1

Linear Elasticity

The purpose of this chapter is to recall the theory of linear elasticity which is the general framework of the following chapters. We consider in the following deformable solids in quasi-static equilibrium (no inertia forces). We introduce hereafter the notations and the vocabulary of a theory which is supposed to be known by the reader.

1.1. Notations

Tensors will be used to represent the physical quantities which describe an elastic solid such as the displacement vector, the strain tensor, the stress tensor, etc. The physical space is endowed with an orthonormal reference $\left(O, \underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3\right)$ where O is the origin and $\underline{\boldsymbol{e}}_i$ is the base vector in direction i. A geometrical point M of the physical space is represented by its coordinates in this reference, that is the components of vector $\underline{OM}$ in the base $\left(\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3\right)$. The following notations will be used to represent the tensors and their components in the base $\left(\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3\right)$:

– latin letters in italic represent scalars: x, y, z, X, Y, Z, ...etc.;

– 2D or 3D vectors, i.e. first-order tensors, are underlined. Latin indices, i, j, k, l... go through $1, 2, 3$ whereas Greek indices, $\alpha, \beta, \gamma, \delta$,... go through $1, 2$. So, $\underline{\boldsymbol{x}} = (x_1, x_2) = (x_\alpha)$ is a 2D vector and $\underline{\boldsymbol{x}} = (x_1, x_2, x_3) = (x_i)$ is a 3D vector. The following equivalent notations of the same vector will be used:

$$\underline{\boldsymbol{x}} = (x_1, x_2, x_3) = (x_i) = x_1\underline{\boldsymbol{e}}_1 + x_2\underline{\boldsymbol{e}}_2 + x_3\underline{\boldsymbol{e}}_3 = x_i\underline{\boldsymbol{e}}_i,$$

where the Einstein convention of summation over repeated indices has been used. This convention will be used in all the continuation;

– the 2D or 3D second-order tensors are underlined with a tilde. So, $\underset{\sim}{\boldsymbol{\sigma}} = (\sigma_{\alpha\beta})$ is a 2D second-order tensor and $\underset{\sim}{\boldsymbol{\sigma}} = (\sigma_{ij})$ is a 3D second-order tensor. All the following notations of the same second-order tensor are equivalent:

$$\underset{\sim}{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = (\sigma_{ij}) = \sum_{i,j=1,2,3} \sigma_{ij} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j = \sigma_{ij} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j,$$

where $\underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j$ is the tensorial (or dyadic) product of vector $\underline{\boldsymbol{e}}_i$ with vector $\underline{\boldsymbol{e}}_j$. We recall that the tensorial product of vector $\underline{\boldsymbol{a}}$ with vector $\underline{\boldsymbol{b}}$ is the second-order tensor $\underline{\boldsymbol{a}} \otimes \underline{\boldsymbol{b}} = (a_i b_j)$;

– fourth-order tensors are underlined with two tildes:

$$\underset{\approx}{\boldsymbol{C}} = C_{ijkl} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j \otimes \underline{\boldsymbol{e}}_k \otimes \underline{\boldsymbol{e}}_l.$$

The following contraction operations will be used:

$$\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}} = x_i y_i, \quad \underset{\sim}{\boldsymbol{\sigma}} . \underline{\boldsymbol{n}} = (\sigma_{ij} n_j), \quad \underset{\sim}{\boldsymbol{p}} \cdot \underset{\sim}{\boldsymbol{q}} = (p_{ik} q_{kj}),$$

$$\underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\boldsymbol{\varepsilon}} = \sigma_{ij} \varepsilon_{ji}, \quad \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\boldsymbol{\varepsilon}} = (C_{ijkl} \varepsilon_{lk});$$

– the norm of a vector or a second-order tensor is denoted as:

$$\left|\underline{\boldsymbol{x}}\right| = \sqrt{\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{x}}} = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \left|\underset{\sim}{\boldsymbol{\sigma}}\right| = \sqrt{{}^T\underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\boldsymbol{\sigma}}}.$$

Here, the (ij) components of the transpose tensor ${}^T\underset{\sim}{\boldsymbol{\sigma}}$ are equal to the (ji) components of $\underset{\sim}{\boldsymbol{\sigma}}$;

– let $X(\underline{\boldsymbol{x}}) = X(x_1, x_2, x_3)$ be a scalar field. The partial derivative of X with respect to x_i is denoted by:

$$\frac{\partial X}{\partial x_i} = X_{,i}.$$

The gradient of X is the vector

$$\boldsymbol{\nabla} X = (X_{,i});$$

– this notation is extended to vector fields. Let $\underline{\boldsymbol{\xi}}(\underline{\boldsymbol{x}}) = \big(\xi_1(\underline{\boldsymbol{x}}), \xi_2(\underline{\boldsymbol{x}}), \xi_3(\underline{\boldsymbol{x}})\big)$ be a vector field. Then, its gradient, denoted as $\boldsymbol{\nabla}\underline{\boldsymbol{\xi}}$, is the second-order tensor $(\xi_{i,j})$. The symmetric part of the gradient, denoted by $\boldsymbol{\nabla}^s\underline{\boldsymbol{\xi}}$, is the second-order tensor given by:

$$\boldsymbol{\nabla}^s\underline{\boldsymbol{\xi}} = \left(\frac{1}{2}\left(\xi_{i,j} + \xi_{j,i}\right)\right).$$

1.2. Stress

A solid body occupying the smooth domain V in an equilibrium state is subjected to internal cohesive forces which maintain its integrity under the action of external forces. According to the Cauchy continuum model theory, the internal forces in the solid can be represented by a second-order tensor so-called *stress field*, usually denoted by $\underset{\sim}{\boldsymbol{\sigma}}\left(\underline{\boldsymbol{x}}\right)$, $\underline{\boldsymbol{x}} \in V$ or more simply $\underset{\sim}{\boldsymbol{\sigma}}$, which is assumed to be sufficiently smooth. The physical interpretation of $\underset{\sim}{\boldsymbol{\sigma}}$ is the following: consider a fictitious plane surface of infinitesimal area δa, centered at point $\underline{\boldsymbol{x}}$, and oriented by the unit normal vector $\underline{\boldsymbol{n}}$ which separates into two sides the material located in the immediate vicinity of point $\underline{\boldsymbol{x}}$: a side + in the direction of $\underline{\boldsymbol{n}}$ and a side – in the opposite direction. Such a surface is called a *facet*. Then, the elementary vector

$$\delta\underline{\boldsymbol{f}} = \underset{\sim}{\boldsymbol{\sigma}}\left(\underline{\boldsymbol{x}}\right)\cdot\underline{\boldsymbol{n}}\delta a \qquad [1.1]$$

represents the resultant force which is applied by the matter situated on the side + of the facet on those situated on the side – (Figure 1.1). Hence, $\underset{\sim}{\boldsymbol{\sigma}}(\underline{\boldsymbol{x}})\cdot\underline{\boldsymbol{n}}$ appear as the limit as δa goes to zero of the ratio of $\delta\underline{\boldsymbol{f}}$ and δa. It should be noted that the Cauchy model assumes that the norm of the resultant moment of the forces exerted by the matter situated on the side + of the facet on those situated on the side – can be neglected with respect to $\left|\delta\underline{\boldsymbol{f}}\right|\sqrt{\delta a}$. Cauchy showed that, under this assumption, the equilibrium of the tetrahedron of vertex $\underline{\boldsymbol{x}}$ and the infinitesimal sides $\delta x_1, \delta x_2, \delta x_3$ imposes the symmetry of the stress tensor $\underset{\sim}{\boldsymbol{\sigma}}$: $\sigma_{ij} = \sigma_{ji}$ for all i, j, or equivalently

$${}^T\underset{\sim}{\boldsymbol{\sigma}} = \underset{\sim}{\boldsymbol{\sigma}}. \qquad [1.2]$$

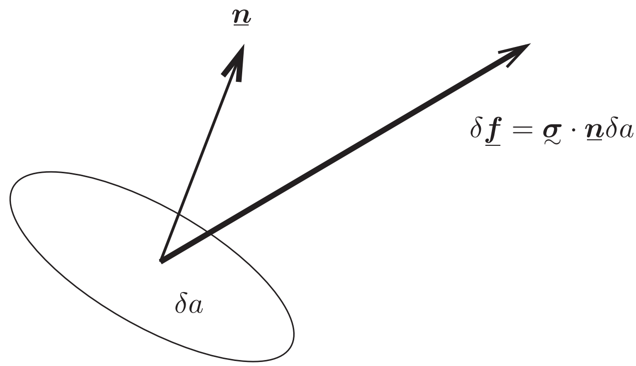

Figure 1.1. *Elementary force $\delta\underline{\boldsymbol{f}}$, Cauchy stress $\underset{\sim}{\boldsymbol{\sigma}}$, and elementary facet $\underline{\boldsymbol{n}}\delta a$*

