Chapter 1

Parametric Sensitivity of Damped Systems

Changes of the eigenvalues and eigenvectors of a linear vibrating system due to changes in system parameters are of wide practical interest. Motivation for this kind of study arises, on the one hand, from the need to come up with effective structural designs without performing repeated dynamic analysis, and, on the other hand, from the desire to visualize the changes in the dynamic response with respect to system parameters. Furthermore, this kind of sensitivity analysis of eigenvalues and eigenvectors has an important role to play in the area of fault detection of structures and modal updating methods. Sensitivity of eigenvalues and eigenvectors is useful in the study of bladed disks of turbomachinery where blade masses and stiffness are nearly the same, or deliberately somewhat altered (mistuned), and one investigates the modal sensitivities due to this slight alteration. Eigensolution derivatives also constitute a central role in the analysis of stochastically perturbed dynamical systems. Possibly, the earliest work on the sensitivity of the eigenvalues was carried out by Rayleigh [RAY 77]. In his classic monograph, he derived the changes in natural frequencies due to small changes in system parameters. Fox and Kapoor [FOX 68] have given exact expressions for the sensitivity of eigenvalues and eigenvectors with respect to any design variables. Their results were obtained in terms of changes in the system property matrices and the eigensolutions of the structure in its current state, and have been used extensively in a wide range of application areas of structural dynamics. Nelson [NEL 76] proposed an efficient method to calculate an eigenvector derivative, which requires only the eigenvalue and eigenvector under consideration. A comprehensive review of research on this kind of sensitivity analysis can be obtained in Adelman and Haftka [ADE 86]. A brief review of some of the existing methods for calculating sensitivity of the eigenvalues and eigenvectors is given in section 1.6 (Chapter 1, [ADH 14]).

The aim of this chapter is to consider parametric sensitivity of the eigensolutions of damped systems. We first start with undamped systems in section 1.1. Parametric sensitivity of viscously damped systems is discussed in section 1.2. In section 1.3, we discuss the sensitivity of eigensolutions of general non-viscously damped systems. In section 1.4, a summary of the techniques introduced in this chapter is provided.

1.1. Parametric sensitivity of undamped systems

The eigenvalue problem of undamped or proportionally damped systems can be expressed by

$$\mathbf{K}(\mathbf{p})\mathbf{x}_j = \lambda_j \mathbf{M}(\mathbf{p})\mathbf{x}_j \qquad [1.1]$$

where λ_j and $\mathbf{x}_j$ are the eigenvalues and the eigenvectors of the dynamic system. $\mathbf{M}(\mathbf{p}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$ and $\mathbf{K}(\mathbf{p}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$, the mass and stiffness matrices, are assumed to be smooth, continuous and differentiable functions of a parameter vector $\mathbf{p} \in \mathbb{R}^m$. Note that $\lambda_j = \omega_j^2$ where ω_j is the jth undamped natural frequency. The vector $\mathbf{p}$ may consist of material properties, e.g. mass density, Poisson's ratio and Young's modulus; or geometric properties, e.g. length, thickness and boundary conditions. The eigenvalues and eigenvectors are smooth differentiable functions of the parameter vector $\mathbf{p}$.

1.1.1. *Sensitivity of the eigenvalues*

We rewrite the eigenvalue equation as

$$[\mathbf{K} - \lambda_j \mathbf{M}]\,\mathbf{x}_j = \mathbf{0} \qquad [1.2]$$

$$\text{or} \quad \mathbf{x}_j^T\,[\mathbf{K} - \lambda_j \mathbf{M}]\,. \qquad [1.3]$$

The functional dependence of $\mathbf{p}$ is removed for notational convenience. Differentiating the eigenvalue equation [1.2] with respect to the element p of the parameter vector we have

$$\left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p}\right]\mathbf{x}_j + [\mathbf{K} - \lambda_j \mathbf{M}]\,\frac{\partial \mathbf{x}_j}{\partial p} = \mathbf{0}. \qquad [1.4]$$

Premultiplying by $\mathbf{x}_j^T$, we have

$$\mathbf{x}_j^T\left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p}\right]\mathbf{x}_j + \mathbf{x}_j^T\,[\mathbf{K} - \lambda_j \mathbf{M}]\,\frac{\partial \mathbf{x}_j}{\partial p} = \mathbf{0}. \qquad [1.5]$$

Using the identity in [1.3], we have

$$\mathbf{x}_j^T \left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j = \mathbf{0} \tag{1.6}$$

$$\text{or} \quad \frac{\partial \lambda_j}{\partial p} = \frac{\mathbf{x}_j^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j}{\mathbf{x}_j^T \mathbf{M} \mathbf{x}_j}. \tag{1.7}$$

Note that when the modes are mass normalized, $\mathbf{x}_j^T \mathbf{M} \mathbf{x}_j = 1$. Equation [1.7] shows that the derivative of a given eigenvalue depends only on eigensolutions corresponding to that particular eigenvalue. Next, we show that this fact is not true when we consider the derivative of the eigenvectors.

1.1.2. *Sensitivity of the eigenvectors*

Different methods have been developed to calculate the derivatives of the eigenvectors. One way to express the derivative of an eigenvector is by a linear combination of all the eigenvectors

$$\frac{\partial \mathbf{x}_j}{\partial p} = \sum_{r=1}^{N} \alpha_{jr} \mathbf{x}_r. \tag{1.8}$$

This can always be done as $\mathbf{x}_r, r = 1, 2, \cdots, N$ forms a complete basis. It is necessary to find expressions for the constant α_{jr} for all $r = 1, 2, \cdots N$. Substituting this in equation [1.4], we have

$$\left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j + \sum_{r=1}^{N} \left[\mathbf{K} - \lambda_j \mathbf{M} \right] \alpha_{jr} \mathbf{x}_r = \mathbf{0}. \tag{1.9}$$

Premultiplying by $\mathbf{x}_k^T$, we have

$$\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j + \sum_{r=1}^{N} \mathbf{x}_k^T \left[\mathbf{K} - \lambda_j \mathbf{M} \right] \alpha_{jr} \mathbf{x}_r = 0 \tag{1.10}$$

We consider $r = k$ and the orthogonality of the eigenvectors

$$\mathbf{x}_k^T \mathbf{K} \mathbf{x}_r = \lambda_k \delta_{kr} \quad \text{and} \quad \mathbf{x}_k^T \mathbf{M} \mathbf{x}_r = \delta_{kr}. \tag{1.11}$$

Using these, we have

$$\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j + (\lambda_k - \lambda_j)\, \alpha_{jik} = 0. \qquad [1.12]$$

From this, we obtain

$$\alpha_{jik} = -\frac{\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j}{\lambda_k - \lambda_j}, \quad \forall\, k \neq j. \qquad [1.13]$$

To obtain the jth term α_{jj}, we differentiate the mass orthogonality relationship in [1.11] as

$$\frac{\partial(\mathbf{x}_j^T \mathbf{M} \mathbf{x}_j)}{\partial p} = 0 \quad \text{or} \quad \frac{\partial \mathbf{x}_j^T}{\partial p} \mathbf{M} \mathbf{x}_j + \mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j + \mathbf{x}_j^T \mathbf{M} \frac{\partial \mathbf{x}_j}{\partial p} = 0. \qquad [1.14]$$

Considering the symmetry of the mass matrix and using the expansion of the eigenvector derivative, we have

$$\mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j + 2\mathbf{x}_j^T \mathbf{M} \frac{\partial \mathbf{x}_j}{\partial p} = 0 \quad \text{or} \quad \sum_{r=1}^{N} 2\mathbf{x}_j^T \mathbf{M} \alpha_{jr} \mathbf{x}_r = -\mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j. \qquad [1.15]$$

Utilizing the othonormality of the mode shapes, we have

$$\alpha_{jj} = -\frac{1}{2} \mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j. \qquad [1.16]$$

The complete eigenvector derivative is therefore given by

$$\frac{\partial \mathbf{x}_j}{\partial p} = -\frac{1}{2} \left(\mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j \right) \mathbf{x}_j + \sum_{k=1 \neq j}^{N} \frac{\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j}{\lambda_j - \lambda_k} \mathbf{x}_k. \qquad [1.17]$$

From equation [1.17], it can be observed that when two eigenvalues are close, the modal sensitivity will be higher as the denominator of the right-hand term will be very small. Unlike the derivative of the eigenvalues given in [1.7], the derivative of an eigenvector requires all the other eigensolutions. This can be computationally demanding for large systems. The method proposed by Nelson [NEL 76] can address this problem. We will discuss Nelson's method in the context of damped systems in the following sections.

