PART 1

Particles and Rigid Bodies

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1

Galileo's Principle of Relativity

1.1. Events and space–time

DEFINITION 1.1.– An *event* $\mathbf{X}$ is just an occurrence at a specific moment and at a specific place. The *space-time* (or *universe*) is the set $\mathcal{U}$ of all the events.

Lightning striking a tree, a crash, the battle of Fontenoy, a birthday, the reception of an e-mail by a computer are some examples of events. Most events are relatively blurred, without either beginning or end or precisely defined localization. The events which, within the limits imposed by our measuring instruments, seem instantaneous and pointwise are called *punctual events*. In the following, when talking about events, readers are referred only to punctual events.

DEFINITION 1.2.– A *particle* is an object appearing as a pointwise phenomenon endowed with some time persistence.

We can see it as a sequence of events. A trace can be kept, for instance, due to a film consisting of frames recorded by a camera. Of course, this kind of observation has a discontinuous feature. If a high-speed camera is used, the observed events are closer. If we imagine that the time resolution can be arbitrarily reduced, a continuous sequence of events is obtained.

DEFINITION 1.3.– A *trajectory* is the continuous sequence of events revealing the persistence of a particle and represented by a continuous map $t \mapsto \mathbf{X}(t)$.

1.2. Event coordinates

1.2.1. *When?*

The *clock* is an instrument allowing us to measure the *durations*.

DEFINITION 1.4.– By the choice of a reference event $\mathbf{X}_0$ to which the time $t_0 = 0$ is assigned, an observer can assign to any event $\mathbf{X}$ a number t called the *date*, equal to the duration between $\mathbf{X}_0$ and $\mathbf{X}$, if $\mathbf{X}$ succeeds to $\mathbf{X}_0$, and to its opposite, if $\mathbf{X}$ precedes $\mathbf{X}_0$.

Conversely, the duration elapsed between two events $\mathbf{X}_1$ and $\mathbf{X}_2$ is calculated as the date difference $\Delta t = t_2 - t_1$. We assume that all the clocks are synchronized, i.e. they measure the same duration between any events:

$$\Delta t = \Delta t'. \tag{1.1}$$

This means each clock measures the durations with the same unit (for instance, the second). This also entails that if a clock assigns a date t' to some event, the other one assigns to the same event a date $t = t' + \tau_0$ where τ_0 depends only on both clocks.

DEFINITION 1.5.– Two events are *simultaneous* if, measured with the same clock, their dates are identical.

Clearly, if two events are simultaneous for a clock, it is so for any other one.

1.2.2. *Where?*

The most common measuring instrument for a *distance* is the *graduated ruler*. Of course, there exist less accurate instruments (the land-surveyor's string or measuring tape), while others are much more accurate (especially due to the lasers) but, for the simplicity of the presentation, the readers are only referred to the rulers as distance measuring instruments.

Whatever, we have just to know that the ruler allows us to measure the distance Δs between two *simultaneous events* $\mathbf{X}_1$ and $\mathbf{X}_2$. We assume that all the rulers are standardized in the sense that they measure the same distance between events:

$$\Delta s = \Delta s'. \tag{1.2}$$

This means each ruler measures the distances with the same unit (for instance, the meter). Let us have a break now to explain the meaning of the simultaneity between events. When they fit the ruler graduations, the observer is informed by light signals. The essential point is – as mentioned before – these signals arrive at the observer with an infinite velocity, and then instantaneously.

As we assigned to each event a date, we would like to assign it a position. Without entering into the details of the measurement method, which is not useful to our discussion, let us say only that – in addition to the rulers – instruments are required to measure the angles, for instance *set squares* and *protractors*. We admit that the measurement method allows an observer to assign to any event $\mathbf{X}$ three coordinates x^1, x^2, x^3. The column vector gathering them:

$$x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

is called *position* of $\mathbf{X}$.

