*

It is possible to calculate the field and the induction, as well as the capacities and influence coefficients, on the basis of data relating to the system of conductors in the vacuum, in both the following cases, where the dielectric only partially fills the space between the conductors:

– the dielectric is bounded by equipotentials (Figure 1.10);

– the dielectric is bounded by force tubes (Figure 1.11).

With the first hypothesis, we have the same problem as for the previous condenser. The field and induction are normal to the surface of the dielectric. If this surface carries no real charge, we have $D'_N=D_N$. If the conductors carry the same charges, the induction everywhere is the same as if there were no dielectric, because in the vicinity of the conductors it is always $D=\sigma$. On the contrary, the field is smaller in the dielectric; instead of being D/ε_0, it becomes D/ε. With constant charges, the differences in potential (DP) between the conductors are smaller. The problem is equivalent to that found for condensers with cascaded arrangements of dielectrics and vacuums.

With the second hypothesis, the field and induction are tangents to the surface of the dielectric, so $E_T=E'_T$. If the conductors exhibit the same DPs, the field everywhere is the same is if there were no dielectric. Conversely, the induction is greater in the dielectric: it is $\varepsilon.E$ instead of $\varepsilon_0.E$. The charges and capacities of the conductors are greater with constant potentials. The problem is equivalent to that of parallel condensers.

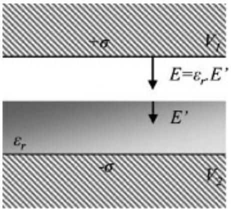

Figure 1.10. *Interstice parallel to the equipotentials*

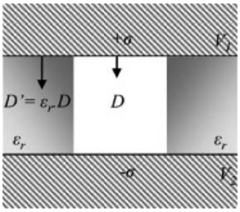

Figure 1.11. *Interstice parallel to the induction tube*

It is important to note that in any crack in the dielectric transversal to the lines of force, there appears a field $E'=\varepsilon_r.E$. The gas which fills the crack is therefore far more susceptible to ionization than that which surrounds the dielectric outside. This explains why the introduction of a cylindrical insulating column between the casings of an air-insulated flat condenser can considerably hasten breakdown in air along the column. Although in principle the field is unchanged, it is considerably increased in the interstitial space remaining between the bases of the column and the casings, and breakdown results from this. We can verify that this occurs with a voltage that decreases with an increasing value of the dielectric constant of the column. Conversely, if the air is replaced by a liquid whose constant is greater than that of the solid, no such phenomenon occurs.

Box 1.1. *Careful of cracks!*

1.2.8. *Depolarizing field*

When a piece of a perfect dielectric is immersed in an electrical field E, we must not allow ourselves to believe that the polarization is $P=\chi.\varepsilon_0.E$, and the induction $D=\varepsilon.E$, because the (bound) polarization charges which appear on the surface of the dielectric greatly modify the primitive field. This modification is always a decrease. For this reason, we say that any dielectric creates a depolarizing field (similar to the demagnetizing field with soft iron). Only with dielectrics of very low susceptibility (gases) can the depolarizing field be discounted. Conversely, when the dielectric constant is high, the influence of the depolarizing field is very significant.

The simplest example of a depolarizing field is provided by conductors (carrying fixed charges) surrounded by a dielectric with constant ε_r (Figure 1.12). If, before the introduction of the dielectric, the intensity of the field at a given point is E, we would find – if we ignore the depolarizing field – that the induction should become $\varepsilon.E$ in the presence of the dielectric. In actual fact we know that this result is entirely false, and that in reality the field is divided by ε_r, such that D does not change when we introduce the dielectric.

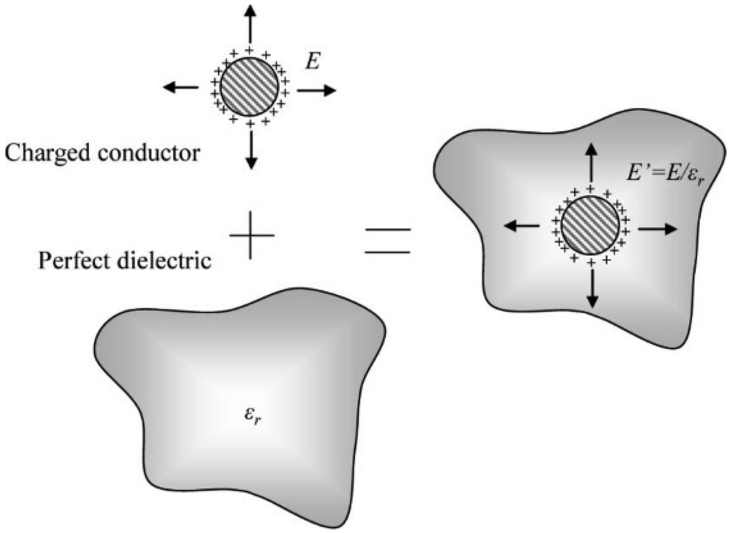

Figure 1.12. *Charged conductive sphere immersed in a perfect dielectric*

The effect of the depolarizing field is more difficult to calculate when the dielectric fills only some of the space in which the field is.

As our first example, let us consider the case of a dielectric sphere with constant ε_r (Figure 1.13) or (Figure 1.15).

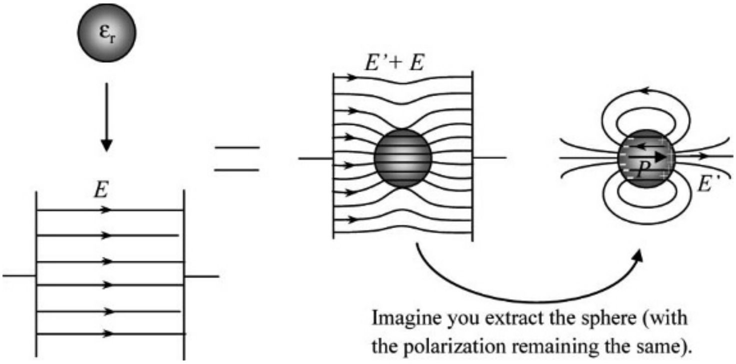

Figure 1.13. *Dielectric sphere immersed in a uniform field and then removed*

If we ignored the depolarizing field E', we would say simply that the sphere acquires a polarization $P=\chi.\varepsilon_0.E$, but this approach is wrong.

Calculating the depolarizing field in a dielectric sphere

If we make the analogy suggested by Figure 1.14, we obtain:

$$\vec{E}' = \frac{-\rho}{3.\varepsilon_0}.\frac{\vec{L}}{2} + \frac{\rho}{3.\varepsilon_0}.\frac{-\vec{L}}{2} = -\frac{\rho.\vec{L}}{3.\varepsilon_0} = -\frac{\vec{P}}{3.\varepsilon_0}$$

(on the axis of polarization)

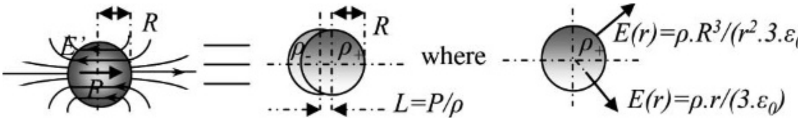

Figure 1.14. *The field of a polarized sphere is the sum of the fields produced by two uniformly-charged spheres placed a distance L apart*

Calculating the polarization and the induction in a dielectric sphere

Uniform polarization P produces a depolarizing field:

$$\vec{E}' = -\frac{\vec{P}}{3.\varepsilon_0}$$

We can now write that:

$$\vec{P} = \chi.\varepsilon_0.(\vec{E} + \vec{E}') = \chi.\varepsilon_0.\left(\vec{E} - \frac{\vec{P}}{3.\varepsilon_0}\right) \Rightarrow \vec{P} = \frac{\chi.\varepsilon_0.\vec{E}}{1+\frac{\chi}{3}} = \frac{\varepsilon_0.\vec{E}}{\frac{1}{\chi}+\frac{1}{3}}$$

However:

$$\chi = \varepsilon_r - 1$$

Hence:

$$\vec{P} = \frac{\varepsilon_0.\vec{E}}{\frac{1}{\varepsilon_r - 1}+\frac{1}{3}}$$

In addition:

$$\overrightarrow{D} = \varepsilon_0.(\vec{E} + \vec{E}') + \vec{P} = \varepsilon_0.\left(\vec{E} - \frac{\vec{P}}{3\varepsilon_0}\right) + \vec{P}$$

$$\Rightarrow \overrightarrow{D} = \varepsilon_0.\vec{E} + \frac{2}{3}\vec{P}$$

$$\Rightarrow \overrightarrow{D} = \varepsilon_0.\vec{E} + \frac{2}{3}\frac{\varepsilon_0.\vec{E}}{\frac{1}{\varepsilon_r - 1}+\frac{1}{3}}$$

$$\Rightarrow \overrightarrow{D} = \vec{E}.\varepsilon_0.(1 + \frac{2.(\varepsilon_r - 1)}{\varepsilon_r + 2})$$

so that:

$$\overrightarrow{D} = \vec{E}.\varepsilon_0.\left(\frac{3.\varepsilon_r}{2 + \varepsilon_r}\right)$$

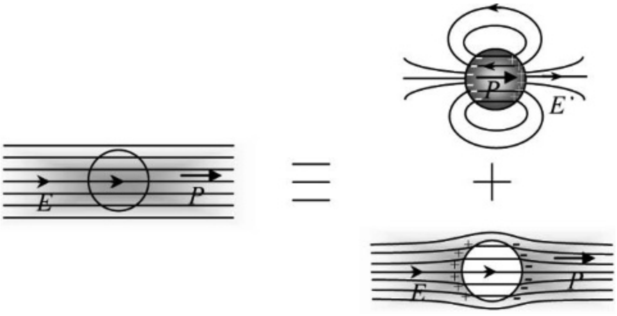

Figure 1.15. *Dielectric sphere extracted from a uniformly-polarized medium*

The formulae relating to P and D demonstrate the influence of the depolarizing field. If χ is small in comparison to the unit, the value of ε_r is close to 1. Obviously $P=\chi.\varepsilon_0.E$ and P is significantly smaller than $\varepsilon_0.E$. However, once ε_r reaches the value of a few units, then P is of the same order as $\varepsilon_0.E$; for instance, if $\varepsilon_r=4$, $P=3/2.\varepsilon_0.E$ and $D=2.E.\varepsilon_0$. If ε_r is very large, the induction – instead of being great – is simply equal to $3.E.\varepsilon_0$.