1.2. Parametric sensitivity of viscously damped systems

The analytical method in the preceding section is for undamped systems. For damped systems, unless the system is proportionally damped (see section 2.4, Chapter 2 of [ADH 14]), the mode shapes of the system will not coincide with the undamped mode shapes. In the presence of general non-proportional viscous damping, the equation of motion in the modal coordinates will be coupled through the off-diagonal terms of the modal damping matrix, and the mode shapes and natural frequencies of the structure will, in general, be complex. The solution procedures for such non-proportionally damped systems follow mainly two routes: the state-space method and approximate methods in the configuration space, as discussed in Chapters 2 and 3 [ADH 14]. The state-space method (see [NEW 89, GÉR 97], for example) although exact in nature, requires significant numerical effort for obtaining the eigensolutions as the size of the problem doubles. Moreover, this method also lacks some of the intuitive simplicity of traditional modal analysis. For these reasons, there has been considerable research effort in analyzing non-proportionally damped structures in the configuration space. Most of these methods either seek an optimal decoupling of the equation of motion or simply neglect the off-diagonal terms of the modal damping matrix. It may be noted that following such methodologies, the mode shapes of the structure will still be real. The accuracy of these methods, other than the light damping assumption, depends upon various factors, for example frequency separation between the modes and driving frequency (see [PAR 92a, GAW 97] and the references therein for discussions on these topics). A convenient way to avoid the problems that arise due to the use of real normal modes is to incorporate complex modes in the analysis. Apart from the mathematical consistency, by conducting experimental modal analysis, we also often identify complex modes: as Sestieri and Ibrahim [SES 94] have put it "... it is ironic that the real modes are in fact not real at all, in that in practice they do not exist, while complex modes are those practically identifiable from experimental tests. This implies that real modes are pure abstraction, in contrast with complex modes that are, therefore, the only reality!" But surprisingly, most of the current application areas of structural dynamics, which utilize the eigensolution derivatives, e.g. modal updating, damage detection, design optimization and stochastic finite element methods, do not use complex modes in the analysis but rely on the real undamped modes only. This is partly because of the problem of considering an appropriate damping model in the structure and partly because of the unavailability of complex eigensolution sensitivities. Although, there have been considerable research efforts toward damping models, sensitivity of complex eigenvalues and eigenvectors with respect to system parameters appears to have received less attention.

In this section, we determine the sensitivity of complex natural frequencies and mode shapes with respect to some set of design variables in non-proportionally damped discrete linear systems. It is assumed that the system does not possess repeated eigenvalues. In section 2.5 (Chapter 2, [ADH 14]), the mathematical background on linear multiple-degree-of-freedom discrete systems needed for further derivations has already been discussed. Sensitivity of complex eigenvalues is derived in section 1.2.1 in terms of complex modes, natural frequencies and changes in the system property matrices. The approach taken here avoids the use of state-space formulation. In section 1.2.2, sensitivity of complex eigenvectors is derived. The derivation method uses state-space representation of equation of motion for intermediate calculations and then relates the eigenvector sensitivities to the complex eigenvectors of the second-order system and to the changes in the system property matrices. In section 1.2.2.3, a two degree-of-freedom system that shows the “curve-veering” phenomenon has been considered to illustrate the application of the expression for rates of changes of complex eigenvalues and eigenvectors. The results are carefully analyzed and compared with presently available sensitivity expressions of undamped real modes.

1.2.1. *Sensitivity of the eigenvalues*

The equation of motion for free vibration of a linear damped discrete system with N degrees of freedom can be written as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0} \qquad [1.18]$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are mass, damping and stiffness matrices, $\mathbf{q}(t) \in \mathbb{R}^N$ is the vector of the generalized coordinates and $t \in \mathbb{R}^+$ denotes time. The eigenvalue problem associated with equation [1.18] is given by

$$s_j^2\mathbf{M}\mathbf{z}_j + s_j\mathbf{C}\mathbf{z}_j + \mathbf{K}\mathbf{z}_j = \mathbf{0}, \quad \forall\, j = 1, 2, \cdots 2N \qquad [1.19]$$

where $\mathbf{z}_j$ are the mode shapes and the natural frequencies s_j are defined by $s_j = \mathrm{i}\lambda_j$. Unless system [1.18] is proportionally damped, i.e. $\mathbf{C}$ is simultaneously diagonalizable with $\mathbf{M}$ and $\mathbf{K}$ (conditions were derived by Caughey and O’Kelly [CAU 65]), in general, s_j and $\mathbf{z}_j$ will be complex in nature. The calculation of complex modes and natural frequencies is discussed in detail in Chapters 2 and 6 [ADH 14].

Complex modes and frequencies can be exactly obtained by the state-space (first-order) formalisms. Transforming equation [1.18] into state-space form, we obtain

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) \qquad [1.20]$$

where $\mathbf{A} \in \mathbb{R}^{2N\times 2N}$ is the system matrix and $\mathbf{u}(t) \in \mathbb{R}^{2N}$ is the response vector in the state space given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}; \qquad \mathbf{u}(t) = \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix}. \qquad [1.21]$$

In the above equation, $\mathbf{O} \in \mathbb{R}^{N\times N}$ is the null matrix and $\mathbf{I} \in \mathbb{R}^{N\times N}$ is the identity matrix. The eigenvalue problem associated with the above equation is now in terms of an asymmetric matrix and can be expressed as

$$\mathbf{A}\boldsymbol{\phi}_j = s_j\boldsymbol{\phi}_j, \qquad \forall j = 1, \cdots, 2N \tag{1.22}$$

where s_j is the jth eigenvalue and $\boldsymbol{\phi}_j \in \mathbb{C}^{2N}$ is the jth *right* eigenvector that is related to the eigenvector of the second-order system as

$$\boldsymbol{\phi}_j = \left\{ \begin{array}{c} \mathbf{z}_j \\ s_j\mathbf{z}_j \end{array} \right\}. \tag{1.23}$$

The *left* eigenvector $\boldsymbol{\psi}_j \in \mathbb{C}^{2N}$ associated with s_j is defined by the equation

$$\boldsymbol{\psi}_j^T\mathbf{A} = s_j\boldsymbol{\psi}_j^T \tag{1.24}$$

where $(\bullet)^T$ denotes matrix transpose. For distinct eigenvalues, it is easy to show that the right and left eigenvectors satisfy an orthogonality relationship, that is

$$\boldsymbol{\psi}_j^T\boldsymbol{\phi}_k = 0; \qquad \forall j \neq k \tag{1.25}$$

and we may also normalize the eigenvectors so that

$$\boldsymbol{\psi}_j^T\boldsymbol{\phi}_j = 1. \tag{1.26}$$

The above two equations imply that the dynamic system defined by equation [1.20] possesses a set of biorthonormal eigenvectors. As a special case, when all eigenvalues are distinct, this set forms a *complete* set. Henceforth in our discussion, it will be assumed that all the system eigenvalues are distinct.

Suppose the structural system matrices appearing in [1.18] is a function of a parameter p. This parameter can be an element of a larger parameter vector. This can denote a material property (such as Young's modulus) or a geometric parameter (such as thickness). We wish to find the sensitivity of the eigenvalues and eigenvectors with respect to this general parameter. We aim to derive expressions of derivative of eigenvalues and eigenvectors with respect to p without going into the state space.