Similarly, the equilibrium of the parallelepiped rectangle centered at $\underline{\boldsymbol{x}}$ of infinitesimal sides $\delta x_1, \delta x_2, \delta x_3$ leads to the equilibrium equation:

$$\boldsymbol{\nabla}_x \cdot \underset{\sim}{\boldsymbol{\sigma}} + \underline{\boldsymbol{f}}^{\text{ext}} = 0, \qquad [1.3]$$

where $\underline{\boldsymbol{f}}^{\text{ext}}(\underline{\boldsymbol{x}})$ is the volumic density of at distance external body forces such as gravity. The divergence of $\underset{\sim}{\boldsymbol{\sigma}}$, noted $\boldsymbol{\nabla}_x \cdot \underset{\sim}{\boldsymbol{\sigma}}$, is the vector whose i-th component is $\sigma_{ij,j}$. In components, the equilibrium equation can be written as:

$$\sigma_{ij,j} + f_i^{\text{ext}} = 0.$$

In most cases, $\underset{\sim}{\boldsymbol{\sigma}}(\underline{\boldsymbol{x}})$ is piece-wise continuously differentiable and its divergence is understood in the classical meaning to which the following condition must be added. Let Γ be a surface discontinuity of $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underline{\boldsymbol{n}}$ its normal vector. Then, the equilibrium of a facet situated at Γ of normal $\underline{\boldsymbol{n}}$ imposes the continuity of the stress vector $\underset{\sim}{\boldsymbol{\sigma}}(\underline{\boldsymbol{x}}) \cdot \underline{\boldsymbol{n}}$ (and not all the components of $\underset{\sim}{\boldsymbol{\sigma}}$!) when $\underline{\boldsymbol{x}}$ goes through Γ (Figure 1.2). A weak formulation of the equilibrium equation [1.3] is obtained by performing the scalar product of [1.3] by a smooth field of virtual velocity vectors, $\underline{\boldsymbol{v}}(\underline{\boldsymbol{x}})$, and then integrating over the domain V:

$$\int_V \left(\boldsymbol{\nabla}_x \cdot \underset{\sim}{\boldsymbol{\sigma}} + \underline{\boldsymbol{f}}^{\text{ext}}\right) \cdot \underline{\boldsymbol{v}}\, dV = 0.$$

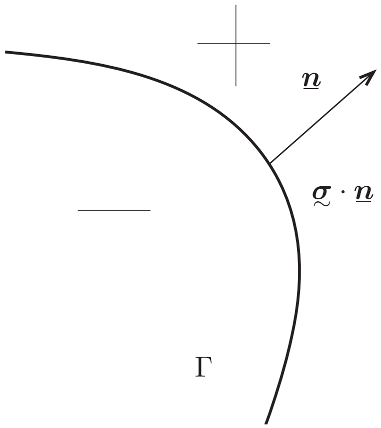

Figure 1.2. *Continuity of the stress vector*

Using the following integration by parts formula:

$$\int_V \left(\boldsymbol{\nabla}_x \cdot \underset{\sim}{\boldsymbol{\sigma}}\right) \cdot \underline{\boldsymbol{v}}\ dV = -\int_V \underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\nabla}_x^s \underline{\boldsymbol{v}}\ dV + \int_{\partial V} \left(\underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{n}}\right) \cdot \underline{\boldsymbol{v}}\ da,$$

where ∂V is the boundary of V of outer normal $\underline{\boldsymbol{n}}$, we get:

$$\int_V \underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\nabla}_x^s \underline{\boldsymbol{v}}\ dV = \int_V \underline{\boldsymbol{f}}^{\text{ext}} \cdot \underline{\boldsymbol{v}}\ dV + \int_{\partial V} \left(\underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{n}}\right) .\underline{\boldsymbol{v}}\ da \qquad [1.4]$$

for all smooth vector field $\underline{\boldsymbol{v}}$. Hence, the left-hand side of this equation appears as the internal power in the virtual velocity field $\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{T}} = \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{n}}$ appears as the external surfacic force applied at the boundary of the domain.

Considering in the above equation rigid body velocity vectors of the form:

$$\underline{\boldsymbol{v}}\left(\underline{\boldsymbol{x}}\right) = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}} \times \underline{\boldsymbol{x}}, \qquad [1.5]$$

where $\underline{\boldsymbol{a}}$ is an arbitrary velocity vector and $\underline{\boldsymbol{b}}$ is an arbitrary rotation (pseudo) vector, we find that $\boldsymbol{\nabla}_x^s \underline{\boldsymbol{v}}$ is null and that the equilibrium equation imposes that the resultant external forces and moments must be null:

$$\int_V \underline{\boldsymbol{f}}^{\text{ext}}\ dV + \int_{\partial V} \underline{\boldsymbol{T}}\ da = 0, \quad \int_V \underline{\boldsymbol{x}} \times \underline{\boldsymbol{f}}^{\text{ext}}\ dV + \int_{\partial V} \underline{\boldsymbol{x}} \times \underline{\boldsymbol{T}}\ da = 0. \qquad [1.6]$$

1.3. Linearized strains

Assume that the solid occupies the smooth domain V_0 in the initial configuration, i.e. before the application of the external body forces $\underline{\boldsymbol{f}}^{\text{ext}}$ and the external forces per unit surface $\underline{\boldsymbol{T}}$. The material point initially located at $\underline{\boldsymbol{X}} \in V_0$ is now located in $\underline{\boldsymbol{x}} \in V$ in the current configuration with:

$$\underline{\boldsymbol{x}} = \underline{\boldsymbol{X}} + \underline{\boldsymbol{\xi}}\left(\underline{\boldsymbol{X}}\right).$$

Here, $\underline{\boldsymbol{\xi}}(\underline{\boldsymbol{X}})$ is the displacement field defined on V_0 (Figure 1.3). We assume throughout this book that the deformation of the solid is infinitesimal which means that:

$$\left|\boldsymbol{\nabla}_X \underline{\boldsymbol{\xi}}\right| \ll 1 \Leftrightarrow \forall i, j = 1, 2, 3 \quad \left|\frac{\partial \xi_i}{\partial X_j}\right| \ll 1. \qquad [1.7]$$

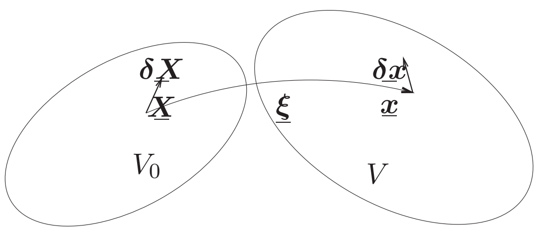

Figure 1.3. *Deformation of a solid*

Consider the segment of material connecting point $\underline{\boldsymbol{X}}$ to point $\underline{\boldsymbol{X}} + \delta\underline{\boldsymbol{X}}$ in the initial configuration where $\delta\underline{\boldsymbol{X}}$ is an infinitesimal vector. This segment is transformed into the segment connecting point $\underline{\boldsymbol{x}}$ image of $\underline{\boldsymbol{X}}$ in the current configuration to point $\underline{\boldsymbol{x}} + \boldsymbol{\delta}\underline{\boldsymbol{x}}$ image of $\underline{\boldsymbol{X}} + \delta\underline{\boldsymbol{X}}$ in the current configuration. We get:

$$\underline{\boldsymbol{x}} + \boldsymbol{\delta}\underline{\boldsymbol{x}} = \underline{\boldsymbol{X}} + \delta\underline{\boldsymbol{X}} + \underline{\boldsymbol{\xi}}\left(\underline{\boldsymbol{X}} + \delta\underline{\boldsymbol{X}}\right) \approx \underline{\boldsymbol{X}} + \delta\underline{\boldsymbol{X}} + \underline{\boldsymbol{\xi}}\left(\underline{\boldsymbol{X}}\right) + \left(\boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}}\right) \cdot \delta\underline{\boldsymbol{X}}$$

at the first-order in $\left|\delta\underline{\boldsymbol{X}}\right|$. Hence,

$$\boldsymbol{\delta}\underline{\boldsymbol{x}} = \delta\underline{\boldsymbol{X}} + \left(\boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}}\right) \cdot \delta\underline{\boldsymbol{X}} = \left(\underset{\sim}{\boldsymbol{\delta}} + \boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}}\right) \cdot \delta\underline{\boldsymbol{X}},$$

where $\underset{\approx}{\boldsymbol{\delta}}$ is the second-order unit tensor having diagonal components equal to 1 and 0 otherwise. Using [1.7], it can be seen that the relative extension of segment $\left[\underline{\boldsymbol{X}}, \underline{\boldsymbol{X}} + \boldsymbol{\delta}\underline{\boldsymbol{X}}\right]$ in this transformation is given (at first-order in $\left|\boldsymbol{\nabla}_X\, \underline{\boldsymbol{\xi}}\right|$) by the formula:

$$\frac{\left|\delta\underline{\boldsymbol{x}}\right| - \left|\delta\underline{\boldsymbol{X}}\right|}{\left|\delta\underline{\boldsymbol{X}}\right|} \approx \frac{\delta\underline{\boldsymbol{X}}}{\left|\delta\underline{\boldsymbol{X}}\right|} \cdot \underset{\approx}{\boldsymbol{\varepsilon}}\left(\underline{\boldsymbol{X}}\right) \cdot \frac{\delta\underline{\boldsymbol{X}}}{\left|\boldsymbol{\delta}\underline{\boldsymbol{X}}\right|}, \tag{1.8}$$

where

$$\underset{\approx}{\boldsymbol{\varepsilon}}\left(\underline{\boldsymbol{X}}\right) = \boldsymbol{\nabla}_X^s\, \underline{\boldsymbol{\xi}} \tag{1.9}$$

is the symmetric second-order tensor of linearized strains.