DEFINITION 1.6.– To each event $\mathbf{X}$, an observer can assign a time t – in the sense prescribed by definition 1.4 – and a column $x \in \mathbb{R}^3$, called the *position*, by the choice of a reference event $\mathbf{X}_0$ with position $x_0 = 0$ and in such a way that for any distinct but *simultaneous* events $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ of respective positions x_1, x_2 and x_3:

– if $\Delta x = x_2 - x_1$, we can calculate the distance between the first two by:

$$\Delta s = \| \Delta x \| ;$$

– and the angle θ between the segments Δx and $\Delta' x = x_3 - x_1$ by:

$$\cos\theta = \Delta x \cdot \Delta' x \;/\; \| \Delta x \| \, \| \Delta' x \| .$$

In short, any observer has available instruments measuring durations, distances and angles. This allows him or her to assign to each event $\mathbf{X}$ a date t and a position x. In the following, we adopt the following convention:

CONVENTION 1.1.– Coordinate labels:

– Latin indices i, j, k and so on run over the special coordinate labels, usually, $1, 2, 3$ or x, y, z.

– Greek indices α, β, γ and so on run over the four space-time coordinate labels $0, 1, 2, 3$ or t, x, y, z.

DEFINITION 1.7.– To each event $\mathbf{X}$, a column $X \in \mathbb{R}^4$:

$$X = \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

is assigned by an observer. Their components $X^0 = t$, $X^i = x^i$ are called *coordinates* of the event. The assignment is one-to-one. Each observer creates her or his own *coordinate system*.

Hence, an observer can record the trajectory of a particle $t \mapsto \mathbf{X}(t)$ due to an assignment $t \mapsto X(t)$ in her or his own coordinate system.

Additionally, the length and angle measures allow us to calculate the areas and volumes, at least for simple geometrical objects.

DEFINITION 1.8.– The positions being determined by an observer for simultaneous events:

– the positions of three of its vertices being x_1, x_2, x_3, the *area* of a parallelogram is calculated by:

$$S = \parallel \Delta x \times \Delta' x \parallel,$$

with $\Delta x = x_2 - x_1$ and $\Delta' x = x_3 - x_1$;

– three adjoining faces of it being defined by four of its vertices x_1, x_2, x_3, x_4, the *oriented volume* of a parallelepiped is calculated by:

$$V = (\Delta x \times \Delta' x) \cdot \Delta'' x,$$

with $\Delta'' x = x_4 - x_1$.

1.3. Galilean transformations

1.3.1. *Uniform straight motion*

Newton's first law claims the velocity of a particle or a body remains constant unless the body is acted upon by an external force. This assumes we know what a force is, at least intuitively. We prefer to take it as starting point to define the forces.

DEFINITION 1.9.– A *force* is a phenomenon modifying the velocity of a particle. Hence, a *free particle* force moves in a straight line at uniform velocity. This is the *uniform straight motion* (*USM*). If the velocity is null, the particle is said to be *at rest* in the considered coordinate system.

The problem is that gravity is a large-scale force affecting all matter equally, so there are no completely free particles, even in deep space. On the Earth, experiences of USM can be carried out only in reduced regions of the space-time, for instance during a small enough duration or with objects moving without friction on a horizontal plane. The motion of a free particle is given by:

$$x = x_0 + v\, t,$$

where the initial position $x_0 \in \mathbb{R}^3$ at $t = 0$ and the uniform velocity $v \in \mathbb{R}^3$ are constant. The event "the particle is passing through x_0 at $t = 0$" is represented in the considered coordinate system by:

$$X_0 = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}.$$

Introducing the 4-column:

$$U = \begin{pmatrix} 1 \\ v \end{pmatrix},$$

the event "the particle is in x at t" is represented by:

$$X = X_0 + Ut. \qquad [1.3]$$

DEFINITION 1.10.– With respect to a given family of coordinate systems, a characteristic of an object or a quantity is *invariant* if its representation in all the systems of the family is identical. We also talk about the *invariance* of the characteristic or the quantity and say that the coordinates changes of the family preserve the considered characteristic or quantity.

For instance, let us consider the family of the coordinate systems of observers for which the motion of the same particle is straight and uniform. We would like to ask the following question: what are the coordinate changes $X' \mapsto X$ of this family?