The hypothesis that the depolarizing field is uniform is borne out by the fact that a density in the form $\sigma_0.cos(\theta)$ on a sphere causes a uniform field inside it. Indeed, it is the density which appears on a conductive sphere immersed in a uniform field.

The simplicity of the above calculation is due to the fact that the depolarizing field is uniform. This property is common to all dielectric quadrics immersed in a uniform field. If the external field is not directed along one of the axes of symmetry of the quadric, the depolarizing field forms an angle with it but retains its uniformity.

For each axis of the quadric, we can define a coefficient of depolarizing field A, which gives us the depolarizing field E'

created by a polarization P parallel to that axis (equation [1.26]):

$$\vec{E}' = -A.\frac{\vec{P}}{\varepsilon_0} \qquad [1.26]$$

The sum of the coefficients for the 3 axes of the quadric is equal to 1. This property enables us to immediately find the coefficient of the 1/3 sphere because of symmetry.

As a particular example of a quadric, we can take a blade with parallel faces perpendicular to the field (Figure 1.16a). Thus, the depolarizing field is $E' = -P/\varepsilon_0$ and $D = \varepsilon_0.E$. Hence, the effect of the depolarizing field is maximum.

Similarly, a circular cylinder whose generatrices are perpendicular to the external field (Figure 1.16b) will yield $E' = -P/2.\varepsilon_0$ and $D = 2.\varepsilon_0.E$.

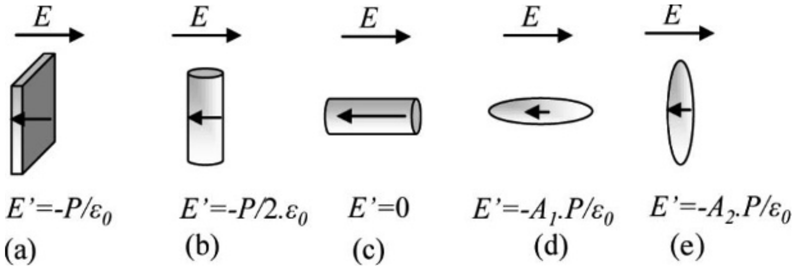

Figure 1.16. *Depolarizing field in different (quadric) geometric forms: a) slip; b) cylinder ⊥E; c) cylinder //E; d) ellipsoid //E; e) ellipsoid ⊥E*

Conversely, if the same (infinite) cylinder is parallel to the field (Figure 1.16c), the depolarizing field is null and $D = \varepsilon_0.E$. The coefficient 1/2 is found immediately by thinking that it is valid for 2 axes of the quadric because of the revolution symmetry, whereas, for the axis of revolution, the coefficient is zero.

It is easy to calculate the depolarizing field of an ellipsoid of revolution whose large axis is parallel to the external field (Figure 1.16d). Since we know that the polarization is uniform and parallel to the axis, we need only calculate the coefficient of depolarizing field A_1 in the center. With uniform polarization P, we find [1.27]:

$$A_1 = \frac{b^2}{2.a^2.e^3}\left[\log\left(\frac{1+e}{1-e}\right) - 2e\right] \qquad [1.27]$$

(a = large axis, b = small axis, e = excentricity). The coefficient of depolarizing field A_2 for a field perpendicular to the axis of revolution (Figure 1.16e) is obtained by writing that the sum of the three coefficients is 1. Hence, it is equal to equation [1.28]:

$$A_2 = \frac{1}{2}\left[1 - \frac{b^2}{2.a^2.e^3}\left[\log\left(\frac{1+e}{1-e}\right) - 2e\right]\right] \qquad [1.28]$$

When we are no longer dealing with a quadric, the depolarizing field is no longer uniform, even if the primitive field is uniform. The calculation is generally more difficult, because it is a problem much like that of equilibrium of conductors. It is precisely because the problem of equilibrium of conductive quadrics is simple that we also have simple results for dielectric quadrics.

A difficult problem, for example, is that of a dielectric bar with a circular cross-section and of finite length, immersed in a uniform field parallel to its axis. We can easily see that the depolarizing field cannot be uniform and that the real polarization must decrease towards the extremities, forming a considerable angle with the axis, so that the surface polarizing charges cover not only the end surfaces, but also the lateral surface near to the extremities, as we can see in real life. However, if the bar is relatively long and narrow, we can accept that the polarization is essentially uniform throughout the majority of its volume. This is equivalent to viewing it as an ellipsoid with greatly elongated revolution.

The depolarizing field coefficient can then be calculated approximately in the center as being $(1-cos(\theta))$ with $\theta=r/2.l$ or noticeably $(r^2/(2.l^2))/(4)$ (remember that r is the radius and l is the length of the bar).

Yet it may be remarked upon that this result does not coincide with the coefficient relating to the ellipsoid with supposed greatly-elongated revolution. It is likely that the coefficient of the ellipsoid represents a better approximation, because – for both the bar and the ellipsoid – there are charges on the lateral surface, which are not taken into account by the simplified formulation, and which increase the depolarizing field.

All these developments on the depolarizing field in the dielectrics are mathematically identical to those encountered in magnetism with regard to the demagnetizing field.

Numerically speaking, there are certain differences in the orders of magnitude. The dielectric constants are generally less than the magnetic permeability of the iron, but there are no phenomena resembling saturation, except in the rather rare case of “ferro-electric” substances (titanates, Rochelle salt).

1.3. Forces exerted on polarized dielectrics

1.3.1. *Forces exerted on a rigidly-polarized solid dielectric*

To begin with, we shall assume that the polarization is rigid, meaning that it is independent of the external field applied.

We shall also suppose that there are no real charges, for which we know the calculations.

In a uniform field, a piece of dielectric is subject only to torque force. Indeed, as the total polarization charge is null, the resultant of the electrical forces is also null (equation [1.29]):

$$\Sigma F = \Sigma q.\vec{E} = \vec{E}.\Sigma q = 0 \qquad [1.29]$$

Each dipole m is subjected to torque (equation [1.30]):

$$\vec{T} = \vec{m} \wedge \vec{E} \qquad [1.30]$$

The geometric sum of these torques is given by equation [1.31]:

$$\Sigma(\vec{m} \wedge \vec{E}) = \Sigma(\vec{m}) \wedge \vec{E} \qquad [1.31]$$

The vector Σm thus represents the electric moment of the piece of dielectric.

In a non-uniform field, there are both torque and a resultant.

We can easily verify that the potential energy of a dipole of moment m in an electrical field of intensity E is: $W=-m.E$. The resultant R of the forces acting on the dipole is thus (equation [1.32]):

$$R_x = -\frac{\partial W}{\partial x} = \frac{\partial \vec{m}.\vec{E}}{\partial x} = \vec{m}.\frac{\partial \vec{E}}{\partial x} \qquad [1.32]$$

For a piece of dielectric, we would thus have (equation [1.33]):

$$R_x = \iiint \vec{P}.\frac{\partial \vec{E}}{\partial x}.dv \qquad [1.33]$$

It is easy to show that we can replace the dipoles or polarization with fictitious densities ρ' and σ' in calculating the forces.

The energy of the dielectric in the field E is [1.34]:

$$W = -\iiint \vec{P}.\vec{E}.dv = \iiint \vec{P}.\overrightarrow{grad}\,V.dv$$

however:

$$div(V.\vec{P}) = V.div\vec{P} + \overrightarrow{grad}\,V.\vec{P}$$

thus:

$$W = \iiint div(V.\vec{P}).dv - \iiint V.div\vec{P}.dv$$

$$W = \iint \vec{n}.V.\vec{P}.dS - \iiint V.div\vec{P}.dv$$

hence:

$$W = \iint V.\sigma'.dS - \iiint V.\rho'.dv \qquad [1.34]$$

W is the energy in the field of potential V with charges of densities ρ' and σ'.

Thus, we need only calculate the forces exerted by the field E on the elementary charges $\rho'.dv$ and $\sigma'.dS$.

The above results relate only to solid dielectrics. Only in this case are we able to add torques, consider a resultant and say that the global energy determines the forces entirely. In addition, only solids can be rigidly polarized.

1.3.2. *Forces exerted on a solid perfect dielectric*

If we now look at a solid dielectric – this time not rigidly polarized but perfect – the results are different, although the action of the external field on each dipole remains the same. Indeed, it is now the diminished external field of the depolarizing field which determines the polarization.

In a uniform field, a piece of solid perfect dielectric can only be subject to a torque force, but this torque is zero whenever the depolarizing field is colinear with the external field. Such is the case of a sphere or a quadric immersed in a uniform field parallel to one of its axes. On the other hand, if an ellipsoid, for instance, is subjected to an oblique field, its polarization is not parallel to the field because the coefficients of the depolarizing field are not the same for all three axes. This results in a torque tending to bring the large axis parallel to the field. The same result is seen with a cylindrical bar.

These phenomena are absolutely comparable to those observed with ellipsoids or soft iron bars in a magnetic field, or even simply with iron filings.

In the case of electrostatic, iron filings mimic the behavior of textile fibers or animal hairs which undergo high-energy orientation effects in a field and form chains between the electrodes. This phenomenon is damaging at high voltage levels, because these filaments greatly encourage sparking. The effect of fiber orientation is largely exploited in electrostatic flocking, a technique whereby a (sticky) surface is covered with perpendicular fibers, which produces a highly desirable fake velvet effect.

If m is the moment of the piece of dielectric in the uniform external field E, the torque is always $T=m \wedge E$. For instance, in the case of a dielectric ellipsoid, we can take the axes to be those of the ellipsoid. In addition to the three components E_x,

E_y and E_z, there are three corresponding components P_x, P_y and P_z determined by the coefficients of the depolarizing field A_x, A_y and A_z. Remember that:

$$P_x = \chi.\varepsilon_0.(E_x + E_x') = \chi.\varepsilon_0.\left(E_x - A_x.\frac{P_x}{\varepsilon_0} \right) \quad [1.35]$$

$$\text{and thus} \begin{cases} P_x = \dfrac{\chi.\varepsilon_0}{1+\chi.A_x}.E_x \\ P_y = \dfrac{\chi.\varepsilon_0}{1+\chi.A_y}.E_y \\ P_z = \dfrac{\chi.\varepsilon_0}{1+\chi.A_z}.E_z \end{cases}$$

The torque components are calculated as follows:

$$T_x = m_y.E_z - m_z.E_y$$

Yet the dipole moment of the ellipsoid of volume v is $m=P.v$ because P is uniform, so:

$$T_x = P_y.v.E_z - P_z.v.E_y$$

$$\Rightarrow T_x = \frac{\chi.\varepsilon_0}{1+\chi.A_y}.E_y.v.E_z - \frac{\chi.\varepsilon_0}{1+\chi.A_z}.E_z.v.E_y$$

$$\Rightarrow T_x = \chi.\varepsilon_0.E_y.v.E_z \left(\frac{1}{1+\chi.A_y} - \frac{1}{1+\chi.A_z} \right)$$

$$\Rightarrow T_x = \frac{\chi^2.\varepsilon_0.E_y.v.E_z(A_z - A_y)}{(1+\chi.A_y)(1+\chi.A_z)}$$

These components are obviously all zero if the field is directed along one of the axes of the ellipsoid. If the field is in one of the ellipsoid's planes of symmetry, the torque is perpendicular to it; it is maximal when the field forms a 45° angle with the axes.