Indeed, we have:

$$\begin{aligned}\left|\delta\underline{\boldsymbol{x}}\right|^2 &= \delta\underline{\boldsymbol{X}} \cdot \left(\underset{\approx}{\boldsymbol{\delta}} + {}^T\boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}}\right) \cdot \left(\underset{\approx}{\boldsymbol{\delta}} + \boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}}\right) \cdot \delta\underline{\boldsymbol{X}} \\ &= \delta\underline{\boldsymbol{X}} \cdot \left(\underset{\approx}{\boldsymbol{\delta}} + {}^T\boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}} + \boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}} + {}^T\boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}} \cdot \boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}}\right) \cdot \delta\underline{\boldsymbol{X}}\end{aligned}$$

Neglecting the term ${}^T\boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}} \cdot \boldsymbol{\nabla}_X\underline{\boldsymbol{\xi}}$ which is of order $\left|\boldsymbol{\nabla}_X\, \underline{\boldsymbol{\xi}}\right|^2$, we obtain:

$$\left|\delta\underline{\boldsymbol{x}}\right|^2 \approx \left|\delta\underline{\boldsymbol{X}}\right|^2 + 2\delta\underline{\boldsymbol{X}} \cdot \underset{\approx}{\boldsymbol{\varepsilon}} \cdot \delta\underline{\boldsymbol{X}}.$$

Then, taking the square root of $\left|\delta\underline{\boldsymbol{x}}\right|^2 / \left|\delta\underline{\boldsymbol{X}}\right|^2$ and taking into account [1.7], which implies $\left|\underset{\approx}{\boldsymbol{\varepsilon}}\right| \ll 1$, [1.8] is obtained.

It can be shown that, for simply connex domains, the necessary and sufficient conditions on $\underset{\approx}{\boldsymbol{\varepsilon}}$ to be the symmetric part of the gradient of a displacement field [1.9] are:

$$2\varepsilon_{23,23} = \varepsilon_{33,22} + \varepsilon_{22,33} \text{ with circular permutation of the indices}, \tag{1.10}$$

and

$$\varepsilon_{13,23} + \varepsilon_{32,31} = \varepsilon_{12,33} + \varepsilon_{33,21} \text{ with circular permutation of the indices.} \tag{1.11}$$

Moreover, the rigid body displacements of the form

$$\underline{\xi}\left(\underline{x}\right) = \underline{a} + \underline{b} \times \underline{x} \qquad [1.12]$$

are the only one that generate null linearized strain fields.

1.4. Small perturbations

As it has been mentioned in the above sections, the stress field is defined on the current configuration which is unknown *a priori*. The equilibrium equation is written in this configuration on domain V (Euler variable, $\underline{x}$) whereas the strain field is defined by [1.9] in the initial configuration V_0 (Lagrange variable, $\underline{X}$). The small perturbations assumption stipulates that, besides the infinitesimal transformation assumption [1.7], we have:

$$\left|\underline{\xi}\right| / L \ll 1 \qquad [1.13]$$

where L is a typical length of the domain V_0, as its diameter, for instance. This assumption enables us to identify the initial and the current geometries. Thus, the stress field $\underset{\sim}{\sigma}(\underline{x})$ defined on V is identified with the field $\underset{\sim}{\sigma}(\underline{X})$ defined on V_0, obtained by substituting $\underline{X}$ for $\underline{x}$. Using assumptions [1.7] and [1.13], the equilibrium equation [1.3] on V is replaced by the following equation on V_0:

$$\boldsymbol{\nabla}_X \cdot \underset{\sim}{\sigma} + \underline{f}^{\text{ext}}\left(\underline{X}\right) = 0. \qquad [1.14]$$

In all the continuation, we adopt the small perturbations assumption: initial and current configurations V_0 and V are identified, as well as the derivations with respect to variables $\underline{x}$ and $\underline{X}$.

1.5. Linear elasticity

Under the assumption of small perturbations with constant temperature from a free-stress initial configuration (i.e. the stress field is identically null in the absence of external loads), the *linear elastic* constitutive law can be written as:

$$\underset{\sim}{\sigma}\left(\underline{x}\right) = \underset{\approx}{C}\left(\underline{x}\right) : \underset{\sim}{\varepsilon}\left(\underline{x}\right), \quad \text{or equivalently} \quad \sigma_{ij} = C_{ijkl}\varepsilon_{lk}, \qquad [1.15]$$

where $\underset{\approx}{\boldsymbol{C}}(\underline{\boldsymbol{x}})$ is the fourth-order *elastic stiffness tensor* which represents a local physical property of the material located in the vicinity of point $\underline{\boldsymbol{x}}$. When $\underset{\approx}{\boldsymbol{C}}$ does not vary with $\underline{\boldsymbol{x}}$ we say that the material is *homogeneous*, otherwise it is *heterogeneous*. The symmetries of $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underset{\sim}{\boldsymbol{\varepsilon}}$ imply the following *minor symmetries* on the components of $\underset{\approx}{\boldsymbol{C}}$:

$$C_{ijkl} = C_{jikl} = C_{ijlk}.$$

For thermodynamic reasons, the tensor $\underset{\approx}{\boldsymbol{C}}$ must fulfill the following *major symmetry* on its components:

$$C_{ijkl} = C_{klij},$$

and be positive definite in the sense of the following quadratic form in $\underset{\sim}{\boldsymbol{\varepsilon}}$, which is called the strain elastic energy density:

$$w\left(\underset{\sim}{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{x}}\right) = \frac{1}{2}\underset{\sim}{\boldsymbol{\varepsilon}} : \underset{\approx}{\boldsymbol{C}}\left(\underline{\boldsymbol{x}}\right) : \underset{\sim}{\boldsymbol{\varepsilon}} = \frac{1}{2}C_{ijkl}\left(\underline{\boldsymbol{x}}\right)\varepsilon_{ij}\varepsilon_{kl} \geq 0$$

for all $\underset{\sim}{\boldsymbol{\varepsilon}}$, with $w(\underset{\sim}{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{x}}) = 0 \Rightarrow \underset{\sim}{\boldsymbol{\varepsilon}} = 0$. Actually, $w\,dV$ is the elastic (i.e reversible) energy stored in the infinitesimal volume element dV when it is brought from the initial configuration to the deformed current configuration with a strain $\underset{\sim}{\boldsymbol{\varepsilon}}$. This energy is a quadratic form of $\underset{\sim}{\boldsymbol{\varepsilon}}$ whose components (i.e the components of $\underset{\approx}{\boldsymbol{C}}$) are physical characteristics of the material located at $\underline{\boldsymbol{x}}$. Due to both minor and major symmetries, there are only 21 independent components of $\underset{\approx}{\boldsymbol{C}}$. In the presence of *material symmetries*, subject of section 8 of this chapter, this number can be reduced up to two for an *isotropic* material which behaves in the same way in all the directions of the space.

Besides, the constitutive relation [1.15] can be inverted giving:

$$\underset{\sim}{\boldsymbol{\varepsilon}}\left(\underline{\boldsymbol{x}}\right) = \underset{\approx}{\boldsymbol{S}}\left(\underline{\boldsymbol{x}}\right) : \underset{\sim}{\boldsymbol{\sigma}}\left(\underline{\boldsymbol{x}}\right),$$

where

$$\underset{\approx}{\boldsymbol{S}}\left(\underline{\boldsymbol{x}}\right) = \left(S_{ijkl}\left(\underline{\boldsymbol{x}}\right)\right) = \underset{\approx}{\boldsymbol{C}}^{-1}\left(\underline{\boldsymbol{x}}\right)$$

is the fourth-order tensor of elastic compliance at point $\underline{\boldsymbol{x}}$. This tensor has the same minor and major symmetries as $\underset{\approx}{\boldsymbol{C}}$ and it defines the following quadratic form which is called the stress elastic energy density:

$$w^*\left(\underset{\sim}{\boldsymbol{\sigma}}, \underline{\boldsymbol{x}}\right) = \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{S}}\left(\underline{\boldsymbol{x}}\right) : \underset{\sim}{\boldsymbol{\sigma}} = \frac{1}{2}S_{ijkl}\left(\underline{\boldsymbol{x}}\right)\sigma_{ij}\sigma_{kl} \geq 0$$

for all $\underset{\sim}{\boldsymbol{\sigma}}$, with $w^*(\underset{\sim}{\boldsymbol{\sigma}}, \underline{\boldsymbol{x}}) = 0 \Rightarrow \underset{\sim}{\boldsymbol{\sigma}} = 0$.

