PART 1

Dynamics of Solids

Chapter 1

Motion within Solids

The concept of stress and strain waves emerges from the equations of motion in elastic continuum. Uniaxial propagation is particularly well-studied because of its practical importance. The waves can be altered within their propagation by dispersion and dissipation. For viscoelastic solids, we can address the effects of behavior sensitive to strain in a linear framework.

1.1. Representation of the medium

1.1.1. *Framework of continuum mechanics*

The problem for the engineer is to describe the position (or displacement) of solids and fluids. This mechanism is of a macroscopic scale. On this macroscopic scale, solid or fluid matter can be seen as continuous, which is not the case at the microscopic level of particles, molecules and atoms. The macroscopic scale is not the same for all materials: a fraction of a millimeter for a metal, a few inches for geomaterials such as rock or concrete.

In this book, we only refer to classical mechanical engineering knowledge of continuum mechanics and structural strength. This chapter intends to recall the basics of continuum motion when these

can be described by linear equations, such as in the context of small strains and elastic or viscoelastic material behavior.

1.1.2. *Representation of motion*

The motions in matter are identified in an affine Euclidean space. A point in matter which occupies position M_0 at time 0, defined by the vector $\underline{X}$, is in position M_t at time t, defined by the vector $\underline{x}$. A certain amount of matter that occupies the simply connected domain Ω_0 at time 0, also occupies the simply connected domain Ω_t at time t (Figure 1.1).

$$\underline{OM}^0 = \underline{X}(X_1, X_2, X_3) \qquad \underline{OM}^t = \underline{x}(x_1, x_2, x_3) \qquad \underline{U} = \underline{x} - \underline{X} \qquad [1.1]$$

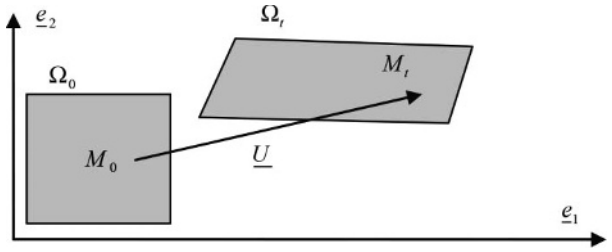

Figure 1.1. *Material field in its initial position and after transformation*

The description of these movements can be given from two points of view: one is called "Lagrangian" and the other is called "Eulerian". The Lagrangian description involves following the matter points in their motion. The current position, or displacement, is expressed depending on the initial position and t, using a continuous vector function [1.2] that defines the trajectories (this is a bijection of Ω_0 to Ω_t, due to the continuity of displacement of the system):

$$\underline{x} = \underline{\varphi}(\underline{X}, t) \qquad [1.2]$$

The Eulerian description involves the knowledge of the velocity field at each moment relative to the current position. The Eulerian

description of the velocity field involves specifying the velocity of the particle passing position $\underline{x}$ at time t [1.3]:

$$\underline{V}(\underline{x},t) \qquad [1.3]$$

If the Eulerian velocity field does not depend on time, the motion is stationary. Eulerian representation is mainly used for fluids and materials undergoing very large strains. In the remaining chapter, we use the Lagrangian point of view. The strain is characterized by the Green–Lagrange strain tensor [1.4]:

$$\begin{aligned} &\underline{\underline{E}} = \tfrac{1}{2}\left({}^{t}\underline{\underline{\nabla\varphi}}\,\underline{\underline{\nabla\varphi}} - \underline{\underline{1}} \right) \quad \underline{\underline{E}} = \tfrac{1}{2}\left(\underline{\underline{\nabla U}} + {}^{t}\underline{\underline{\nabla U}} + {}^{t}\underline{\underline{\nabla U}}\,\underline{\underline{\nabla U}} \right) \\ &E_{ij} = \tfrac{1}{2}\left(U_{i,j} + U_{j,i} + U_{k,i}U_{k,j} \right) \end{aligned} \qquad [1.4]$$

In many cases for solids, displacements and strains are very small (0.001% elongation, for example). A linearization is then performed by retaining only the first-order infinitesimal. This is called the linearized strain tensor [1.5]:

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\underline{\nabla U}} + {}^{t}\underline{\underline{\nabla U}} \right) \qquad \varepsilon_{ij} = \frac{1}{2}\left(U_{i,j} + U_{j,i} \right) \qquad [1.5]$$

It is this tensor that will be most widely used. Each component has a simple physical significance: ε_{ii} represents the relative elongation in direction i. ε_{ij} ($i \neq j$) represents the angular distortion relative to the two directions i and j. The tensor trace, equal to the divergence of the displacement vector, represents the relative change in volume. The partition of the strain tensor in its spherical part, which is related to the change in volume, and its deviatoric part, which is related to the change of form [1.6], is used:

$$\begin{aligned} &\theta = trace\,\underline{\underline{\varepsilon}} = div\underline{U} \qquad \underline{\underline{e}} = \underline{\underline{\varepsilon}} - \frac{\theta}{3}\underline{\underline{\mathfrak{I}}} \\ &\underline{\underline{\varepsilon}} = \frac{\theta}{3}\underline{\underline{\mathfrak{I}}} + \underline{\underline{e}} \qquad \varepsilon_{ij} = \frac{\theta}{3}\delta_{ij} + e_{ij} \end{aligned} \qquad [1.6]$$

1.1.3. *Representation of internal forces*

In a continuum, internal forces can be represented either by a scalar field, pressure ($p(\underline{x},t)$), or by a tensorial field, the Cauchy stress tensor. In an orthonormal frame, this tensor is represented by a symmetric matrix [1.7]. Figure 1.2 shows a representation of the components of the stress tensor of a material whose faces are normal to the reference axes wherein the said tensor is expressed:

$$\underline{\underline{\sigma}}(\underline{x},t) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \qquad [1.7]$$

For solids, it is customary to partition the stress tensor into a so-called "spherical" or isotropic part, characterized by pressure p, and a so-called "deviatoric" part [1.8]:

$$\begin{aligned} p &= -\frac{1}{3} trace\, \underline{\underline{\sigma}} \qquad \underline{\underline{S}} = \underline{\underline{\sigma}} + p\,\underline{\underline{\Im}} \\ \underline{\underline{\sigma}} &= -p\,\underline{\underline{\Im}} + \underline{\underline{S}} \qquad \sigma_{ij} = -p\,\delta_{ij} + S_{ij} \end{aligned} \qquad [1.8]$$

The representation of internal forces by pressure concerns, *a priori*, non-viscous fluids. However, when shocks occur in solids, the pressure can become very large and the terms of the deviatoric part (shear stress) may be negligible compared to the pressure. In these cases, it is possible to retain only pressure to represent internal forces. This is a hydrodynamic model of a material.

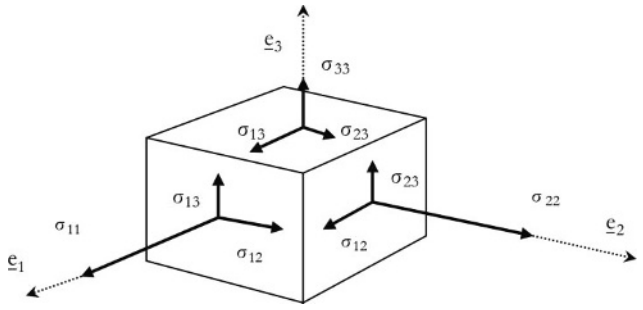

Figure 1.2. *Representation of the components of the Cauchy stress tensor on a piece of material*

Writing the behavior or resistance of a material logically involves stress tensor invariants. Besides the main constraints (σ_I, σ_{II}, σ_{III}) that are the eigenvalues of the matrix representing the stress tensor, various invariants of this tensor are used:

– The first invariant of the stress tensor, which defines the pressure [1.9]:

$$I_1 = trace\,\underline{\underline{\sigma}} = \sigma_{kk} = -3p \qquad [1.9]$$

– The second invariant of the deviator tensor, which defines the von Mises equivalent stress [1.10]:

$$J_2 = \frac{1}{2}\,\underline{\underline{S}} : \underline{\underline{S}} = \frac{1}{2} S_{ij} S_{ij} \qquad \sigma_{eq} = \sqrt{3J_2} = \sqrt{\frac{3}{2}\underline{\underline{S}} : \underline{\underline{S}}} = \sqrt{\frac{3}{2} S_{ij} S_{ij}} \qquad [1.10]$$

– The third invariant, and the Lode angle χ [1.11]:

$$J_3 = \tfrac{1}{3}\left(\mathrm{trace}\left(\underline{\underline{\sigma}}^3\right)\right)^{1/3} = \tfrac{1}{3}\left(\sigma_{ij}\,\sigma_{jk}\,\sigma_{ki}\right)^{1/3} \qquad \cos(3\chi) = \frac{3\sqrt{3}\,J_3}{2J_2^{3/2}} \qquad [1.11]$$

Use of the fundamental principle of mechanics and its application to the continuum leads to equilibrium equations. We consider a continuum in a field Ω with a density ρ. We denote volume forces by $\underline{f}$. Then, the equations are as follows:

– with a pressure field [1.12]:

$$-\underline{grad}\; p + \underline{f} = \rho \underline{\ddot{U}} \qquad p_{,i} + f_i = \rho U_{i,tt} \qquad [1.12]$$

– or with a stress field [1.13]:

$$\underline{div}\;\underline{\underline{\sigma}} + \underline{f} = \rho \underline{\ddot{U}} \qquad \sigma_{ij,j} + f_i = \rho U_{i,tt} \qquad [1.13]$$

In general, these equations are insufficient to predict motion across the field from external forces data. It is necessary to involve a relationship between the internal forces and the geometric transformation (strain). This relationship, which cannot be deduced from the fundamental principles of mechanics, is called behavior. In this chapter, we consider isotropic linear elasticity and linear

viscoelasticity. Nonlinear behaviors and their consequences on motion will be discussed in Chapter 3.