Sometimes, we use a dielectric ellipsoid to measure the dielectric constant by looking for the maximum torque in a given field.

If the solid perfect dielectric is subjected to a non-uniform field, we can always find the resultant moment and the resultant of the forces by looking for the polarization at each point, or the fictitious charge equivalent to polarization. For a given state of polarization, the mechanical action of the external field is the same as if the dielectric were rigidly polarized. In order to find these forces, we need only look at what has already been said about them.

Very frequently, at all points of the dielectric, we know the total field (final, internal, macroscopic) rather than the external field (initial, applied).

Unlike what takes place, for instance, with a distribution in volume of real charges, where the force per unit volume is $dF=\rho.E.dv$, E being the total field and ρ the charge density, we cannot say that the force exerted on an element with volume dv made of the solid perfect dielectric of polarization P is $P.(dE/dx).dv$ or the torque $P\wedge E.dv$, where E is the total field. This gives us absurd responses. Thus, the torque $P\wedge E.dv$ would always be zero because $P=\chi.\varepsilon_0.E$, but in general this is not the case. The reason is that the contribution of the element of volume dv to the total field does not tend toward zero with decreasing dimensions of the element.

Indeed, if the element is spherical, of radius a, its moment is approximately $P.a^3$, and the field that it creates is around $P.a^3/a^3$, or in other words, around P. Conversely, an element with volume containing a density ρ has a charge of around

$\rho.a^3$ and creates a field of around $\rho.a^3/a^2=\rho.a$, which tends toward zero as a does. Hence, the forces exerted on an element of polarized volume cannot be calculated on the basis of the total field, which contains an unwanted contribution of finite value, from the element in question itself.

> The total field cannot be used to calculate the action of the rest of the dielectric (and of other electrified substances) on the element.

Box 1.2. *Limitation of the total field*

The only method that we can use is the energy method. In order to calculate the resultant and the resultant moment of the forces applied to a piece of solid perfect dielectric, we calculate the total energy of the system [1.36]:

$$W=\iiint \frac{1}{2}.\varepsilon.E^2.dv \qquad [1.36]$$

and we evaluate the variations dW/dx of that energy in an infinitesimal translational (dx) or rotational movement ($d\alpha$) of the dielectric. If the potentials are constant during that movement, we would have:

$$R_x=+\frac{\partial W}{\partial x}$$

$$T=+\frac{\partial W}{\partial \alpha}$$

and if the charges are constant, we use the same formulae but with minus signs.

Calculating the forces exerted on a dielectric slip contained in a flat condenser

A flat condenser has a dielectric slip, which can slide between the casings with no friction (see Figure 1.17).

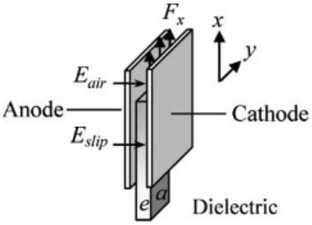

Figure 1.17. *Flat condenser with a moving dielectric slip*

$$W = \frac{1}{2}.\varepsilon.E^2.v_{slip} + \frac{1}{2}.\varepsilon_0.E^2.v_{air}$$

If the slip moves horizontally, the energy does not change; thus, the resulting force has no horizontal component. With an upward vertical movement Δx, we have $\Delta v_{slip}=a.e.\Delta x$ and $\Delta v_{slip}=\Delta v_{air}$. Hence:

$$\Delta W = \frac{1}{2}.E^2.(\varepsilon.\Delta v_{slip} - \varepsilon_0.\Delta v_{slip}) = \frac{1}{2}.E^2.(\varepsilon - \varepsilon_0).a.e.\Delta x$$

That is, a vertical upward force:

$$F = \frac{1}{2}.a.e.(\varepsilon - \varepsilon_0).E^2$$

Thus, the slip is attracted to those regions where the field is strongest – this is an extremely common phenomenon.

1.3.3. *Forces exerted on a liquid dielectric*

Liquid dielectrics are always perfect (conduction apart), so we are dealing with the same conditions as in the previous case. However, we cannot compound the forces applied to a liquid; as the liquid is supposed to be in a state of equilibrium, the electrical forces result in a pressure additional to that caused by gravity, capillary forces, etc. This pressure can also be calculated by an energy method.

Calculation of the forces exerted on a dielectric liquid in a flat condenser

A flat condenser with horizontal casings is partially filled with a dielectric liquid. When a potential difference is applied, the level of the liquid tends to rise. The pressure, which – in the immediate vicinity of the surface – was equal to atmospheric pressure, then becomes less.

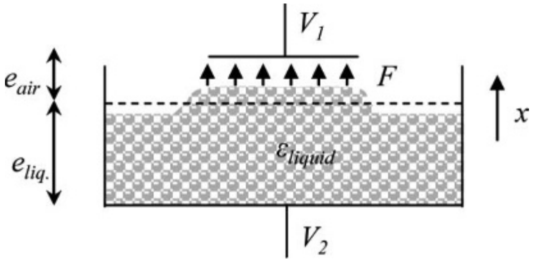

Figure 1.18. *Force exerted on a liquid in a flat condenser*

In order to calculate this drop in pressure, we need only imagine a movement of the surface – e.g. a rise. If e_{air} and e_{diel} are the thicknesses of air and liquid between the casings, the capacity of the condenser is:

$$C = \frac{\varepsilon_0 . S}{e_{air} + \frac{e_{diel}}{\varepsilon_r}}$$

and the energy:

$$W = \frac{1}{2} . C . V^2$$

If the liquid rises by Δx, the energy changes by:

$$\Delta W = \frac{V^2}{2}.\frac{dC}{dx}.\Delta x$$

so we have the pressure:

$$p = \frac{\Delta W}{\Delta v} = \frac{\Delta W}{S.\Delta x} = \frac{V^2}{2.S}.\frac{dC}{dx}$$

However:

$$\frac{dC}{dx} = -\frac{\varepsilon_0.S}{\left(e_{air} + \frac{e_{diel}}{\varepsilon_r}\right)^2}.\left(-1 + \frac{1}{\varepsilon_r}\right) = \frac{\varepsilon_0.S}{\left(e_{air} + \frac{e_{diel}}{\varepsilon_r}\right)^2}.\left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right)$$

Hence:

$$p = \frac{V^2}{2.S}.\frac{\varepsilon_0.S}{\left(e_{air} + \frac{e_{diel}}{\varepsilon_r}\right)^2}.\left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right)$$

Yet the field in the air is equal to:

$$E_{air} = \frac{V}{e_{air} + \frac{e_{diel}}{\varepsilon_r}}$$

Thus:

$$p = \frac{E^2{}_{air}}{2}.\varepsilon_0.\left(\frac{\varepsilon_r - 1}{2}\right) = \frac{E^2{}_{diel}}{2}.\varepsilon_r.\varepsilon_0.(\varepsilon_r - 1)$$

If the dielectric is free to rise, it fills the whole of the condenser. Were we to balance out this pressure with a drop in level by a height h, we would have $h.\rho.g=p$.

1.3.4. *Electrostriction*

In our calculations, we have thus far assumed solid dielectrics to be non-deformable and liquids incompressible. In reality, this is not the case. Solids are elastic, and liquids are compressible, and the dielectric constant changes with such deformations. From this, it results that the compression of a liquid, for instance, leads to a change in energy, and that consequently, we see in an electrical field an additional pressure relating to compressibility, which is not accounted for by our previous calculations. This pressure tends to cause a contraction of the liquid, which we call electrostriction.

Quantification of the phenomenon of electrostriction of a liquid in a flat condenser

A flat condenser is filled with liquid; this liquid fills a tube which ends in an area where there is no field. When voltage is applied, the liquid contracts and moves back down the tube. The pressure inside the condenser has thus increased.

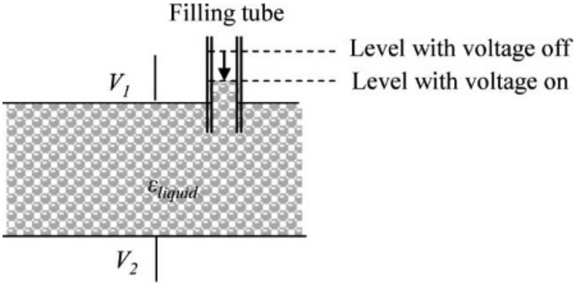

Figure 1.19. *Illustration of the phenomenon of electrostrition of a liquid in a flat condenser*

To find out how much the pressure has increased by, as before, we need only write:

$$-p.\Delta v = \Delta W$$

However:

$$\Delta W = \frac{1}{2}.\Delta\varepsilon.E^2.v$$

where v is the volume of the condenser.

Thus:

$$p = -\frac{E^2}{2}.v.\frac{d\varepsilon}{dv}$$

The increase in pressure p is the same as the electrostatic pressure:

$$p \sim \frac{\varepsilon.E^2}{2}$$

which is a tiny fraction of an atmosphere.

The actual contraction of the liquid, therefore, is extremely slight.

Electrostrition in solids has significant applications, because it enables us to transform electrical energy into elastic deformation and construct oscillating electrostatic motors which can be rather powerful (e.g. quartz or titanate ultrasound generators).

1.4. Dielectric losses

Dielectric losses occur when E is not a unique function of D. We have seen that the energy supplied to a dielectric per unit volume is equal to [1.37]:

$$W/v = \int_0^{\vec{D}} \vec{E}.d\vec{D} \qquad [1.37]$$

If we describe a closed cycle, this integral is zero only if *E* assumes the same values with the same values of *D*. Otherwise, there is ultimately an energy exchange between the dielectric and the voltage source. Experience shows us that the energy is absorbed by the dielectric, and that almost all of it is given off in the form of heat.