We have

$$w^* \left(\underset{\approx}{\boldsymbol{\sigma}}, \underline{\boldsymbol{x}}\right) = w \left(\underset{\approx}{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{x}}\right)$$

for all couples $(\underset{\approx}{\boldsymbol{\sigma}}, \underset{\approx}{\boldsymbol{\varepsilon}})$ complying with the linear elastic constitutive law [1.15]. Let us indicate that the quadratic form $\underset{\approx}{\boldsymbol{\varepsilon}} \rightarrow w(\underset{\approx}{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{x}})$, or $\underset{\approx}{\boldsymbol{\sigma}} \rightarrow w^*(\underset{\approx}{\boldsymbol{\sigma}}, \underline{\boldsymbol{x}})$, completely defines the constitutive law at point $\underline{\boldsymbol{x}}$ since tensor $\underset{\approx}{\boldsymbol{C}}(\underline{\boldsymbol{x}})$, respectively tensor $\underset{\approx}{\boldsymbol{S}}(\underline{\boldsymbol{x}})$, can be obtained by taking the second-order derivative of w with respect to $\underset{\approx}{\boldsymbol{\varepsilon}}$, respectively $\underset{\approx}{\boldsymbol{\sigma}}$:

$$C_{ijkl} \left(\underline{\boldsymbol{x}}\right) = \frac{\partial^2 w}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}, \quad S_{ijkl} \left(\underline{\boldsymbol{x}}\right) = \frac{\partial^2 w^*}{\partial \sigma_{ij} \partial \sigma_{kl}}.$$

The quadratic formes w and w^* are actually dual in the sense of the Legendre–Fenchel transform:

$$w^* \left(\underset{\approx}{\boldsymbol{\sigma}}\right) = \sup_{\underset{\approx}{\boldsymbol{\varepsilon}}} \left\{\underset{\approx}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{\varepsilon}} - w \left(\underset{\approx}{\boldsymbol{\varepsilon}}\right)\right\} \text{ and}$$

$$w \left(\underset{\approx}{\boldsymbol{\varepsilon}}\right) = \sup_{\underset{\approx}{\boldsymbol{\sigma}}} \left\{\underset{\approx}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{\varepsilon}} - w^* \left(\underset{\approx}{\boldsymbol{\sigma}}\right)\right\}, \qquad [1.16]$$

where the dependence on $\underline{\boldsymbol{x}}$ has been omitted. Thus, for all couples $(\underset{\approx}{\boldsymbol{\sigma}}, \underset{\approx}{\boldsymbol{\varepsilon}})$,

$$w \left(\underset{\approx}{\boldsymbol{\varepsilon}}\right) + w^* \left(\underset{\approx}{\boldsymbol{\sigma}}\right) \geq \underset{\approx}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{\varepsilon}} \qquad [1.17]$$

with equality if, and only if, $\underset{\approx}{\boldsymbol{\sigma}}$ and $\underset{\approx}{\boldsymbol{\varepsilon}}$ comply with the constitutive law $\underset{\approx}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{C}} : \underset{\approx}{\boldsymbol{\varepsilon}}$.

1.6. Boundary value problem in linear elasticity

Let us summarize the problem to be solved: the smooth domain V (identical to V_0) is given and the fields of elasticity stiffness tensor $\underset{\approx}{\boldsymbol{C}}$ and body forces $\underline{\boldsymbol{f}}^{\text{ext}}$ are also given on V. The unknowns are the displacement field $\underline{\boldsymbol{\xi}}$, the strain field $\underset{\approx}{\boldsymbol{\varepsilon}}$ and the stress field $\underset{\approx}{\boldsymbol{\sigma}}$. They are solution of the partial differential equations on V, [1.3], [1.9] and [1.15]:

$$\begin{cases} \boldsymbol{\nabla} \cdot \underset{\approx}{\boldsymbol{\sigma}} + \underline{\boldsymbol{f}}^{\text{ext}} = 0, \\ \underset{\approx}{\boldsymbol{\varepsilon}} = \boldsymbol{\nabla}^s \underline{\boldsymbol{\xi}}, \\ \underset{\approx}{\boldsymbol{\sigma}} \left(\underline{\boldsymbol{x}}\right) = \underset{\approx}{\boldsymbol{C}} \left(\underline{\boldsymbol{x}}\right) : \underset{\approx}{\boldsymbol{\varepsilon}} \left(\underline{\boldsymbol{x}}\right). \end{cases} \qquad [1.18]$$

Yet, the resolution of this system requires boundary conditions on ∂V, the boundary of V: on each portion of ∂V of outer normal $\underline{\boldsymbol{n}}$, three components must be prescribed among the six components of both the displacement vector $\underline{\boldsymbol{\xi}}$ and the stress vector $\underline{\boldsymbol{T}} = \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{n}}$. For instance, restraint boundary conditions impose that the three components of the displacement vector $\underline{\boldsymbol{\xi}}$ must vanish at the boundary (Dirichlet boundary conditions). Free boundary conditions mean that the three components of the stress vector $\underline{\boldsymbol{T}}$ are null (Neumann boundary conditions). Uniform pressure conditions prescribe that the stress vector $\underline{\boldsymbol{T}}$ is equal to $-p\underline{\boldsymbol{n}}$ where $p > 0$ is the given pressure. No friction bilateral contact conditions can be written as: $\underline{\boldsymbol{\xi}} \cdot \underline{\boldsymbol{n}} = 0$, $\underline{\boldsymbol{m}}_1 \cdot \underline{\boldsymbol{T}} = 0$ and $\underline{\boldsymbol{m}}_2 \cdot \underline{\boldsymbol{T}} = 0$ where $(\underline{\boldsymbol{n}}, \underline{\boldsymbol{m}}_1, \underline{\boldsymbol{m}}_2)$ is a local orthonormal basis at the considered point of the boundary.

It can be proved that, under suitable regularity conditions, the system (1.18) + boundary conditions has unique stress and strain solutions, $\underset{\sim}{\boldsymbol{\varepsilon}}$ and $\underset{\sim}{\boldsymbol{\sigma}}$, the uniqueness of displacement field $\underline{\boldsymbol{\xi}}$ being not guaranteed unless boundary conditions eliminate rigid body displacements (i.e those displacements fields which produce null strain field). In all the continuation, we will consider such well-posed linearized elasticity problems.

1.7. Variational formulations

In this section, we recall the main variational formulations of problem (1.18) + boundary conditions. Let us be more specific about boundary conditions although other boundary conditions can be considered in a very similar way. We assume that ∂V, the boundary of V, is divided into two parts: ∂V_ξ where the displacement vector $\underline{\boldsymbol{\xi}}$ is prescribed and ∂V_T where the stress vector $\underline{\boldsymbol{T}}$ is prescribed as:

$$\underline{\boldsymbol{\xi}} = \underline{\boldsymbol{\xi}}^d \text{ on } \partial V_\xi \quad \text{and} \quad \underline{\boldsymbol{T}} = \underline{\boldsymbol{T}}^d \text{ on } \partial V_T. \qquad [1.19]$$

Here, $\underline{\boldsymbol{\xi}}^d$ and $\underline{\boldsymbol{T}}^d$ are given functions.

1.7.1. *Compatible strains and stresses*

We introduce the set of kinematically compatible strain fields: it is the set $KC\left(\underline{\boldsymbol{\xi}}^d\right)$ of strain fields on V which are generated by *regular enough* displacement fields complying with the kinematic boundary condition on ∂V_{ξ}:

$$KC\left(\underline{\xi}^d\right) = \left\{\underset{\sim}{\varepsilon},\ \exists \text{ regular enough } \underline{\xi},\ \underset{\sim}{\varepsilon} = \nabla^s \underline{\xi} \text{ on } V,\ \underline{\xi} = \underline{\xi}^d \text{ on } \partial V_{\xi}\right\}. \quad [1.20]$$

Let $\underset{\sim}{\varepsilon}'$ be an arbitrary strain field in $KC\left(\underline{\xi}^d\right)$. Then, $\underset{\sim}{\varepsilon} \in KC\left(\underline{\xi}^d\right)$ is equivalent to $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}' + \underset{\sim}{\varepsilon}^0$ with $\underset{\sim}{\varepsilon}^0 \in KC^0$ where KC^0 is the set $KC\left(\underline{\xi}^d\right)$ obtained by setting $\underline{\xi}^d = 0$:

$$KC^0 \equiv KC\left(\underline{\xi}^d = 0\right). \quad [1.21]$$

The set $SC\left(\underline{f}^{\text{ext}}, \underline{T}^d\right)$ of statically compatible stress fields is the set of *regular enough* stress fields complying with both equilibrium equation with $\underline{f}^{\text{ext}}$ and static boundary conditions on ∂V_T :

$$SC\left(\underline{f}^{\text{ext}}, \underline{T}^d\right) = \left\{\underset{\sim}{\sigma} \text{ regular enough},\right.$$
$$\left.\nabla \cdot \underset{\sim}{\sigma} + \underline{f}^{\text{ext}} = 0 \text{ on } V,\ \underline{T} = \underline{T}^d \text{ on } \partial V_T\right\}. \quad [1.22]$$

Similarly, let $\underset{\sim}{\sigma}'$ be an arbitrary stress field in $SC\left(\underline{f}^{\text{ext}}, \underline{T}^d\right)$. Then, $\underset{\sim}{\sigma} \in SC\left(\underline{f}^{\text{ext}}, \underline{T}^d\right)$ is equivalent to $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}' + \underset{\sim}{\sigma}^0$ with $\underset{\sim}{\sigma}^0 \in SC^0$ where SC^0 is the set $SC\left(\underline{f}^{\text{ext}}, \underline{T}^d\right)$ obtained by setting $\left(\underline{f}^{\text{ext}}, \underline{T}^d\right) = (\underline{\mathbf{0}}, \underline{\mathbf{0}})$:

$$SC^0 \equiv SC\left(\underline{f}^{\text{ext}} = 0, \underline{T}^d = 0\right). \quad [1.23]$$