1.2. Elastodynamic equations

1.2.1. *Navier equations*

To obtain the equations governing motion in a continuum, we must associate the equilibrium equations and material behavior. Elastic behavior has been widely studied [ACH 93, GRA 75]. Isotropic linear elastic behavior is reflected in the equations linking stress to strain. The stress tensor can be expressed in terms of strain and the strain tensor can also be expressed in terms of stress [1.14]:

$$\begin{aligned} &\underline{\underline{\sigma}} = \lambda\ trace\left(\underline{\underline{\varepsilon}}\right)\ \underline{\underline{\mathfrak{I}}} + 2\mu\ \underline{\underline{\varepsilon}} \qquad \sigma_{ij} = \lambda\ \varepsilon_{kk}\delta_{ij} + 2\mu\,\varepsilon_{ij} \\ &p = -K\theta \qquad \underline{\underline{S}} = 2G\ \underline{\underline{e}} \qquad S_{ij} = 2G\,e_{ij} \\ &\underline{\underline{\varepsilon}} = \frac{1+\nu}{E}\,\underline{\underline{\sigma}} - \frac{\nu}{E}\,trace\left(\underline{\underline{\sigma}}\right)\ \underline{\underline{\mathfrak{I}}} \qquad \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\,\sigma_{kk}\delta_{ij} \end{aligned} \qquad [1.14]$$

λ and μ are Lamé coefficients, E is the Young modulus and ν is the Poisson coefficient. K is the bulk modulus and G is the shear modulus. The relationship between these different coefficients is given in [1.15]:

$$\begin{aligned} &E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \qquad \nu = \frac{\lambda}{2(\lambda+\mu)}, \qquad K = \frac{E}{3(1-2\nu)} \\ &\lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}, \qquad \mu = G = \frac{E}{2(1+\nu)} \end{aligned} \qquad [1.15]$$

An elasticity problem is formulated by combining the fundamental principle of the mechanics equation and the behavior relationship [1.16]:

$$\begin{aligned} &\underline{div}\,\underline{\underline{\sigma}} + \underline{f} = \rho\ddot{\underline{U}} \qquad \sigma_{ij,j} + f_i = U_{i,tt} \\ &\underline{\underline{\varepsilon}} = \tfrac{1}{2}\left(\underline{\underline{\nabla U}} + {}^{t}\underline{\underline{\nabla U}}\right) \qquad \varepsilon_{ij} = \tfrac{1}{2}\left(U_{i,j} + U_{j,i}\right) \\ &\underline{\underline{\sigma}} = \lambda\left(trace\ \underline{\underline{\varepsilon}}\right)\underline{\underline{\mathfrak{I}}} + 2\mu\,\underline{\underline{\varepsilon}} \qquad \sigma_{ij} = \lambda\,\varepsilon_{kk}\delta_{ij} + 2\mu\,\varepsilon_{ij} \end{aligned} \qquad [1.16]$$

The Navier equations are the displacement equations obtained by eliminating stress and strain. First, we can express stress as a function of displacement, and thus obtain its gradient [1.17]:

$$\begin{aligned} &\underline{\underline{\sigma}} = \lambda\left(div\underline{U}\right)\underline{\underline{\Im}} + \mu\left(\underline{\nabla U} + {}^{t}\underline{\nabla U}\right)\underline{\underline{\varepsilon}} \quad \sigma_{ij} = \lambda\, U_{i,i}\delta_{ij} + \mu\left(U_{i,j} + U_{j,i}\right) \\ &\underline{div}\,\underline{\underline{\sigma}} = \left(\lambda+\mu\right)\underline{\nabla}\left(div\underline{U}\right) + \mu\,\underline{\Delta U} \quad \sigma_{ij,j} = \lambda\, U_{i,ij} + \mu\left(U_{i,jj} + U_{j,ij}\right) \end{aligned} \quad [1.17]$$

Then, we can obtain the equations governing the motion of an elastic continuum, involving only the displacement field. Two vector equations are possible, using either the Laplacian or the curve of the displacement field [1.18]:

$$\begin{aligned} &\mu\,\underline{\Delta U} + \left(\lambda+\mu\right)\underline{\nabla}\left(div\underline{U}\right) + \underline{f} = \rho\underline{\ddot{U}} \quad \left(\mu U_{i,jj} + \left(\lambda+\mu\right)U_{j,ji} + f_i = U_{i,tt}\right) \\ &\left(\lambda+2\mu\right)\,\underline{\nabla}\left(div\underline{U}\right) - \mu\,\underline{rot}\left(\underline{rot}\,\underline{U}\right) + \underline{f} = \rho\underline{\ddot{U}} \end{aligned} \quad [1.18]$$

By deriving and summing equations [1.18], a scalar equation governing volume variations is obtained. This is the expansion equation [1.19]:

$$\left(\lambda+2\mu\right)\,\Delta\left(div\underline{U}\right) + div\,\underline{f} = \rho\, div\underline{\ddot{U}} \quad \left(\lambda+2\mu\right)U_{i,jji} + f_{i,i} = \rho\ddot{U}_{i,i} \quad [1.19]$$

1.2.2. *Strain waves*

1.2.2.1. *Helmholtz decomposition*

The Helmholtz decomposition theorem states that any vector field, differentiable twice, can be written in a unique manner as the sum of a gradient field and a rotational field [1.20]:

$$\exists\ \underline{U}^{P}\ \ and\ \ \underline{U}^{S}\ \ \underline{U} = \underline{U}^{P} + \underline{U}^{S} \quad \begin{cases} \underline{U}^{P} = \underline{\nabla}\,\boldsymbol{\Phi} \\ \underline{U}^{S} = \underline{rot}\,\underline{\boldsymbol{\Psi}} \end{cases} \quad [1.20]$$

The field $\underline{U}^{P}$ is the primary irrotational field [1.21]:

$$\underline{rot}\,\underline{U}^{P} = \underline{rot}\left(\underline{\nabla}\,\boldsymbol{\Phi}\right) = 0 \quad [1.21]$$

The field $\underline{U}^S$ is the second field, which is said to be solenoid, and corresponds to motion that does not cause changes in volume [1.22]:

$$div\,\underline{U}^S = div\left(\underline{rot}\,\underline{\Psi}\right) = 0 \qquad [1.22]$$

1.2.2.2. *P-waves*

If we only consider the primary field, the equation of motion is written as [1.23]:

$$\left(\lambda + 2\mu\right)\,\underline{\nabla}\left(div\,\underline{U}^P\right) = \rho\,\underline{\ddot{U}}^P \qquad [1.23]$$

This is a propagation equation. This primary field corresponds to the propagation of P-waves [1.24]:

$$\underline{\Delta}\,\underline{U}^P = \frac{\rho}{\left(\lambda + 2\mu\right)}\,\underline{\ddot{U}}^P \qquad U^P_{i,jj} = \frac{\rho}{\left(\lambda + 2\mu\right)}\,U^P_{i,tt} \qquad [1.24]$$

The propagation velocity of these P-waves is C_p [1.25]:

$$C_P = \sqrt{\frac{\left(\lambda + 2\mu\right)}{\rho}} \qquad = \sqrt{\frac{E\left(1-\nu\right)}{\rho\left(1-2\nu\right)\left(1+\nu\right)}} \qquad [1.25]$$

Motions of P-wave propagation are extensions and compressions.

