Part I

Deterministic Signals

Chapter 1

Signal Fundamentals

Although this work is mainly focused on discrete-time signals, a discussion of continuous-time signals cannot be avoided, for at least two reasons:

– the first reason is that the quantities we will be using – taken from numeric sequences – are taken from continuous-time signal sampling. What is meant is that the numeric value of a signal, such as speech, or an electroencephalogram reading, etc., is measured at regular intervals;

– the second reason is that for some developments, we will have to use mathematical tools such as *Fourier series* or *Fourier transforms* of continuous-time signals.

The objective is not an extensive display of the knowledge needed in the field of deterministic signal processing. Many other books have already done that quite well. We will merely give the main definitions and properties useful to further developments. We will also take the opportunity to mention *systems* in a somewhat restricted meaning, this word referring to what are called *filters*.

1.1 The concept of signal

A *deterministic* continuous-time signal is defined as a function of the real *time* variable t:

$$\text{Signal} = \text{function } x(t),\ t \in \mathbb{R}$$

The space made up of these functions is completed by the *Dirac pulse* distribution, or $\delta(t)$ function. Actually a *distribution* (a *linear functional*), this object can be handled just like a function without any particular problems in the exercises we will be dealing with.

The following functions spaces are considered:

– $L_1(\mathbb{R})$ is the vector space of summable functions such that $\int_{\mathbb{R}} |x(t)|dt < +\infty$;

– $L_1(a,b)$ is the vector space (vector sub-space of $L_1(\mathbb{R})$) of functions such that $\int_a^b |x(t)|dt < +\infty$;

– $L_2(\mathbb{R})$ is the vector space of *finite energy functions* such that $\int_{\mathbb{R}} |x(t)|^2 dt < +\infty$;

– $L_2(a,b)$ is the vector space (vector sub-space of $L_2(\mathbb{R})$) of functions such that $\int_a^b |x(t)|^2 dt < +\infty$;

– the set of "finite power" functions characterized by:

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt < +\infty$$

$L_2(0,T)$ has the structure of what is called a *Hilbert space* structure with respect to the scalar product $\int x(t)y^*(t)dt$, a property that is often used for decomposing functions, for example in the case of *Fourier series.*

In the course of our work, we will need to deal with a particular type of signal, in sets that have already been defined, taken from $\mathbb{R}^+$.

Definition 1.1 (Causal and anticausal signals) *Signals $x(t)$ such that $x(t) = 0$ for $t < 0$ are said to be causal. Signals $x(t)$ such that $x(t) = 0$ for $t \geq 0$ are said to be anticausal.*

1.1.1 A few signals

We will often be using particular functions characteristic of typical behaviors. Here are some important examples:

– the *unit step* function or *Heaviside function* is defined by:

$$u(t) = \mathbf{1}(t \in (0, +\infty[) \tag{1.1}$$

Its value at the origin, $t = 0$, is arbitrary. Most of the time, it is chosen equal to 1/2. The unit step can be used to show causality: $x(t)$ is causal if $x(t) = x(t)u(t)$;

– the *sign* function is defined using the unit step by $\text{sign}(t) = 2u(t) - 1$;

– the *gate* or *rectangle* function is defined by:

$$\text{rect}_T(t) = \mathbf{1}(t \in (-T/2, T/2)) = u(t + T/2) - u(t - T/2) \tag{1.2}$$

It will be used to express the fact that a signal is observed over a finite time horizon, with a duration of T. The phrases *rectangular windowing* and *rectangular truncation* of $x(t)$ are also used: $x_T(t) = x(t)\text{rect}_T(t)$;

– the *pulse*, or *Dirac function*, has the following properties which serve the purpose of calculation rules:

1. $\int_{\mathbb{R}} \delta(t)dt = 1$ and $\int_{\mathbb{R}} \delta(t)x(t)dt = x(0)$.
2. $x(t) = \int_{\mathbb{R}} x(u)\delta(t-u)du = (x \star \delta)(t)$ ($\star$ is the *convolution operation*).
3. $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$.
4. $(x(u) \star \delta(u-t_0))(t) = (x \star \delta)(t-t_0) = x(t-t_0)$.
5. $\delta(at) = \delta(t)/|a|$ for $a \neq 0$.
6. $\forall t$, $\int_{-\infty}^{t} \delta(u)du = \mathbf{1}(t \in (0,+\infty)) = u(t)$ and therefore $du(t)/dt = \delta(t)$. This result makes it possible to define the derivative of a function with a jump discontinuity at a time t_0. Let $x(t) = x_0(t) + au(t-t_0)$ where $x_0(t)$ is assumed to be differentiable. We have $dx(t)/dt = dx_0(t)/dt + a\delta(t-t_0)$;

