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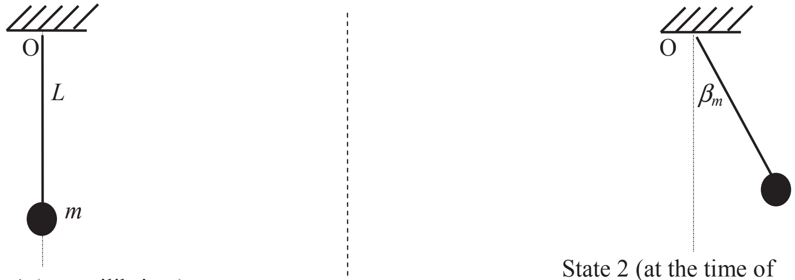
Principle of Conservation of Energy and Rayleigh's Principle

The well-known principle of conservation of energy forms the basis of some common convenient analytical techniques in Mechanics. According to this principle, the total energy of a closed system remains unchanged. This means that in the absence of any losses due to friction etc., the sum of the total potential energy and the kinetic energy of a vibratory system will be a constant. Although in practice there will always be some damping, and hence dissipation of energy, for many mechanical systems such losses may be neglected. Such systems are called conservative systems.

The natural frequencies of conservative systems may be obtained by equating the maximum kinetic energy (T_m) to the maximum total potential energy (V_m) associated with vibration. The meaning of these energy terms is very important. To illustrate the principle of conservation of energy, and the meaning of the energy terms let us study some simple vibratory systems.

1.1. A simple pendulum

Consider the oscillatory motion of the simple pendulum consisting of a bob of mass m and a massless string of length L as shown in Figure 1.1. It would be at rest in a vertical configuration under gravity field. If it is given a small disturbance β_m and then released, it will tend to vibrate about this equilibrium state. The restoring action of the gravity force will initiate a motion toward the equilibrium state but as the bob approaches the lowest point in its motion it has a velocity and therefore carries on swinging up on the other side until the gravity force causes it to come to a halt momentarily. In the absence of any damping forces, this motion would go on forever, but in reality the damping forces will help to put an end to this vibration after some time.



State 1 (at equilibrium)

State 2 (at the time of maximum excursion)

Figure 1.1. *Simple pendulum*

Assuming that energy loss associated with mechanical friction and aerodynamic resistance is negligible, we have two types of energy term to consider. These are the kinetic energy (denoted by T_1, T_2) where the subscripts 1 and 2 refer to states 1 and 2 respectively, and the potential energy (denoted by V_1, V_2). The kinetic energy is proportional to the square of the velocity and the potential energy is dependent on the vertical position of the bob.

The pendulum will have the maximum kinetic energy as the bob passes through the equilibrium state (state 1) at which time it will have the lowest potential energy. At the time of maximum excursion (state 2), the bob will be at its highest point, and therefore the system will have the maximum potential energy, but since it has no velocity its kinetic energy will be minimum. The potential energy can be defined arbitrarily by selecting a datum. In our example, the increase in the potential energy as the system changes from state 1 to state 2 is entirely associated with the vibration, and will be referred to as the maximum potential energy hereafter. As the bob returns to state 1 from state 2, it loses potential energy and gains kinetic energy. The maximum kinetic energy associated with vibration is the kinetic energy at state 1 minus the kinetic energy at state 2. (The latter is not necessarily absolutely zero, as the support point may have a velocity. In rotating systems care must be taken to ensure that the kinetic energy terms are calculated correctly.) Since the total energy is conserved, the maximum kinetic energy associated with vibration must be equal to the maximum total potential energy associated with vibration. The inclusion of the phrase "associated with vibration" is used here since terms such as "maximum" and "total" can otherwise cause confusion.

From the principle of conservation of energy:

$$V_1 + T_1 = V_2 + T_2$$

i.e. $V_2 - V_1 = T_1 - T_2$

The gain in potential energy as the bob moves from state 1 to state 2 is the maximum potential energy associated with vibration and may be denoted by V_m.

$$V_2 - V_1 = V_m$$

Similarly the maximum kinetic energy associated with vibration is:

$$T_1 - T_2 = T_m \qquad [1.1]$$

From the above equations we have $V_m = T_m$

In applying the principle of conservation of energy for vibratory systems, it is sufficient to equate the maximum potential and kinetic energy terms associated with vibration.