For the jth set, equation [1.19] can be rewritten as

$$\mathbf{F}_j\mathbf{z}_j = 0 \tag{1.27}$$

where the regular matrix pencil is

$$\mathbf{F}_j = s_j^2\mathbf{M} + s_j\mathbf{C} + \mathbf{K}. \qquad [1.28]$$

Note that complex frequencies can be obtained by substituting $s_j = \mathrm{i}\lambda_j$. Premultiplication of equation [1.27] by $\mathbf{z}_j^T$ yields

$$\mathbf{z}_j^T\mathbf{F}_j\mathbf{z}_j = 0. \qquad [1.29]$$

Differentiating the above equation with respect to p_j, we obtain

$$\frac{\partial \mathbf{z}_j}{\partial p}^T \mathbf{F}_j\mathbf{z}_j + \mathbf{z}_j^T\frac{\partial \mathbf{F}_j}{\partial p}\mathbf{z}_j + \mathbf{z}_j^T\mathbf{F}_j\frac{\partial \mathbf{z}_j}{\partial p} = 0 \qquad [1.30]$$

where $\frac{\partial \mathbf{F}_j}{\partial p}$ stands for $\frac{\partial \mathbf{F}_j}{\partial p_j}$, and can be obtained by differentiating equation [1.28] as

$$\frac{\partial \mathbf{F}_j}{\partial p} = \left[\frac{\partial s_j}{\partial p}\left(2s_j\mathbf{M} + \mathbf{C}\right) + s_j^2\frac{\partial \mathbf{M}}{\partial p} + s_j\frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]. \qquad [1.31]$$

Now taking the transpose of equation [1.27] and using the symmetry property of $\mathbf{F}_j$, it can shown that the first and third terms of the equation [1.30] are zero. Therefore, we have

$$\mathbf{z}_j^T\frac{\partial \mathbf{F}_j}{\partial p}\mathbf{z}_j = 0. \qquad [1.32]$$

Substituting $\frac{\partial \mathbf{F}_j}{\partial p}$ from equation [1.31] into the above equation, we obtain

$$-\frac{\partial s_j}{\partial p}\mathbf{z}_j^T\left(2s_j\mathbf{M} + \mathbf{C}\right)\mathbf{z}_j = \mathbf{z}_j^T\left[s_j^2\frac{\partial \mathbf{M}}{\partial p} + s_j\frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j. \qquad [1.33]$$

From this, we have

$$\frac{\partial s_j}{\partial p} = -\frac{\mathbf{z}_j^T\left[s_j^2\frac{\partial \mathbf{M}}{\partial p} + s_j\frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T\left(2s_j\mathbf{M} + \mathbf{C}\right)\mathbf{z}_j} \qquad [1.34]$$

which is the derivative of the jth complex eigenvalue. For the undamped case, when $\mathbf{C} = 0$, $s_j \to \mathrm{i}\omega_j$ and $\mathbf{z}_j \to \mathbf{x}_j$ (ω_j and $\mathbf{x}_j$ are undamped natural frequencies and modes satisfying $\mathbf{K}\mathbf{x}_j = \omega_j^2\mathbf{M}\mathbf{x}_j$), with usual mass normalization, the denominator $\to 2\mathrm{i}\omega_j$, and we obtain

$$-\mathrm{i}\frac{\partial\omega_j}{\partial p} = \frac{\mathbf{x}_j^T\left[\frac{\partial\mathbf{K}}{\partial p} - \omega_j^2\frac{\partial\mathbf{M}}{\partial p}\right]\mathbf{x}_j}{2\mathrm{i}\omega_j} \quad \text{or} \quad \frac{\partial\omega_j^2}{\partial p} = \mathbf{x}_j^T\left[\frac{\partial\mathbf{K}}{\partial p} - \omega_j^2\frac{\partial\mathbf{M}}{\partial p}\right]\mathbf{x}_j. \qquad [1.35]$$

This is exactly the well-known relationship derived by Fox and Kapoor [FOX 68] for the undamped eigenvalue problem. Thus, equation [1.34] can be viewed as a generalization of the familiar expression of the sensitivity of undamped eigenvalues to the damped case. Following observations may be noted from this result:

– The derivative of a given eigenvalue requires the knowledge of only the corresponding eigenvalue and eigenvector under consideration, and thus a complete solution of the eigenproblem, or from the experimental point of view, eigensolution determination for *all* the modes is not required.

– Changes in mass and/or stiffness introduce more change in the real part of the eigenvalues whereas changes in the damping introduce more change in the imaginary part.

Since $\frac{\partial s_j}{\partial p}$ is complex in equation [1.34], it can be effectively used to determine the sensitivity of the modal damping factors with respect to the system parameters. For small damping, the modal damping factor for the jth mode can be expressed in terms of complex frequencies as $\zeta_j = \Im(\lambda_j)/\Re(\lambda_j)$, with $\Re(\bullet)$ and $\Im(\bullet)$ denoting real and imaginary parts, respectively. As a result, the derivative can be evaluated from

$$\frac{\partial\zeta_j}{\partial p} = \frac{\partial\Im(\lambda_j)/\Re(\lambda_j)}{\partial p} = \left[\frac{\Im(\frac{\partial\lambda_j}{\partial p})\Re(\lambda_j) - \Im(\lambda_j)\Re(\frac{\partial\lambda_j}{\partial p})}{\Re(\lambda_j)^2}\right]. \qquad [1.36]$$

This expression may turn out to be useful since we often directly measure the damping factors from experiment.

1.2.2. *Sensitivity of the eigenvectors*

1.2.2.1. *Modal approach*

We use the state-space eigenvectors to calculate the derivative of the eigenvectors in the configuration space. Since $\mathbf{z}_j$ is the first N rows of $\boldsymbol{\phi}_j$ (see equation [1.23]), we first try to derive $\frac{\partial\boldsymbol{\phi}_j}{\partial p}$ and subsequently obtain $\frac{\partial\mathbf{z}_j}{\partial p}$ using their relationships.

Differentiating [1.22] with respect to p_j, we obtain

$$(\mathbf{A} - s_j)\frac{\partial \boldsymbol{\phi}_j}{\partial p} = -\left(\frac{\partial \mathbf{A}}{\partial p} - \frac{\partial s_j}{\partial p}\right)\boldsymbol{\phi}_j. \quad [1.37]$$

Since it has been assumed that $\mathbf{A}$ has distinct eigenvalues, the right eigenvectors, $\boldsymbol{\phi}_j$, form a complete set of vectors. Therefore, we can expand $\frac{\partial \boldsymbol{\phi}_j}{\partial p}$ as

$$\frac{\partial \boldsymbol{\phi}_j}{\partial p} = \sum_{l=1}^{2N} a_{jl}\boldsymbol{\phi}_l \quad [1.38]$$

where $a_{jl}, \forall l = 1, \cdots 2N$ are set of complex constants to be determined. Substituting $\frac{\partial \boldsymbol{\phi}_j}{\partial p}$ in equation [1.37] and premultiplying by the left eigenvector $\boldsymbol{\psi}_k^T$, we obtain the scalar equation

$$\sum_{l=1}^{2N}(\boldsymbol{\psi}_k^T\mathbf{A}\boldsymbol{\phi}_l - s_j\boldsymbol{\psi}_k^T\boldsymbol{\phi}_l)\, a_{jl} = -\boldsymbol{\psi}_k^T\frac{\partial \mathbf{A}}{\partial p}\boldsymbol{\phi}_j + \frac{\partial s_j}{\partial p}\boldsymbol{\psi}_k^T\boldsymbol{\phi}_j. \quad [1.39]$$

Using the orthogonality relationship of left and right eigenvectors from the above equation, we obtain

$$a_{jk} = \frac{\boldsymbol{\psi}_k^T\frac{\partial \mathbf{A}}{\partial p}\boldsymbol{\phi}_j}{s_j - s_k}; \qquad \forall k = 1, \cdots, 2N; \neq j. \quad [1.40]$$

The a_{jk} as expressed above is not very useful since it is in terms of the left and right eigenvectors of the first-order system. In order to obtain a relationship with the eigenvectors of second-order system, we assume

$$\boldsymbol{\psi}_j = \left\{\begin{array}{c}\boldsymbol{\psi}_{1j}\\ \boldsymbol{\psi}_{2j}\end{array}\right\} \quad [1.41]$$

where $\boldsymbol{\psi}_{1j}, \boldsymbol{\psi}_{2j} \in \mathbb{C}^N$. Substituting $\boldsymbol{\psi}_j$ in equation [1.24] and taking transpose, we obtain

$$\begin{aligned} s_j\boldsymbol{\psi}_{1j} &= -\mathbf{K}\mathbf{M}^{-1}\boldsymbol{\psi}_{2j}\\ s_j\boldsymbol{\psi}_{2j} &= \boldsymbol{\psi}_{1j} - \mathbf{C}\mathbf{M}^{-1}\boldsymbol{\psi}_{2j}\\ \text{or} \quad \boldsymbol{\psi}_{1j} &= \left[s_j\mathbf{I} + \mathbf{C}\mathbf{M}^{-1}\right]\boldsymbol{\psi}_{2j}. \end{aligned} \quad [1.42]$$

