1

The Need for Time–Frequency Analysis

Most real signals are non-stationary where the frequency can vary with time. The classic Fourier transform analyzes the frequency content of the signal without any time information. It emphasizes the importance of time–frequency transforms designed to detect the frequency changes of the signal over time. Moreover, it allows extracting relevant features to classify signal signatures. This chapter presents the stationary and non-stationary concepts and the representations of the signal in time or frequency domains. The limitations of these representations and the need of the time–frequency domain are also introduced and discussed.

1.1. Introduction

From a theoretical point of view, signals can be divided into two main groups: deterministic and random. Deterministic signals are well known mathematically (analytically describable), so the future values of the signal can be calculated from the past values with complete certainty. However, random signals cannot be described as a mathematical expression and cannot be predicted with a total certainty, which leads to the study of their statistical properties (average, variance, covariance, etc.) in order to have an idea about their structure.

In a deterministic or random framework, a signal as an abstraction of physical quantities of a process can be classified intuitively into two main classes: stationary and non-stationary signals. This qualitative classification is based mainly on information variation of a signal over time. In the case of random signals, for example, the stationary signals have constant statistical properties over time while non-stationary signals are characterized by the variation of their statistical properties during the interval of observation. In a deterministic framework, stationary signals can be defined as a sum of discrete sinusoids that have an invariant frequency over time, otherwise they are considered as non-stationary.

Most real-life signals are non-stationary and contain random components that can be caused by the measurement instruments (random noise, spike, etc.) and/or by the nature of the physical process under study. For example, in the acquisition of the heart sound signal, which is a non-stationary signal by nature, several factors affect the quality of the acquired signal: the type of electronic stethoscope, the patient's position during auscultation and the surrounding noises. Moreover, the heart sound as an abstraction of the mechanical activity of the heart contains by nature random components such as murmurs. Another example is the power quality signals and their disturbances that have negative impacts on power systems and make the electric signal random and non-stationary. These two examples of non-stationary signals will be the main applications in this book (Chapters 3 and 4).

The aim of this chapter is to present the stationary and non-stationary concepts briefly. The different signal representations will be introduced and the limitations of time or frequency representations in the case of non-stationary signals will be shown. This will lead us to introduce some essential concepts such as the uncertainty principle and the instantaneous frequency (IF) measure.

1.2. Stationary and non-stationary concepts

1.2.1. *Stationarity*

1.2.1.1. *Deterministic signal*

A deterministic signal is said to be stationary if it can be written as a sum of sinusoidal components [AUG 05]. In other words, the signal is stationary if it has a constant instantaneous amplitude and frequency over time. Let us consider a deterministic signal $x(t)$ that can be written as:

$$x(t) = \sum_k A_k \cos(2\pi f_k t + \varphi_k) \qquad [1.1]$$

where A_k, f_k and φ_k are real constant[1] coefficients that correspond to the amplitude, frequency and phase of $x(t)$, respectively.

EXAMPLE 1.1.– Consider an example of a multicomponent sinusoidal signal:

$$x(t) = \sin(2\pi f_1 t) + 0.7\sin(2\pi f_2 t)$$

where $f_1 = 10$ Hz and $f_2 = 20$ Hz.

1 If one of these coefficients is random, then the signal becomes random.

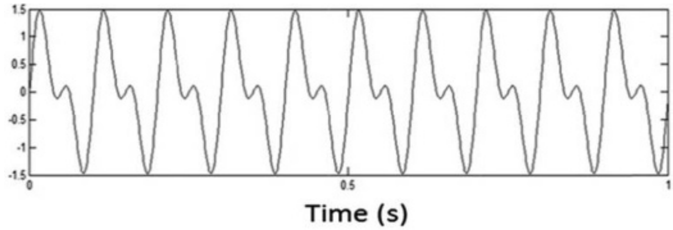

Figure 1.1. *Example of deterministic signal: sum of two sinusoidal signals*

It is clear that it is possible to know the future values of the signal from the past values with complete certainty since its mathematical equation is well known.