1.2.2.3. *S-waves*

If we only consider the secondary field, the equation of motion is written as [1.26]:

$$\mu\;\underline{\Delta U}^S = \rho\,\underline{\ddot{U}}^S \qquad \mu\,U^S_{i,jj} = \rho\,U^S_{i,tt} \qquad [1.26]$$

This is also a propagation equation. The propagation of this secondary field corresponds to S-waves, with a propagation velocity C_s [1.27]:

$$C_s = \sqrt{\frac{\mu}{\rho}} \qquad = \sqrt{\frac{E}{2\rho(1+\nu)}} \qquad [1.27]$$

The propagation of S-waves corresponds to shear movements without volume change. In an elastic system, the velocity ratio of P- and S-waves depends only on the Poisson coefficient [1.28]:

$$\left(\frac{C_p}{C_s}\right)^2 = \frac{\lambda + 2\mu}{\mu} = 2\frac{1-\nu}{1-2\nu} \qquad [1.28]$$

1.2.2.4. *Plane waves*

If the displacement depends on only one variable in space ($\underline{U}(x_1)$), the motion is that of plane waves. Propagation is in one direction. The equations governing the motion are [1.29]:

$$\begin{cases} \dfrac{\partial^2 U_1}{\partial {x_1}^2} = \dfrac{1}{{C_P}^2}\ddot{U}_1 \\ \dfrac{\partial^2 U_2}{\partial {x_1}^2} = \dfrac{1}{{C_S}^2}\ddot{U}_2 \\ \dfrac{\partial^2 U_3}{\partial {x_1}^2} = \dfrac{1}{{C_S}^2}\ddot{U}_3 \end{cases} \qquad [1.29]$$

The longitudinal components of displacement, along the axis of propagation, correspond to P-waves and thus propagate at velocity C_P. The transversal displacements, orthogonal to the direction of propagation, correspond to S-waves at velocity C_S. Figure 1.3 gives an illustration of the movements associated with these strain waves.

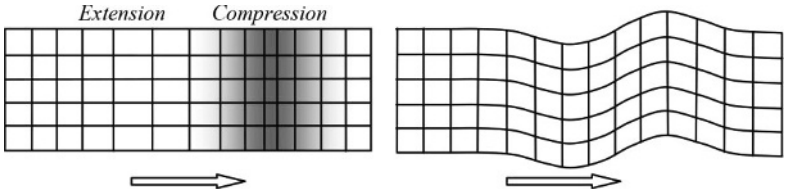

Figure 1.3. *Movements and strains associated with P-waves (left) and S-waves (right)*

1.2.2.5. *P-waves in spherical symmetry*

Consider a pressure wave emitted from a point within a large solid. This could represent the effect of an underground explosion. We naturally take the spherical coordinates, because the movement follows the symmetry [1.30]:

$$\underline{U}\begin{pmatrix} U_r = u(r) \\ U_\theta = 0 \\ U_\varphi = 0 \end{pmatrix} \qquad [1.30]$$

The propagation equation in the radial direction is then [1.31]:

$$\Delta u - \left(\frac{\rho}{\lambda + 2\mu}\right)\frac{\partial^2 u}{\partial t^2} = 0 \qquad [1.31]$$

This equation can also be written as [1.32]:

$$\frac{1}{r}\frac{\partial^2 (ru)}{\partial r^2} - \frac{1}{{C_P}^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad [1.32]$$

If we insert $z = ru$, the equation takes the form [1.33]:

$$\frac{\partial^2 z}{\partial r^2} - \frac{1}{{C_P}^2}\frac{\partial^2 z}{\partial t^2} = 0 \qquad [1.33]$$

With this change of variable, we see that the propagation occurs at P-wave velocity and the range of motion decreases in proportion to *1/r*.

1.3. One-dimensional waves

1.3.1. *Uniaxial stress state*

Uniaxial stress state is an approximation that is made when the considered solid is of slender shape (wire or bar undergoing tension or compression). The solid is then represented geometrically by a straight

line, and the displacement field depends only on the abscissa along the line (Figure 1.4). Stress and strain tensors at a point then take a simplified form [1.34]:

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix} \qquad \underline{\underline{\sigma}} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [1.34]$$

The equations of motion take the form of system [1.35] (note that, for the ease of writing, $x_1 = x \quad \sigma_1 = \sigma \quad \varepsilon_1 = \varepsilon \quad U_1 = U$):

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 U}{\partial t^2} \qquad \varepsilon = \frac{\partial U}{\partial x} \qquad \sigma(\varepsilon) \qquad [1.35]$$

In the case of a linear elastic medium, the propagation equation is [1.36]:

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{C_0^2} \frac{\partial^2 U}{\partial t^2} = 0 \quad \text{where} \quad C_0 = \sqrt{\frac{E}{\rho}} \qquad [1.36]$$

C_0 is the wave velocity in uniaxial stress state.

1.3.2. *Uniaxial strain state*

Uniaxial strain state corresponds to a situation where movement is in one direction only (axial). There is no radial strain (Figure 1.4). This may correspond to strain on the area near the impact in a solid. There exists a test called “impact plate” where a material is tested in a state of stress. Stress and strain tensors at a point then take a simplified form [1.37]:

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\underline{\sigma}} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix} \qquad [1.37]$$

Elastic behavior gives the relationship between the axial stress and radial stress [1.38]:

$$\sigma_2 = \frac{\nu}{(1-\nu)}\sigma_1 \qquad \sigma_1 = (\lambda + 2\mu)\varepsilon_1 = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\varepsilon_1 \qquad [1.38]$$

The propagation equation is [1.39]:

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{C_P^2}\frac{\partial^2 U}{\partial t^2} = 0 \qquad [1.39]$$

Compression motion in a uniaxial strain state corresponds to P-waves.

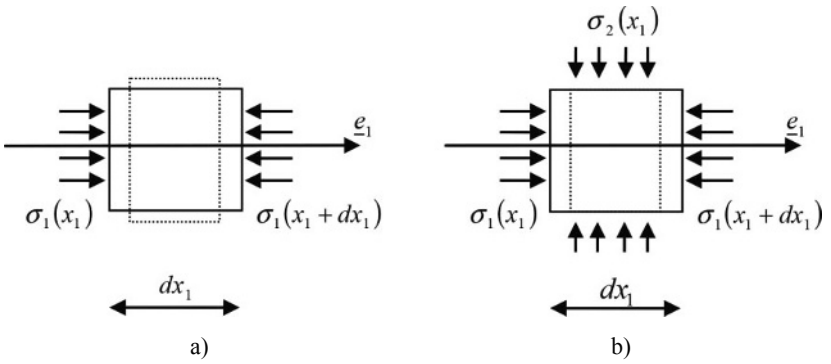

Figure 1.4. *Volume element in a state of uniaxial stress a), or in a state of uniaxial strain b)*

Table 1.1 presents the values of elastic properties, and uniaxial compression waves and P-waves propagation velocities for common building materials.

	ρ (kg/m^3)	E (GPa)	ν	C_0 (m/s)	C_P (m/s)
Steel	7,870	210	0.3	5,160	5,990
Concrete	2,200	20	0.2	3,010	3,170
Glass	2,500	69	0.18	5,250	5,470
Wood*	500	10	0.35	4,470	5,660
Brick	1,800	14	0.2	2,790	2,940

*Average characteristics of glulam.

Table 1.1. *Elastic characteristics of some building materials and velocity values for waves in uniaxial stress state and uniaxial strain state*

1.3.3. *The d'Alembert solution*

There are several ways of explaining the wave equation solutions that we have just seen. The propagation equation of type [1.36] or [1.39] can also be written as [1.40] (note the speed C):

$$\left(\frac{\partial}{\partial x}-\frac{1}{C}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}+\frac{1}{C}\frac{\partial}{\partial t}\right)U=0 \qquad [1.40]$$

It is possible to make the following change in variables [1.41]:

$$U(\xi,\varsigma),\ \xi=t-\frac{x}{C}\ \text{and}\ \varsigma=t+\frac{x}{C} \qquad [1.41]$$

In this case, equation [1.40] leads to [1.42]:

$$\frac{\partial^2 U}{\partial\,\xi\,\partial\,\varsigma}=0 \qquad [1.42]$$

The solution can be written as the sum of two functions of one variable:

$$U(\xi,\varsigma)=F(\xi)+G(\varsigma),\ U(x,t)=f\left(t-\frac{x}{C}\right)+g\left(t+\frac{x}{C}\right) \qquad [1.43]$$

Functions *f* and *g* are determined by the initial conditions. Figure 1.5 shows strain fields at two instants if the motion is described by a function of type *f*. This function is called a progressive wave, as it describes a displacement field or strain, which moves in a positive direction at speed *C*. A type function *g* is called a regressive wave, because it corresponds to a field that moves in the negative direction.