Elimination of $\boldsymbol{\psi}_{1_j}$ from the above two equation yields

$$\begin{aligned} s_j\left(s_j\boldsymbol{\psi}_{2_j}+\mathbf{C}\mathbf{M}^{-1}\boldsymbol{\psi}_{2_j}\right) &= -\mathbf{K}\mathbf{M}^{-1}\boldsymbol{\psi}_{2_j} \\ \text{or} \quad \left[s_j^2\mathbf{M}+s_j\mathbf{C}+\mathbf{K}\right]\left(\mathbf{M}^{-1}\boldsymbol{\psi}_{2_j}\right) &= 0. \end{aligned} \qquad [1.43]$$

By comparison of this equation with equation [1.19], it can be seen that the vector $\mathbf{M}^{-1}\boldsymbol{\psi}_{2_j}$ is parallel to $\mathbf{z}_j$; that is, there exists a non-zero $\beta_j \in \mathbb{C}$ such that

$$\mathbf{M}^{-1}\boldsymbol{\psi}_{2_j} = \beta_j\mathbf{z}_j \quad \text{or} \quad \boldsymbol{\psi}_{2_j} = \beta_j\mathbf{M}\mathbf{z}_j. \qquad [1.44]$$

Now substituting $\boldsymbol{\psi}_{1_j}$, $\boldsymbol{\psi}_{2_j}$ and using the definition of $\boldsymbol{\phi}_j$ from equation [1.23] into the normalization condition [1.26], the scalar *constant* β_j can be obtained as

$$\beta_j = \frac{1}{\mathbf{z}_j^T\left[2s_j\mathbf{M}+\mathbf{C}\right]\mathbf{z}_j}. \qquad [1.45]$$

Using $\boldsymbol{\psi}_{2_j}$ from equation [1.44] into the second equation of [1.42], we obtain

$$\boldsymbol{\psi}_j = \beta_j\mathbf{P}_j\boldsymbol{\phi}_j; \quad \text{where} \quad \mathbf{P}_j = \begin{bmatrix} s_j\mathbf{M}+\mathbf{C} & \mathbf{O} \\ \mathbf{O} & \frac{\mathbf{M}}{s_j} \end{bmatrix} \in \mathbb{C}^{2N\times 2N}. \qquad [1.46]$$

The above equation along with the definition of $\boldsymbol{\phi}_j$ in [1.23] completely relates the left and right eigenvectors of the first-order system to the eigenvectors of the second-order system.

The derivative of the system matrix $\mathbf{A}$ can be expressed as

$$\begin{aligned} \frac{\partial\mathbf{A}}{\partial p} &= \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \frac{\partial\mathbf{M}^{-1}\mathbf{K}}{\partial p} & \frac{\partial\mathbf{M}^{-1}\mathbf{C}}{\partial p} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ -\mathbf{M}^{-2}\frac{\partial\mathbf{M}}{\partial p}\mathbf{K}+\mathbf{M}^{-1}\frac{\partial\mathbf{K}}{\partial p} & -\mathbf{M}^{-2}\frac{\partial\mathbf{M}}{\partial p}\mathbf{C}+\mathbf{M}^{-1}\frac{\partial\mathbf{C}}{\partial p} \end{bmatrix} \end{aligned} \qquad [1.47]$$

from which after some simplifications, the numerator of the right-hand side of equation [1.40] can be obtained as

$$\boldsymbol{\psi}_k^T\frac{\partial\mathbf{A}}{\partial p}\boldsymbol{\phi}_j = -\beta_k\mathbf{z}_k^T\left\{-\mathbf{M}^{-1}\frac{\partial\mathbf{M}}{\partial p}\left[\mathbf{K}+s_j\mathbf{C}\right]+\frac{\partial\mathbf{C}}{\partial p}+\frac{\partial\mathbf{K}}{\partial p}\right\}\mathbf{z}_j. \qquad [1.48]$$

Since $\mathbf{I} = \mathbf{M}\mathbf{M}^{-1}$, $\frac{\partial \mathbf{I}}{\partial p} = \frac{\partial \mathbf{M}}{\partial p}\mathbf{M}^{-1} + \mathbf{M}\left[-\mathbf{M}^{-2}\frac{\partial \mathbf{M}}{\partial p}\right] = \mathbf{O}$ or $\frac{\partial \mathbf{M}}{\partial p}\mathbf{M}^{-1} = \mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial p}$, that is $\mathbf{M}^{-1}$ and $\frac{\partial \mathbf{M}}{\partial p}$ commute in product. Using this property and also from [1.19] noting that $s_j^2 \mathbf{z}_j = -\mathbf{M}^{-1}\left[s_j \mathbf{C} + \mathbf{K}\right]\mathbf{z}_j$, we finally obtain

$$a_{jk} = -\beta_k \frac{\mathbf{z}_k \left[s_j^2 \frac{\partial \mathbf{M}}{\partial p} + s_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j}{s_j - s_k}; \; \forall k = 1, \cdots, 2N; \neq j. \qquad [1.49]$$

This equation relates the a_{jk} with the complex modes of the second-order system.

To obtain a_{jj}, we begin with differentiation of the normalization condition [1.26] with respect to p and obtain the relationship

$$\frac{\partial \boldsymbol{\psi}_j}{\partial p}^T \boldsymbol{\phi}_j + \boldsymbol{\psi}_j^T \frac{\partial \boldsymbol{\phi}_j}{\partial p} = 0. \qquad [1.50]$$

Substitution of $\boldsymbol{\psi}_j$ from equation [1.46] further leads to

$$\beta_j \left\{ \frac{\partial \boldsymbol{\phi}_j}{\partial p}^T \mathbf{P}_j^T \boldsymbol{\phi}_j + \boldsymbol{\phi}_j^T \frac{\partial \mathbf{P}_j}{\partial p}^T \boldsymbol{\phi}_j + \boldsymbol{\phi}_j^T \mathbf{P}_j^T \frac{\partial \boldsymbol{\phi}_j}{\partial p} \right\} = 0 \qquad [1.51]$$

where $\frac{\partial \mathbf{P}_j}{\partial p}$ can be derived from equation [1.46] as

$$\frac{\partial \mathbf{P}_j}{\partial p} = \begin{bmatrix} \frac{\partial s_j}{\partial p}\mathbf{M} + s_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p} & \mathbf{O} \\ \mathbf{O} & -\frac{\mathbf{M}}{s_j^2}\frac{\partial s_j}{\partial p} + \frac{\frac{\partial \mathbf{M}}{\partial p}}{s_j} \end{bmatrix}. \qquad [1.52]$$

Since $\mathbf{P}_j$ is a symmetric matrix, equation [1.51] can be rearranged as

$$2\left(\beta_j \boldsymbol{\phi}_j^T \mathbf{P}_j\right) \frac{\partial \boldsymbol{\phi}_j}{\partial p} = -\beta_j \boldsymbol{\phi}_j^T \frac{\partial \mathbf{P}_j}{\partial p} \boldsymbol{\phi}_j. \qquad [1.53]$$

Note that the term within the bracket is $\boldsymbol{\psi}_j^T$ (see equation [1.46]). Using the assumed expansion of $\frac{\partial \boldsymbol{\phi}_j}{\partial p}$ from [1.40], this equation reads

$$2\boldsymbol{\psi}_j^T \sum_{l=1}^{2N} a_{jl} \boldsymbol{\phi}_l = -\beta_j \boldsymbol{\phi}_j^T \frac{\partial \mathbf{P}_j}{\partial p} \boldsymbol{\phi}_j. \qquad [1.54]$$

The left-hand side of the above equation can be further simplified

$$\begin{aligned}\boldsymbol{\phi}_j^T \frac{\partial \mathbf{P}_j}{\partial p}\boldsymbol{\phi}_j = &\; \mathbf{z}_j^T \left[\frac{\partial s_j}{\partial p}\mathbf{M} + s_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right]\mathbf{z}_j + \\ & \mathbf{z}_j^T s_j \left[-\frac{\mathbf{M}}{s_j^2}\frac{\partial s_j}{\partial p} + \frac{\frac{\partial \mathbf{M}}{\partial p}}{s_j}\right] s_j \mathbf{z}_j = \mathbf{z}_j^T \left[2 s_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right]\mathbf{z}_j.\end{aligned} \quad [1.55]$$

Finally, using the orthogonality property of left and right eigenvectors, from equation [1.54], we obtain

$$a_{jj} = -\frac{1}{2}\frac{\mathbf{z}_j^T \left[2 s_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T \left[2 s_j \mathbf{M} + \mathbf{C}\right]\mathbf{z}_j}. \quad [1.56]$$