Let $\underset{\sim}{\varepsilon}$ be in $KC\left(\underline{\xi}^d\right)$ and $\underset{\sim}{\sigma}$ be in $SC\left(\underline{f}^{\text{ext}}, \underline{T}^d\right)$. Then, due to formula [1.4], we have:

$$\int_V \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon}\, dV = \Phi\left(\underline{\xi}\right) + \Phi^*\left(\underset{\sim}{\sigma}\right), \quad [1.24]$$

where the functionals Φ and Φ^* are respectively defined by:

$$\Phi\left(\underline{\boldsymbol{\xi}}\right) = \int_V \underline{\boldsymbol{f}}^{\text{ext}} \cdot \underline{\boldsymbol{\xi}} \, dV + \int_{\partial V_T} \underline{\boldsymbol{T}}^d . \underline{\boldsymbol{\xi}} \, da \qquad [1.25]$$

and

$$\Phi^*\left(\underset{\sim}{\boldsymbol{\sigma}}\right) = \int_{\partial V_\xi} \underline{\boldsymbol{T}} . \underline{\boldsymbol{\xi}}^d \, da. \qquad [1.26]$$

In particular, for $\underset{\sim}{\boldsymbol{\varepsilon}} \in KC^0$ and $\underset{\sim}{\boldsymbol{\sigma}} \in SC^0$, we will have:

$$\int_V \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\boldsymbol{\varepsilon}} \, dV = 0, \qquad [1.27]$$

which means that KC^0 and SC^0 are orthogonal in the sense of the above equation. Let $\underline{\boldsymbol{x}} \to \underset{\sim}{\boldsymbol{\varepsilon}}(\underline{\boldsymbol{x}})$ and $\underline{\boldsymbol{x}} \to \underset{\sim}{\boldsymbol{\sigma}}(\underline{\boldsymbol{x}})$ be two smooth fields of symmetric second-order tensors on V. We introduce the following two functionals:

$$W\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right) = \int_V w\left(\underset{\sim}{\boldsymbol{\varepsilon}}\left(\underline{\boldsymbol{x}}\right), \underline{\boldsymbol{x}}\right) dV = \frac{1}{2}\int_V \underset{\sim}{\boldsymbol{\varepsilon}}\left(\underline{\boldsymbol{x}}\right) : \underset{\approx}{\boldsymbol{C}}\left(\underline{\boldsymbol{x}}\right) : \underset{\sim}{\boldsymbol{\varepsilon}}\left(\underline{\boldsymbol{x}}\right) dV \qquad [1.28]$$

and

$$W^*\left(\underset{\sim}{\boldsymbol{\sigma}}\right) = \int_V w^*\left(\underset{\sim}{\boldsymbol{\sigma}}\left(\underline{\boldsymbol{x}}\right), \underline{\boldsymbol{x}}\right) dV = \frac{1}{2}\int_V \underset{\sim}{\boldsymbol{\sigma}}\left(\underline{\boldsymbol{x}}\right) : \underset{\approx}{\boldsymbol{S}}\left(\underline{\boldsymbol{x}}\right) : \underset{\sim}{\boldsymbol{\sigma}}\left(\underline{\boldsymbol{x}}\right) dV. \qquad [1.29]$$

1.7.2. *Principle of minimum of potential energy*

Let the displacement field $\underline{\boldsymbol{\xi}}^s$, the strain field $\underset{\sim}{\boldsymbol{\varepsilon}}^s$ and the stress field $\underset{\sim}{\boldsymbol{\sigma}}^s$ be the solutions of the elasticity problem [1.18–1.19]. Then, $\left(\underline{\boldsymbol{\xi}}^s, \underset{\sim}{\boldsymbol{\varepsilon}}^s\right)$ minimizes the potential energy

$$W\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right) - \Phi\left(\underline{\boldsymbol{\xi}}\right) \qquad [1.30]$$

over all kinematically compatible fields in $KC\left(\underline{\boldsymbol{\xi}}^d\right)$. Indeed, since W is quadratic in $\underset{\sim}{\boldsymbol{\varepsilon}}$, we have by simple algebra:

$$W\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right) = W\left(\underset{\sim}{\boldsymbol{\varepsilon}}^s\right) + W\left(\underset{\sim}{\boldsymbol{\varepsilon}} - \underset{\sim}{\boldsymbol{\varepsilon}}^s\right) + \int_V \underset{\sim}{\boldsymbol{\sigma}}^s\left(\underline{\boldsymbol{x}}\right) : \left(\underset{\sim}{\boldsymbol{\varepsilon}}\left(\underline{\boldsymbol{x}}\right) - \underset{\sim}{\boldsymbol{\varepsilon}}^s\left(\underline{\boldsymbol{x}}\right)\right) dV, \qquad [1.31]$$

where the equation

$$\underset{\sim}{\boldsymbol{\sigma}}^s\left(\underline{\boldsymbol{x}}\right) = \underset{\approx}{\boldsymbol{C}}\left(\underline{\boldsymbol{x}}\right) : \underset{\sim}{\boldsymbol{\varepsilon}}^s\left(\underline{\boldsymbol{x}}\right)$$

has been used. Using

$$\int_V \underset{\sim}{\boldsymbol{\sigma}}^s\left(\underline{\boldsymbol{x}}\right) : \left(\underset{\sim}{\boldsymbol{\varepsilon}}\left(\underline{\boldsymbol{x}}\right) - \underset{\sim}{\boldsymbol{\varepsilon}}^s\left(\underline{\boldsymbol{x}}\right)\right) dV = \Phi\left(\underline{\boldsymbol{\xi}}\right) - \Phi\left(\underline{\boldsymbol{\xi}}^s\right), \qquad [1.32]$$

the positiveness of $W\left(\underset{\sim}{\boldsymbol{\varepsilon}} - \underset{\sim}{\boldsymbol{\varepsilon}}^s\right)$ guarantees the result.

1.7.3. *Principle of minimum of complementary energy*

Similarly, it can be shown that the stress $\underset{\sim}{\boldsymbol{\sigma}}^s$ minimizes the complementary energy

$$W^*\left(\underset{\sim}{\boldsymbol{\sigma}}\right) - \Phi^*\left(\underset{\sim}{\boldsymbol{\sigma}}\right) \qquad [1.33]$$

over all statically compatible stress fields in $\mathcal{SC}\left(\underline{\boldsymbol{f}}^{\text{ext}}, \underline{\boldsymbol{T}}^d\right)$.

Moreover, we have the following two remarkable properties: the first property stipulates that the value at the solution of the potential energy is opposite the value of the complementary energy at this solution:

$$W\left(\underset{\sim}{\boldsymbol{\varepsilon}}^s\right) - \Phi\left(\underline{\boldsymbol{\xi}}^s\right) + W^*\left(\underset{\sim}{\boldsymbol{\sigma}}^s\right) - \Phi^*\left(\underset{\sim}{\boldsymbol{\sigma}}^s\right) = 0. \qquad [1.34]$$

This motivates the use of the term "complementary energy" for the functional $W^* - \Phi^*$. The second is the Clapeyron formula which stipulates that the elastic energy stored in the material is equal to half the work of external forces:

$$W\left(\underset{\sim}{\boldsymbol{\varepsilon}}^s\right) = W^*\left(\underset{\sim}{\boldsymbol{\sigma}}^s\right) = \frac{1}{2}\int_V \underset{\sim}{\boldsymbol{\sigma}}^s\left(\underline{\boldsymbol{x}}\right) : \underset{\sim}{\boldsymbol{\varepsilon}}^s\left(\underline{\boldsymbol{x}}\right) dV = \frac{1}{2}\left(\Phi\left(\underline{\boldsymbol{\xi}}^s\right) + \Phi^*\left(\underset{\sim}{\boldsymbol{\sigma}}^s\right)\right). \qquad [1.35]$$

1.7.4. *Two-energy principle*

Let $\underset{\sim}{\varepsilon}$ be in $KC\left(\underline{\boldsymbol{\xi}}^d\right)$ and $\underset{\sim}{\boldsymbol{\sigma}}$ be in $SC\left(\underline{\boldsymbol{f}}^{\text{ext}}, \underline{\boldsymbol{T}}^d\right)$, then we have:

$$W\left(\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^s\right) + W^*\left(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}^s\right) = W\left(\underset{\sim}{\varepsilon} - \underset{\approx}{\boldsymbol{S}} : \underset{\sim}{\boldsymbol{\sigma}}\right) = W^*\left(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\varepsilon}\right). \quad [1.36]$$

Indeed, the last right-hand equation in the above formula is easy to obtain by simple algebra. Moreover, we have:

$$\underset{\sim}{\boldsymbol{\sigma}} - \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\varepsilon} = \underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}^s + \underset{\sim}{\boldsymbol{\sigma}}^s - \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\varepsilon} = \underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}^s - \underset{\approx}{\boldsymbol{C}} : \left(\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^s\right). \quad [1.37]$$

Hence,

$$W^*\left(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\varepsilon}\right) = W^*\left(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}^s\right) + W^*\left(\underset{\approx}{\boldsymbol{C}} : \left(\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^s\right)\right) - \int_V \left(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}^s\right) : \left(\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^s\right) \quad [1.38]$$

Noticing that $\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^s \in KC^0$ and $\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}^s \in SC^0$, the last right-hand term in the above equation is null due to the orthogonality of KC^0 and SC^0. Finally, the result is obtained because we have $W^*\left(\underset{\approx}{\boldsymbol{C}} : \left(\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^s\right)\right) = W\left(\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^s\right)$ by simple algebra.