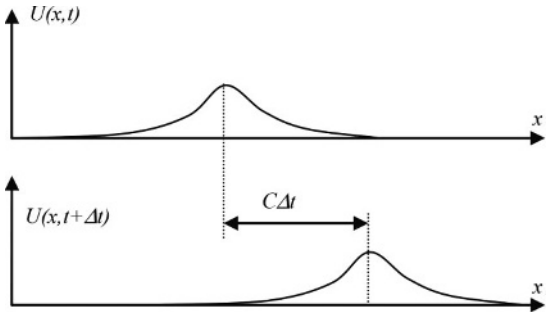

Figure 1.5. *Example of a displacement field at two instances separated by Δt, for a progressive wave represented by an f (t – x/C) function*

This form of writing can be used for spherical *P-waves*. A wave that corresponds to a movement from the origin of the coordinate can be written as [1.44]:

$$u(r,t) = \frac{1}{r} f\left(t - \frac{r}{C_P}\right) \qquad [1.44]$$

1.4. Harmonic waves

1.4.1. *Definitions*

Propagation equation solutions can also be sought in harmonic form. That is, the functions *f* and *g* from the previous section are written as sums of harmonic functions [1.45]:

$$\begin{aligned} &U(x,t) = A\ \cos\left(kx - \omega t\right) = A\cos k\left(x - Ct\right) \text{ or} \\ &U(x,t) = A\ \Re e\left(e^{i(kx-\omega t)}\right) \end{aligned} \qquad [1.45]$$

A number of variables associated with the propagation of the harmonic wave are defined as follows:

– phase $(kx - \omega t)$;

– phase velocity $C = \frac{\omega}{k}$;

– wavelength $\lambda = \frac{2\pi}{k}$;

– number of waves $n = \frac{1}{\lambda}$.

Generally, any function can be represented as a sum of harmonic functions via the Fourier transformation [1.46]:

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U^*(x,\omega)e^{-i\omega t}d\omega \qquad [1.46]$$

The previously seen unidirectional waves can be written as the sum of harmonic waves. Another example for writing waves in this form is that of Rayleigh waves (Figure 1.6). These waves correspond to vertical surface motion. The motion is sinusoidal at the surface and absorbed exponentially with depth [1.47]:

$$\underline{U}\begin{pmatrix} U_1 = Ae^{-x_2+ik(x_1-C_R t)} \\ U_2 = Be^{-x_2+ik(x_1-C_R t)} \\ U_3 = 0 \end{pmatrix} \qquad [1.47]$$

The speed of Rayleigh waves is calculated by expressing stress conditions on the surface as zero. This leads to equation [1.48]. The Rayleigh wave speed is slightly less than that of S-waves:

$$\left(2 - \frac{C_R^2}{C_S^2}\right)^2 - 4\sqrt{\left(1 - \frac{C_R^2}{C_P^2}\right)\left(1 - \frac{C_R^2}{C_S^2}\right)} = 0$$

$$0{,}862 < \frac{C_R}{C_S} < 0{,}955 \quad \left(\frac{C_R}{C_S} \approx \frac{0{,}862 + 1{,}14\,\nu}{1+\nu}\right) \qquad [1.48]$$

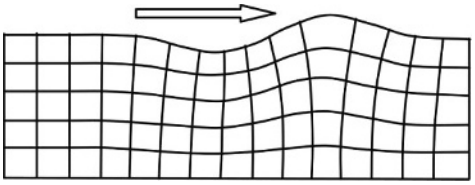

Figure 1.6. *Motion and strains associated with Rayleigh waves*

1.4.2. *Wave dispersion*

When expressed as the superposition of harmonic waves, it is possible that phase velocity is constant (as in the examples we have just seen) or that it depends on wavelength or pulse. The dispersion relation is the function that connects phase velocity to wavelength or pulse ($C(\omega)$ or $C(\lambda)$ or $k(\omega)$). The wave is then formulated according to expression [1.49]:

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U^*(x,\omega)e^{-i\omega t}d\omega \quad U^*(x,\omega) = W^*(\omega)e^{ik(\omega)x} \qquad [1.49]$$

If the phase velocity is not constant, independent of pulse or wavelength, a movement propagates by "deforming" itself because all its harmonic components do not propagate at the same speed. This is called dispersion. As previously seen in uniaxial plane waves, there is no dispersion because wave velocity is constant. For uniaxial stress waves, we get the dispersion equation by introducing the harmonic solution [1.45] into the equation of motion [1.36]. Then, we obtain equation [1.50], which shows that the phase velocity is constant:

$$Ek^2 - \rho\omega^2 = 0 \quad C = \frac{\omega}{k} = \sqrt{\frac{E}{\rho}} \qquad [1.50]$$

The dispersion phenomenon occurs when movements are the solutions of equations that are a little more complex than those we have already seen. These situations occur when the motion in certain solids is studied. In Chapter 8, we will see the case of beam flexion. An interesting case, important for its practical application, is the case of strain waves in rods.

1.4.3. *Dispersion of waves in a rod*

The strain wave rod model, previously seen with the uniaxial stress state hypothesis, is an approximation to the motion occurring in an actual rod. In practice, this model is widely used. The description of motion can be improved by taking into account the radial strain that occurs under the Poisson effect [BAN 41, MIN 60]. A first approach is, quite simply, the "Love-Rayleigh" wave. A detailed study of waves in rods was given by Pochhammer [POC 76] and Chree [CHR 89]. These descriptions lead to dispersed waves (Figure 1.7).

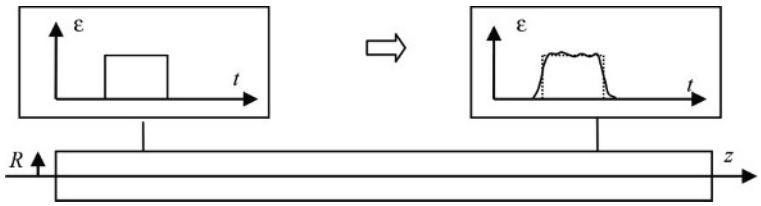

Figure 1.7. *Modifying the shape of a wave by dispersion effect. Example of two-point strain measurements, due to the propagation of a pressure wave in a rod of radius R*

1.4.3.1. *The "Love-Rayleigh" solution*

The idea is to take lateral movement into account by considering its contribution in the expression of kinetic energy. To do this, a hypothesis on the velocity field is made [1.51]. A radial displacement is associated with axial strain (along z), proportional to the radius:

$$\underline{U}\left(\underline{X},t\right)=\underline{U}\left(r,\theta,z,t\right)=\begin{pmatrix}U_r\left(r,z,t\right)\\U_\theta=0\\U_z\left(z,t\right)\end{pmatrix}\qquad U_r=-\nu r\frac{\partial U_z}{\partial z}\qquad[1.51]$$

The kinetic energy of a rod section of unit length dz can be calculated by expression [1.52]:

$$E_c=\frac{1}{2}\int_0^R 2\pi\rho\left(\frac{\partial U_r}{\partial t}\right)^2 rdr+\frac{1}{2}\rho\pi R^2\left(\frac{\partial U_z}{\partial t}\right)^2\qquad[1.52]$$

Taking the above kinetic hypothesis into account, we eliminate radial displacement of the expression and obtain the expression of energy [1.53]:

$$E_c = \frac{1}{2}\int_0^R 2\pi\rho v^2 \left(\frac{\partial^2 U_z}{\partial z \partial t}\right)^2 r^3 dr + \frac{1}{2}\rho\pi R^2 \left(\frac{\partial U_z}{\partial t}\right)^2 \qquad [1.53]$$

Integration gives expression [1.54]:

$$E_c = \frac{1}{4}\pi\rho v^2 R^4 \left(\frac{\partial^2 U_z}{\partial z \partial t}\right)^2 + \frac{1}{2}\rho\pi R^2 \left(\frac{\partial U_z}{\partial t}\right)^2 \qquad [1.54]$$

The kinetic energy theorem indicates that the power of the forces applied to the element is equal to the kinetic energy time derivative. The power of the forces on a single slice is expressed as [1.55]:

$$\Pi_e = \pi R^2 \frac{\partial \sigma}{\partial z}\frac{\partial U_z}{\partial t} = E\pi R^2 \frac{\partial^2 U_z}{\partial z^2}\frac{\partial U_z}{\partial t} \qquad [1.55]$$

The kinetic energy time derivative is expressed as [1.56]:

$$\frac{\partial E_c}{\partial t} = \frac{1}{2}\pi\rho v^2 R^4 \frac{\partial^2 U_z}{\partial z \partial t}\frac{\partial^3 U_z}{\partial z \partial t^2} + \rho\pi R^2 \frac{\partial U_z}{\partial t}\frac{\partial^2 U_z}{\partial t^2} \qquad [1.56]$$

Expression of the first term is changed by noting that we can consider relationship [1.57]:

$$\frac{\partial^2 U_z}{\partial z \partial t}\frac{\partial^3 U_z}{\partial z \partial t^2} = \int \frac{\partial^3 U_z}{\partial z^2 \partial t}\frac{\partial^3 U_z}{\partial z \partial t^2} dz + \int \frac{\partial^2 U_z}{\partial z \partial t}\frac{\partial^4 U_z}{\partial z^2 \partial t^2} dz \qquad [1.57]$$

Integration by parts of the last term of expression [1.57] leads to equation [1.58]:

$$\frac{\partial^2 U_z}{\partial z \partial t}\frac{\partial^3 U_z}{\partial z \partial t^2} = \frac{\partial U_z}{\partial t}\frac{\partial^4 U_z}{\partial z^2 \partial t^2} \qquad [1.58]$$

Taking this result into account, application of the kinetic energy theorem leads to the equation of motion [1.59]:

$$\frac{\rho\nu^2R^2}{2}\frac{\partial^4U_z}{\partial z^2\partial t^2}+\rho\frac{\partial^2U_z}{\partial t^2}=E\frac{\partial^2U_z}{\partial z^2} \qquad [1.59]$$

The equation of wave dispersion associated with this equation of motion takes the form [1.60]:

$$\frac{\rho\nu^2R^2}{2}k^2\omega^2+\rho\omega^2=Ek^2 \qquad [1.60]$$

With this modeling, there is wave dispersion, since speed depends on the number of waves according to formula [1.61]:

$$C=\frac{C_0}{\sqrt{1+\frac{\nu^2R^2k^2}{2}}} \qquad [1.61]$$

The change in velocity as a function of wave number is shown in Figure 1.7.