THEOREM 1.1.– The coordinate changes preserving:

– the USMs;

– the durations;

– the distances and angles;

– the oriented volumes;

are regular affine maps of the following form:

$$X = PX' + C, \quad C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \qquad [1.4]$$

where $\tau_0 \in \mathbb{R}$, $k \in \mathbb{R}^3$, $u \in \mathbb{R}^3$ and $R \in \mathbb{SO}(3)$ (see Comment 1, section 1.4).

PROOF.– Parametrization [1.3] of the trajectory being affine, the coordinate change in $\mathbb{R}^4$ preserves straight lines and the middle of segments. As a parallelogram is a quadrilateral whose the diagonals meet in their middle, the coordinate change preserves parallelograms and, reasoning by recurrence, parallelepipeds and parallelotopes. So, the coordinate change is affine:

$$X = P X' + C, \tag{1.5}$$

where $C \in \mathbb{R}^4$ and the 4×4 matrix P are constant. As the coordinate systems define one-to-one assignments from X into the event $\mathbf{X}$, the coordinate change is also one-to-one. Considering the difference of the columns representing two events $\mathbf{X}_1$ and $\mathbf{X}_2$ in the considered coordinate systems:

$$\Delta X = X_2 - X_1 = \begin{pmatrix} \Delta t \\ \Delta x \end{pmatrix}, \quad \Delta X' = X'_2 - X'_1 = \begin{pmatrix} \Delta t' \\ \Delta x' \end{pmatrix},$$

we obtain a linear relation:

$$\Delta X = P \Delta X'. \tag{1.6}$$

Next, we put:

$$C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}, \quad P = \begin{pmatrix} \alpha & w^T \\ u & F \end{pmatrix},$$

where $\alpha, \tau_0 \in \mathbb{R}$, $u, w, k \in \mathbb{R}^3$ and F is a 3×3 matrix. Hence, [1.6] gives:

$$\Delta t = \alpha \Delta t' + w^T \Delta x'.$$

Identifying it with condition [1.1] ensuring the invariance of the duration gives:

$$\alpha = 1, \quad w = 0.$$

Hence, we have:

$$P = \begin{pmatrix} 1 & 0 \\ u & F \end{pmatrix}. \tag{1.7}$$

As P is regular, F must be so. Hence, [1.6] gives for simultaneous events:

$$\Delta x = F \Delta x' \quad .$$

Invariance [1.2] of the distance reads:

$$(\Delta x')^T F^T F \Delta x' = (\Delta x')^T \Delta x.$$

The column $\Delta\, x'$ being arbitrary, we obtain:

$$F^T\, F = 1_{\mathbb{R}^3}.$$

The matrix F is orthogonal. Taking into account that oriented volumes [1.8] are transformed as:

$$V' = \det(F) V,$$

their invariance entails that F is a rotation that we denote by R afterward. As $\det(P) = \det(R) = 1$, P is regular and so is the affine map $X' \mapsto X$. ■

DEFINITION 1.11.– The coordinate changes [1.4] are called *Galilean transformations*. Any of them can be obtained composing elementary ones from amongst:

– *clock change* τ_0 (with $k = u = 0,\ R = 1_{\mathbb{R}^3}$): $t = t' + \tau_0$, $x = x'$;

– *spatial translation* k: $t = t'$, $x = x' + k$;

– *rotation* R: $t = t'$, $x = R\, x'$;

– *Galilean boost* or *velocity of transport* u: $t = t'$, $x = x' + ut$.

A general Galilean transformation reads:

$$x = R\, x' + u\, t' + k, \quad t = t' + \tau_0, \qquad [1.8]$$

or in matrix form:

$$C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \qquad [1.9]$$

1.3.2. *Principle of relativity*

If a particle is in USM for an observer, it is a also so for any other observer. Hence, all the coordinate systems in the sense defined by definition 1.6 are equivalent, including the ones in which the particle is at rest. In other words, we admit in particular the equivalence between the motion and rest. Galileo Galilei proposed in his famous "Dialogue concerning the two chief world systems" (1632) this point of view according to which the observations of physical phenomena do not allow us to know whether we are in motion or at rest, provided that the motion is straight and uniform. *Galileo's principle of relativity* turns this from a negative to a positive statement:

PRINCIPLE 1.1.– The statement of the physical laws of the classical mechanics is the same in all the coordinate systems in the sense of definition 1.6.