The energy dissipated per cycle per unit volume is therefore equal to the integral [1.38]:

$$W_{cycle} / v = \int_{cycle} \vec{E}.d\vec{D} \qquad [1.38]$$

However, as [1.39]:

$$\vec{D} = \varepsilon_0.\vec{E} + \vec{P} \qquad [1.39]$$

this integral is reduced to [1.40]:

$$W_{cycle} / v = \int_{cycle} \vec{E}.d\vec{P} \qquad [1.40]$$

If *E* and *P* are colinear, we can represent their variations on the coordinates plane *(E,P)*, obtaining a hysteresis cycle whose area represents the energy dissipated per cycle.

The hysteresis cycle is generally elliptical in shape, except if the losses are due to the ionization of gas vacuoles or if we are dealing with titanates, which give curves similar to those for ferromagnetic media.

We can also characterize the imperfections of a dielectric by way of another concept: the loss angle.

Suppose we have an imperfect-dielectric condenser, subject to an AC field with pulsation $\omega=2\pi.f$. We have the relation: $I''=V/Z''$. I'' is the quadrature component of the current, Z'' is the purely imaginary component of the condenser's impedance, given by $Z''=1/(C'.\omega)$, where C' is the condenser's capacity expressed in [*Farads*]. From this, we can deduce the strength of the current running through the condenser [1.41]:

$$I''=C'.V.\omega \text{ and } I'=C'.V.\omega.\sin\varphi \qquad [1.41]$$

The power required by such a condenser is [1.42]:

$$P=V.I.\cos\varphi \qquad [1.42]$$

where φ is the phase-shifting of the voltage in relation to the intensity. With a perfect dielectric $\varphi=-\pi/2$, and in this latter case, $P=0$ – in other words, the condenser integrally restores the energy that it receives.

If the dielectric is imperfect, $\varphi=\delta-\pi/2$, then we have $P=V.I.\sin(\delta)\neq 0$, where δ characterizes the losses, and is called the loss angle (Table 1.4). However, the imaginary component I' of the current I is $I.\cos(\delta)=V.C'.\omega$, which gives us equation [1.43]:

$$P=C'.V^2.\omega.\tan\delta=C'.V^2.2\pi.f.\tan\delta \qquad [1.43]$$

As the RMS of the voltage V and the maximum voltage V_m are linked by the relation $V_m=V.\sqrt{2}$, the losses also have the following expression ([1.44]):

$$P=C'.{V_m}^2.\pi.f.\tan\delta \qquad [1.44]$$

Finally, if we use W_m to denote the maximum energy of the condenser and W_p the energy lost during one cycle, we have: $P=f.W_p$ and $W_m=1/2.C'.V_m^2$, which gives us the energy lost in a cycle [1.45]:

$$W_p = W_m.2\pi.\tan\delta \qquad [1.45]$$

PolyVinyl Chloride	**Epoxy**	**Phlogopite mica**	**Jarylec®**	**PolyPropylene**
$\delta = 10^{-1}$	$\delta \approx 10^{-2}$	$\delta \approx 10^{-3}$	$\delta \approx 10^{-3}$	$\delta \approx 10^{-4}$

Table 1.4. *Order of magnitude of the loss angles δ [radians] of certain dehydrated dielectrics at f=50Hz and T=25°C*

Calculation of the order of magnitude of the losses

If δ is around 0.1, the energy dissipated per cycle is 0.62 times the maximum energy. The losses may therefore reach a considerable value.

Consider a 200 [μF] condenser, operating at V_m=100 [V]. Its maximum energy W_m is such that:

$$W_m = \frac{1}{2}.200.10^{-6}.100^2 = 1\,[J]$$

If its insulator has a loss angle of around 0.15, it dissipates one [*Joule*] per cycle. Over the course of 50 periods, this loss adds up to 50 [*Watts*].

Let us now consider a cable whose dielectric is characterized by ε_r=5; suppose that E_m=1 [MV/m]. Per m^3 of material, we have:

$$W_m = \frac{1}{2}.\varepsilon.E_m{}^2 = \frac{1}{2}.\frac{5}{36\pi.10^9}.(10^6)^2 = 22\,[J]$$

If the cross-section of the dielectric is S=10 [cm^2], a meter of cable represents a volume of 10^3 [cm^3]. Hence, W_m=0.022 [J]. If δ=0.15, $Wp \approx W_m$=0.022 [J], and if f=50 [Hz], P=1.1 [W]. For 1 [km], therefore, the losses are equal to 1.1 [kW].

Dielectric hysteresis has been demonstrated with radio, where the frequencies are high. Currently, dielectric losses are observed in numerous applications, and generally have an unfavorable effect. The dielectric losses are high with strong fields and high frequencies.

Since the loss angle generally increases with temperature, as the temperature rises its value increases, and the risk of breakdown is increased. This is one of the reasons why the issue of insulation is more difficult with AC voltage than with DC.

At high frequencies, even if the field is slight, the losses are still considerable because a very large number of cycles take place each second (f=10^5 [Hz] at 10^{10} [Hz] in practice).

If the power is negligible, there is no heating, but we witness a damping effect which is harmful to the quality of the circuit (resonance). This is why all radio condensers use air or mica strips, to minimize losses.

If the power is high, we see a heating effect which may continue until the insulation melts or is destroyed. In ultra-short-wave devices where the frequencies are extremely high, of around 10^{10} [Hz], dielectric losses are extremely problematic; hence, we use solid insulators as little as possible.

On the other hand, dielectric hysteresis has useful applications. It is sometimes necessary to heat an insulator, developing heat uniformly throughout its mass, because given that insulators are generally very poor heat conductors, it is difficult to heat them quickly using conventional methods without subjecting certain points to extremely high temperatures. Thus, we can quickly vulcanize large pieces of rubber, quickly melt plastic material, dry wood, etc., to say nothing of microwave ovens!

1.5. Residual charges

Residual charges in condensers are also caused by dielectric hysteresis. Consider a flat condenser using an imperfect dielectric. When the charge is released, by running a conductor between its casings, the DP becomes null, but the dielectric remains polarized: its polarization is P. On the two faces of the insulating strip, it gives rise to polarization densities $+P$ and $-P$, which create a field P/ε_0 inside. Given that the DP between the casings is null, they must generate an equal and opposite field and have opposite densities. Hence, each casing bears a $P.S$ if S is its surface. With all means of communication between the casings having been removed, if we wait a certain amount of time, the polarization gradually disappears. The charges of the casings, which have been retained, create a field which is no longer compensated by that of the polarization charges and, consequently, a DP between these casings. The condenser is once again charged and will give rise to a residual discharge when the casings are joined by a conductor.

Residual charges are particularly marked with solid dielectrics. Condensers can remain charged for days after an installation has been shut down, even when we believe all danger has been averted. We remedy this situation by shunting high-energy condensers with a fairly high discharge resistance so as not to cause any appreciable dissipation of service.

It is very important to point out that the phenomenon is only possible because the charge of the condensers no longer determines its equilibrium state, and that with a non-null charge their corresponding DP may be null or non-null depending on the polarization of the dielectric. Thus we can see how, if the insulator is imperfect, it is no longer enough to know the charge or the potential of each conductor to determine the equilibrium state of the system.

1.6. Electrets

The curious phenomena presented by electrets, which were discovered in the 1930s, offer a particularly striking example of residual charges.

Electrets are obtained by allowing certain waxes to solidify in an electrical field. Thereby we obtain pars which (apparently) exhibit permanent and constant polarization (like a magnet) – hence the name.

In fact, we cannot be dealing with real polarization of the dielectric, because of the constant presence of ions in the air which tend to neutralize it. In addition, when exposed to a highly conductive medium (e.g. humidity, ionized air, etc.), the electret seems to lose these properties, but it regains them if it is left in a sheltered environment for a certain amount of time, and particularly if we remove its external field by short-circuiting its poles, as we do with a magnet with its casing.

These recovery capabilities of the electret show that it is constantly changing, and that the apparent constancy of its polarization in favorable conditions simply results from the fact that its effective external field is limited to a set value by the fixed ionization of the air, which becomes very intense when we draw near to 3 $[MV/m]$. If the electret keeps its field in spite of the ions in the air and regains it after exposure to a conductive medium, it is because its true polarization is constantly decreasing, so that definitive equilibrium between the field from the electret and the field from the ions fixed to its surface is never reached.

The total amount of electricity that the electret can exchange throughout its lifetime can be measured by definitively destroying it by melting it between two plates connected by an electrometer. This quantity is infinitely greater than the amount of electricity required to create the electret's external field. In addition, it is noteworthy that the permanent dipole moment of a molecule is approximately 3.3×10^{-30} [$C.m$] (the elementary charge is 1.6×10^{-19} [C] and the interatomic distance is around 10^{-10} [m]) and that 1 [m^3] of matter with density 1 and molecular weight 200 contains a moment $(10^6/200).6\times10^{23}\times3.3\times10^{-30}$, so a polarization of $P=1.10^{-2}$ [C/m^2]. However, a polarization $P=2.67\times10^{-5}$ [C/m^2] is sufficient to generate the critical field for air, $E=P/\varepsilon_0=2.67\times10^{-5}\times36\pi\times10^9=3.02\times10^6$ [V/m]. A gradual decrease in the initial polarization is therefore sufficient to maintain the external field in spite of the presence of ions.

1.7. Characteristics of an insulator

From the above discussion, we can see that the characteristics which it is helpful to specify for industrial insulators are: resistivity, dielectric rigidity, the dielectric constant, the loss angle and the mechanical qualities.

– Resistivity: in general, it is sufficient, with the exception of poor insulators (woods) and many liquids.

– Dielectric rigidity: this is the maximum electrical field which the material can withstand before breakdown. This field depends on the conditions of use. In addition, we can only estimate it to within a certain order of magnitude, so have to leave a safety margin.

Ionization of the dielectric does not occur for a specifically determined value of the field. It also depends on the way in which it is applied. If the voltage is applied suddenly, the dielectric generally deals with it less well than if it is applied gradually. With repeated electric shocks (where voltage is applied for a period of time of around a microsecond), breakdown of the dielectric occurs at a level of voltage which is often very low if subjected to a great many shocks. Hence, it is impossible to state that a dielectric will break down at a given level of voltage or strength of field. What we do is to quote the rigidity – for example for alternating current of 50 [*Hz*] applied for a determinate length of time.

If the dielectric is solid, it is destroyed upon passage of a spark, which creates a permanent conductive channel. In an organic dielectric, it is carbon; in a ceramic, it is a fracture.

Liquid or gaseous dielectrics regenerate quickly when the ions have recombined (such is the case in compressed air circuit breakers). With liquids, we have to take account of the products of decomposition created by the spark, which may be problematic when there is repeated initialization.