1.4.3.2. *The Pochhammer and Chree solution*

A solution of the Navier equation in a cylindrical domain of radius R was obtained by Pochhammer and Chree. The solution is sought as a sum of harmonics according to [1.49], but in three-dimensional space, taking into account the symmetry of revolution [1.62]. This is done using the Fourier transformation (denoted by *):

$$\underline{U}(\underline{X},t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\underline{U}^*(r,z,\omega)e^{-i\omega t}d\omega \quad \underline{U}^*(r,z,\omega)=\underline{W}^*(r,\omega)e^{ik(\omega)z} \qquad [1.62]$$

The elastic behavior can be expressed in the frequency domain [1.63]:

$$\underline{\underline{\sigma}}^*(\omega)=\lambda\, tr(\underline{\underline{\varepsilon}}^*(\omega))\underline{\underline{1}}+2\mu\,\underline{\underline{\varepsilon}}^*(\omega) \qquad [1.63]$$

The equation of motion becomes [1.64]:

$$\mu \underline{\Delta U}^* + (\lambda + \mu) \underline{\nabla} (div \underline{U}^*) + \underline{f} = -\rho \omega^2 \underline{U}^* \qquad [1.64]$$

The Helmholtz decomposition is used [1.65]:

$$\underline{U}^*(\underline{X}, \omega) = \nabla \Phi^*(\underline{X}, \omega) + rot \underline{\Psi}^*(\underline{X}, \omega) \qquad [1.65]$$

This decomposition leads to two propagation equations [1.66]:

$$\begin{aligned} &\nabla^2 \boldsymbol{\phi}^*(\underline{X}, \boldsymbol{\omega}) + \frac{\boldsymbol{\omega}^2}{C_1^2} \boldsymbol{\phi}^*(\underline{X}, \boldsymbol{\omega}) = 0 \quad \text{and} \\ &\nabla^2 \underline{H}^*(\underline{X}, \boldsymbol{\omega}) + \frac{\boldsymbol{\omega}^2}{C_2^2} \underline{H}^*(\underline{X}, \boldsymbol{\omega}) = 0 \end{aligned} \qquad [1.66]$$

The "potential" functions are logically formulated using a cylindrical coordinate system [1.67]:

$$\begin{aligned} &\Phi^*(\underline{X}, \omega) = \varphi(r, \theta, \omega)\, e^{ikz} \\ &\underline{\Psi}^*(\underline{X}, \omega) = \left(\psi_r(r, \theta, \omega) \underline{e_r} + \psi_\theta(r, \theta, \omega) \underline{e_\theta} + \psi_z(r, \theta, \omega) \underline{e_z}\right) e^{ikz} \end{aligned} \qquad [1.67]$$

Given the symmetry of the revolution problem, it is possible to simplify these expressions and write the potentials as [1.68]:

$$\begin{aligned} &\Phi^*(\underline{X}, \omega) = \varphi(r, \omega)\, e^{ikz} \\ &\underline{\Psi}^*(\underline{X}, \omega) = \psi_\theta(r, \omega)\, e^{ikz} \underline{e_\theta} \end{aligned} \qquad [1.68]$$

Taking the limiting factors into account (zero stress on the cylindrical surface of rod $r = R$), the resolution leads to writing the analytical expression of these functions as [1.69]:

$$\begin{aligned} &\Phi^*(\underline{X}, \omega) = A(\omega) J_0(\alpha r) e^{ikz} \\ &\underline{\Psi}^*(\underline{X}, \omega) = B(\omega) J_1(\beta r) e^{ikz} \end{aligned} \quad \text{where} \quad \begin{aligned} \alpha^2 &= \frac{\rho \omega^2}{\lambda + 2\mu} - k^2 \\ \beta^2 &= \frac{\rho \omega^2}{\mu} - k^2 \end{aligned} \qquad [1.69]$$

J_0 and J_1 are Bessel functions of order 0 and 1. By introducing the displacement field obtained from these functions into the Navier equation, we obtain the dispersion equation [1.70]:

$$\begin{aligned}&(2\alpha/R)(\beta^2+k^2)J_1(\alpha.R)J_1(\beta.R)\\&-(\beta^2-k^2)^2J_0(\alpha.R)J_1(\beta.R)\ -4k^2\alpha.\beta.J_1(\alpha.R)J_0(\beta.R)=0\end{aligned} \qquad [1.70]$$

Figure 1.8 shows the variation in wave speed depending on wave number and radius of the rod.

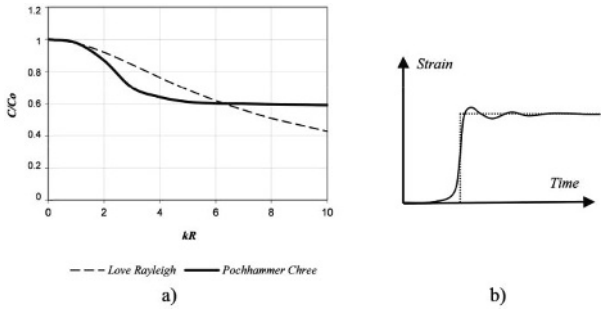

Figure 1.8. *a) Change in wave speed in a rod of radius R, according to the parameter kR, for the Love-Rayleigh model (dotted line) and for the Pochhammer Chree model (solid line); b) mapping the effect of dispersion on a strain wave that was originally a discontinuity*

Figure 1.8(b) shows, qualitatively, the effect of this dispersion. If the temporal evolution of strain at the end of a bar, subsequent to a shock, is represented by a discontinuity (dashed line). The temporal evolution of strain at another point, far from the end, is shown by the solid line.

1.5. Viscoelasticity

1.5.1. *Representation of rheological behavior*

1.5.1.1. *Representation and models*

When considering the problem of dynamics of solids, it is logical to question the possible dependence of the material behavior in physical time, through rate of strain. In this section, we discuss some

ways to take this effect into account when it exists. Linear behavior is developed using models and common tools. An application will be made in order to acquire equations of motion for viscoelastic bodies. Nonlinear effects will be discussed in Chapter 3. The functional representation of behavior is the most general way to express stress as a function of strain and the history of the material [1.71]. We can obviously express strain based on the history of stress using the inverse function:

$$\underline{\underline{\sigma}}(t) = \Re\left(\underline{\underline{\varepsilon}}(\tau)\right) \qquad \underline{\underline{\varepsilon}}(t) = \Re^{-1}\left(\underline{\underline{\sigma}}(\tau)\right) \qquad \tau \in]-\infty, t] \qquad [1.71]$$

Each variable is determined by its value at an initial time and the loading history between the initial time and time t. The material "remembers" all the stresses that have been imposed on it in the past. This memory may be tempered by progressive "oversight", meaning that stress $\sigma(\tau)$ has increasingly less influence on the current strain as τ gets further away from t. The formula should respect the principle of non-duality (or causality). That is, the stress at a given time may depend on the rate of strain at the same time. However, the strain at a given time may not depend on the stress rate at the same time. Thus, we find that a discontinuity in the rate of strain can cause a discontinuity in stress, but a discontinuity in the rate of stress cannot cause a discontinuity in strain. In practice, a functional representation of this is difficult because of the need to remember strain history. Attempts have been made to replace strain history by taking into account stress and strain derivative values at a considered time (differential equations and rheological models), or internal variables (variables of state). One idea is to write stress as a function of strain and its derivatives. Unfortunately, this does not describe some rheological aspects. More generally, it is possible to write an equation linking stress, strain and their derivatives [1.72]. This formula is less stringent than the functional formula and not all equations are able to represent a behavior:

$$L\left(\sigma, \dot{\sigma}, ..., \varepsilon, \dot{\varepsilon}, ...\right) = 0 \qquad [1.72]$$

1.5.1.2. *Construction of rheological models*

In rheological models, the material is compared to an assembly of simple mechanical elements [PER 69]. These analogous elements are only symbols and are unrelated to the constitution of the material. Figure 1.9 shows three typical rheological models to represent viscoelasticity. The purpose is to describe a phenomenological behavior by the construction of its thermodynamically relevant model. The mechanical elements may be translated by linear or nonlinear relationships. When behavior is linear, it is often useful to replace the functions of time (stress and strain) by their operational images using the Laplace–Carson transformation. The Laplace–Carson transformation is very similar to the Laplace transformation. The image of a function $f(t)$ is the Laplace transformation of the time derivative thereof [1.73]. It will be denoted by $f(p)$ (p complex):

$$\overline{f}(p) = p\int_0^{\infty} e^{-pt} f(t)\,dt \qquad [1.73]$$

This transformation has the same properties as the Laplace transformation. The Heaviside function, which is often used in rheology, has the operational image of the unit.