For the moment, this principle is formulated in rather general words but we will soon make it clearer in applications. By classical mechanics, let us recall that we consider phenomena for which the velocity of the light is so huge that it may be considered as infinite.

1.3.3. *Space–time structure and velocity addition*

Up to now, the space-time was a set of which the elements – the events – were parametrized by four coordinates. Considering only USMs, we need only affine transformations [1.5] for the coordinate changes. In other words, the space-time $\mathcal{U}$ may be perceived as an *affine space* of 4 dimensions and the coordinates of an event $\mathbf{X}$ change according to the transformation law for the component of one of its points. Hence, the structure of the space-time must not be imposed *a priori* but is deduced from the physical observations (the USM).

Have a look now at our starting point, the USM. In the old coordinate system, it reads:

$$X' = X'_0 + U't'.$$

Combining it with the Galilean transformation [1.4] gives:

$$X = P(X'_0 + U't') + C.$$

Taking into account [1.8], we recover [1.3], provided that:

$$X_0 = P(X'_0 - U'\tau_0) + C, \qquad [1.10]$$

$$U = PU'. \qquad [1.11]$$

What do these relations teach us?

– Without clock change, the first one reads:

$$X_0 = PX'_0 + C,$$

which is nothing other than the transformations law for the components of a point of $\mathcal{U}$. For more general transformations, the additional term in [1.10] takes into account the clock change.

– The second relation, [1.11], is the transformation law for the components of a vector $\vec{\mathbf{U}}$ of the vector space attached to $\mathcal{U}$. It will be called the 4-*velocity*.

Let us consider, for instance, a particle of velocity v' in the coordinate system X'. In another one X obtained from X' by a Galilean transformation [1.9], the 4-velocity is given by [1.11]:

$$U = \frac{dX}{dt} = \begin{pmatrix} 1 \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix} \begin{pmatrix} 1 \\ v' \end{pmatrix}. \quad [1.12]$$

Thus, the velocity in the new coordinate system is:

$$\boxed{v = u + R\,v'.} \quad [1.13]$$

In particular, for a Galilean boost u, we have:

$$v = u + v'.$$

This is the *velocity addition formula*. Also, combining two Galilean boosts u_1 and u_2, we verify that the resulting velocity of transport is:

$$u = u_1 + u_2.$$

1.3.4. *Organizing the calculus*

For convenience, an affine transformation $X' \mapsto X = PX' + C$ can be denoted by $a = (C, P)$. Applying successively a_1 and a_2 gives a new affine transformation a_3:

$$a(X) = a_2(a_1(X)) = a_2(C_1 + P_1 X) = C_2 + P_2(C_1 + P_1 X),$$

hence:

$$a_3 = a_2 a_1 = (C_2, P_2)(C_1, P_1) = (C_2 + P_2 C_1, P_2 P_1).$$

This product is associative and has an identity transformation $e = (0, 1_{\mathbb{R}^4})$ such that $e\,a = a\,e = a$. Each affine transformation $a = (C, P)$ has an inverse transformation $a^{-1} = (-P^{-1}C, P^{-1})$ such that $a^{-1}a = a\,a^{-1} = e$. It is straightforward to verify that the combination of two Galilean transformations a_2 and a_1 is also a Galilean transformation a given by:

$$u = u_2 + R_2 u_1, \quad R = R_2 R_1, \quad \tau_0 = \tau_2 + \tau_1, \quad k = k_2 + R_2 k_1 + u_2 \tau_1. \quad [1.14]$$