The sudden disappearance of an electrical current (for instance sudden discharge of a condenser after gradual accumulation of charge) has the same dangerous effect as a shock: after a certain number of these "negative" shocks, the dielectric experiences breakdown, although it could withstand the same level of voltage indefinitely if it were kept constant.

The issue of the tenacity of an insulator is thus a highly complicated one and, in practice, must be examined on a case-by-case basis. The results obtained in a given case tell us nothing concrete about the behavior of the same dielectric in different circumstances. For instance, we would need to look at the behavior with direct current and alternating current at low, medium, high and very high frequencies, and with shock voltage (here specifying the law of establishment of the voltage).

Generally, it is with constant DC supply that the performances of insulators are best.

– The dielectric constant: this determines the capacity of devices insulated with the dielectric (cables, condensers, etc.). It is generally very low in high-voltage insulation and high in condensers with an energy storage function.

– The loss angle: this characterizes the losses from hysteresis. When dealing with AC, it is crucial to know the loss angle – all the more so at higher frequencies. At low frequency (LF), we only need to worry about the loss angle for condensers, cables, etc., where the field is strong. At high frequency (HF), all the insulators of a device need to be studied from this point of view. The loss angle also depends on the frequency – it generally passes through a certain number of maxima, and then decreases as the frequency increases. Thus, we have to know it for various frequencies.

– The mechanical qualities: the mechanical qualities such as toughness, resilience, etc., are also greatly important.

Name	ε_r **(*f*=50*Hz*)**	E_s **[*MV/m*] (*DC*)**	δ**[*rad*] (*f*=50*Hz*)**
PolyPropylene	2.2	600	10^{-4}
PolyLactic Acid	2.5	500	–
PolyEthylene	2.3	300	10^{-4}
Polyimide (Kapton®)	3.5	250	10^{-3}
Air (30 *atm*)	1	70	0
Phlogopite mica	6 to 8	60	10^{-3}
Epoxy	3.5	19	10^{-2}
Air (1 *atm*)	1	3	0

Table 1.5. *Dielectric characteristics of dehydrated insulators a few [mm] in thickness at T=25°C.* E_s *is the breakdown threshold field strength*

1.8. Pyro and piezo-electricity

Hitherto, we have examined the polarization of insulators by the electrical field. Some crystalline substances are permanently polarized in the absence of any field. Each of their molecules forms a dipole and the moments of these dipoles are oriented the same way, so that every volumetric element has a non-zero polarization.

Yet it is not possible to immediately see this polarization. Indeed, the air constantly contains a small number of ions (free electrical charges) due to ionization by cosmic rays and the natural radioactivity of all things (10–20 ion-electron pairs per second per cm^3). These ions are subjected to the field produced by the polarized crystal; regions with a positive polarization density $\sigma' = P.\cos(\alpha)$ receive negative ions; negative regions receive positive ions. Thus, the ions form a layer of electricity which destroys the field produced outside by polarization, and the phenomenon continues until compensation is perfect, and the ions in the surrounding air are no longer subjected to any force.

Thus, a permanently-polarized substance has its polarization masked by the fixation of ions from the air, which it is impossible to get rid of. However, if by any means, we alter the polarization, this alteration will be perceptible until the substance has received the additional ions needed to re-establish compensation – i.e. for a few hours at least.

Apart from the action of an external electrical field, there are two causes which can alter permanent polarization:

– a variation in temperature (pyroelectricity);

– a mechanical action (piezoelectricity).

We can easily explain these effects by saying that these factors, by dilatation or compression, alter the density of molecules, i.e. the number of dipoles per unit volume. The total moment per unit volume, or polarization, varies at the same time.

1.8.1. *Pyroelectricity*

This phenomenon, which has long been known, has no practical advantage. It is to be seen in tourmaline, for instance. A splinter of tourmaline, when heated, attracts light substances.

As we might imagine, the polarization – or more precisely, the variation in polarization – is proportional to the variation in temperature; it changes sign with it. Conversely, if we polarize a pyroelectric substance, its temperature changes.

This thermal effect is a consequence of the second law of thermodynamics; we can work back to this law by considering that the temperature variation tends to be opposed to the cause which gives rise to it – the change in polarization due to the field. Of course, the thermal effect changes direction as the field does.

1.8.2. *Piezoelectricity*

Unlike pyroelectricity, piezoelectricity – which was discovered fairly recently (in the late 19th Century) – is of crucial technical importance.

The crystal used should have no center of symmetry; thus in principle, it should exhibit the optical phenomenon of rotary polarization. In earlier years, only quartz was used. Nowadays, we also use Rochelle salt (potassium sodium tartrate), and many different synthetic compounds (e.g. titanates with high dielectric constants). Piezoelectricity is closely linked to electrostrition.

An oblong block of piezoelectric quartz, cut to appropriate proportions (see Figure 1.20), produces the following phenomena:

If a traction force is applied to the faces *ABCD* and *A'B'C'D'*, we witness the appearance of a polarization *P* perpendicular to faces *ADD'A'* and *BCC'B'*. This polarization is proportional to the traction force and changes direction if the traction becomes compression. We see the same results for pressure exerted on the faces *ADD'A'* and *BCC'D'*. The polarization P caused by a pressure of *1* [*kg.cm*^{2}] is 2.07× 10^{-7} [C/m^2]. If the pressure is applied to the faces *ABCD* and *A'B'C'D'*, we have to multiply F by the ratio of *AA'* to *AB*. Finally, a force applied to the faces *ABB'A'* and *DCC'D'* has no effect.

Conversely, if the quartz is placed in an electrical field perpendicular to the faces *ADD'A'* and *BCC'B'*, polarization of it leads to a contraction in the direction of *AB* and a dilatation in the direction of *AA'*, or vice versa, depending on the direction of the field, which is the same is electrostrition.

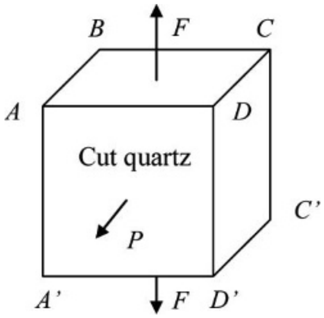

Figure 1.20. *Representation of a block of piezoelectric quartz*

As the polarization P is proportional to the pressure F/S, if P and F are colinear (which is often the case), the fictitious charge $Q'=P.S$ and the force F are proportional. This gives us the "piezoelectric charge coefficient" $K_Q = Q'/F = const$ [C/N]. For quartz, it is 2.10^{-12}; for titanates, it might be 100 times greater.

If the transducer is insulated, it gives us a voltage for which the various expressions stem from equation [1.46]:

$$U = e.\frac{P}{\varepsilon} \qquad [1.46]$$

Hence: $U = \frac{e}{\varepsilon}.\frac{Q'}{S} = \frac{e}{\varepsilon .S}.K_Q.F = \frac{Q'}{C}$

C is the capacity of the transducer and e the thickness.

We also write that ([1.47]):

$$U = K_U.\frac{e}{S}.F \text{ where } K_U = \frac{K_Q}{\varepsilon}[\text{V.m/N}] \qquad [1.47]$$

With ceramics, $K_U \approx 10^{-2}$. Thus, we can easily obtain several hundred volts with lamellae of a few millimeters thick, subject to a force of a few [Kg/mm^2].

Materials with a large ε have the advantage of having a proper capacity C which is large in comparison to that C' of the associated equipment, so the voltage obtained may practically be that generated by the transducer alone, so that [1.48]:

$$U = \frac{P.e}{\varepsilon .(1 + C'/C)} \approx \frac{P.e}{\varepsilon} = K_U.\frac{e}{S}.F \text{ where } C'/C << 1 \qquad [1.48]$$

Piezoelectricity is used to convert a mechanical force to an electrical force. As piezoelectric substances have marked electrostrictive properties, we can use them to produce a mechanical force from a DP, which is in fact an application of electrostrition.

1.8.2.1. *Pressure measurement*

It is clear that quartz could be used to measure all sorts of mechanical forces. It has the great advantage of being a dynamometer with very low inertia and requiring only a very slight movement of the point of application of the force being measured. These two remarkable qualities make it an ideal dynamometer, infinitely superior to spring dynamometers or dead-weight dynamometers. It is used in innumerable devices, especially when the forces being measured change rapidly, as is the case when recording the pressure in combustion engines.

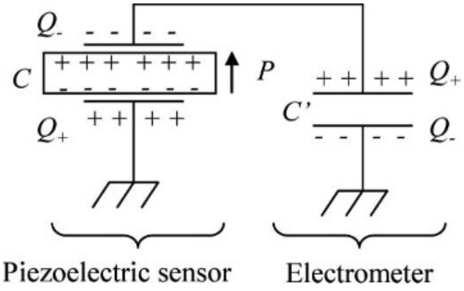

Figure 1.21. *Principle of measurement of polarization (and indirectly of pressure)*

To make use of quartz, we place it between the casings of a condenser, one of which is grounded and the other is connected to an electrometer (Figure 1.21). On both faces of the quartz, the polarization P creates densities $\pm P$ which cause a field P/ε and a DP $P.e/\varepsilon$, where e is the thickness. For this to be the case, the capacity C' of the electrometer must be very small in relation to that of the quartz condenser C. Indeed, this condenser and the electrometer have the same charge Q. We have $Q=C'.V$, with V representing the DP measured by the electrometer. The charge Q of the condenser gives rise to a DP: $V=Q/C'$, which results from that $P.e/\varepsilon$ produced by the quartz. Hence, the electrometer measures (equation [1.49]):

$$V = \frac{P.e}{\varepsilon} - \frac{q}{C} = \frac{P.e}{\varepsilon} - \frac{C'}{C}.V \tag{1.49}$$

This means that [1.50]:

$$V = \frac{P.e}{\varepsilon.(1 + C'/C)} \tag{1.50}$$

The ratio of C' to C must therefore be very small for maximum sensitivity. Hence, we can see the interest of very low-capacity electrometers.

1.8.2.2. *Space charge measurement*

A space charge, when trapped in an elastic medium and subject to a pulsed field, produces a characteristic acoustic signal around itself. This is essentially the phenomenon upon which the Pulsed Electro-Acoustic (PEA) method is based.

In order to give a clearer idea of the principle behind this method, consider a sample of thickness d with a layer of negative charge ρ_x of thickness Δx at depth x (see Figure 1.22). By total influence, this layer causes charges ρ_d and ρ_0 such that $\rho_d=(-x/d)\cdot\rho_x$ and $\rho_0=((x-d)/d)\cdot\rho_x$. The application of a pulsed potential $u_p(t)$, by way of "Coulombian force", causes a transitory displacement of the space charges around their abscissa position x. Waves of elementary pressure $p_\Delta(t)$ from each charged zone, the amplitude of which is proportional to the local density of the charges, then propagate at the speed of sound in the sample v_p and then in the output electrode v_e. Under the influence of these variations in pressure, the piezoelectric sensor delivers a voltage $v_s(t)$ which is characteristic of the range of pressures encountered. The distribution of the space charge within the sample can then be calculated using equation [1.51].