1.8. Anisotropy

Before recalling the concepts of anisotropy and material symmetries, the objective of this section, it is useful to recall the Voigt matrix representation of elasticity tensors.

1.8.1. *Voigt notations*

In some cases, it is convenient to represent the second-order symmetric tensors of stress $\underset{\sim}{\boldsymbol{\sigma}} = (\sigma_{ij})$ and strain $\underset{\sim}{\varepsilon} = (\varepsilon_{ij})$ by the two six-component columns $\left[\underset{\sim}{\boldsymbol{\sigma}}\right]$ and $\left[\underset{\sim}{\varepsilon}\right]$, respectively:

$$\left[\underset{\approx}{\boldsymbol{\sigma}}\right] = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{31} \\ \sqrt{2}\sigma_{12} \end{bmatrix}, \quad \left[\underset{\approx}{\boldsymbol{\varepsilon}}\right] = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{31} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix}.$$

Introducing multiplier $\sqrt{2}$ in the non-diagonal components ensures the identification of the scalar product $\left[\underset{\approx}{\boldsymbol{\sigma}}\right] \cdot \left[\underset{\approx}{\boldsymbol{\varepsilon}}\right]$ with the contraction product $\underset{\approx}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{\varepsilon}}$. Indeed, we have:

$$\left[\underset{\approx}{\boldsymbol{\sigma}}\right] \cdot \left[\underset{\approx}{\boldsymbol{\varepsilon}}\right] = \underset{\approx}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{\varepsilon}} = \sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{23}\varepsilon_{23} + 2\sigma_{31}\varepsilon_{31} + 2\sigma_{12}\varepsilon_{12}$$

Using these notations, one can easily check that the constitutive equation can be written in the following matrix form:

$$\underset{\approx}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{C}} : \underset{\approx}{\boldsymbol{\varepsilon}} \Longleftrightarrow \left[\underset{\approx}{\boldsymbol{\sigma}}\right] = \left[\underset{\approx}{\boldsymbol{C}}\right] \cdot \left[\underset{\approx}{\boldsymbol{\varepsilon}}\right]$$

where $\left[\underset{\approx}{\boldsymbol{C}}\right]$ is the symmetric 6×6–matrix given in terms of C_{ijkl} by:

$$\left[\underset{\approx}{\boldsymbol{C}}\right] = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & \sqrt{2}C_{1123} & \sqrt{2}C_{1131} & \sqrt{2}C_{1112} \\ & C_{2222} & C_{2233} & \sqrt{2}C_{2223} & \sqrt{2}C_{2231} & \sqrt{2}C_{2212} \\ & & C_{3333} & \sqrt{2}C_{3323} & \sqrt{2}C_{3331} & \sqrt{2}C_{3312} \\ & & & 2C_{2323} & 2C_{2331} & 2C_{2312} \\ & \text{SYM} & & & 2C_{3131} & 2C_{3112} \\ & & & & & 2C_{1212} \end{bmatrix}.$$

The inversion of the previous equation gives:

$$\underset{\approx}{\boldsymbol{\varepsilon}} = \underset{\approx}{\boldsymbol{S}} : \underset{\approx}{\boldsymbol{\sigma}} \Longleftrightarrow \left[\underset{\approx}{\boldsymbol{\varepsilon}}\right] = \left[\underset{\approx}{\boldsymbol{S}}\right] \cdot \left[\underset{\approx}{\boldsymbol{\sigma}}\right]$$

where $\left[\underset{\approx}{\boldsymbol{S}}\right]$ is the symmetric 6×6-matrix inverse of $\left[\underset{\approx}{\boldsymbol{C}}\right]$:

$$\left[\underset{\approx}{\boldsymbol{S}}\right] = \left[\underset{\approx}{\boldsymbol{C}}\right]^{-1}.$$

Its components in terms of S_{ijkl} are:

$$\left[\underset{\approx}{\boldsymbol{S}}\right] = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & \sqrt{2}S_{1123} & \sqrt{2}S_{1131} & \sqrt{2}S_{1112} \\ & S_{2222} & S_{2233} & \sqrt{2}S_{2223} & \sqrt{2}S_{2231} & \sqrt{2}S_{2212} \\ & & S_{3333} & \sqrt{2}S_{3323} & \sqrt{2}S_{3331} & \sqrt{2}S_{3312} \\ & & & 2S_{2323} & 2S_{2331} & 2S_{2312} \\ & \text{SYM} & & & 2S_{3131} & 2S_{3112} \\ & & & & & 2S_{1212} \end{bmatrix}.$$

The elastic strain and stress energy densities can be expressed using the matrix representations of the elasticity tensors as follows:

$$w = \frac{1}{2}\underset{\sim}{\boldsymbol{\varepsilon}} : \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\boldsymbol{\varepsilon}} = \frac{1}{2}\left[\underset{\sim}{\boldsymbol{\varepsilon}}\right] \cdot \left[\underset{\approx}{\boldsymbol{C}}\right] \cdot \left[\underset{\sim}{\boldsymbol{\varepsilon}}\right] \quad \text{and}$$

$$w^* = \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{S}} : \underset{\sim}{\boldsymbol{\sigma}} = \frac{1}{2}\left[\underset{\sim}{\boldsymbol{\sigma}}\right] \cdot \left[\underset{\approx}{\boldsymbol{S}}\right] \cdot \left[\underset{\sim}{\boldsymbol{\sigma}}\right].$$

1.8.2. *Material symmetries*

An orthogonal transformation is a linear application mapping any orthonormal basis into an orthonormal basis. Hence, it is a second-order tensor, $\underset{\sim}{\boldsymbol{Q}}$, which can be represented by its components in the basis $\left(\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3\right)$ by $\underset{\sim}{\boldsymbol{Q}} = (O_{ij})$, with the following property:

$$^T\underset{\sim}{\boldsymbol{Q}} = \underset{\sim}{\boldsymbol{Q}}^{-1}$$

where $^T\underset{\sim}{\boldsymbol{Q}}$ is the transpose of $\underset{\sim}{\boldsymbol{Q}}$. The above property is equivalent to $^T\underset{\sim}{\boldsymbol{Q}} \cdot \underset{\sim}{\boldsymbol{Q}} = \underset{\sim}{\boldsymbol{\delta}}$ which can be written in components $O_{ki}O_{kj} = \delta_{ij}$ where δ is the Kronecker symbol, equal to 1 if $i = j$ and to 0 otherwise, and $\underset{\sim}{\boldsymbol{\delta}}$ is identity second-order tensor.

Let us consider a medium occupied by a homogeneous material of elasticity stiffness tensor $\underset{\approx}{\boldsymbol{C}}$. Let $\underset{\sim}{\boldsymbol{\varepsilon}}$ be an uniform strain of the medium. We recall that the relative extension of an infinitesimal segment along the normal vector $\underline{\boldsymbol{n}}$ is given by $\underline{\boldsymbol{n}} \cdot (\underset{\sim}{\boldsymbol{\varepsilon}} \cdot \underline{\boldsymbol{n}}) = \varepsilon_{ij}n_in_j$. By definition, the image of $\underset{\sim}{\boldsymbol{\varepsilon}}$ by the orthogonal transformation $\underset{\sim}{\boldsymbol{Q}}$ is the unique strain tensor $\underset{\sim}{\varepsilon}^\dagger$ which preserves the relative extension of any infinitesimal segment in the transformation $\underset{\sim}{\boldsymbol{Q}}$. So, we have:

$$\underline{\boldsymbol{n}} \cdot \left(\underset{\sim}{\varepsilon} \cdot \underline{\boldsymbol{n}}\right) = \underline{\boldsymbol{n}}^\dagger \cdot \left(\underset{\sim}{\varepsilon}^\dagger \cdot \underline{\boldsymbol{n}}^\dagger\right)$$

for all $\underline{\boldsymbol{n}}$ and $\underline{\boldsymbol{n}}^\dagger = \underset{\sim}{\boldsymbol{Q}} \cdot \underline{\boldsymbol{n}}$. It can be deduced that:

$$\underset{\sim}{\varepsilon}^\dagger = \underset{\sim}{\boldsymbol{Q}} \cdot \underset{\sim}{\varepsilon} \cdot {}^T\underset{\sim}{\boldsymbol{Q}}. \qquad [1.39]$$

Or, in components:

$$\varepsilon^\dagger_{ij} = O_{ik} O_{jl} \varepsilon_{kl}. \qquad [1.40]$$

We say that the orthogonal transformation $\underset{\sim}{\boldsymbol{Q}}$ is a *material symmetry* of the elastic medium and that this medium is invariant under the action of $\underset{\sim}{\boldsymbol{Q}}$ if, and only if, for all $\underset{\sim}{\varepsilon}$, the strain elastic energy density of the medium is preserved:

$$w\left(\underset{\sim}{\varepsilon}\right) = w\left(\underset{\sim}{\varepsilon}^\dagger\right). \qquad [1.41]$$