To find the circular natural frequency ω of an undamped system, the motion may be assumed to be simple harmonic.

i.e. $\beta = \beta_m \sin(\omega t + \alpha)$, where t is time and α is a phase shift angle.

Then $\dfrac{d\beta}{dt} = \dot{\beta} = \omega\beta_m \cos(\omega t + \alpha)$

The maximum velocity is therefore $= L\omega\beta_m$

This means the maximum velocity is equal to the amplitude of vibration times the frequency. This statement is true for any natural mode, since at natural modes the vibration is simple harmonic.

Hence $T_m = m\dfrac{(L\omega\beta_m)^2}{2}$

The potential energy is due to the change in position of the gravity force mg.

Thus $V_m = mgL(1 - \cos\beta_m)$

Substituting these into equation [1.1] gives:

$$m\frac{(L\omega\beta_m)^2}{2} = mgL(1-\cos\beta_m)$$

For small amplitude vibration, $(1-\cos\beta_m) = \frac{\beta_m^2}{2}$

This gives: $m\frac{(L\omega\beta_m)^2}{2} = mgL\frac{\beta_m^2}{2}$

This actually gives us two possible solutions. One is that $\beta_m = 0$. This implies there will not be any motion and is therefore a trivial solution. The other solution is:

$$\omega^2 = g/L$$

in which case $\beta_m \neq 0$ and vibration is possible. That is to say, in the absence of any external force, the system can vibrate freely at a frequency of $\omega = \sqrt{g/L}$ rad/s. This is therefore the circular natural frequency of the pendulum. The natural frequency in Hz (cycles/s) is $f = \omega/2\pi = \sqrt{g/L}/2\pi$. From this point onwards, for simplicity, we will refer to circular natural frequencies as natural frequencies.

The simple pendulum is a "single degree of freedom" system. This means it can only vibrate in one specific mode, in this case, the string and the bob rotating about the equilibrium state. In cases where the mode is defined, the above method yields the exact value of the natural frequency. We will soon see why it is not always possible or convenient to get the exact frequency.

1.2. A spring-mass system

Consider the motion of a simple spring-mass system shown in Figure 1.2. A rigid body of mass *m* is connected to a linear elastic spring of stiffness *k*. Assume that the system is free to vibrate only axially (in the direction of the spring). If the mass is displaced from the equilibrium state by distance $\hat{u}$ which induces a force in the spring and then released, it would tend to return to its equilibrium state. However as it approaches the equilibrium state it has a velocity and this velocity causes the mass to move away from the equilibrium state now on to the opposite side. Then as the spring force develops, the motion comes to an end momentarily and the mass then returns to the equilibrium state and the cycle repeats as in the case of the pendulum.

The maximum vibratory potential energy in the spring is given by $V_m = \dfrac{k\hat{u}^2}{2}$ and the maximum kinetic energy is $T_m = \dfrac{m\omega^2\hat{u}^2}{2}$.

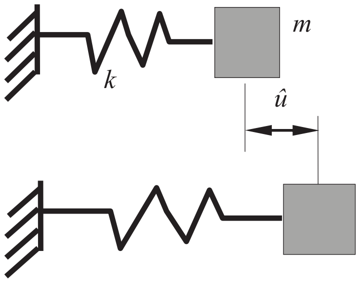


Figure 1.2. *A spring-mass system*

Using equation [1.1], $\dfrac{k\hat{u}^2}{2} = \dfrac{m\omega^2\hat{u}^2}{2}$

For a non-trivial solution, we have the natural frequency given by $\omega = \sqrt{k/m}$.

For multidegree of freedom systems and continuous systems, calculation of the energy terms requires an assumption about their modes. This method results in exact values of natural frequencies only if the exact modes are used in the calculations. If the exact mode is not known, it has been shown that the use of any other mode shape that does not violate any geometric constraints of the system results in a frequency that cannot be lower than the exact fundamental natural frequency. This is one of the many interesting proofs that Lord Rayleigh gives in his famous book *The Theory of Sound* [RAY 45a]. This is best illustrated through some examples. We will start with a two degree of freedom system, as its exact solution is easily understood.