Nevertheless, a calibration procedure needs to be carried out. The aim of this is to eliminate the unknown $H(v)$ which represents the transfer function in the chain of measurements (waveguide + piezoelectric transducer + amplifiers – see Figure 1.23). For this purpose, we have an uncharged sample to which we apply a constant voltage U; the surface density of charge per electrode is thereby discovered. To this we add an electrical field pulse $u_p(t)$, with the hypothesis that the capacitive charges due to the pulsed field are negligible in comparison to all the other charges, and thus we record $v_{s1}(t)$.

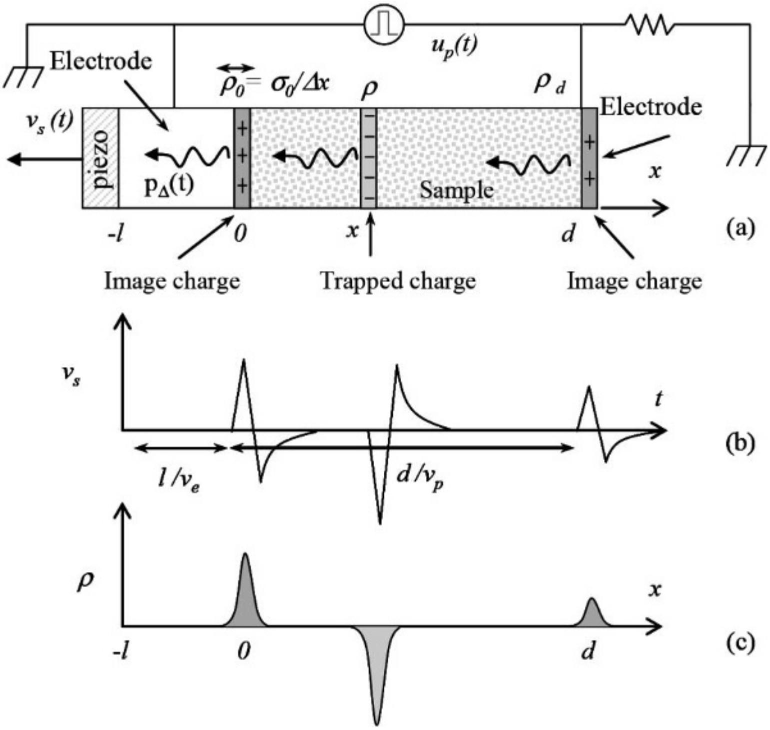

Figure 1.22. *Principle of measurement of the space charge using the PEA method; a) when subjected to a pulse field, the charged zones generate acoustic waves; b) the piezoelectric sensor therefore delivers a voltage* $v_s(t)$*; c) appropriate signal processing then yields a spatial distribution of the image and trapped charges*

1.8.2.3. *Pressure generation*

Piezoelectric substances serve to produce mechanical effects, using their electrostrition. Thus, they are used in the production of ultrasound. A generator produces an A/C difference in potential which is applied to a quartz or titanate condenser; the dielectric which is alternately polarized undergoes synchronous contractions and dilatations by electrostrition, and emits the corresponding sound. As the losses are very slight, the yield is good and it is easy to produce sounds at 50 [*kHz*] with powers of around a [*kW*]. These ultrasounds are greatly absorbed by air, but far less by water; they may be used for explorations by sounding, for instance.

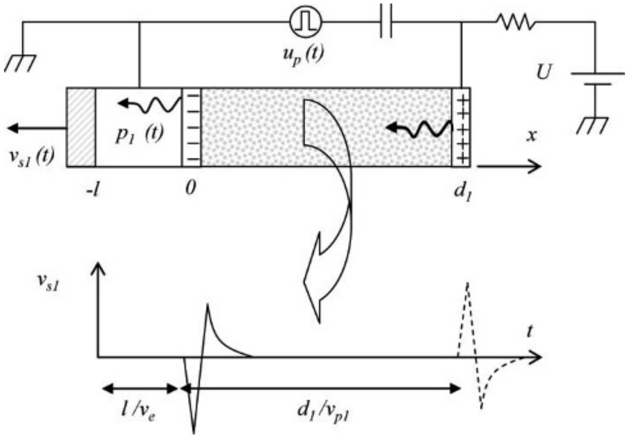

Figure 1.23. *Principle of the reference procedure (calibration). The reference signal corresponds to the electro-acoustic signal produced by the capacitive charges located on the electrode adjacent to the piezoelectric sensor*

$$\rho(x) = \mathrm{F}^{-1}[R(\nu)] = \mathrm{F}^{-1}\left[\frac{\varepsilon_1.U}{v_p.d_1}.\frac{v_s(\nu)}{v_{s1}(\nu)}\right] \qquad [1.51]$$

1.8.2.4. *Timebase generation*

Quartz is likely to vibrate mechanically like any elastic substance (e.g. springs, tuning forks, etc.), but is able to produce extremely high frequencies (a million periods per second), and its damping is very slight. Exploiting its piezoelectric properties, we can use it in a lamp or a transistor which sustain its oscillations by electrostrition. Thereby, we obtain quartz oscillators which use little power but whose frequency is very well defined; these oscillators serve as “pilots” for the more powerful oscillators used in telecommunications networks. As the frequency is perfectly constant at a given temperature, the period of a quartz crystal can serve as a time marker. We see quartz clocks, wherein an oscillator is used, with a frequency demultiplier, to drive a synchronous motor, which in turn drives the hands round. These clocks give a definition of the time which is at least equal and possibly superior to that obtained by the rotation of the Earth. They are found, notably, as timebases in computers and watches.

1.9. Currents in extended conductors

In many cases, the current, instead of circulating only in the wires, spreads out throughout the entire mass of a conductor extended in all three dimensions: electrolysis baths, grounding, etc. In addition, the mathematical properties of the distribution of current in conductors have very important applications in electrical engineering, aerodynamics and hydrodynamics.

1.9.1. *Current density*

Hitherto, we have only looked at the intensity of currents, i.e. the amount of charge passing through the section of a conductor in one second. This intensity is similar to the flowrate in a water pipe. We now introduce the concept of current density, which is similar to the velocity of water in the pipe. Suppose for simplicity's sake that only one form of electricity (positive or negative charge) is mobile (as happens in metals); consider ρ the volumetric density of that electricity and v its velocity vector (this velocity varies from case to case, from a few $[\mu m/s]$ to a few $[mm/s]$). If we consider an element with surface ds within the conductor, the “volume of electricity” passing through ds in one second ($f=1[Hz]$) is, by definition, the flow of v through ds, i.e. $f.v.ds$, and the amount of electricity which would have passed through will be equal to that volume multiplied by the volumetric density ρ, so $f.\rho.v.ds$. This amount of electricity passing through ds in one second is the intensity dI through that element.

We can say that the intensity dI of current passing through an element with a given surface is equal to the flow, through that element, of the vector $\rho.v$. This vector is called the "current density", and is denoted by a lowercase i, with the notation I being reserved for the intensity of a current, which is a scalar rather than a vector (because it is a flow).

If both types of electricity are mobile, with volumetric densities ρ and ρ', and velocities v and v', the total amount of electricity passing through an element will be the algebraic sum of the flow of + and – charges. This means that the current density will be: $i=\rho.v+\rho'.v'$. The volumetric densities ρ and ρ' have opposite signs, but the same is true of v and v', and it is clear that the currents produced by the movement of both forms of electricity are added to one another.

In summary, the intensity of current passing over a surface S traced in a conductor is equal to the flow across that surface of the current density vector i. This density has the dimensions of a current divided by a surface, and can be expressed in $[A/m^2]$. In industry, it is also common to use $[A/mm^2]$.

We use the term "streamlines" for the lines tangent to the current density at all their points, and "streamtube" for a tube similar to a force tube formed of such lines. The outer surface of a conductor in a permanent regime is a streamtube: the velocity v of the electricity remains tangent to it, in the same way as the velocity of a fluid remains tangent to the walls of the pipe within which it is flowing.

1.9.1.1. *Application: conductivity of an electrolyte*

In an electrolytic solution, we accept that there are positive and negative ions, with each of them representing an electricity density ρ and having mobility K. This mobility is the velocity assumed by the ion in the unit electrical field. Hence, we have $v=K.E$.

In the case of a binary electrolyte such as *NaCl*, there are only two sorts of ions to deal with. In addition, the total density of electricity is quite obviously null throughout the entire volume of the element (the slightest deviation from this law would produce extremely intense electrical fields, which are incompatible with the conductivity of the medium). Hence, we have $i=(K_+.\rho_+ + K_-.\rho_-).E$. Yet $\rho_+=-\rho_-=\rho$. Remembering that the two mobilities have opposite signs, we get: $i=(K_+ + |K_-|).\rho.E=\sigma.E$, where $\sigma[S/m]$ is the conductivity.

In water, the mobility of the ions is around 10^{-8} $[m^2/(s.V)]$. With a solution containing one molecule-gram of salt per liter of totally ionized solvent, we have a density of ionic charge of around 10^{+8} $[C/m^3]$. The conductivity is therefore approximately 2 $[S/m]$, or the resistivity is 0.5 $[\Omega.m]$. The greatest conductivity is seen in acids, and the H^+ ion has exceptional mobility in water.

1.9.2. *Expression of Ohm's law in an extended conductor*

Consider a cylindrical wire traversed by a current. The streamlines are parallel to the generatrices. The field E in the wire is uniform and also parallel to the generatrices, because the potential varies proportionally to the length, according to Ohm's law. Take S to be the section of the wire, I the intensity of current, and V_A and V_B the potentials at the two ends. Ohm's law is written as $V_A-V_B=R.I$; yet $i=I/S$, meaning that $V_A-V_B=R.i.S$; in the conductor, E is uniform, parallel to the direction of movement of the electrical charges: V_A-V_B=$E.l$, so $i=E.l/(R.S)$. Hence, $R=E.l/i.S$ but $i=E/\rho$, so $R=\rho.l/S$ where ρ represents the resistivity of the medium.

At all points on a wire, according to Ohm's law, therefore, E and i are proportional to one another. As they are governed by the same linear equation, we can write $i=E/\rho$, and if we set $\sigma=1/\rho$ (conductivity), we get $i=\sigma.E$.