In the above equation, a_{jj} is expressed in terms of the complex modes of the second-order system. Now recalling the definition of $\boldsymbol{\phi}_j$ in [1.23], from the first N rows of equation [1.38], we can write

$$\begin{aligned}\frac{\partial \mathbf{z}_j}{\partial p} &= a_{jj}\mathbf{z}_j + \sum_{k \neq j}^{2N} a_{jk}\mathbf{z}_k = -\frac{1}{2}\frac{\mathbf{z}_j^T \left[2 s_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T \left[2 s_j \mathbf{M} + \mathbf{C}\right]\mathbf{z}_j}\mathbf{z}_j \\ &- \sum_{k \neq j}^{2N} \beta_k \frac{\mathbf{z}_k \left[s_j^2 \frac{\partial \mathbf{M}}{\partial p} + s_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j}{s_j - s_k}\mathbf{z}_k.\end{aligned} \quad [1.57]$$

We know that for any real symmetric system, first-order eigenvalues and eigenvectors appear in complex conjugate pairs. Using usual definition of natural frequency, that is $s_k = \mathrm{i}\lambda_k$ and consequently $s_k^* = -\mathrm{i}\lambda_k^*$, where $(\bullet)^*$ denotes complex conjugate, the above equation can be rewritten in a more convenient form as

$$\begin{aligned}\frac{\partial \mathbf{z}_j}{\partial p} &= -\frac{1}{2}\frac{\mathbf{z}_j^T \left[\frac{\partial \mathbf{M}}{\partial p} - \mathrm{i}\frac{\partial \mathbf{C}}{\partial p}/2\lambda_j\right]\mathbf{z}_j}{\mathbf{z}_j^T \left[\mathbf{M} - \mathrm{i}\mathbf{C}/2\lambda_j\right]\mathbf{z}_j}\mathbf{z}_j \\ &+ \sum_{k \neq j}^{N} \left[\frac{\alpha_k (\mathbf{z}_k^T \frac{\partial \tilde{\mathbf{F}}}{\partial p}\mathbf{z}_j)\mathbf{z}_k}{\lambda_j - \lambda_k} - \frac{\alpha_k^* ({\mathbf{z}_k^*}^T \frac{\partial \tilde{\mathbf{F}}}{\partial p}^* \mathbf{z}_j^*)\mathbf{z}_k^*}{\lambda_j + \lambda_k^*}\right]\end{aligned} \quad [1.58]$$

where

$$\frac{\partial \tilde{\mathbf{F}}}{\partial p} = \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j^2 \frac{\partial \mathbf{M}}{\partial p} + \mathrm{i}\lambda_j \frac{\partial \mathbf{C}}{\partial p} \right]$$

$$\text{and} \quad \alpha_k = \mathrm{i}\beta_k = \frac{1}{\mathbf{z}_k^T \left[2\lambda_k \mathbf{M} - \mathrm{i}\mathbf{C} \right] \mathbf{z}_k}.$$

This result is a generalization of the known expression of the sensitivity of real undamped eigenvectors to complex eigenvectors. The following observations can be made from this result:

– Unlike the eigenvalue derivative, the derivative of a given complex eigenvector requires the knowledge of all the other complex eigenvalues and eigenvectors.

– The sensitivity depends very much on the modes whose frequency is close to that of the considered mode.

– Like eigenvalue derivative, changes in mass and/or stiffness introduce more changes in the real part of the eigenvector whereas changes in damping introduce more changes in the imaginary part.

From equation [1.58], it is easy to see that in the undamped limit $\mathbf{C} \to 0$, and consequently $\lambda_k, \lambda_k^* \to \omega_k$; $\mathbf{z}_k, \mathbf{z}_k^* \to \mathbf{x}_k$; $\frac{\partial \tilde{\mathbf{F}}}{\partial p}, \frac{\partial \tilde{\mathbf{F}}}{\partial p}^* \to \left[\frac{\partial \mathbf{K}}{\partial p} - \omega_j^2 \frac{\partial \mathbf{M}}{\partial p} \right]$ and also with usual mass normalization of the undamped modes $\alpha_k, \alpha_k^* \to \frac{1}{2\omega_k}$ reduces the above equation exactly to the corresponding well-known expression derived by Fox and Kapoor [FOX 68] for derivative of undamped modes.

1.2.2.2. *Nelson's method*

The method outlined in the previous section obtained the eigenvector derivative as a linear combination of all the eigenvectors. For large-scale structures, with many degrees of freedom, obtaining all the eigenvectors is a computationally expensive task. Nelson [NEL 76] introduced the approach, extended here, where only the eigenvector of interest was required. Lee *et al.* [LEE 99a] calculated the eigenvector derivatives of self-adjoint systems using a similar approach to Nelson. This section extends Nelson's method to non-proportionally damped systems with complex modes. This method has the great advantage that only the eigenvector of interest is required.

The eigenvectors are not unique, in the sense that any scalar (complex) multiple of an eigenvector is also an eigenvector. As a result, their derivatives are also not unique. It is necessary to normalize the eigenvector for further mathematical derivations. There are numerous ways of introducing a normalization to ensure uniqueness. For undamped systems, mass normalization is the most common. A useful normalization for damped systems that follows from equation [2.211] (Chapter 2, [ADH 14]) is

$$\mathbf{z}_j^T \left[s_j\mathbf{M} + (1/s_j)\,\mathbf{K}\right] \mathbf{z}_j = \mathbf{z}_j^T \left[2s_j\mathbf{M} + \mathbf{C}\right] \mathbf{z}_j = 1. \qquad [1.59]$$

Differentiating the equation governing the eigenvalues [1.19] with respect to the parameter p, gives

$$\left[s_j^2 \frac{\partial \mathbf{M}}{\partial p} + s_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right] u_j + \left[2s_j\mathbf{M} + \mathbf{C}\right] u_j \frac{\partial s_j}{\partial p} + \left[s_j^2\,\mathbf{M} + s_j\,\mathbf{C} + \mathbf{K}\right] \frac{\partial u_j}{\partial p} = 0. \qquad [1.60]$$

Rewriting this, we see that the eigenvector derivative satisfies

$$\left[s_j^2\,\mathbf{M} + s_j\,\mathbf{C} + \mathbf{K}\right] \frac{\partial \mathbf{u}_j}{\partial p} = \mathbf{h}_j \qquad [1.61]$$

where the vector $\mathbf{h}_j$ consists of the first two terms in equation [1.60], and all these quantities are now known. Equation [1.61] cannot be solved to obtain the eigenvector derivative because the matrix is singular. For distinct eigenvalues, this matrix has a null space of dimension 1. Following Nelson's approach, the eigenvector derivative is written as

$$\frac{\partial \mathbf{u}_j}{\partial p} = \mathbf{v}_j + d_j\mathbf{u}_j \qquad [1.62]$$

where $\mathbf{v}_j$ and d_j have to be determined. These quantities are not unique since any multiple of the eigenvector may be added to $\mathbf{v}_j$. A convenient choice is to identify the element of maximum magnitude in $\mathbf{u}_j$ and make the corresponding element in $\mathbf{v}_j$ equal to zero. Although other elements of $\mathbf{v}_j$ could be set to zero, this choice is most likely to produce a numerically well-conditioned problem. Substituting equation [1.62] into equation [1.61], gives

$$\left[s_j^2\,\mathbf{M} + s_j\,\mathbf{C} + \mathbf{K}\right]\,\mathbf{v}_j = \mathbf{F}_j\,\mathbf{v}_j = \mathbf{h}_j. \qquad [1.63]$$

This may be solved, including the constraint on the zero element of $\mathbf{v}_j$, by solving the equivalent problem,

$$\begin{bmatrix} \mathbf{F}_{j11} & \mathbf{0} & \mathbf{F}_{j13} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{F}_{j31} & \mathbf{0} & \mathbf{F}_{j33} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_{j1} \\ x_{j2}\,(=0) \\ \mathbf{v}_{j3} \end{Bmatrix} = \begin{Bmatrix} \mathbf{h}_{j1} \\ 0 \\ \mathbf{h}_{j3} \end{Bmatrix}. \qquad [1.64]$$

where the $\mathbf{F}_j$ is defined in equation [1.63], and has the row and column corresponding to the zeroed element of $\mathbf{v}_j$ replaced with the corresponding row and column of the identity matrix. This approach maintains the banded nature of the structural matrices, and hence is computationally efficient.