1.3. A two degree of freedom system

Consider the vibration of the spring-mass system shown in Figure 1.3. The masses are assumed to be free to move only in the axial direction. Since the two masses can be moved independently of each other there are two degrees of freedom. The relevant values for the stiffness and mass are shown in the figure. Let the dynamic displacement of the masses be u_1, u_2 and their amplitudes be $\hat{u}_1$, $\hat{u}_2$. The various energy terms may be written in terms of $\hat{u}_1$ and $\hat{u}_2$.

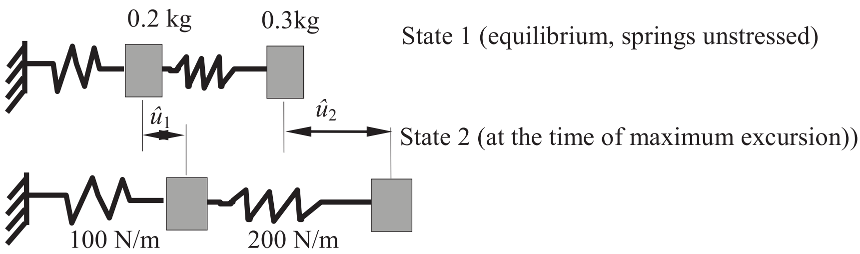


Figure 1.3. *A two degree of freedom system*

If $u_1 = \hat{u}_1 \sin(\omega t+\alpha)$ and $u_2 = \hat{u}_2 \sin(\omega t+\alpha)$, then the velocities are given by:

$$\dot{u}_1 = \omega \hat{u}_1 \cos(\omega t+\alpha) \text{ and } \dot{u}_2 = \omega \hat{u}_2 \cos(\omega t+\alpha).$$

The maximum potential energy due to vibration is

$$V_m = (1/2)100\,(\hat{u}_1)^2 + (1/2)200\,(\hat{u}_2 - \hat{u}_1)^2$$

The maximum kinetic energy is:

$$T_m = (1/2)(0.2)\,\omega^2\,\hat{u}_1^2 + (1/2)(0.3)\,\omega^2\,\hat{u}_2^2$$

Using the principle of conservation of energy, $V_m = T_m$.

This gives:

$$(1/2)100\,(\hat{u}_1)^2 + (1/2)200\,(\hat{u}_2 - \hat{u}_1)^2 = (1/2)(0.2)\,\omega^2\,\hat{u}_1^2 + (1/2)(0.3)\,\omega^2\,\hat{u}_2^2$$

$$\text{i.e. } \omega^2 = [(1/2)100\,\hat{u}_1^2 + (1/2)200\,(\hat{u}_2 - \hat{u}_1)^2]/[\,(1/2)(0.2)\,\hat{u}_1^2 + (1/2)(0.3)\,\hat{u}_2^2]$$

This equation, which is the Rayleigh Quotient, can be condensed into the following form:

$$\omega^2 = (100\,\eta^2 + 200\,(1-\eta)^2)/(0.3 + 0.2\,\eta^2), \qquad [1.2]$$

where $\eta = \hat{u}_1/\,\hat{u}_2$ (the mode). This is equivalent to normalizing the amplitude of the displacements of the masses with respect to $\hat{u}_2$. Note that modes give a relationship between the displacements of the degrees of freedom in the system.

Unlike the first example, this does not immediately yield a value for the natural frequency. However, it may be solved by assuming a relationship between $\hat{u}_1$

and $\hat{u}_2$ (i.e. assuming a mode). The exact natural frequencies and modes found by solving the two equations of motion (obtainable by applying Newton's second law) are:

$\omega_1 = 12.97$ rad/sec, $\hat{u}_1/\hat{u}_2 = 0.75$ and $\omega_2 = 44.72$ rad/sec, $\hat{u}_1/\hat{u}_2 = -2$

Substituting the fundamental mode into the frequency equation gives the exact result for the fundamental natural frequency. *Any other value for η results in a higher value for the frequency* (see Figure 1.4). This is an important observation and will be discussed later.