This means that the velocity of the charges is proportional to the applied force, so the friction is viscous. This is the general expression of Ohm's law, which is valid in extended media. Hence, the lines of force and the streamlines are one and the same.

In order to apply the relation $E=\rho.i$, we need to use consistent units. In the practical system, the unit of DP is the [V]; if we take the [m] to be the unit of length, the unit of field is the [V/m]; the unit of density will be [A/m^2] and the unit of resistivity [$\Omega.m$].

Thus, let us look for the prevailing electrical field in a copper wire for a density of 3 [A/mm^2] (the average density used in industry). Hence, we have: i=3.10^6 [A/m^2] and ρ=2.10^8 [$\Omega.m$], so E=6.10^{-2} [V/m]. This field is absolutely tiny in relation to those which may exist in insulators, caused by electrostatic phenomena, which are around 100 [kV/m], so 10^7 times larger. This explains why, in ordinary electrodynamic experiments, with small circuits, the electrical fields and the DPs are extremely small in comparison to those which electrostatic phenomena are capable of generating.

In addition, the above calculation shows that, for a copper line which is normally used, with current density 3 [A/mm^2], the voltage loss is 6×10^{-2} [V/m]. As the voltage loss must be a small fraction of the total voltage in order for the performance of an electricity transport line to be acceptable, we can see that the voltage in such an electricity transport line must be proportional to its length, if we want to make proper use of it. If we allow for 1 [*kV*] per [*km*] of cable, we limit the voltage loss to 6% with a single-wire line, and 12% with a two-wire line. We can, of course, use lower voltages than those prescribed by the 1 [kV/km] rule. If we do so, we have to increase the cross-section of the copper in order to limit the current density and therefore the voltage loss.

1.9.3. *Conservation of current density*

In the permanent regime, the current density is a conservative vector. If we consider an enclosed surface traced in the conductor, the total amount of electricity contained in that surface is constant, and the total current through that surface is zero. Thus, the flow of current density traversing any enclosed surface is zero; the current density is a conservative vector, meaning that $div(i)=0$. Yet because i is proportional to E according to Ohm's law, we also have $div(E)=0$. Thus, the electrical field is conservative, in an ohmic conductor and in a vacuum. If there is no electromotive force in the conductor, E is derived from a potential, V; hence, it has all the properties of a static field in the vacuum, i.e. it is derived from a potential and is conservative. It is worth noting that the equation $div(E)=0$ means that the algebraic density of charges is null inside the conductor, which seems incongruous with the fact that there is a current running through it. This is known as the Kirchhoff paradox. Its explanation has already been given.

1.9.4. *Distribution of current in an extended medium*

In general, the current is fed into the medium by metallic electrodes. If the resistivity of these electrodes is far less than that of the medium (electrolysis, for instance) the DPs between the points of the same electrode are practically negligible in comparison to the DPs between the electrodes. Thus, we can agree that the surface of an electrode is an equipotential. Hence, if the conductive medium is externally delimited by a surface S, the surfaces of the electrodes being S_1, S_2, etc., the field will be normal to the surface of the electrodes, because they are equipotentials. The problem of the distribution of current in the medium is very similar to the problem of equilibrium of the conductors; it even becomes identical if the surface S is rejected at infinity.

We are going to demonstrate a premise which is very similar to that already seen with equilibrium of conductors: if, for each electrode, we know either the potential or the total current that it brings to the medium, then the field, current density and potential at all points in the medium are completely determined uniquely.

The demonstration is the same as in electrostatic. We begin with equation [1.52]:

$$div(V.\vec{E}) = V.div\vec{E} + \vec{E}.grad(V) = -E^2 \text{ because } div\vec{E} = 0 \qquad [1.52]$$

and we apply Green's theorem to the vector $V.E$, the integration volume being the volume filled by the conductive medium. Thus, we have [1.53]:

$$\iiint_v -E^2.dv = \iiint_v div(V.\vec{E}).dv = \iint_{S+S_1+S_2} \vec{n}.(V.\vec{E}).ds \qquad [1.53]$$

The flow $V.E$ across S is null, because E, like i, is tangent to S. We still have the flows across the surfaces of the electrodes, which – from what we have always seen – are $(V_1.Q_1+V_2.Q_2+\ldots)/\varepsilon$, where V_1 and Q_1 are the potential and the static charge of electrode 1. If we use the notation E_1 for the total current originating in that electrode, there is a very simple relation between I_1 and Q_1. Indeed, according to Gauss' theorem, $Q=\varepsilon.\varphi(E)$, with $\varphi(E)$ being the flow of E across the surface of the electrode. However, because $E=\rho.i$

(from Ohm's law), we have $\varphi(E)=\rho.\varphi(i)$, and because the flow of i is the total current I originating in the electrode, we get: $Q_1=\rho.\varepsilon.I_1$, and hence [1.54]:

$$\iiint_v E^2.dv=(V_1.Q_1+V_2.Q_2+\ldots)/\varepsilon=\rho(V_1.I_1+V_2.I_2+\ldots) \qquad [1.54]$$

If we suppose for each electrode that either V=0 or I=0, the triple integral $E^2.dv$ is null; hence E and i are null everywhere. If V or I is given for each electrode, the distribution of E and i is unique. Indeed, if there were two solutions, the difference between them would be a distribution corresponding to a null value of V or I for each electrode, i.e. identically null.

We have seen that the distribution of E is identical to that which would be realized in the vacuum, the electrodes being insulated conductors, if the medium is extended to infinity, or at least to very great distances in relation to the dimensions of the system of electrodes. There is another case where this identity is strictly correct: where we choose a surface S delimiting the conductive medium to be a surface which is universally tangential to the field existing in the vacuum between the electrodes. At that time, this field satisfies all the conditions which the field must fulfill in the conductive medium: being normal to the electrodes and being tangent to S. Because the solution is unique (for given values of the potentials), it is the sought solution. Hence, the two fields are identical. Consider, for instance, the field between two cylinders with any base, which are parallel and infinite. This field is universally normal to the direction of the generatrices. If we fill the gap between two planes perpendicular to that direction with a conductive medium, we shall see that the field is universally tangent to those planes. The distribution of the field is thus the same, in the slice of the conductive medium contained between the two planes, and between the two infinite insulated cylinders.

1.9.5. *Power dissipated by the Joule effect*

In electrostatic, the quantity $½.\varepsilon.E^2$ represents the energy contained by each [m^3] of insulator. Here, the analogous quantity is E^2/ρ, and as we shall see, it represents the power dissipated by the Joule effect per [m^3] of conductor.

Consider an element of streamtube with section *ds* and length *dl*. The DP between the extremities of the element is $dV=E.dl$ and the current intensity is $I=i.ds$. The dissipated power is therefore [1.55]:

$$dP = dV.dI = E.i.ds.dl = E.i.dv \qquad [1.55]$$

dv being the volume of the element. As $i=E/\rho$, we have $P=(E^2/\rho).dv$, this means that the power per [m^3] is indeed E^2/ρ. The integral [1.56] thus represents the total power dissipated by the Joule effect in the conductive medium:

$$P = \frac{1}{\rho}\iiint E^2 dv \qquad [1.56]$$

This power is therefore equal to $V_1.I_1+V_2.I_2+\ldots$, which is, of course, evident to begin with.

1.9.6. *Resistance of the conductive medium*

Suppose for simplicity's sake that there are only two electrodes. The resistance *R* of the medium is defined by $V_1–V_2=R.I$. Yet, as we know from our earlier discussion, $I=Q/(\rho.\varepsilon)$, with *Q* being the charge of an electrode, meaning that $R=(V_1–V_2)/I=(V_1–V_2).\rho.\varepsilon/Q=\rho.\varepsilon/C$, because $C=Q/(V_1–V_2)$.

In general, we have a network of resistance R_{ij} connecting the conductors *i* and *j* – this network is equivalent to the extended network $R_{ij}.T_{ij}=\rho.\varepsilon$, and T_{ij} being the capacity coefficients, which are all positive.

The applications of the above results are many and important. We shall now give some examples.

1.9.6.1. *Application: electrolysis bath*

Suppose the electrodes are cylindrical and coaxial, with radii R_1 and R_2. If the height of the liquid is h, the capacity C is [1.57]:

$$C = \frac{2.\pi.\varepsilon.h}{\ln(R_2 / R_1)} \qquad [1.57]$$

which gives us the resistance [1.58]:

$$R = \frac{\rho}{2.\pi.h}.\ln(R_2 / R_1) \qquad [1.58]$$

If the electrodes are two cylinders with radius a, separated by a distance d, their capacity is [1.59]:

$$C = \frac{4.\pi.\varepsilon.h}{\ln(d / a)} \qquad [1.59]$$

Hence, we get the resistance [1.60]:

$$R = \frac{\rho}{4\pi.h}.\ln(d / a) \qquad [1.60]$$

Note that this result is valid only if the dimensions of the bath are extremely large, because the field of two parallel cylinders outside of one another extends to infinity. If the bath is of limited dimensions, the resistance is higher because the space available to the current is more restricted.

1.9.6.2. *Application: grounding*

The medium can be viewed as being infinite if the depth of the electrode is extremely great in comparison to its dimensions. If it is a sphere with radius a, the resistance is

$\rho/(4.\pi.a)$. Thus, this resistance is independent of the distance from the other grounding rod. This remarkable result accounts for the practical advantage of grounding mechanisms. Their resistance is independent of the distance, whereas with a wire it is proportional. Given that the resistivity of the ground is around 100 [$\Omega.m$], let us calculate the resistance of a grounding mechanism comprising a disk one meter in radius. Its capacity is 70 [pF], so the resistance is $\rho.\varepsilon/C$ or 12.5[Ω]. In spite of the low conductivity of soil, which is nearly 10^{10} times less than that of copper, the resistance of the ground-return circuit is less than that of a wire-return circuit over distances of more than a few dozen [km]. In fact, there are two grounding mechanisms, which means the resistance of an ground-return circuit is 25 [Ω]. Note that the formula is valid only for a depth of the probe at least ten times less than the dimensions of the electrode. If the probe is at a lesser depth, the resistance is increased. The resistance would be doubled if the electrode only penetrated to half the depth in the ground, because the current would only traverse half of the lines of force. Grounding mechanisms are used less nowadays than they used to be, but are still in use for the protection of people and property.

1.9.6.3. *Application: insulation resistance of a cable*

If we know the capacity of a cable (irrespective of the dielectric power of the insulator) and the resistivity of its insulator, we can immediately calculate its insulation resistance. Take a cable with capacity 2 [μF]; if the dielectric constant ε_r=2, the capacity C in the formula is 1 [μF]. If the resistivity of the insulator is 10^{11} [$\Omega.m$] the insulation resistance is $10^{11}/(36.\pi.10^9.10^{-6})$, so around 1 [$M\Omega$].