It only remains to compute the scalar constant d_j to obtain the eigenvector derivative. For this, the normalization equation must be used. Differentiating equation [1.59], substituting equation [1.62] and rearranging produces

$$d_j = -\mathbf{u}_j^T \left[2s_j\mathbf{M} + \mathbf{C}\right] \mathbf{v}_j - \frac{1}{2}\mathbf{u}_j^T \left[2\mathbf{M}\frac{\partial s_j}{\partial p} + 2s_j\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right] \mathbf{u}_j. \qquad [1.65]$$

1.2.2.3. *Example: two degree-of-freedom system*

Sensitivity of complex frequencies

A two degree-of-freedom system has been considered to illustrate a possible use of the expressions developed so far. Figure 1.1 shows the example taken together with the numerical values. When eigenvalues are plotted against a system parameter, they create a family of "root loci". When two loci approach together, they may cross or rapidly diverge. The latter case is called "curve veering". The veering of the real part of the complex frequencies for the system considered is shown in Figure 1.2. During veering, rapid changes take place in the eigensolutions, as Leissa [LEI 74] pointed out "... the (eigenfunctions) must undergo violent change − figuratively speaking, a dragonfly one instant, a butterfly the next, and something indescribable in between". Thus, this is an interesting problem for applying the general results derived in this section.

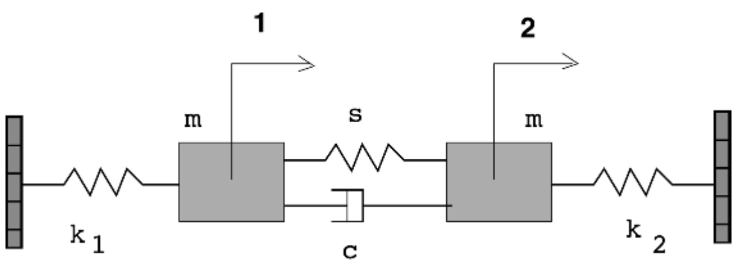

Figure 1.1. *Two degree-of-freedom system shows veering,* $m = 1$ *kg,* $k_1 = 1{,}000$ *N/m,* $c = 4.0$ *Ns/m*

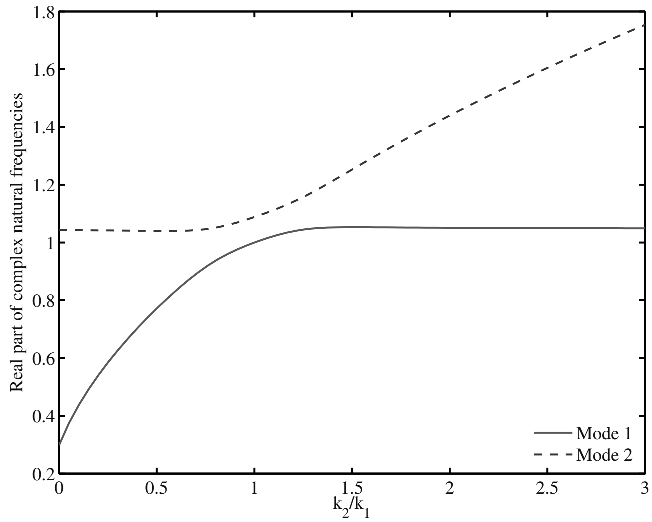

Figure 1.2. *Real part of the complex frequencies of the two modes as a function of k_2 showing the veering phenomenon*

Figure 1.3 shows the imaginary part (normalized by dividing with $\sqrt{k_1/m}$) of the derivative of first natural frequency with respect to the damping parameter "c" over a parameter variation of k_2 and s. This plot was obtained by programming of equation [1.34] in `Matlab`™, substituting $s_j = \mathrm{i}\lambda_j$. The imaginary part has been chosen to be plotted here because a change in damping is expected to contribute a significant change in the imaginary part. The sharp rise of the rate in the low-value region of k_2 and s could be intuitively guessed because there the damper becomes the only "connecting element" between the two masses and so any change made there is expected to have a strong effect. As we move near to the veering range ($k_2 \approx k_1$ and $s \approx 0$), the story becomes quite different. In the first mode, the two masses move in the same direction, in fact in the limit the motion approaches a "rigid body mode". Here, the change no longer remains sensitive to the changes in connecting the element (i.e. only the damper since $s \approx 0$) as hardly any force transmission takes place between the two masses. For this reason, we expect a sharp fall in the derivative as can be noted along the $s \approx 0$ region of the figure. For the region when s is large, we also observe a lower value of derivative, but the reason there is different. The stiffness element "s" shares most of the force being transmitted between the two masses and hence does not depend much on the change of the value of the damper. A similar plot has been shown in Figure 1.4 for the second natural frequency. Unlike the previous case, here the derivative increases in the veering range. For the second mode, the masses move in the opposite direction and in the veering range the difference between them becomes maximal. Since $s \approx 0$, only the damper is being stretched and as a result of this, a small change there produces a large effect. Thus, the use of equation [1.34] can provide good physical insight into the problem and can effectively be used in modal updating, damage detection and for design purposes by taking the damping matrix together with the mass and stiffness matrices, improving the current practice of using the mass and stiffness matrices only.

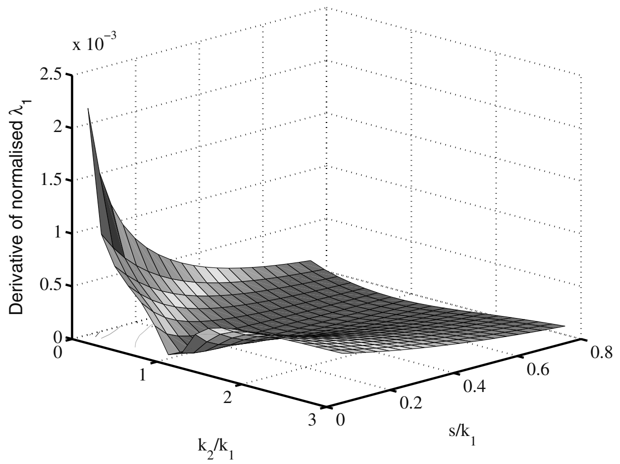

Figure 1.3. *Imaginary part of the derivative of the first natural frequency, λ_1, with respect to the damping parameter, c*

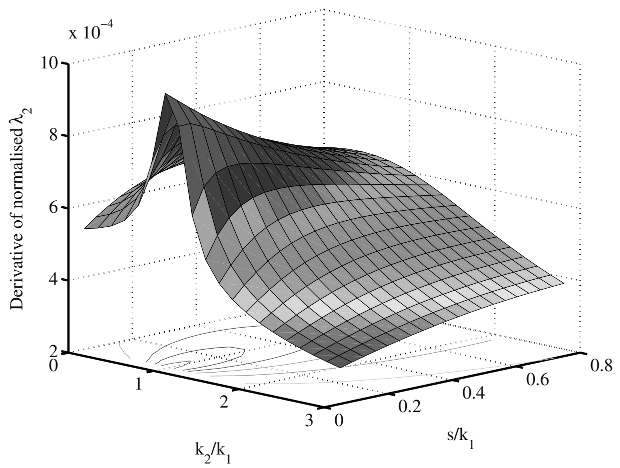

Figure 1.4. *Imaginary part of the derivative of the second natural frequency, λ_2, with respect to the damping parameter, c*

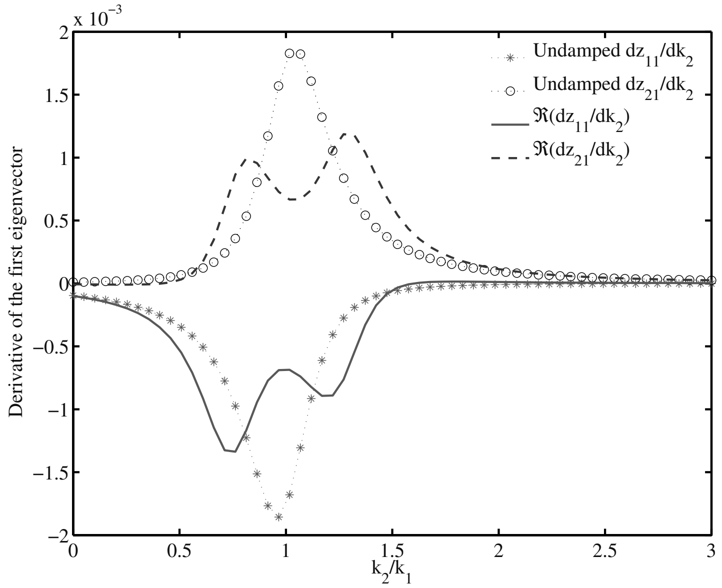

Figure 1.5. *Real part of the derivative of the first eigenvector with respect to the stiffness parameter* k_2