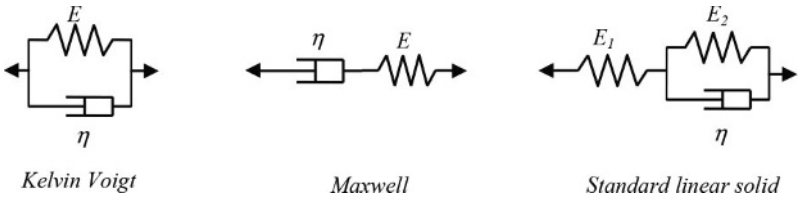

Figure 1.9. *Representation of three classical rheological models*

1.5.2. *Creep and relaxation functions*

1.5.2.1. *Formulation*

The creep function $f(t)$ is the strain response of a solicitation represented by a stress slot [1.74]:

$$f(t) = \varepsilon(t) \;\; if \;\; \sigma(t) = H(t) \qquad [1.74]$$

With operational images, creep function is expressed as a ratio [1.75]:

$$\bar{f}(p) = \frac{\bar{\varepsilon}(p)}{\bar{\sigma}(p)} \qquad [1.75]$$

Behavior can be obtained from the creep function. We obtain the Boltzmann equations [1.76] (square brackets indicate discontinuity):

$$\begin{aligned} \varepsilon(t) &= \sigma(0) f(t) + \int_0^t f(t-\tau) \frac{d\sigma}{d\tau}(\tau) d\tau + \sum_{i=1}^{n} f(t-t_i) [[\sigma(t_i)]] \\ \varepsilon(t) &= \sigma(t) f(0) + \int_0^t \sigma(t-\tau) \frac{df}{d\tau}(\tau) d\tau + \sum_{i=1}^{n} f(t-t_i) [[\sigma(t_i)]] \end{aligned} \qquad [1.76]$$

The relaxation function $r(t)$ is the stress response to a solicitation represented by a strain slot [1.77]:

$$r(t) = \sigma(t) \quad if \quad \varepsilon(t) = H(t) \qquad [1.77]$$

The operational image of the relaxation function is expressed by the ratio [1.78]:

$$\bar{r}(p) = \frac{\bar{\sigma}(p)}{\bar{\varepsilon}(p)} \qquad [1.78]$$

Use of the relaxation function is symmetrical to that of the creep function, and we obtain the behavioral relationships [1.79]:

$$\begin{aligned} \sigma(t) &= \varepsilon(0) r(t) + \int_0^t r(t-\tau) \frac{d\varepsilon}{d\tau}(\tau) d\tau + \sum_{i=1}^{n} r(t-t_i) [[\varepsilon(t_i)]] \\ \sigma(t) &= \varepsilon(t) r(0) + \int_0^t \varepsilon(t-\tau) \frac{dr}{d\tau}(\tau) d\tau + \sum_{i=1}^{n} r(t-t_i) [[\varepsilon(t_i)]] \end{aligned} \qquad [1.79]$$

1.5.2.2. *Three-dimensional aspect*

In the case of isotropy, the definition of relaxation (or creep) involves only two independent scalar functions. Relaxation is written as [1.80]:

$$\underline{\underline{\boldsymbol{\sigma}}}(t) = \boldsymbol{\lambda}(t) \left(trace \underline{\underline{\boldsymbol{\varepsilon}}}_0 \right) \underline{\underline{1}} + 2\boldsymbol{\mu}(t) \underline{\underline{\boldsymbol{\varepsilon}}}_0 \quad if \quad \underline{\underline{\boldsymbol{\varepsilon}}}(t) = \underline{\underline{\boldsymbol{\varepsilon}}}_0 H(t) \qquad [1.80]$$

– $2\mu(t)$ is the relaxation function in simple shear;

– $\lambda(t)+2\mu(t)$ is the relaxation function in simple extension.

Therefore, it returns to loading, regardless of formula [1.81]:

$$\begin{aligned}\underline{\underline{\sigma}}(t) &= \lambda(t)\left(trace\,\underline{\underline{\varepsilon}}(0)\right)\underline{\underline{\mathfrak{I}}} + 2\mu(t)\,\underline{\underline{\varepsilon}}(0) \\ &+ \int_0^t \left(\frac{d\lambda}{d\tau}(t-\tau)\left(trace\left(\underline{\underline{\varepsilon}}(\tau)\right)\right)\underline{\underline{\mathfrak{I}}} + 2\frac{d\mu}{d\tau}(t-\tau)\,\underline{\underline{\varepsilon}}(\tau)\right) d\tau\end{aligned} \quad [1.81]$$

This equation is much simpler when written in operational form [1.82]:

$$\overline{\underline{\underline{\sigma}}} = \overline{\lambda}\left(trace\,\overline{\underline{\underline{\varepsilon}}}\right)\underline{\underline{1}} + 2\overline{\mu}\;\overline{\underline{\underline{\varepsilon}}} \quad [1.82]$$

Independent creep functions are also two in number and we can write the inverse relationship [1.83]:

$$\overline{E}\,\overline{\underline{\underline{\varepsilon}}} = (1+\overline{\nu})\,\overline{\underline{\underline{\sigma}}} - \overline{\nu}\;trace\left(\overline{\underline{\underline{\sigma}}}\right)\underline{\underline{1}} \quad [1.83]$$

Since the definition of behavior requires data from two relaxation (or creep) functions, we can use the following two functions:

– $K(t)$ relaxation function under isotropic (hydrostatic) load;

– $G(t)$ relaxation function under split load.

The spherical part and the deviatoric part of behavior are then separately identified. K and G functions can be from different rheological models [1.84]. Generally, spherical behavior is solid-like, while deviatoric behavior can be fluid-like:

$$\begin{aligned}&\overline{p} = 3\overline{K}\,\overline{\theta} \quad \Leftrightarrow \quad p = 3\theta(0)K(t) + 3\int_0^t \frac{dK}{d\tau}(t-\tau)\,\theta(\tau)\,d\tau \quad \left(\theta = trace\underline{\underline{\varepsilon}}\right) \\ &\overline{s}_{ij} = 2\overline{G}\,\overline{e}_{ij} \quad \Leftrightarrow \quad s_{ij} = 2e_{ij}(0)\,G(t) + 2\int_0^t \frac{dG}{d\tau}(t-\tau)\,e_{ij}(\tau)\,d\tau\end{aligned} \quad [1.84]$$

1.5.3. *Rheological models*

1.5.3.1. *Kelvin–Voigt model*

This model is obtained by connecting a spring and a shock absorber in parallel (Figure 1.9). The differential equation of behavior and its operational image are immediate [1.85]:

$$\sigma(t) = E\varepsilon(t) + \eta\,\dot{\varepsilon}(t) \quad \text{and} \quad \bar{\sigma}(p) = (E + p\eta)\,\bar{\varepsilon}(p) \qquad [1.85]$$

It is easy to solve this equation to express strain as a function of stress [1.86]:

$$\varepsilon(t) = \varepsilon_0\, e^{-\frac{E}{\eta}(t-t_0)} + \frac{1}{\eta}\int_{t_0}^{t} e^{-\frac{E}{\eta}(t-\tau)}\sigma(\tau)\,d\tau \qquad [1.86]$$

In the simple model, the creep function is obtained from its operational image. A characteristic creep time τ appears [1.87]:

$$\bar{f}(p) = \frac{\bar{\varepsilon}(p)}{\bar{\sigma}(p)} = \frac{1}{E + \eta\, p}$$

$$f(t) = E\left(1 - e^{-\frac{t}{\tau}}\right) \qquad \text{where} \quad \tau = \frac{\eta}{E} \qquad [1.87]$$

We can build a generalized model with multiple creep times, which is to imagine putting several simple Kelvin–Voigt models in series, with different characteristics (E_i, η_i). The creep function can be written as [1.88]:

$$f(t) = \frac{1}{E_0} + \sum_{i=1}^{n}\frac{1}{E_i}\left(1 - e^{-\frac{t}{\tau_i}}\right) + \frac{t}{\eta_\infty} \qquad \text{with} \quad \tau_i = \frac{\eta_i}{E_i} \qquad [1.88]$$

The first term exists if there is a spring without a shock absorber in series, and the last term is present if there is a shock absorber without a spring in series (in the latter case, we get fluid-like behavior). This model is characterized by a spectrum of E_i stiffness values associated with characteristic times τ_i. A more general model can be considered

using a continuous distribution of stiffness versus creep time $j(\tau)$. The creep function takes the form [1.89]:

$$f(t) = \frac{1}{E_0} + \int_0^{+\infty} j(\tau)\left(1 - e^{-\frac{t}{\tau}}\right) d\tau + \frac{t}{\eta_\infty} \qquad [1.89]$$

The function $j(\tau)$ can be deduced from the experimental measure of $f(t)$.