1.9.6.4. *Application: studying the electrical field in a vacuum*

Nowadays, the study of the distribution of the electrical field in the vicinity of a system of conductors has become extremely important in technological engineering. Indeed, it is through knowledge of this field that we are able to tell whether the insulator surrounding the conductors will be capable of withstanding. The distribution of the electrical field plays an even more important role in all electronic devices such as amplifier tubes, electron cannons, accelerators, microscopes, etc.) where it controls the movement of charged corpuscles (electrons or ions) and, generally, in all issues to do with corpuscular optics. Yet the complete mathematical solution to the problem, except in the simple cases discussed above, is often too complicated. However, it is possible to get around the difficulty by replacing the system of conductors to be studied with a system of similar electrodes immersed in a bath of conductive fluid (ordinary water). On condition that the bath is big enough, the distribution of the electrical field around the conductors in a vacuum or around the electrodes in the bath is the same. Between the electrodes, we establish DPs proportional to those which should exist between the conductors with an AC source so as to avoid electrolytic polarization. We then use a needle or probe that is mobile in the three dimensions, which enables us to measure the potential at every point in the bath. This potential is measured with a potentiometer, by way of a zero method. When we move the probe so the indicator on the potentiometer remains at zero, it describes an equipotential. Mechanical devices can be used to show the probe's position on a graph at all times. Thus, we can easily trace the different equipotentials. The field can be deduced from this easily, because the DP ΔV between two neighboring equipotentials is equal to the field E multiplied by their distance Δx: $\Delta V = E.\Delta x$.

1.9.6.5. *Application: release of heat in a bath*

We frequently use the Joule effect to heat electrolysis baths, or *bains-marie* (double boilers). The heat released per second per [m^3] of the bath is E^2/ρ, so we can see that it is far greater in the vicinity of the electrodes, where E is high, than in the middle of the bath, and that it is greatest at the points of the electrodes, where E is very high (point power). This shows that the heating thereby obtained is very irregular, and that we cannot expect a uniform temperature unless we use electrodes forming a field E which is more or less constant.

1.9.6.6. *Application: electrolysis*

The amount of material washed away or deposited on an electrode is proportional to the amount of electricity which has passed through it. The quantity of material involved per [cm^2] – that is, the thickness of the material – is proportional to the amount of electricity which is passed per [cm^2] in a given time-period, i.e. the current density. Hence, in electrolysis, the current density determines the rate of deposit or of erosion. It is the current density which it is essential to know for successful electrolysis. As i is proportional to E, we can see that the rate of deposit or erosion is, at each point of the electrode, proportional to the density of surface charge which exists on that electrode. The thickness of metal deposited or removed will therefore be very great at the points and the corners (point power) and null in hollow areas (hollow conductors). Hence, the deposition of material will be highly irregular. We regularize it by using multiple electrodes and stirring the pieces in the bath. If we wish to deposit material on the inside of a cavity, this can obviously not be done, even if the cavity walls are pierced with holes. The only way to do so is to introduce a special electrode which goes right to the center of the cavity.

We can illustrate these phenomena with the following experiment: a soluble anode has the form of a stick. It starts being eroded at its end, where the values of the field and density are greatest. However, as the erosion attack continues, the point size and the field increase further, the point becomes sharp and the erosion then occurs only at that point. The stem therefore gradually becomes shorter, whilst remaining sharp. The same is true in batteries, where the zinc is in the shape of a cylindrical stick. If the erosion is most marked at a given point of the zinc, the stick shrinks at that point and, as the field increases with an increase in curvature, the erosion is concentrated more and more at that point, and the zinc stick ends up being broken in two.

One very important application of point power in electrolysis is electrolytic polishing. The part needing to be polished serves as an anode in an appropriate electrolysis bath – usually a highly viscous liquid such as a mixture of acetic anhydride and perchloric acid. As the current density is far greater on the outcrops than in the hollows, the outcrops are the first to be eroded, and the piece is polished in very little time.

1.9.6.7. *Application: hydrodynamics and aerodynamics*

As we have seen, the velocity vector in the flow of an incompressible fluid is conservative, and therefore satisfies $div(v)=0$, like the electrical field in a conductor. In practice, it is not only water and other liquids that can be regarded as incompressible: air can too, so long as its velocity does not surpass 100 $[m/s]$. In addition, it often happens that the velocity vector derives from a potential (velocity potential). Thus, we have a total similarity between this velocity vector and the electrical field in a conductor. In order to study the movement of air around the wing of an airplane, we would make a model of the wing and immerse it in a vessel filled with a conductive liquid, in which two flat, parallel electrodes are producing a uniform field. This field represents the velocity of a uniform stream of air. When we place the model of the wing into the vessel, the streamlines are deformed, as they have to remain tangent to the insulators, and mimic the streamlines of air around the wing. By tracing the equipotentials, we can obtain these lines and calculate the electrical field, i.e. the velocity of air proportional to it. Knowing this velocity at all points enables us to calculate the pushing force exerted on the wing by the moving air. This method is of crucial importance in fluid mechanics. The accuracy of these results is, however, limited by the fact that air is no longer incompressible at velocities approaching, or greater than, the speed of sound. In hydrodynamics, we do not have this drawback, so we can determine the flow around turbine blades, ship rudders, etc.

1.9.6.8. *Application: grounding*

We have previously seen how grounding enabled us to achieve lower resistances than those using a wire, no matter what the distance. Ground return was commonly used on telegraph lines. On telephone lines it is not used, because in the ground there are natural currents constantly circulating which are very slight, but nevertheless appreciable. These currents cause DPs between the different points of the ground, and these DPs would give rise to considerable parasitic currents in telephonic devices, which are highly sensitive. In the *Aurora Borealis* (the Northern Lights), these currents are amplified and could themselves interfere with telegraphic transmissions. The return of intense industrial currents through the ground is not practical, and is even forbidden. The reason is that these currents, circulating in the ground, would produce considerable DPs between the different points. Thus, between the different grounding rods in a telegraph network, there would be significant DPs (of around one [V]), which would create intense parasitic currents in the devices. In distribution networks, there is always one pole connected to the ground, so that the DP with the ground is fixed and cannot reach its dangerous value (under the influence of atmospheric electricity), but the systematic use of the ground as a return conductor is formally prohibited. In addition, using a system of balanced currents (e.g. a three-phase star arrangement) for energy transport, it is always possible to obtain the economic advantages of ground return without any of its disadvantages.

Let us remember that ground telegraphy operation (referred to as TPS for télégraphie par le sol) was used by the armed forces during WWI (1914–1918), before the development of wireless communication. The transmitter was composed of a musical frequency alternator, connected to two grounding rods A and B, a few dozen meters apart. In the ground, the electrical field created by A and B was comparable to that of two equal and opposite point charges. At the receiver post, on average about a kilometer away, there were another two grounding rods C and D, connected to a telephone. The electrical field created by A and B caused a maximum DRP between C and D, when AB and CD were on the line running between the two posts. This DP caused a current in the telephone. It was advantageous to augment the bases AB and CD as much as possible, so as to increase the electrical field and the DP that it created. Note that the power consumption at the transmitter was more or less independent of the distance between A and B, as the resistance does not depend on the distance between the two rods, which are assumed to be relatively far apart. TPS was abandoned and replaced by wireless communication. In military telephony, ground return is often used, in order to spare the wire. The drawback to this is that it means an enemy, even far away, can listen to the telephonic conversation using two grounding rods linked to an amplifying device. It should be noted that these considerations offer an advantage only in the type of positional war such as that fought between 1914 and 1918.

1.9.7. *Deviation of the electrical current when crossing the surface separating two conductors*

Not content with studying the field E in a vacuum, we can also study it in dielectrics. This problem is interesting with regard to the construction of cables and insulators.

Consider two media with conductivity σ and σ' separated by the surface S (Figure 1.24). For the current density and the field, we shall define normal and tangential components, as we did for the field and induction with regard to dielectrics. E derives from a potential if there is no electromotive force. Thus, we can use the same reasoning as for dielectrics, and we find that $E_T = E'_T$.

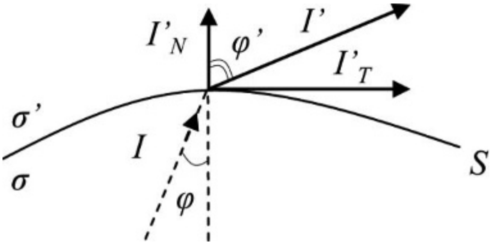

Figure 1.24. *Illustration of the deviation of electrical current on passing from one conductive medium σ to another σ'*

Let us now look at the normal components of i, which is conservative because the regime is permanent. By the same reasoning process which shows the equality of D_N and D'_N, we find $i_N = i'_N$. Thus, we have the relations: $E_T = E'_T$ and $i_N = i'_N$, and because $i = \sigma.E$, we also have $i_T = \sigma.E_T$, $i_N = \sigma.E_N$, $i'_T = \sigma'.E'_T$ and $i'_N = \sigma'.E'_N$, similar to $D_T = \varepsilon.E_T$, etc.

Furthermore, if φ' is the angle that the current density i' forms with the normal to the surface of separation and φ is the similar angle between i and the normal, we have the equations:

$$\tan\varphi' = \frac{i'_T}{i'_N} \text{ and } \tan\varphi = \frac{i_T}{i_N}$$

From this we deduce [1.61]:

$$\frac{\tan\varphi'}{\tan\varphi} = \frac{i'_T}{i'_N}.\frac{i_N}{i_T} = \frac{i'_T}{i_T} = \frac{\sigma'.E'_T}{\sigma.E_T}$$

Thus:

$$\frac{\tan\varphi'}{\tan\varphi} = \frac{\sigma'}{\sigma} \qquad [1.61]$$

This result is similar to that which we found for dielectrics [1.62]:

$$\frac{\tan\varphi'}{\tan\varphi} = \frac{\varepsilon'}{\varepsilon} \qquad [1.62]$$

This is of interest when studying an electrical field where there are insulators other than a vacuum – e.g. the field in a cable insulator placed in air. We then consider a cable, raised to the potential V. We immerse it in a liquid with conductivity σ. Beside it, we place a model of the insulator, with conductivity σ', such that $\sigma'/\sigma=\varepsilon'_r/\varepsilon_r$, with ε'_r being the dielectric constant of the insulator and ε_r that of air (that is, *1*).