It is easy to verify that the inverse transformation $X \mapsto X' = P^{-1}X + C'$ is a Galilean transformation represented by (see Comment 2 section 1.4):

$$C' = \begin{pmatrix} \tau_0' \\ k' \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 \\ -R^T u & R^T \end{pmatrix}, \qquad [1.15]$$

putting:

$$\tau_0' = -\tau_0, \qquad k' = -R^T(k - u\tau_0).$$

It is often convenient to organize the matrix calculation by working rather in $\mathbb{R}^5$, representing the column X and the affine transformation $a = (C, P)$, respectively, by:

$$\tilde{X} = \begin{pmatrix} 1 \\ X \end{pmatrix} \in \mathbb{R}^5 \qquad \tilde{P} = \begin{pmatrix} 1 & 0 \\ C & P \end{pmatrix}, \qquad [1.16]$$

so affine transformation [1.4] looks like a simple regular linear transformation:

$$\tilde{X} = \tilde{P}\tilde{X}', \qquad [1.17]$$

where, taking into account [1.9], the Galilean transformation a is represented by the 5×5 matrix decomposed by blocks:

$$\tilde{P} = \begin{pmatrix} 1 & 0 & 0 \\ \tau_0 & 1 & 0 \\ k & u & R \end{pmatrix}. \qquad [1.18]$$

In a similar way, owing to [1.15], the inverse transformation is represented by:

$$\tilde{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \tau_0' & 1 & 0 \\ k' & -R^T u & R^T \end{pmatrix}. \qquad [1.19]$$

1.3.5. *About the units of measurement*

There is still a long way to go to cover the mechanics of continua but let us stop for a moment to have a look at the conversion of units. Let the event $\mathbf{X}$ be represented in a coordinate system by $\bar{X}$ where durations and times are measured with new units. Let us say that the time and length units in the old coordinate system are equal, respectively, to T and L in the new one. The conversion of units is given by the scaling:

$$\bar{t} = T\,t, \qquad \bar{x} = Lx,$$

or, in matrix form:

$$\bar{X} = P_u X \qquad [1.20]$$

with:

$$P_u = \begin{pmatrix} T & 0 \\ 0 & L1_{\mathbb{R}^3} \end{pmatrix}. \qquad [1.21]$$

Similarly, let us apply the scaling:

$$\bar{X}' = P_u X' \qquad [1.22]$$

Combining Galilean transformation [1.4] and scalings [1.20] and [1.22] leads to:

$$\bar{X} = \bar{P}\bar{X}' + \bar{C},$$

with:

$$\bar{P} = P_u P P_u^{-1}, \qquad \bar{C} = P_u C.$$

Using [1.9] and [1.21] shows that

$$\bar{C} = \begin{pmatrix} \bar{\tau}_0 \\ \bar{k} \end{pmatrix}, \quad \bar{P} = \begin{pmatrix} 1 & 0 \\ \bar{u} & \bar{R} \end{pmatrix}, \qquad [1.23]$$

with $\bar{C}$ being a simple scaling of C:

$$\bar{\tau}_0 = T\,\tau_0, \qquad \bar{k} = Lk,$$

and:

$$\bar{u} = (L/T)\,u, \qquad \bar{R} = R,$$

As result of the conversion of units, the rotation is invariant while the boost u is scaled as a velocity. It is worth observing that, in a conversion of units, a Galilean transformation $a = (C, P)$ turns into a Galilean transformation $\bar{a} = (\bar{C}, \bar{P})$ (see Comment 3, section 1.4). The conversion does not affect the Galilean feature of an affine transformation. Of course, calculations can be organized with 5×5 matrices:

$$\tilde{\bar{P}} = \tilde{P}_u \tilde{P} \tilde{P}_u^{-1} \qquad \text{where} \qquad \tilde{P}_u = \begin{pmatrix} 1 & 0 \\ 0 & P_u \end{pmatrix}.$$

1.4. Comments for experts

COMMENT 1.– This theorem is related to the Toupinian structure of the space-time which gives a theoretical framework to the universal or absolute time and space (see section 16.1).

COMMENT 2.– In fact, the set of all the Galilean transformations is a Lie group of 10 dimensions called Galileo's group.

COMMENT 3.– Conversely, the normalizer of Galileo's group in the affine group is composed of the Galilean transformations themselves and the conversions of units [1.20] (see section 16.2).