1.2.1.2. *Random (stochastic) signal*

A stochastic signal $x(t)$ is said to be stationary if its expectation is independent of time and its autocorrelation function $E\left[x(t_1)x^*(t_2)\right]$ depends only on the time difference $t_2 - t_1$:

$$\forall t, T : E\left[x(t)\right] = E\left[x(t+T)\right] = m_x \qquad [1.2]$$

where m_x is a constant,

and

$$\forall t_1, t_2, T : E\left[x(t_1)x^*(t_2)\right] = E\left[x(t_1+T)x^*(t_2+T)\right] \qquad [1.3]$$

EXAMPLE 1.2.– An example of a stationary random signal is white Gaussian noise (Figure 1.2).

In this case, we cannot describe the signal using an analytical equation. However, the signal can be characterized by a probability density function (pdf), which is a normal (Gaussian) distribution in this example (see Figure 1.3).

On the other hand, the signal is said to be stationary because its statistical properties are unchanged during the time of observation.

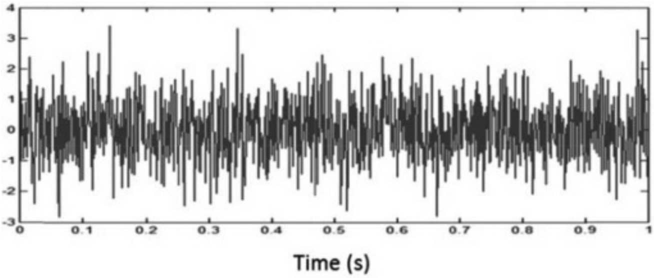

Figure 1.2. *Example of stationary random signal: white Gaussian noise*

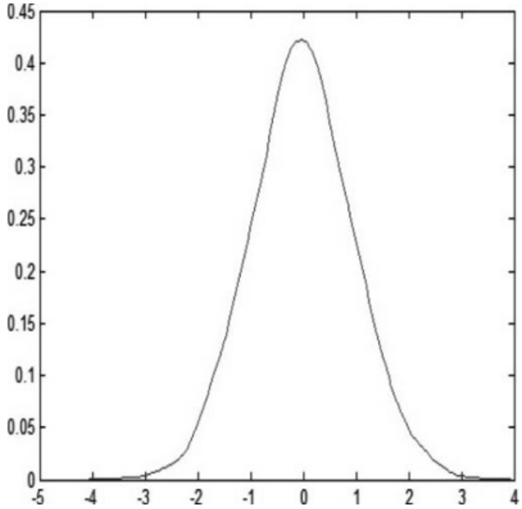

Figure 1.3. *The pdf estimated from the signal in Figure 1.2*

1.2.2. *Non-stationarity*

Non-stationarity as a "non-property" is validated if the assumptions of stationarity are no longer valid [AUG 05]. In other words, a signal is considered to be non-stationary if its frequency and statistical properties vary during the time of observation. A linear chirp and a multitone sine wave (Example 1.3) are familiar synthetic examples of non-stationary signals. Otherwise, most real signals such as human speech and biomedical signals are non-stationary.

EXAMPLE 1.3.– Let us consider a multicomponent sinusoidal signal composed of four components. Let the signal be composed of one component with frequency

f_1 = 10 Hz in the first interval, two components with frequency f_1 and f_2 = 50 Hz in the second interval, three components with frequency f_1, f_2 and f_3 = 100 Hz in the third interval and finally one component with frequency f_4 = 20 Hz in the last interval. The signal $x(t)$ described above can mathematically be given as:

$$\begin{cases} x(t) = \cos(2\pi f_1 t) \quad 0 \le t \le 200 \text{ ms} \\ x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) \quad 200 \le t \le 400 \text{ ms} \\ x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t) \quad 400 \le t \le 600 \text{ ms} \\ x(t) = \cos(2\pi f_4 t) \quad 600 \le t \le 800 \text{ ms} \end{cases}$$

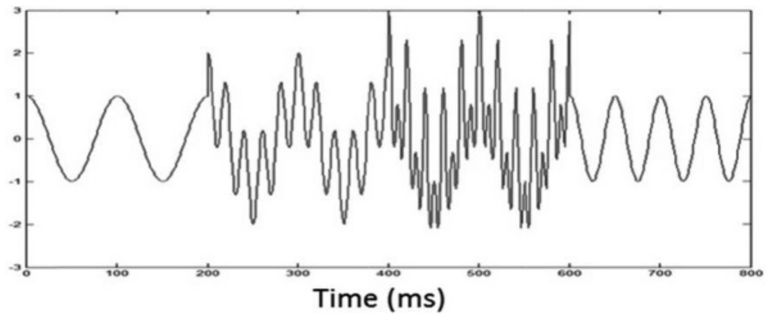

Figure 1.4. *Non-stationary sinusoidal signal composed of frequencies 10, 20, 50 and 100 Hz*

Unlike Example 1.1, such a signal (Figure 1.4) can be considered as non-stationary since its frequency varies over time.