Sensitivity of eigenvectors

Sensitivity of eigenvectors for the problem shown in Figure 1.1 can be directly obtained from equation [1.58]. Here, we have focused on the calculation of the sensitivity of eigenvectors with respect to the parameter k_2. Figure 1.5 shows the real part of the sensitivity of the first eigenvector normalized by its $\mathcal{L}^2$ norm (that is $\Re\left\{\frac{\mathrm{d}\mathbf{z}_1}{\mathrm{d}k_2}\right\} / \parallel \mathbf{z}_1 \parallel$) plotted over a variation of k_2/k_1 from 0 to 3 for both the coordinates. The value of the spring constant for the connecting spring is kept fixed at $s = 100$ N/m. The real part of the sensitivity of complex eigenvectors has been chosen mainly for two reasons: (1) any change in stiffness is expected to have made more changes in the real part; and (2) to compare it with the corresponding changes of the real undamped modes. Derivative of the first eigenvector (normalized by its $\mathcal{L}^2$ norm) with respect to k_2 corresponding to the undamped system (i.e. removing the damper) is also shown in the same figure (see the figure legend for details). This is calculated from the expression derived by Fox and Kapoor [FOX 68]. Similar plots for the second eigenvector are shown in Figure 1.6. Both of these figures reveal a common feature: around the veering range, i.e. $0.5 < k_2/k_1 < 1.5$, the damped and the undamped sensitivities show considerable differences while outside this region they almost trace each other. A physical explanation of this phenomenon can be given. For the problem considered here, the damper acts as an additional "connecting element" between the two masses together with the spring "s". As a result, it "prevents" the system from being closed to show a "strong" veering effect (i.e. when $k_2 = k_1$ and the force transmission between the masses is close to zero) and thus reduces the sensitivity of both the modes. However, for the first mode, both masses move in the same direction and the damper has less effect compared to the second mode where the masses move in the opposite directions and have much greater effect on the sensitivities.

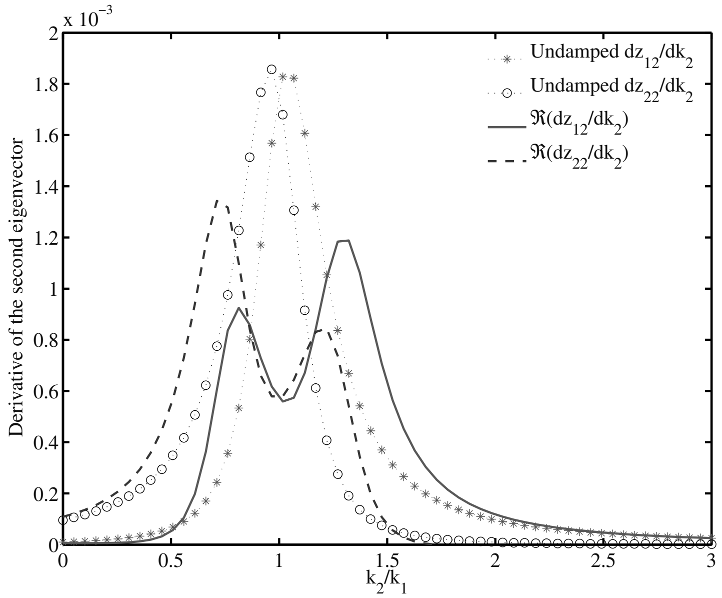

Figure 1.6. *Real part of the derivative of the second eigenvector with respect to the stiffness parameter* k_2

To analyze the results from a quantitative point of view, at this point it is interesting to look at the variation of the modal damping factors shown in Figure 1.7. For the first mode, the damping factor is quite low (in the order of $\approx 10^{-4}$ near the veering range) but still the sensitivities of the undamped mode and that of the the real part of the complex mode for both coordinates are quite different. Again, away from the veering range, $k_2/k_1 > 2$, the damping factor is high but the sensitivities of the undamped mode and that of real part of the complex mode are quite similar. This is the opposite to what we normally expect, as the common belief is that, when the damping factors are low, the undamped modes and the real part of complex modes should behave similarly and vice versa. For the second mode, the damping factor does not change very much due to a variation of k_2 except that it becomes slightly higher in the vicinity of the veering range. But the difference between the sensitivities of the undamped mode and that of the real part of the complex mode for both coordinates changes much more significantly than the damping factor. This demonstrates that even when the damping factors are similar, the sensitivity of the undamped modes and that of the real part of the complex modes can be significantly different. Thus, the use of the expression for the derivatives of undamped mode shapes can lead to a significant error even when the damping is very low. The expressions derived in this section should be used for any kind of study involving such a sensitivity analysis.

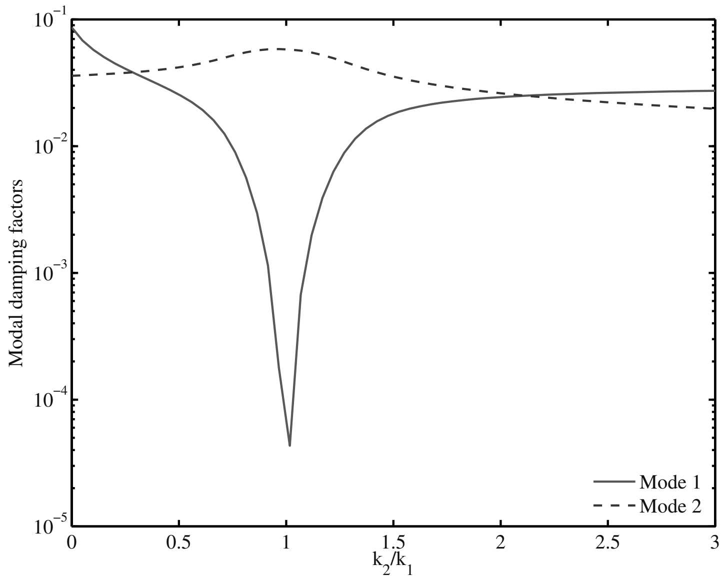

Figure 1.7. *Modal damping factors for both the modes*

Since the expression in equations [1.34] and [1.58] has been derived exactly, the numerical results obtained here are also exact within the precision of the arithmetic used for the calculations. The only instance for arriving at an approximate result is when approximate complex frequencies and modes are used in the analysis. However, for this example, it was verified that the use of approximate methods to obtain complex eigensolutions in the configuration space discussed in Chapter 2 of [ADH 14] and the exact eigensolutions obtained from the state-space method produces negligible discrepancy. Since in most engineering applications we normally do not encounter very high value of damping, we can use approximate methods to obtain eigensolusions in the configuration space in conjunction with the sensitivity expressions derived here. This will allow the analyst to study the sensitivity of eigenvalues and eigenvectors of non-classically damped systems in a similar way to those of undamped systems.

1.3. Parametric sensitivity of non-viscously damped systems

The studies so far have only considered viscous damping models. However, it is well known that the viscous damping is not the only damping model within the scope of linear analysis, examples are: damping in composite materials [BAB 94], energy dissipation in structural joints [EAR 66, BEA 77], damping mechanism in composite beams [BAN 91], to mention only a few. We consider a class of non-viscous damping models in which the damping forces depend on the past history of motion via convolution integrals over some kernel functions (see Chapters 4 and 5 of [ADH 14]). The equation of motion describing free vibration of an N degree-of-freedom linear system with such damping can be expressed by

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \int_{-\infty}^{t} \boldsymbol{\mathcal{G}}(t-\tau)\,\dot{\mathbf{q}}(\tau)\,\mathrm{d}\tau + \mathbf{K}\mathbf{q}(t) = \mathbf{0} \qquad [1.66]$$

where $\mathbf{M}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are the mass and stiffness matrices, $\boldsymbol{\mathcal{G}}(t) \in \mathbb{R}^{N\times N}$ is the matrix of kernel functions and $\mathbf{0}$ is an $N \times 1$ vector of zeros. In the special case when $\boldsymbol{\mathcal{G}}(t-\tau) = \mathbf{C}\,\delta(t-\tau)$, equation [1.66] reduces to the case of viscously damped systems. The damping model of this kind is a further generalization of the familiar viscous damping. The central aim of this section is to extend the eigensensitivity analysis to non-viscously damped systems of the form [1.66]. In the subsequent sections, the derivative of eigenvalues and eigenvectors is derived. Unlike viscously damped systems, the conversion of equation [1.66] into the state-space form may not be advantageous because the eigenvalue problem in the state space cannot be presented in the form of the conventional matrix eigenvalue problem involving *constant* matrices. For this reason, the approach adopted here does not employ the state-space formulation of the equation of motion. An application of the derived expressions for the derivative of eigensolutions is illustrated by considering a two degree-of-freedom system with nonviscous damping.