– the *sine function* is defined by:

$$x(t) = x_0 \sin(\omega_0 t + \phi) = x_0 \sin(2\pi f_0 t + \phi) \tag{1.3}$$

where x_0 is the peak amplitude of the signal, ω_0 its angular frequency (in radians/s), ϕ its phase at the origin, $f_0 = \omega_0/2\pi$ its frequency (in Hz) and $T = 1/f_0$ its period;

– the *complex exponential function* is defined by:

$$x(t) = x_0 \exp(2j\pi f_0 t + j\phi) \tag{1.4}$$

– the *sine cardinal* is defined by $\text{sinc}(t) = \sin(\pi t)/\pi t$. It is equal to 0 for all integers except $t = 0$ (hence its name). We have $\int_{\mathbb{R}} \text{sinc}(t)dt = 1$, $\int_{\mathbb{R}} \text{sinc}(u)\text{sinc}(u-t)du = \text{sinc}(t)$ and the following orthogonality property, for $n \in \mathbb{N}$:

$$\int_{\mathbb{R}} \text{sinc}(u)\text{sinc}(u-n)du = \begin{cases} 1 & \text{with} \quad n = 0 \\ 0 & \text{with} \quad n \neq 0 \end{cases}$$

1.1.2 Spectral representation of signals

Fourier series

A periodic signal with a period of $T = 1/f_0$ may be decomposed as a sum of complex exponentials, a sum we will refer to as *Fourier series*[1]:

$$\begin{cases} x(t) \overset{FS}{=} \displaystyle\sum_{k=-\infty}^{+\infty} X_k e^{2j\pi k f_0 t} \\ X_k = \dfrac{1}{T} \displaystyle\int_0^T x(t) e^{-2j\pi k f_0 t} dt \end{cases} \tag{1.5}$$

[1] We will only be using the complex exponential decomposition, since it easily leads to the one with the sine and cosine functions.

$f_0 = 1/T$ is called *fundamental* frequency, and its multiples are called *harmonic* frequencies. A few comments should be made:

- a signal with a bounded support on (t_1, t_2) is also expandable in a Fourier series, but the series converges to the periodized function outside of the (t_1, t_2) interval;
- expression 1.5 indicates that X_k is the k-th component of $x(t)$ in the orthonormal basis of the complex exponentials $\{T^{-1/2}e^{2j\pi k f_0 t}\}_{k\in\mathbb{Z}}$ in the *Hilbert space* $L_2(0, T)$;
- $x_M(t) = \sum_{k=-M}^{M} X_k e^{2j\pi kt/T}$ is the best length M approximation of $x(t)$ in the sense of the least squares. Sometimes $\stackrel{L_2}{=}$ can be used instead of $\stackrel{FS}{=}$ to indicate that the Fourier series convergence is ensured and is not uniform, which results in:

$$\int_{(T)} \left| x(t) - \sum_{k=-M}^{M} X_k e^{2j\pi kt/T} \right|^2 dt \to 0 \text{ when } M \to +\infty \tag{1.6}$$

- when $x(t)$ is *continuous*, $x_M(t)$ converges uniformly to $x(t)$ for any t, when $M \to +\infty$;
- if $x(t)$ shows first order discontinuities, $x_M(t)$ will converge to the half-sum of the left and right limits of $x(t)$. Finally, $x_M(t)$ can show some non-evanescent oscillations in the neighborhoods of all discontinuities. This phenomenon is referred to as the *Gibbs phenomenon*;
- we have *Parseval's relation*:

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k\in\mathbb{Z}} |X_k|^2 \tag{1.7}$$

  Because the first member of 1.7 is by definition the signal's power, the sequence $\{|X_k|^2\}$ can be interpreted as the power distribution along the frequency axis. It is also called *power spectral density*, or *psd*.

Fourier transform

The spectral contents $X(f)$ of the function $x(t) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ can be represented by an integral that uses complex exponentials, an integral we will call *Fourier transform*:

$$X(f) = \int_{\mathbb{R}} x(t) e^{-2j\pi ft} dt \quad \longleftrightarrow \quad x(t) = \int_{\mathbb{R}} X(f) e^{2j\pi ft} df \tag{1.8}$$

$|X(f)|$ is called *spectrum* of $x(t)$. The Fourier transform's main properties are summarized in Appendix A1.

The convolution property 11.1 leads to *Parseval's formula*:

$$\int_{\mathbb{R}} |x(t)|^2 dt = \int_{\mathbb{R}} |X(f)|^2 df \tag{1.9}$$

Because the left member of 1.9 is, by definition, the signal's energy, $|X(f)|^2$ can be interpreted as the energy distribution along the frequency axis. It is also called *energy spectral density*, or *esd*.