1.3. Temporal representations

Signals can be represented in many different ways. The temporal representations of signals are the most natural representation that gives information about the instance durations and the energy of the different components. The duration of the first and the second heart sounds, for example, (see Example 1.4) can be an accurate feature to distinguish between some pathological cases and normal cases. Also the energy of murmurs can be an indicator of the severity of the pathology. Another example is the duration and energy of disturbances in electrical signals, which gives an idea of the nature of disturbance and the quality of the electrical network. Many real signals are produced by a time-varying process: heart sounds, electrical signals, speech signals, electromagnetic fields, etc.

The total energy of a signal defined by how much energy the signal has or how much energy it takes to be produced [COH 95] is obtained by integrating the instantaneous power $|x(t)|^2$ as follows:

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt \qquad [1.4]$$

The two other features that can be calculated based on time domain are the first- and second-order moments. Respectively, they reveal the average time instant where the energy of the signal is localized and the dispersion of the signal around which this time is constituted [AUG 08]. If we consider that $|x(t)|^2$ is a density in time, the first moment or the average time of the signal $x(t)$ can be given as:

$$t_x = \frac{1}{E_x} \int_{-\infty}^{+\infty} t |x(t)|^2 dt \qquad [1.5]$$

and the second-order moment (time spreading) can be given as:

$$\begin{aligned} (\Delta t_x)^2 &= \frac{1}{E_x} \int_{-\infty}^{+\infty} (t - t_x)^2 |x(t)|^2 dt \\ &= \frac{1}{E_x} \int_{-\infty}^{+\infty} t^2 |x(t)|^2 dt - t_x^2 \end{aligned} \qquad [1.6]$$

where the standard deviation is:

$$\Delta t_x = \sqrt{\frac{1}{E_x} \int_{-\infty}^{+\infty} (t - t_x)^2 |x(t)|^2 dt} \qquad [1.7]$$

1.4. Frequency representations of signals

Another domain for the representation of signals is the frequency domain. It gives an idea of the repetition of an event in the signal that is not accessible in the time domain. The concept of frequency is based on the sinusoidal waves. The essential mathematical analysis tool of the frequency domain is the Fourier transform.

1.4.1. *Fourier transform*

The objective of this transform is to change the basis of the signal into sinusoidal basis vectors. The Fourier transform $X(f)$ of signal $x(t)$ is given as:

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \quad [1.8]$$

The inverse Fourier transform is given as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df \quad [1.9]$$

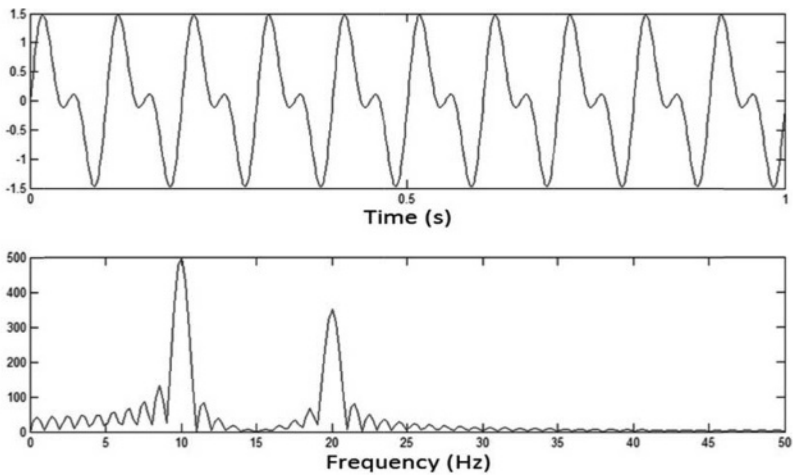

Figure 1.5. *Multicomponent signal with a) temporal representation and b) frequency representation*