1.5.3.2. *Maxwell model*

This model is obtained by connecting a spring and a shock absorber in series (Figure 1.9). The strains of the spring and shock absorber are added. The behavior is formulated as [1.90]:

$$\dot{\varepsilon}(t) = \frac{1}{E}\dot{\sigma}(t) + \frac{1}{\eta}\sigma(t) \qquad p\bar{\varepsilon}(p) = \left(\frac{p}{E} + \frac{1}{\eta}\right)\bar{\sigma}(p) \qquad [1.90]$$

It is also easy to solve and express strain depending on stress and stress as a function of strain [1.91]:

$$\varepsilon(t) = \varepsilon_0 + \frac{1}{E}\left(\sigma(t) - \sigma_0\right) + \frac{1}{\eta}\int_{t_0}^{t} \sigma(\tau)\, d\tau$$

$$\sigma(t) = \left(\sigma_0 - E\varepsilon_0\right) e^{-\frac{E}{\eta}(t-t_0)} - \frac{E^2}{\eta}\int_{t_0}^{t} e^{-\frac{E}{\eta}(t-\tau)}\varepsilon(\tau)\, d\tau \qquad [1.91]$$

The relaxation function is useful to describe the simple model [1.92]:

$$\bar{r}(p) = \frac{1}{\frac{1}{E} + \frac{1}{\eta p}} \quad \Rightarrow \quad r(t) = E\, e^{-\frac{t}{\tau}} \quad avec \quad \tau = \frac{\eta}{E} \qquad [1.92]$$

Let us consider placing several simple models with different features in parallel. We then get a stiffness spectrum based on relaxation time. The relaxation function takes the form [1.93]:

$$r(t) = E_\infty + \sum_{i=1}^{n} E_i e^{-\frac{t}{\tau_i}} \qquad [1.93]$$

The first term exists if we place a spring alone in parallel (we obtain solid-like behavior). A generalization of the model is obtained by considering a stiffness density continuous distribution as a function of relaxation time [1.94]:

$$r(t) = E_\infty + \int_0^{+\infty} g(\tau) e^{-\frac{t}{\tau}} d\tau \qquad [1.94]$$

1.5.3.3. *Standard linear model*

This rather general model is also called the Zener model (Figure 1.9). The behavior is written as [1.95]:

$$\begin{aligned} &(E_1 + E_2 + p\eta)\,\bar{\sigma}(p) = E_1 (E_2 + p\eta)\,\bar{\varepsilon}(p) \\ &\sigma(t) + \frac{\eta}{E_1 + E_2}\dot{\sigma}(t) = \frac{E_1 E_2}{E_1 + E_2}\left(\varepsilon(t) + \frac{\eta}{E_2}\dot{\varepsilon}(t)\right) \end{aligned} \qquad [1.95]$$

This differential equation is often written as [1.96]:

$$\sigma(t) + a\dot{\sigma}(t) = E\left(\varepsilon(t) + b\dot{\varepsilon}(t)\right) \qquad [1.96]$$

But it only represents a behavior under the conditions E, a, $b > 0$ and $a < b$.

A more general linear model is represented by the linear differential equation [1.97] (with some restrictions linked to the causality condition):

$$L_1(\sigma) = L_2(\varepsilon) \quad \Leftrightarrow \quad a_0\sigma + a_1\dot{\sigma} + ...a_i\sigma^{(i)}... = E_0\varepsilon + E_1\dot{\varepsilon} + ...E_i\varepsilon^{(i)}... \qquad [1.97]$$

The behavior can be written in operational form [1.98]:

$$a(p)\,\bar{\sigma}(p) = E(p)\,\bar{\varepsilon}(p) \qquad [1.98]$$

As $a(p)$ and $E(p)$ are polynomials, we can decompose the relaxation and creep functions into simple elements [1.99]:

$$\bar{r}(p)=\frac{E(p)}{a(p)}=E_{\infty}+\sum_{i=1}^{n}E_i\frac{p}{p+\dfrac{1}{\tau_i}}+\eta_0 p$$

$$\bar{f}(p)=\frac{a(p)}{E(p)}=\frac{1}{E_0}+\sum_{i=1}^{n}\frac{1}{E_i}\frac{\dfrac{1}{\tau_i}}{p+\dfrac{1}{\tau_i}}+\frac{1}{p\eta_{\infty}} \qquad [1.99]$$

Returning to the original formulas, it can be noted that an equivalence exists between this model and that of the generalized Kelvin–Voigt and generalized Maxwell models.

1.5.4. *Complex modulus*

With linear behavior, if the stress is periodic, the response is also periodic. If the strain is harmonic, the Fourier transformation [1.100] is used:

$$\varepsilon(t)=\Re e\left(\varepsilon^*(t)\right) \qquad \varepsilon^*(t)=\varepsilon_0 e^{-i\omega t} \qquad \bar{\varepsilon}^*=\varepsilon_0\frac{p}{p-i\omega} \qquad [1.100]$$

If the material is a viscoelastic solid, the response is [1.101]:

$$\bar{\sigma}^*(p)=\varepsilon_0\left(\frac{E_{\infty}p}{p-i\omega}+\sum_{j=1}^{n}E_j\frac{p^2}{\left(p+\dfrac{1}{\tau_j}\right)(p-i\omega)}\right) \qquad [1.101]$$

This expression can easily be developed into rational fractions [1.102]:

$$\bar{\sigma}^*=\varepsilon_0\frac{\bar{r}(i\omega)p}{p-i\omega}+\varepsilon_0\sum_{j=1}^{n}E_j\frac{\left(\dfrac{1}{1+i\omega\tau_j}\right)p}{p+\dfrac{1}{\tau_j}} \qquad [1.102]$$

We then return to the original function [1.103]:

$$\sigma^*(t) = r(i\omega)\,\varepsilon_0\,e^{i\omega t} + \varepsilon_0 \sum_{j=1}^{n} E_j \left(\frac{1}{1+i\omega\tau_j}\right) e^{-\frac{t}{\tau_j}} \qquad [1.103]$$

The second term represents the transitional state that tends toward 0. Steady state is represented by [1.104]:

$$\sigma^*(t) = \sigma_0 e^{i\omega t} \qquad \text{where} \quad \sigma_0 = r(i\omega)\varepsilon_0 \qquad [1.104]$$

The $r(i\omega)$ function is the complex modulus E^*. It can be written as [1.105]:

$$E^*(\omega) = \tilde{E}(\omega) e^{-i\varphi(\omega)} \qquad [1.105]$$

Using harmonic tests and a frequency sweep, we can access curves representing the modulus and the complex modulus phase experimentally. We can also define the dynamic modulus E_d and the loss factor β as [1.106]:

$$E^*(\omega) = E_d(\omega)\left(1 + i\beta(\omega)\right) \qquad [1.106]$$

For the three models described above, the complex modulus parameters can be explained:

– Kelvin–Voigt model [1.107]:

$$E_d = E \qquad \beta = \frac{\eta\omega}{E} \qquad [1.107]$$

– Maxwell model [1.108]:

$$E_d = \frac{E}{1+\left(E/\eta\omega\right)^2} \qquad \beta = \frac{E}{\eta\omega} \qquad [1.108]$$

– Standard linear model [1.109]:

$$E_d = \frac{E_1\left(E_2\left(E_1+E_2\right)+\eta^2\omega^2\right)}{\left(E_1+E_2\right)^2+\eta^2\omega^2} \quad \beta = \frac{\eta\omega E_1}{E_2\left(E_1+E_2\right)+\eta^2\omega^2} \qquad [1.109]$$