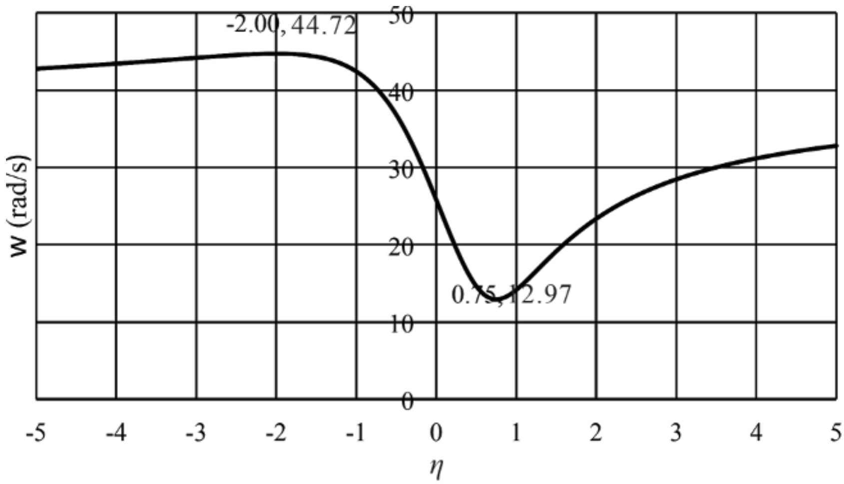


Figure 1.4. *Variation of calculated frequency with trial mode*

It is possible to show that the frequency calculated will not be lower than the fundamental natural frequency. We will do this by expressing the assumed mode in terms of the natural modes.

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix} = G_1 \begin{Bmatrix} 0.75 \\ 1.00 \end{Bmatrix} + G_2 \begin{Bmatrix} -2.0 \\ 1.00 \end{Bmatrix}$$

Then the total potential energy is:

$$V_m = (1/2)(100)(0.75\,G_1 - 2\,G_2)^2 + (1/2)(200)(\,(0.75\,G_1 - 2\,G_2) - (\,G_1 + G_2))^2$$

$$= 34.375\,G_1^{2+}\,1100\,G_2^2$$

$$T_m = (\omega^2/2)(0.2)(0.75\,G_1 - 2.0\,G_2)^2 + (\omega^2/2)(0.3)(G_1 + G_2)^2$$

$$= \omega^2(0.20625\,G_1^2 + 0.55G_2^2)$$

It may be noted that the energy terms do not contain the product G_1G_2 and this is due to the orthogonality of the natural modes:

$V_m = T_m$ gives

$$\omega^2 = \frac{34.375G_1^2 + 1100G_2^2}{0.20625\,G_1^2 + 0.55G_2^2} \qquad [1.3]$$

This may be written as

$$\omega^2 = (K_1\,G_1^2 + K_2\,G_2^2)/(M_1\,G_1^2 + M_2\,G_2^2) \qquad [1.4]$$

where K_1, K_2 and M_1, M_2 are generalized stiffness and mass terms associated with the first and second modes.

Taking only the terms associated with G_1 will result in ω_1 and taking only the terms associated with G_2 will yield ω_2. Any combination of the two modes will result in a frequency which will be an upperbound to ω_1 as shown below.

If the assumed mode is the exact first mode, then equation [1.4] will give the first natural frequency;

i.e. $\omega_1^2 = K_1\,G_1^2/\,M_1G_1^2 = K_1/\,M_1$

Therefore,

$$K_1 = M_1\omega_1^2 \qquad [1.5]$$

Similarly it can be shown that

$$K_2 = M_2\omega_2^2 \qquad [1.6]$$

Substituting equations [1.5] and [1.6] into equation [1.4] gives:

$$\omega^2 = (M_1\,\omega_1^2\,G_1^2 + M_2\,\omega_2^2\,G_2^2)/(\,M_1\,G_1^2 + M_2\,G_2^2)$$

Dividing the numerator and denominator of the RHS of this equation by $\omega_1^2\,G_1^2$ gives:

$$(\omega/\omega_1)^2 = (M_1 + M_2(\omega_2/\omega_1)^2\,(G_2/G_1)^2)/(\,M_1 + M_2(G_2/\,G_1)^2)$$

Since $\omega_2 \geq \omega_1$, on the R.H.S., the numerator ≥ denominator. Hence $\omega \geq \omega_1$.

This means that the calculated frequency will be an upperbound to the fundamental natural frequency. In the next chapter we will extend this proof for a system with any number of degrees of freedom, and discuss Rayleigh's principle. We will then see that this statement of boundedness is true for all conservative systems, as long as the assumed displacement configuration satisfies certain conditions as explained in the next chapter.