The Fourier transform is a complex transform where its amplitude is called the magnitude spectrum, its phase is known as the phase spectrum and the square of the magnitude is the energy spectral density, which describes how the energy of the signal is distributed over frequencies. Thus, the total signal energy is obtained by integrating the energy spectral density $\left|X\left(f\right)\right|^2$ over frequency:

$$E_x = \int_{-\infty}^{+\infty} \left|X\left(f\right)\right|^2 df \qquad [1.10]$$

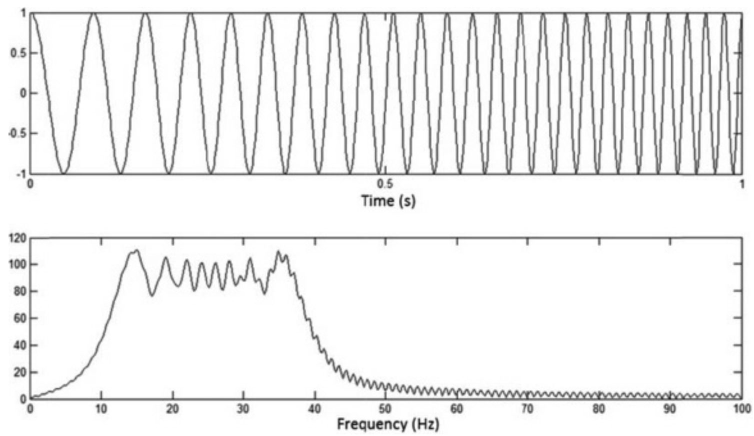

Figure 1.6. *Chirp signal in a) temporal domain and b) frequency domain*

Some of the mathematical properties of the Fourier transform are described as follows:

– *Linearity*

The Fourier transform of a linear combination of signals is equal to the linear combination of their Fourier transforms:

$$TF\left\{\alpha x(t)+\beta y(t)\right\} = \alpha TF\left\{x(t)\right\} + \beta TF\left\{y(t)\right\} \qquad [1.11]$$

$$\int_{-\infty}^{+\infty} \left(\alpha x(t)+\beta y(t)\right) e^{-j2\pi ft} dt = \alpha \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt + \beta \int_{-\infty}^{+\infty} y(t) e^{-j2\pi ft} dt$$

$$= \alpha X(f) + \beta Y(f)$$

– *Time shift*

Shifting the signal $x(t)$ by t_0 in the time domain results in multiplying the Fourier transform with a phase factor:

$$\int_{-\infty}^{+\infty} x(t-t_0)e^{-j2\pi ft}dt = X(f)e^{-j2\pi ft_0} \qquad [1.12]$$

– *Frequency shift*

Modulating the signal with a complex exponential function shifts the Fourier transform $X(f)$ along the frequency axis:

$$\int_{-\infty}^{+\infty} \left(x(t)e^{j2\pi f_0 t}\right)e^{-j2\pi ft}dt = \int_{-\infty}^{+\infty} x(t)e^{j2\pi(f_0-f)t}dt = X(f-f_0) \qquad [1.13]$$

– *Convolution*

The Fourier transform of convolution of two functions $x(t)$ and $y(t)$ is equal to the product of the Fourier transforms of the individual signals:

$$TF\left\{x(t)*y(t)\right\} = X(f)Y(f) \qquad [1.14]$$

On the other hand, the Fourier transform of the product of two signals equals the convolution of their Fourier transforms:

$$TF\left\{x(t).y(t)\right\} = X(f)*Y(f) \qquad [1.15]$$

– *Derivation*

The Fourier transform of the derivative of function $x(t)$ is equal to the product of the Fourier transform $X(f)$ by $2j\pi f$:

$$TF\left\{x'(t)\right\} = 2j\pi fX(f) \qquad [1.16]$$

– *Parseval's theorem*

The total energy calculated from the energy spectral density should be equal to the total energy calculated directly from the time domain signal (instantaneous power energy):