More generally, we have:

$$\int_{\mathbb{R}} x(t)y^*(t)dt = \int_{\mathbb{R}} X(f)Y^*(f)df \tag{1.10}$$

EXAMPLE **1.1 (Analytical signal)**

Let $x(t)$ be a continuous time real signal. The *analytical signal* associated with $x(t)$ is the signal $z(t)$ that has $Z(f) = 2U(f)X(f)$ as its Fourier transform, where $X(f)$ is the Fourier transform of $x(t)$ and $U(f)$ is the function equal to 1 if $f > 0$ and 0 if $f < 0$. $U(0)$ is chosen equal to 1/2.

Using the properties of the continuous-time Fourier transform, show that the real part of $z(t)$ is equal to $x(t)$, and determine its imaginary part called the *Hilbert transform* of $x(t)$.

HINTS: let:

$$p(t) = \text{Re}(z(t)) = (z(t) + z^*(t))/2$$

Using the Fourier transforms, we get:

$$P(f) = (Z(f) + Z^*(-f))/2 = U(f)X(f) + U(-f)X^*(-f)$$

Because $x(t)$ is real, $X(f) = X^*(-f)$, and therefore, $P(f) = X(f)$, which means $p(t) = x(t)$. As a conclusion, $\text{Re}(z(t)) = x(t)$.

Likewise, let:

$$q(t) = \text{Im}(z(t)) = (z(t) - z^*(t))/2j$$

Using the Fourier transforms, we get:

$$\begin{aligned} Q(f) &= (Z(f) - Z^*(-f))/2j = -j(U(f)X(f) - U(-f)X^*(-f)) \\ &= -j(U(f) - U(-f))X(f) \end{aligned}$$

Because $U(f) - U(-f)$ is the $\text{sign}(f)$ function, $Q(f) = -j\text{sign}(f)X(f)$. This equation can be interpreted as filtering (see paragraph 1.2) with the complex gain filter $-j\text{sign}(f)$. Its gain is equal to 1, meaning that the Fourier transforms of the output and input have the same modulus, $|Q(f)| = |X(f)|$. In the literature, the transformation that associates the output signal $y(t)$ of the complex gain filter $-j\text{sign}(f)$ with the real signal $x(t)$ is called the *Hilbert transform*.

As a conclusion, the analytical signal associated with the real signal $x(t)$ is written (Figure 1.1):

$$z(t) = x(t) + j\hat{x}(t)$$

where $\hat{x}(t)$ refers to the Hilbert transform of $x(t)$.

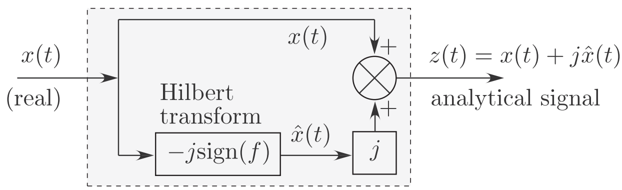

Figure 1.1 – *Analytical signal construction*

1.2 The concept of system

A *system* transforms the signal $x(t)$ and delivers a signal $y(t)$, the result of this alteration. We will refer to this transformation as $y(t) = \mathscr{T}[x(u), t]$, and $x(t)$ and $y(t)$ will be called the input and the output of the system respectively.

Filters

A *filter* with $x(t)$ as the input and $y(t)$ as the output is a system defined by:

$$y(t) = \int_{\mathbb{R}} x(u)h(t-u)du = \int_{\mathbb{R}} x(t-u)h(u)du \tag{1.11}$$

The existence of the integral has to do with how the set $\mathscr{X}$ of considered signals $x(t)$ is chosen. Among the sets that have practical interest, two of them play a fundamental role: the signals that have a Fourier transform and those made up of a linear mix of complex exponentials.

Certain conditions have to be met:

– first, in the case of $\mathscr{X}$ sets that show some practical interest, such a system is *linear*: $\mathscr{T}[a_1x_1(u)+a_2x_2(u), t] = a_1\mathscr{T}[x_1(u), t]+a_2\mathscr{T}[x_2(u), t]$;

– second, it is *time-invariant*: $\mathscr{T}[ax(u), t-t_0] = \mathscr{T}[ax(u-t_0), t]$. Another way of expressing it is to say that the output is independent of the time origin.

EXAMPLE **1.2 (Counterexample)**
The system defined by $y(t) = \int_0^t x(u)du$ is linear but is time-dependent.