In components, [1.41] is equivalent to:

$$\frac{1}{2} C_{ijmn} \varepsilon_{ij} \varepsilon_{mn} = \frac{1}{2} C_{ijmn} \varepsilon^\dagger_{ij} \varepsilon^\dagger_{mn}$$

for all $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\varepsilon}^\dagger$ related by [1.40]. So, we have:

$$\begin{aligned} C_{klop} \varepsilon_{kl} \varepsilon_{op} = C_{ijmn} \varepsilon^\dagger_{ij} \varepsilon^\dagger_{mn} &= C_{ijmn} \left(O_{ik} O_{jl} \varepsilon_{kl}\right) \left(O_{mo} O_{np} \varepsilon_{op}\right) \\ &= \left(O_{ik} O_{jl} O_{mo} O_{np} C_{ijmn}\right) \varepsilon_{kl} \varepsilon_{op}. \end{aligned}$$

for all $\underset{\sim}{\varepsilon}$. Consequently, [1.41] is equivalent to:

$$C_{klop} = O_{ik} O_{jl} O_{mo} O_{np} C_{ijmn}. \qquad [1.42]$$

Notice that any elastic medium is invariant with respect to identity $\underset{\sim}{\boldsymbol{Q}} = \underset{\sim}{\boldsymbol{\delta}}$ and central symmetry $\underset{\sim}{\boldsymbol{Q}} = -\underset{\sim}{\boldsymbol{\delta}}$.

Due to the Legendre–Fenchel duality property [1.16] an equivalent stress version of the invariance property [1.41] can be derived. Let

$$\underset{\sim}{\sigma}^{\dagger} = \underset{\sim}{Q} \cdot \underset{\sim}{\sigma} \cdot {}^{T}\underset{\sim}{Q} \qquad [1.43]$$

be the image of $\underset{\sim}{\sigma}$ by the orthogonal transformation $\underset{\sim}{Q}$. In components, the previous equation can be written as:

$$\sigma^{\dagger}_{ij} = O_{ik} O_{jl} \sigma_{kl}$$

Note that the contraction of a stress tensor with a strain tensor is a scalar, and hence it is invariant by orthogonal transformation:

$$\underset{\sim}{\sigma}^{\dagger} : \underset{\sim}{\varepsilon}^{\dagger} = \sigma^{\dagger}_{ij} \varepsilon^{\dagger}_{ij} = (O_{ik} O_{jl} \sigma_{kl})(O_{im} O_{jn} \varepsilon_{mn}) = \sigma_{kl} \varepsilon_{kl} = \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon}$$

for all $\underset{\sim}{\varepsilon}$ since $O_{ik} O_{im} = \delta_{km}$ and $O_{jl} O_{jn} = \delta_{nl}$.

So, using [1.16], the previous relation and [1.41], we obtain:

$$w^{*}\left(\underset{\sim}{\sigma}\right) = \sup_{\underset{\sim}{\varepsilon}} \left\{\underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} - w\left(\underset{\sim}{\varepsilon}\right)\right\} = \sup_{\underset{\sim}{\varepsilon}} \left\{\underset{\sim}{\sigma}^{\dagger} : \underset{\sim}{\varepsilon}^{\dagger} - w\left(\underset{\sim}{\varepsilon}^{\dagger}\right)\right\}, \qquad [1.44]$$

where $\underset{\sim}{\varepsilon}^{\dagger}$ is given by [1.39] in terms of $\underset{\sim}{\varepsilon}$. For $\underset{\sim}{\varepsilon}$ spaning the space of symmetric second-order tensors, $\underset{\sim}{\varepsilon}^{\dagger}$ spans also the same space. Therefore,

$$w^{*}\left(\underset{\sim}{\sigma}\right) = \sup_{\underset{\sim}{\varepsilon}^{\dagger}} \left\{\underset{\sim}{\sigma}^{\dagger} : \underset{\sim}{\varepsilon}^{\dagger} - w\left(\underset{\sim}{\varepsilon}^{\dagger}\right)\right\} = w^{*}\left(\underset{\sim}{\sigma}^{\dagger}\right). \qquad [1.45]$$

So, if the orthogonal transformation $\underset{\sim}{Q}$ is a material symmetry, then for all $\underset{\sim}{\sigma}$, the stress elastic energy density is preserved by this transformation:

$$w^{*}\left(\underset{\sim}{\sigma}\right) = w^{*}\left(\underset{\sim}{\sigma}^{\dagger}\right). \qquad [1.46]$$

In components, [1.46] is equivalent to:

$$S_{klop} = O_{ik} O_{jl} O_{mo} O_{np} S_{ijmn}. \qquad [1.47]$$

Using the second duality property in [1.16] which express $w\left(\underset{\sim}{\varepsilon}\right)$ as a function of $w^{*}\left(\underset{\sim}{\sigma}\right)$, we establish the reverse implication so that [1.41] and [1.46] are actually equivalent.

1.8.3. *Orthotropy*

We say that the medium is *monoclinic* if it is invariant with respect to a plane-symmetry. For instance, if the plane (1,2) is a material symmetry plane, then we will have:

$$\left[\underset{\approx}{C}\right] = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & \sqrt{2}C_{1112} \\ & C_{2222} & C_{2233} & 0 & 0 & \sqrt{2}C_{2212} \\ & & C_{3333} & 0 & 0 & \sqrt{2}C_{3312} \\ & & & 2C_{2323} & 2C_{2331} & 0 \\ & \text{SYM} & & & 2C_{3131} & 0 \\ & & & & & 2C_{1212} \end{bmatrix}$$

Indeed, in this case:

$$\underset{\sim}{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{\varepsilon}^{\dagger} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & -\varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & -\varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & \varepsilon_{33} \end{pmatrix}.$$

Property [1.41], and its equivalent version [1.42], imply that all the coupling components between ε_{13} and ε_{23} on the one hand, and the other components of $\underset{\sim}{\varepsilon}$, in the other hand, are null. So, a linear elastic monoclinic medium possesses 13 independent elastic constants.

We say that the medium is *orthotropic* when it is invariant by symmetry with respect to three orthogonal planes (1,2), (2,3) and (3,1). Then, in this case, the Voigt matrix representation of $\underset{\approx}{C}$ is:

$$\left[\underset{\approx}{C}\right] = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & 2C_{2323} & 0 & 0 \\ & \text{SYM} & & & 2C_{3131} & 0 \\ & & & & & 2C_{1212} \end{bmatrix}$$

and

$$\left[\underset{\approx}{\boldsymbol{S}}\right] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\mu_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\mu_{31}} & 0 \\ 0 & 0 & 0 & 0 & & \frac{1}{2\mu_{12}} \end{bmatrix},$$

with

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}, \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}$$

because of the symmetry of matrix $\left[\underset{\approx}{\boldsymbol{S}}\right]$.

An orthotropic medium posseses nine independent elastic constants. The shear modulus μ_{ij} is often denoted by G_{ij}. E_i is the Young's modulus in direction i and ν_{ij} is the Poisson's ratio in direction j under uniaxial traction/compression along direction i. For instance, the application of an uniaxial stress along direction 1:

$$\underset{\sim}{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

produces the strain

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \begin{pmatrix} \frac{\sigma_{11}}{E_1} & 0 & 0 \\ 0 & -\nu_{12}\frac{\sigma_{11}}{E_1} & 0 \\ 0 & 0 & -\nu_{13}\frac{\sigma_{11}}{E_1} \end{pmatrix}$$

The necessary and sufficient conditions for the definite positiveness of matrix $\left[\underset{\approx}{\boldsymbol{C}}\right]$ are obtained thanks to the Sylvestre's criteria which stipulates that a symmetric $n \times n$-matrix (M_{ij}) $i, j = 1, ..., n$ is definite positive, if, and only if, for all k going from 1 to n, the determinant of the matrix (M_{ij})

$i, j = 1, ..., k$ is strictly positive. So, applying this criteria gives the following necessary and sufficient conditions:

$$E_1 > 0, \quad \frac{1 - \nu_{12}\nu_{21}}{E_1 E_2} > 0,$$
$$\frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{32}\nu_{21}}{E_1 E_2 E_3} > 0, \quad \mu_{ij} > 0.$$

By circular permutation of the basis-vectors, we have also the necessary conditions:

$$\begin{aligned} &E_i > 0, \mu_{ij} > 0, \\ &1 - \nu_{12}\nu_{21} > 0,\ 1 - \nu_{13}\nu_{31} > 0,\ 1 - \nu_{23}\nu_{32} > 0, \\ &1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{32}\nu_{21} > 0. \end{aligned} \qquad [1.48]$$

1.8.4. *Transverse isotropy*

A transverse isotropic medium of axis 3 is an orthotropic medium which is in addition invariant with respect to any rotation around axis 3. In this case, the directions in the (1,2)-plane are all indistinguishable. By considering the $\pi/2$-rotation around axis 3, the following relations are derived:

$$C_{2222} = C_{1111},\ C_{1133} = C_{2233},\ C_{2323} = C_{3131}$$

and

$$E_1 = E_2, \nu_{12} = \nu_{21}, \nu_{13} = \nu_{23}, \nu_{31} = \nu_{32}.$$

Moreover, considering the rotations of form

$$\underset{\sim}{\boldsymbol{Q}} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and by writing [1.42] and [1.47] for $k = 1, l = 2, o = 1, p = 2$, we can obtain the following relations:

$$C_{1212} = \frac{C_{1111} - C_{1122}}{2} \quad \text{and} \quad \frac{1}{4\mu_{12}} = S_{1212} = \frac{S_{1111} - S_{1122}}{2} = \frac{1 + \nu_{12}}{2E_1}.$$

So, a transverse isotropic medium possess five independent elastic constants.