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df \qquad [1.17]$$

PROOF.–

$$\int_{-\infty}^{+\infty} |X(f)|^2 df = \int_{-\infty}^{+\infty} X(f) X(f)^* df$$

$$= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \right\} . \left\{ \int_{-\infty}^{+\infty} x^*(u) e^{j2\pi fu} du \right\} df = \iiint_{\infty} x(t) x^*(u) e^{j2\pi f(u-t)} dudfdt$$

$$= \iint_{\infty} x(t) x^*(u) \delta(t-u) dudt$$

$$= \int_{-\infty}^{+\infty} x(t) x^*(t) dt$$

where:

$$\int_{-\infty}^{+\infty} e^{2\pi f(u-t)} df = \delta(t-u)$$

1.4.2. *Mean frequency, bandwidth and frequency average*

As for the time domain, a signal can be characterized by its average frequency and the frequency band that it occupies. The average frequency can be given as:

$$f_x = \frac{1}{E_x} \int_{-\infty}^{+\infty} f |X(f)|^2 df \qquad [1.18]$$

and the frequency band or the frequency spreading as:

$$\begin{aligned}(\Delta f_x)^2 &= \frac{1}{E_x}\int_{-\infty}^{+\infty}(f-f_x)^2\left|X(f)\right|^2 df \\ &= \frac{1}{E_x}\int_{-\infty}^{+\infty} f^2\left|X(f)\right|^2 df - f_x^2\end{aligned}$$

Then: [1.19]

$$\Delta f_x = \sqrt{\frac{1}{E_x}\int_{-\infty}^{+\infty}(f-f_x)^2\left|X(f)\right|^2 df}$$

To calculate the average of frequency of signal $x(t)$ without calculating the Fourier transform $X(f)$, we can use the equality [COH 95]:

$$\begin{aligned}f_x &= \frac{1}{E_x}\int_{-\infty}^{+\infty} f\left|X(f)\right|^2 df \\ &= \frac{1}{E_x}\int_{-\infty}^{+\infty} x^*(t)\frac{1}{j}x'(t)\,dt\end{aligned} \qquad [1.20]$$

PROOF.–

$$\begin{aligned}f_x &= \frac{1}{E_x}\iiint_{\infty} f\, x^*(t)\,x(u)\,e^{j2\pi(t-u)f}\,dfdudt \\ &= \frac{1}{E_x 2\pi j}\iiint_{\infty} x^*(t)\,x(u)\frac{\partial}{\partial t}e^{j2\pi(t-u)f}\,dfdudt \\ &= \frac{1}{E_x j}\iint_{\infty} x^*(t)\frac{\partial}{\partial t}\delta(t-u)\,x(u)\,dudt\end{aligned}$$

$$\begin{aligned}&= \frac{1}{E_x}\int_{-\infty}^{+\infty} x^*(t)\frac{1}{j}\frac{\partial}{\partial t}x(t)\,dt \\ &= \frac{1}{E_x}\int_{-\infty}^{+\infty} x^*(t)\frac{1}{j}x'(t)\,dt\end{aligned}$$

EXAMPLE 1.4.– For the Gaussian signal $x(t)$:

$$x(t) = e^{-\alpha^2 t^2}$$

The average localization in time and frequency is given as:

$$\Delta t_x = \frac{1}{\alpha}$$

and

$$\Delta f_x = \frac{\alpha}{2\pi}$$

In addition the inequality related to the uncertainty principle can be calculated as (section 1.5, also see Figure 1.7):

$$\Delta t_x . \Delta f_x = \frac{1}{4\pi}$$

The inequality becomes equality in the case of Gaussian signals. It can be shown that the Gaussian signal is the only signal for which the equality holds [GAB 46].

1.5. Uncertainty principle

The uncertainty principle in signal processing shows that a narrow waveform yields a wide spectrum and a wide waveform yields a narrow spectrum and both the time waveform and frequency spectrum cannot be made arbitrarily small simultaneously [GRÖ 01]. In other words, the more a signal is localized in time, the less it is in frequency and vice versa (see Figure 1.7). When talking about the uncertainty principle in signal processing, several elements have to be taken into consideration: the signal concerned $x(t)$ and its spectrum $X(f)$, the density in time $|x(t)|^2$, the density in frequency $|X(f)|^2$ and the standard deviations of time and frequency, Δt_x and Δf_x, respectively. The uncertainty principle is given by the inequality [1.21]:

$$\Delta t_x . \Delta f_x \geq \frac{1}{4\pi} \qquad [1.21]$$

PROOF.– If we take a signal with a zero mean time and a zero mean frequency (so $t_x = 0$ and $f_x = 0$), this does not affect the generality because the standard deviation of the dispersion around these means is independent of their values. So in this case, by using equation [1.5]:

$$\Delta t_x = \frac{1}{\sqrt{E_x}} \int_{-\infty}^{+\infty} \left| t\, x(t) \right| dt$$