1.5.5. *Waves in viscoelastic media*

1.5.5.1. *Propagation equation*

As we formulated motion in a continuous elastic medium in section 1.2, we can formulate and study motion in a continuous viscoelastic medium [SCH 63, BLA 93]. Generally, we obtain an equation of motion by linking the viscoelastic behavior equation with the dynamics equation [1.110]:

$$\sum_{i=1}^{n} a_i \frac{\partial^i \sigma}{\partial t^i} = \sum_{i=1}^{n} E_i \frac{\partial^i \varepsilon}{\partial t^i} \qquad \frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 U}{\partial t^2} \quad \varepsilon = \frac{\partial U}{\partial x} \qquad [1.110]$$

By eliminating the stress and strain variables, the equation of motion is a differential equation of partial displacement derivatives [1.111]:

$$\sum_{i=1}^{n} E_i \frac{\partial^{i+2} U}{\partial x^2 \partial t^i} = \rho \sum_{i=1}^{n} a_i \frac{\partial^{i+2} U}{\partial t^{i+2}} \qquad [1.111]$$

It is also possible to express the differential equation as a function of stress [1.112]:

$$\sum_{i=1}^{n} E_i \frac{\partial^{i+2} \sigma}{\partial x^2 \partial t^i} = \rho \sum_{i=1}^{n} a_i \frac{\partial^{i+2} \sigma}{\partial t^{i+2}} \qquad [1.112]$$

For the Kelvin–Voigt behavior model, the equations of motion are written as [1.113]:

$$E \frac{\partial^2 U}{\partial x^2} + \eta \frac{\partial^3 U}{\partial x^2 \partial t} = \rho \frac{\partial^2 U}{\partial t^2} \quad \text{and} \quad E \frac{\partial^2 \sigma}{\partial x^2} + \eta \frac{\partial^3 \sigma}{\partial x^2 \partial t} = \rho \frac{\partial^2 \sigma}{\partial t^2} \quad [1.113]$$

For the Maxwell behavior model, the equations of motion are written as [1.113]:

$$\frac{\partial^3 U}{\partial x^2 \partial t} = \frac{\rho}{\eta} \frac{\partial^2 U}{\partial t^2} + \frac{\rho}{E} \frac{\partial^3 U}{\partial t^3} \quad \text{and} \quad \frac{\partial^2 \sigma}{\partial x^2} = \frac{\rho}{\eta} \frac{\partial \sigma}{\partial t} + \frac{\rho}{E} \frac{\partial^2 \sigma}{\partial t^2} \quad [1.114]$$

The equations of motion can be addressed in operational form (using the Laplace transformation). The equation of behavior is written as [1.115]:

$$\sum s^i a_i \,\bar{\sigma} = \sum s^i E_i \,\bar{\varepsilon} \qquad \bar{\sigma} = \bar{E}(s)\bar{\varepsilon} \qquad \bar{E}(s) = \frac{\sum s^i E_i}{\sum s^i a_i} \quad [1.115]$$

The dynamics principle is written as [1.116]:

$$\frac{\partial \bar{\sigma}}{\partial x} = \rho s^2 \bar{U} \quad [1.116]$$

We deduce the equation of motion image [1.117]:

$$\frac{\partial^2 \bar{U}}{\partial x^2} = \frac{\rho s^2}{\bar{E}(s)} \bar{U} \quad [1.117]$$

The general solution of this equation is, in its operational form [1.118]:

$$\bar{U}(x,s) = A(s)\, e^{-sx\sqrt{\frac{\rho}{\bar{E}(s)}}} + B(s)\, e^{sx\sqrt{\frac{\rho}{\bar{E}(s)}}} \quad [1.118]$$

In Chapter 2, we will see some solutions for movement in viscoelastic solids in case of shocks. The viscoelastic properties of a material are often determined by a complex modulus; we may use a Fourier transformation of the equation of motion. Behavior and the dynamics equation can be written using expressions [1.119]:

$$\sigma^* = E^*(\omega)\varepsilon^* \qquad \frac{\partial \sigma^*}{\partial x} = -\rho \omega^2 U^* \quad [1.119]$$

Thus, the equation of motion is [1.120]:

$$\frac{\partial^2 U^*}{\partial x^2} = -\frac{\rho \omega^2}{E^*(\omega)} U^* \quad [1.120]$$

1.5.5.2. *Dispersion equation*

In viscoelastic media, movement expressed in harmonic form must contain a dissipation term, and in the frequency domain, an elementary wave is formulated according to [1.121]:

$$U^*(x,\omega) = A\, e^{\alpha x + i(kx - \omega t)} \qquad [1.121]$$

The dispersion equation, in complex form, is written as [1.122]:

$$E^*(\alpha + ik)^2 U^* = -\rho\omega^2 U^* \qquad [1.122]$$

This complex equation can also be written as a function of the dynamic modulus and loss factor [1.123]. This corresponds to two real equations [1.124]:

$$E_d(1 + i\beta)(\alpha + ik)^2 = -\rho\omega^2 \qquad [1.123]$$

$$\begin{aligned} E_d\left(k^2 - \alpha^2 + 2\beta\alpha k\right) &= \rho\omega^2 \\ \beta\left(k^2 - \alpha^2\right) - 2\alpha k &= 0 \end{aligned} \qquad [1.124]$$

The general solutions to these equations are [1.125]:

$$k = \omega\sqrt{\frac{\rho}{E_d}}\sqrt{\frac{1+\sqrt{1+\beta^2}}{2\left(1+\beta^2\right)}} \qquad \alpha = \beta\omega\sqrt{\frac{\rho}{E_d}}\sqrt{\frac{1}{2\left(1+\beta^2\right)\left(1+\sqrt{1+\beta^2}\right)}} \qquad [1.125]$$

For the Kelvin–Voigt model, the dynamic modulus and loss factor are expressions [1.126]. This leads to the dispersion and dissipation relationships [1.127]:

$$E_d = E \qquad \beta = \frac{\eta\omega}{E} \qquad [1.126]$$

$$\frac{C}{C_0} = \sqrt{\frac{2\left(1+\beta^2\right)}{1+\sqrt{1+\beta^2}}} \qquad A = \frac{\alpha C_0}{\omega} = \frac{\beta}{\sqrt{2\left(1+\beta^2\right)\left(1+\sqrt{1+\beta^2}\right)}} \qquad [1.127]$$

For the Maxwell model, the dynamic modulus and loss factor are expressions [1.128]. This leads to the dispersion and dissipation relationships [1.129]:

$$E_d = \frac{E}{1+\left(E/\eta\omega\right)^2} \qquad \beta = \frac{E}{\eta\omega} \qquad [1.128]$$

$$\frac{C}{C_0} = \sqrt{\frac{2}{1+\sqrt{1+\beta^2}}} \qquad A = \frac{\alpha C_0}{\omega} = \frac{\beta}{\sqrt{2\left(1+\sqrt{1+\beta^2}\right)}} \qquad [1.129]$$

Figures 1.10 and 1.11 show the speed of waves and the absorption parameter A as a function of dimensionless pulse for Kelvin–Voigt and Maxwell models.

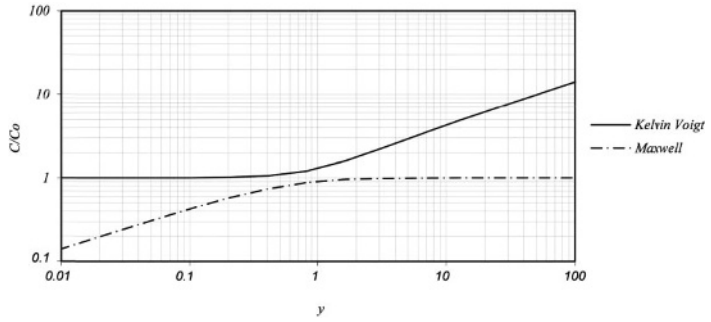

Figure 1.10. *Wave velocity as a function of dimensionless pulse* ($y = \eta\omega/E$) *for the Kelvin–Voigt model and the Maxwell model*

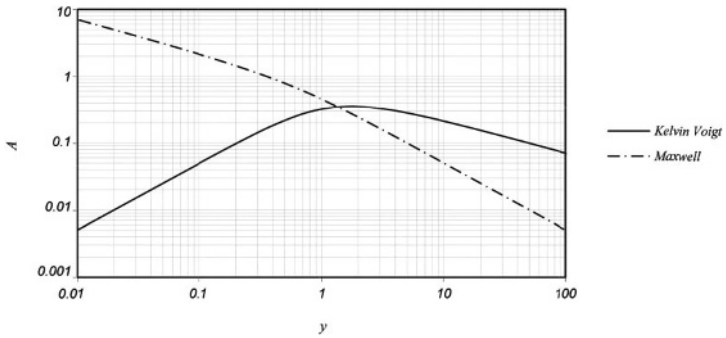

Figure 1.11. *Absorption parameter A as a function of dimensionless pulse* ($y = \eta\omega/E$) *for the Kelvin–Voigt model and the Maxwell model*