Hexagonal material symmetry means that the medium is orthotropic and invariant with respect to rotations around axis 3 of angles $\pm 2\pi/3$. It turns out that in this case, the components of $\underset{\approx}{C}$ and $\underset{\approx}{S}$ comply with the same relations as for transverse isotropy. It should be emphasized that the coincidence of hexagonal symmetry and transverse isotropy is very specific to linear elasticity. This remarkable property does not hold for other constitutive laws (nonlinear elasticity, plasticity, etc).

1.8.5. *Isotropy*

An isotropic medium is invariant with respect to any orthogonal transformation. Therefore, it is transverse isotropic in any basis. Consequently, $\underset{\approx}{C}$ has the following form:

$$\left[\underset{\approx}{C}\right] = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & & & 2\mu & 0 & 0 \\ & \text{SYM} & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix}.$$

Young's modulus E_i in all directions are identical (E), all ν_{ij} are identical (ν) and we have:

$$\lambda = \frac{\nu}{(1+\nu)(1-2\nu)} E, \quad \mu = \frac{E}{2(1+\nu)}.$$

The definite positiveness conditions reduce to:

$$E > 0 \quad \text{and} \quad -1 < \nu < \frac{1}{2}$$

which are equivalent to

$$K = \lambda + \frac{2}{3}\mu > 0 \quad \text{and} \quad \mu > 0.$$

The constants (λ, μ) are the Lamé constants. K is the bulk modulus.

Hence, the constitutive law for an isotropic linear elastic medium can be written as:

$$\underset{\sim}{\boldsymbol{\sigma}} = \lambda tr\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right)\underset{\sim}{\boldsymbol{\delta}} + 2\mu\underset{\sim}{\boldsymbol{\varepsilon}} \Longleftrightarrow \underset{\sim}{\boldsymbol{\varepsilon}} = \frac{1+\nu}{E}\underset{\sim}{\boldsymbol{\sigma}} - \frac{\nu}{E}tr\left(\underset{\sim}{\boldsymbol{\sigma}}\right)\underset{\sim}{\boldsymbol{\delta}},$$

where

$$tr\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right) = \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

is the trace of $\underset{\sim}{\boldsymbol{\varepsilon}}$ and $\underset{\sim}{\boldsymbol{\delta}}$ is the unit second-order tensor.

It is convenient to introduce the fourth-order tensors $\underset{\approx}{\boldsymbol{J}}$ and $\underset{\approx}{\boldsymbol{K}}$, which have the same minor and major symmetries as an elasticity tensor: for all symmetric second-order tensor $\underset{\sim}{\boldsymbol{\varepsilon}}$:

$$\underset{\approx}{\boldsymbol{J}} : \underset{\sim}{\boldsymbol{\varepsilon}} = \frac{1}{3}tr\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right)\underset{\sim}{\boldsymbol{\delta}},$$

$$\underset{\approx}{\boldsymbol{K}} : \underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}} - \frac{1}{3}tr\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right)\underset{\sim}{\boldsymbol{\delta}} = \underset{\sim}{\boldsymbol{\varepsilon}}^{\text{dev}},$$

where $\underset{\sim}{\boldsymbol{\varepsilon}}^{\text{dev}}$ is the deviatoric part $\underset{\sim}{\boldsymbol{\varepsilon}}$.

The above relations write in components:

$$J_{ijkl} : \varepsilon_{lk} = \frac{1}{3}\varepsilon_{mm}\delta_{ij}$$

and

$$K_{ijkl} : \varepsilon_{lk} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{mm}\delta_{ij} = \varepsilon_{ij}^{\text{dev}}.$$

So, we have:

$$\underset{\approx}{\boldsymbol{I}} = \underset{\approx}{\boldsymbol{J}} + \underset{\approx}{\boldsymbol{K}},$$

where $\underset{\approx}{\boldsymbol{I}}$ is the identity fourth-order tensor operating on symmetric second-order tensors. Its components are given by:

$$I_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right).$$

The isotropic linear elastic constitutive law can be written equivalently under the following form:

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\boldsymbol{\varepsilon}} = K tr\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right)\underset{\sim}{\boldsymbol{\delta}} + 2\mu\underset{\sim}{\boldsymbol{\varepsilon}}^{\text{dev}} = \left(3K\underset{\approx}{\boldsymbol{J}} + 2\mu\underset{\approx}{\boldsymbol{K}}\right) : \underset{\sim}{\boldsymbol{\varepsilon}}.$$

Hence,

$$\underset{\approx}{\boldsymbol{C}} = 3K\underset{\approx}{\boldsymbol{J}} + 2\mu\underset{\approx}{\boldsymbol{K}}$$

can simply be denoted by $\underset{\approx}{\boldsymbol{C}} = \{3K, 2\mu\}$. Tensors $\underset{\approx}{\boldsymbol{J}}$ and $\underset{\approx}{\boldsymbol{K}}$ correspond to orthogonal projections on spherical and deviatoric symmetric second-order tensors, respectively. They have the following remarkable properties which are easy to establish:

$$\underset{\approx}{\boldsymbol{J}} : \underset{\approx}{\boldsymbol{J}} = \underset{\approx}{\boldsymbol{J}}, \quad \underset{\approx}{\boldsymbol{K}} : \underset{\approx}{\boldsymbol{K}} = \underset{\approx}{\boldsymbol{K}}, \quad \underset{\approx}{\boldsymbol{J}} : \underset{\approx}{\boldsymbol{K}} = \underset{\approx}{\boldsymbol{K}} : \underset{\approx}{\boldsymbol{J}} = 0.$$

So, these properties enable useful simplifications in the algebraic operations on isotropic fourth-order tensors which reduce to operations on their components according to $\underset{\approx}{\boldsymbol{J}}$ and $\underset{\approx}{\boldsymbol{K}}$. For instance, for any $\underset{\approx}{\boldsymbol{C}} = \{a, b\}$ and $\underset{\approx}{\boldsymbol{C}'} = \{a', b'\}$,

$$\underset{\approx}{\boldsymbol{C}} + \underset{\approx}{\boldsymbol{C}'} = \left\{a + a', b + b'\right\}, \quad \underset{\approx}{\boldsymbol{C}} : \underset{\approx}{\boldsymbol{C}'} = \left\{aa', bb'\right\}, \text{ and}$$
$$\underset{\approx}{\boldsymbol{C}}^{-1} = \left\{a^{-1}, b^{-1}\right\} \text{ if } ab \neq 0.$$

In the isotropic case, the strain and stress elastic energy densities are quadratic functions of the two first invariants of $\underset{\sim}{\boldsymbol{\varepsilon}}$ and $\underset{\sim}{\boldsymbol{\sigma}}$, respectively. We have:

$$w\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right) = \frac{1}{2}\lambda tr^2\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right) + \mu tr\left(\underset{\sim}{\boldsymbol{\varepsilon}}^2\right) = \frac{1}{2}K tr^2\left(\underset{\sim}{\boldsymbol{\varepsilon}}\right) + \mu tr\left(\left(\underset{\sim}{\boldsymbol{\varepsilon}}^{\text{dev}}\right)^2\right) \qquad [1.49]$$

and

$$w^*\left(\underset{\sim}{\boldsymbol{\sigma}}\right) = \frac{1+\nu}{2E} tr\left(\underset{\sim}{\boldsymbol{\sigma}}^2\right) - \frac{\nu}{2E} tr^2\left(\underset{\sim}{\boldsymbol{\sigma}}\right)$$
$$= \frac{1}{18K} tr^2\left(\underset{\sim}{\boldsymbol{\sigma}}\right) + \frac{1}{4\mu} tr\left(\left(\underset{\sim}{\boldsymbol{\sigma}}^{\mathrm{dev}}\right)^2\right),$$

where

$$tr\left(\underset{\sim}{\boldsymbol{\varepsilon}}^2\right) = \underset{\sim}{\boldsymbol{\varepsilon}} : \underset{\sim}{\boldsymbol{\varepsilon}} = \varepsilon_{ik}\varepsilon_{kj}\delta_{ij}$$

and

$$tr\left(\underset{\sim}{\boldsymbol{\sigma}}^2\right) = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\boldsymbol{\sigma}} = \sigma_{ik}\sigma_{kj}\delta_{ij}$$

are the sum of the square of the components of $\underset{\sim}{\boldsymbol{\varepsilon}}$ and $\underset{\sim}{\boldsymbol{\sigma}}$, respectively.