Then:

$$\left(\Delta t_x\right)^2 = \frac{1}{E_x} \int_{-\infty}^{+\infty} \left| t\, x(t) \right|^2 dt$$

and (by using equation [1.19]):

$$\Delta f_x = \frac{1}{\sqrt{E_x}} \int_{-\infty}^{+\infty} \left| f\, X(f) \right| df$$

Then:

$$\left(\Delta f_x\right)^2 = \frac{1}{E_x} \int_{-\infty}^{+\infty} \left| f\, X(f) \right|^2 df$$

Moreover, by using integration by parts, we obtain:

$$\int_{-\infty}^{+\infty} x^2(t) = -2 \int_{-\infty}^{+\infty} t.x(t)\, x'(t) dt$$

$$\left| \int_{-\infty}^{+\infty} x(t)^2\, dt \right| = 2 \int_{-\infty}^{+\infty} \left| t x(t)\, x'(t) \right| dt$$

The Cauchy–Schwarz inequality (which is a special case of Hölder’s inequality) for two functions $f(x)$ and $g(x)$ is given as follows:

$$\int_{-\infty}^{+\infty} \left| f(x)\, g^*(x) \right| dx \le \left(\int_{-\infty}^{+\infty} \left| f(x) \right|^2 dx \right)^{\frac{1}{2}} . \left(\int_{-\infty}^{+\infty} \left| g(x) \right|^2 dx \right)^{\frac{1}{2}}$$

Let $f(t) = tx(t)$ and $g(t) = x'(t)$, we obtain:

$$\int_{-\infty}^{+\infty} \left|x(t)\right|^2 dt \le 2\left(\int_{-\infty}^{+\infty} \left|tx(t)\right|^2 dt\right)^{\frac{1}{2}} . \left(\int_{-\infty}^{+\infty} \left|x^{'}(t)\right|^2 dt\right)^{\frac{1}{2}}$$

$$E_x \le 2\sqrt{E_x}\Delta t_x . \left(\int_{-\infty}^{+\infty} \left|x^{'}(t)\right|^2 dt\right)^{\frac{1}{2}}$$

To evaluate the term $\left(\int_{-\infty}^{+\infty} \left|x^{'}(t)\right|^2 dt\right)^{\frac{1}{2}}$:

$$\int_{-\infty}^{+\infty} \left|x^{'}(t)\right|^2 dt = \iint_{\infty} x^{'}(t)\left(x^{'}(t)\right)^{*} dt$$

By using equation [1.16]:

$$\int_{-\infty}^{+\infty} \left|x^{'}(t)\right|^2 dt = (2j\pi)^2 \iiint_{\infty} fX(f)\omega X^{*}(\omega) e^{j2\pi(f-\omega)t} d\omega df dt$$

$$= (2j\pi)^2 \iint_{\infty} fX(f)\omega X^{*}(\omega)\delta(f-\omega) d\omega df$$

$$= \left|2j\pi\right|^2 \int_{-\infty}^{+\infty} \left|fX(f)\right|^2 df$$

$$= \left|2j\pi\right|^2 E_x \left(\Delta f_x\right)^2$$

and

$$\left(\int_{-\infty}^{+\infty} \left|x^{'}(t)\right|^2 dt\right)^{\frac{1}{2}} = 2\pi\sqrt{E_x}\Delta f_x$$