HINTS: the output corresponding to the signal $x(t-t_0)$:

$$\tilde{y}(t) = \int_0^t x(u-t_0)du = \int_{-t_0}^{t-t_0} x(v)dv$$

is different from:

$$y(t-t_0) = \int_0^{t-t_0} x(u)du$$

which is the output at time $t-t_0$ when $x(t)$ is used as the input signal.

Impulse response

The $h(t)$ function found in 1.11 is called the filter's *impulse response*. The output $y(t)$, convolution product of $x(t)$ and $h(t)$, is denoted $y(t) = (x \star h)(t)$.

A causal system is a system that depends only on the current and previous inputs. This means that a filter is causal if $h(t) = 0$ for $t < 0$.

Frequency response

Let us first consider the case of $x(t)$ signals that have a Fourier transform $X(f)$. Using the convolution product's property leads us to:

$$Y(f) = X(f)H(f)$$

The $H(f)$ function is called the filter's *frequency response* or *complex gain*.

Let us now take a look at signals $x(t)$ that are a linear mix of complex exponentials. Because of the linearity property, all we have to do is calculate the output with $x(t) = \exp(2j\pi f_0 t)$ as the input. We get:

$$y(t) = \int_{\mathbb{R}} \exp(2j\pi f_0(t-u))h(u)du = H(f_0)\exp(2j\pi f_0 t)$$

Therefore, the complex output signal $H(f_0)\exp(2j\pi f_0 t)$ corresponds to the complex exponential $\exp(2j\pi f_0 t)$. In this case, complex exponentials are called the *eigenfunctions* of the filters, the eigenvalue beeing $H(f_0)$.

Stability

A system is said to be *BIBO stable* if for any Bounded Input $x(t)$ the Output $y(t)$ is Bounded, that is $|x(t)| < A \Rightarrow |y(t)| < B$. Stability is an essential system property.

A filter is BIBO stable if and only if:

$$\int_{\mathbb{R}} |h(u)|du < +\infty$$

1.3 Summary

The following table contains some definitions and properties that will be used throughout the next lessons. The properties corresponding to the discrete time are also shown. It must be noted that the Laplace transform is given in its bilateral form. It is most often seen in the form $\int_0^{+\infty} x(t)e^{-st}dt$ in the control field. The same applies for the z transform and its related form $\sum_{n=0}^{+\infty} x(n)z^{-n}$.

Continuous time	Discrete time
Fourier transform $X(f) = \int_{\mathbb{R}} x(t)e^{-2j\pi ft}dt$ $x(t) = \int_{\mathbb{R}} X(f)e^{2j\pi ft}df$	Discrete time Fourier transform $X(f) = \sum_{n\in\mathbb{Z}} x(n)e^{-2j\pi nf}$ $x(n) = \int_{-1/2}^{1/2} X(f)e^{2j\pi nf}df$
Fourier series $X(k) = \frac{1}{T}\int_0^T x(t)e^{-2j\pi kt/T}dt$ $x(t) \stackrel{L_2}{=} \sum_{k\in\mathbb{Z}} X(k)e^{2j\pi kt/T}$	Discrete Fourier transform $X(k) = \sum_{n=0}^{N-1} x(n)e^{-2j\pi kn/N}$ $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{2j\pi nk/N}$
Linear filter $(t \in \mathbb{R})$ $(x \star h)(t) \leftrightarrow X(f)H(f)$ BIBO stability $\Leftrightarrow \int_{\mathbb{R}} \lvert h(t)\rvert dt < +\infty$	Linear filter $(n \in \mathbb{Z})$ $(x \star h)(n) \leftrightarrow X(f)H(f)$ BIBO stability $\Leftrightarrow \sum_{n\in\mathbb{Z}} \lvert h(n)\rvert < +\infty$

Continuous time	Discrete time
Bilateral Laplace transform $X(s) = \int_{\mathbb{R}} x(t) e^{-st} dt$ $x(t) = \frac{1}{2j\pi} \int_{C-j\infty}^{C+j\infty} X(s) e^{st} ds$	z-Transform $X(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n}$ $x(n) = \frac{1}{2j\pi} \oint_{\Gamma} X(z) z^{n-1} dz$
Filter $(t \in \mathbb{R})$ $(x \star h)(t) \leftrightarrow X(s)H(s)$ BIBO stability $\Leftrightarrow$ imaginary axis belongs to the domain of convergence of $H(s)$.	Filter $(n \in \mathbb{Z})$ $(x \star h)(n) \leftrightarrow X(z)H(z)$ BIBO stability $\Leftrightarrow$ unit circle belongs to the domain of convergence of $H(z)$.