The inequality becomes:

$$E_x \le 4\pi \Delta t_x E_x \Delta f_x$$

So finally: $\Delta t_x . \Delta f_x \ge \dfrac{1}{4\pi}$

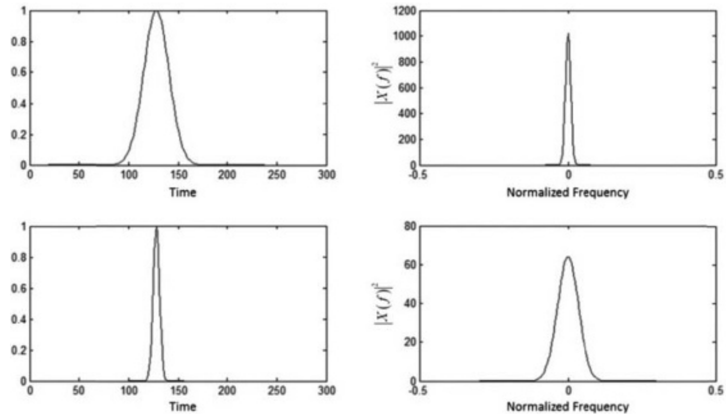

Figure 1.7. *Two different Gaussian signals in the time domain (left) and the frequency domain (right): wide waveforms yield a narrow spectrum and narrow waveforms yield a wide spectrum*

1.6. Limitation of time analysis and frequency analysis: the need for time–frequency representation

Figure 1.8 shows an example of two different chirp signals with the same frequency density content. The Fourier transform integrates the frequency component over time, so the final result will not contain any information about the time localization of the signal. This is a serious limitation in the case of non-stationary signals where following the frequency changes over time become crucial.

A first intuitive solution is to track the frequency instantaneously. This will be presented by the IF concept in the following section.

1.6.1. *Instantaneous frequency*

IF is one of the basic signal descriptors, which provides information about the time-varying spectral changes in non-stationary signals. It can be viewed as the first and most simple solution to deal with the limitations of time or frequency representations. To calculate the IF, we have to define the analytic signal, which is a more advanced illustration of the real signal. The analytic signal gives an idea about amplitude and phase. It can be given as:

$$x_a(t) = A_x e^{j\varphi(t)} \qquad [1.22]$$

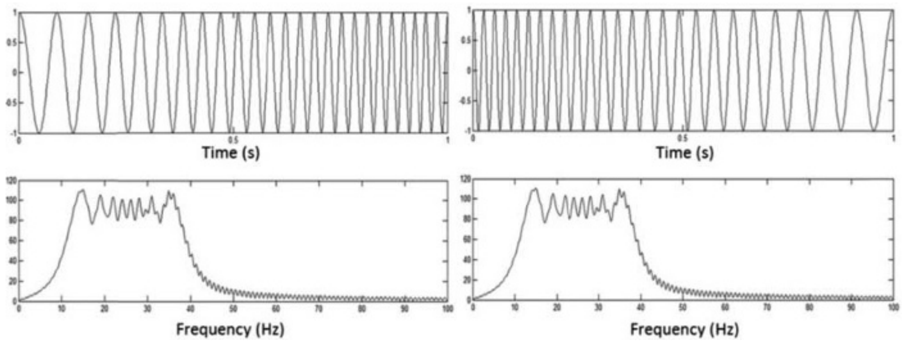

Figure 1.8. *Showing two different signals with the same spectral content*

The complex value associated with the real value can be calculated using the Hilbert transform:

$$x_a(t) = x(t) + jH\{x(t)\} \qquad [1.23]$$

where $H\{x(t)\}$ is the Hilbert transform of signal $x(t)$, which can be calculated as follows:

$$H[x(t)] = x(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{\infty}^{+\infty} \frac{x(\tau)}{\tau - t} d\tau \qquad [1.24]$$

The Hilbert transform of $x(t)$ can be viewed as a convolution of $x(t)$ with the signal $\frac{1}{\pi t}$. It is the response to $x(t)$ of a linear time-invariant filter having impulse response $\frac{1}{\pi t}$.

Ville [VIL 48] defined the IF $f(t)$ of real signal $x(t)$ as the derivation of phase of its analytic signal $x_a(t)$:

$$f(t) = \frac{1}{2\pi} \frac{\partial}{\partial t} \varphi(t) \qquad [1.25]$$

and the instantaneous amplitude as:

$$A_x(t) = \sqrt{x(t)^2 + H\left[x(t)\right]^2} \qquad [1.26]$$

From a spectral point of view, the relationship between the analytic signal and the real signal is given as follows:

$$\begin{cases} X_a\left(f\right) = 0 \ if \ f < 0 \\ X_a\left(f\right) = X\left(f\right) \ if \ f = 0 \\ X_a\left(f\right) = 2X\left(f\right) \ if \ f > 0 \end{cases}$$

where $X_a\left(f\right)$ is the spectrum of the analytic signal and $X\left(f\right)$ is the spectrum of real signal.

The negative frequencies are suppressed in the analytic signal. This does not alter the information content of the signal since we have:

$$X\left(-f\right) = X^*(f) \qquad [1.27]$$

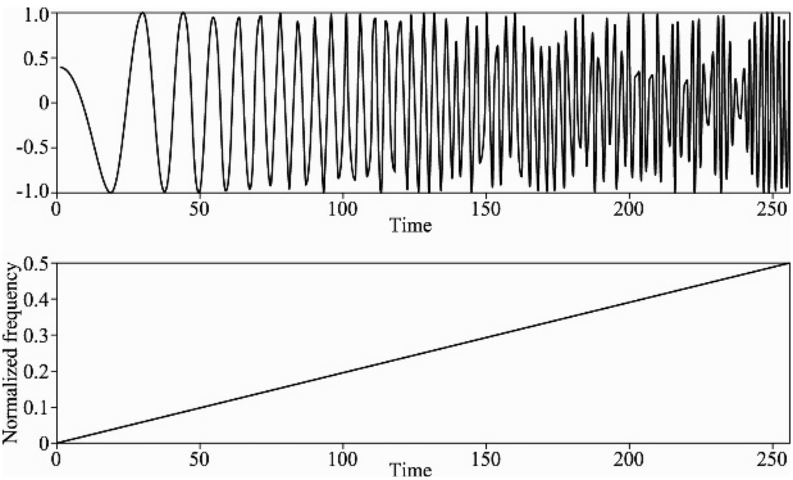

Figure 1.9. *a) Linear chirp in time domain and b) the estimation of its instantaneous frequency*

An example of the IF estimation is shown in Figure 1.9. The major problem with the IF estimation occurs with a multicomponent non-stationary signal where the local spectrum is averaged. This is considered as a serious limitation of IF especially if we want to extract relevant features from each component signature (see Figure 1.10). Hence, a higher dimension is needed to represent the signal more accurately. This can be done by a joint time–frequency representation, which will be the main subject of Chapter 2

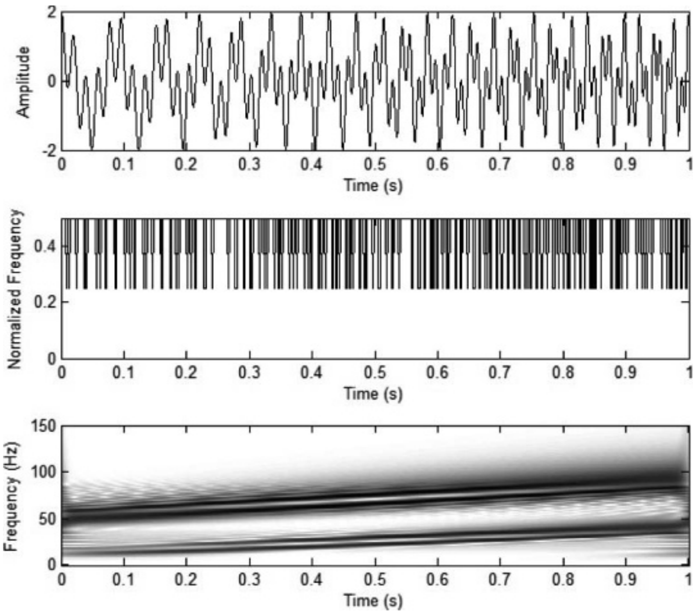

Figure 1.10. *Sum of two linear chirps a), the IF b) and the joint time–frequency (the Stockwell transform) representation c)*

1.7. Conclusion

In this chapter, some concepts of signal theory, such as the stationary and non-stationary processes, have been presented: first, the representation of signals in time or frequency domains, and second, the related mathematical concepts and proofs.

The main objective of this chapter was to show the limitation of time or frequency representation, most notably in the case of multicomponent and non-stationary signals. The need of time–frequency representation is proved by showing the limitations of instantaneous frequency measure. This can be considered as a primary introduction to Chapter 2, which concerns the time–frequency analysis by using the Stockwell transform [STO 96].

1.8. Bibliography

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