The determination of eigenvalues and eigenvectors of general non-viscously damped systems was discussed in Chapter 5 of [ADH 14]. Taking the Laplace transform of equation [1.66], we have

$$s^2\mathbf{M}\bar{\mathbf{q}}(s) + s\,\mathbf{G}(s)\bar{\mathbf{q}}(s) + \mathbf{K}\bar{\mathbf{q}}(s) = \mathbf{0} \quad \text{or} \quad \mathbf{D}(s)\bar{\mathbf{q}}(s) = \mathbf{0} \qquad [1.67]$$

where the *dynamic stiffness matrix* is

$$\mathbf{D}(s) = s^2\mathbf{M} + s\,\mathbf{G}(s) + \mathbf{K} \ \in \mathbb{C}^{N\times N} \qquad [1.68]$$

where $\bar{\mathbf{q}}(s) = \mathcal{L}\left[\mathbf{q}(t)\right] \in \mathbb{C}^{N}$, $\mathbf{G}(s) = \mathcal{L}\left[\boldsymbol{\mathcal{G}}(t)\right] \in \mathbb{C}^{N\times N}$ and $\mathcal{L}\left[\bullet\right]$ denotes the Laplace transform. In the context of structural dynamics, $s = \mathrm{i}\omega$, where $\omega \in \mathbb{R}^{+}$

denotes the frequency. We consider the damping to be "non-proportional" (conditions for proportionality of non-viscous damping were derived in section 5.1, Chapter 5, [ADH 14]), that is, the mass and stiffness matrices as well as the matrix of kernel functions cannot be simultaneously diagonalized by any linear transformation. However, it is assumed that $\mathbf{M}^{-1}$ exists and $\mathbf{G}(s)$ is such that the motion is dissipative. Conditions that $\mathbf{G}(s)$ must satisfy in order to produce dissipative motion were given by Golla and Hughes [GOL 85].

The eigenvalue problem associated with equation [1.66] can be defined from [1.67] as

$$\left[s_j^2\mathbf{M} + s_j\,\mathbf{G}(s_j) + \mathbf{K}\right]\mathbf{z}_j = \mathbf{0} \quad \text{or} \quad \mathbf{D}(s_j)\mathbf{z}_j = \mathbf{0}, \quad \forall j = 1, \cdots, m \qquad [1.69]$$

where $\mathbf{z}_j \in \mathbb{C}^N$ is the jth eigenvector. The eigenvalues, s_j, are roots of the characteristic equation

$$\det\left[s^2\mathbf{M} + s\,\mathbf{G}(s) + \mathbf{K}\right] = 0. \qquad [1.70]$$

We consider that the order of the characteristic equation is m. Following Chapter 5 of [ADH 14], we may group the eigenvectors as (1) elastic modes (corresponding to N complex conjugate pairs of eigenvalues), and (2) non-viscous modes (corresponding to the "additional" $m - 2N$ eigenvalues). The elastic modes are related to the N modes of vibration of structural systems. We assume that *all* m eigenvalues are distinct. Following section 5.6.2 (Chapter 5, [ADH 14]), the eigenvectors can be normalized as

$$\begin{aligned} &\mathbf{z}_j^T \frac{\partial \mathbf{D}(s)}{\partial s}|_{s=s_j}\mathbf{z}_j = \gamma_j \\ \text{or} \quad &\mathbf{z}_j^T\left[2s_j\mathbf{M} + \mathbf{G}(s_j) + s_j\frac{\partial \mathbf{G}(s)}{\partial s}|_{s=s_j}\right]\mathbf{z}_j = \gamma_j, \quad \forall j = 1, \cdots, m \end{aligned} \qquad [1.71]$$

where $\gamma_j \in \mathbb{C}$ is some non-zero constant. Note that equation [1.71] reduces to the corresponding normalization relationship for viscously damped systems (see [VIG 86, SES 94], for example) when $\mathbf{G}(s)$ is constant with respect to s. Numerical values of γ_j can be selected in various ways, see the discussion in section 5.6.2 (Chapter 5, [ADH 14]).

1.3.1. *Sensitivity of the eigenvalues*

Suppose the system matrices in equation [1.66] are functions of some design parameter p. In this section, we intend to obtain an expression of the derivative of the

jth eigenvalue with respect to the design parameter p. Differentiating equation [1.69] with respect to p, we obtain

$$\left[2s_j \frac{\partial s_j}{\partial p}\mathbf{M} + s_j^2 \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial s_j}{\partial p}\mathbf{G}(s_j) + s_j \frac{\partial\left[\mathbf{G}(s_j)\right]}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j + \left[s_j^2\mathbf{M} + s_j\,\mathbf{G}(s_j) + \mathbf{K}\right]\frac{\partial \mathbf{z}_j}{\partial p} = \mathbf{0}. \qquad [1.72]$$

The term $\frac{\partial[\mathbf{G}(s_j)]}{\partial p}$ appearing in the above equation can be expressed as

$$\frac{\partial\left[\mathbf{G}(s_j)\right]}{\partial p} = \frac{\partial s_j}{\partial p}\frac{\partial \mathbf{G}(s)}{\partial s}|_{s=s_j} + \frac{\partial \mathbf{G}(s)}{\partial p}|_{s=s_j}. \qquad [1.73]$$

Premultiplying equation [1.72] by $\mathbf{z}_j^T$ and using the symmetry property of the system matrices, it may be observed that the second term of the equation becomes zero due to [1.69]. Substituting [1.73] into equation [1.72], we obtain

$$\mathbf{z}_j^T\left[s_j^2\frac{\partial \mathbf{M}}{\partial p} + s_j\frac{\partial \mathbf{G}(s)}{\partial p}|_{s=s_j} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j + \mathbf{z}_j^T\left[2s_j\frac{\partial s_j}{\partial p}\mathbf{M} + +\frac{\partial s_j}{\partial p}\mathbf{G}(s_j) + s_j\frac{\partial s_j}{\partial p}\frac{\partial \mathbf{G}(s)}{\partial s}|_{s=s_j}\right]\mathbf{z}_j = 0. \qquad [1.74]$$

Rearranging the preceding equation, the derivative of eigenvalues can be obtained as

$$\frac{\partial s_j}{\partial p} = -\frac{\mathbf{z}_j^T\left[s_j^2\frac{\partial \mathbf{M}}{\partial p} + s_j\frac{\partial \mathbf{G}(s)}{\partial p}|_{s=s_j} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T\left[2s_j\mathbf{M} + \mathbf{G}(s_j) + s_j\frac{\partial \mathbf{G}(s)}{\partial s}|_{s=s_j}\right]\mathbf{z}_j}. \qquad [1.75]$$

Note that the denominator of equation [1.75] is exactly the normalization relationship given by equation [1.71]. In view of this, equation [1.75] can be expressed in a concise form as

$$\frac{\partial s_j}{\partial p} = -\frac{\mathbf{z}_j^T\frac{\partial \mathbf{D}(s)}{\partial p}|_{s=s_j}\mathbf{z}_j}{\mathbf{z}_j^T\frac{\partial \mathbf{D}(s)}{\partial s}|_{s=s_j}\mathbf{z}_j}$$

$$\text{or} \quad \frac{\partial s_j}{\partial p} = -\frac{1}{\gamma_j}\left(\mathbf{z}_j^T\frac{\partial \mathbf{D}(s)}{\partial p}|_{s=s_j}\mathbf{z}_j\right). \qquad [1.76]$$

This is the most general expression for the derivative of eigenvalues of linear dynamic systems. Equation [1.76] can be used to derive the derivative of eigenvalues for various special cases:

1) *Undamped systems* (section 1.1): In this case, $\mathbf{G}(s) = 0$ results

$$\begin{aligned} \mathbf{D}(s) &= s^2\mathbf{M} + \mathbf{K} \\ \text{and} \quad \gamma_j &= 2s_j\mathbf{z}_j^T\mathbf{M}\mathbf{z}_j. \end{aligned} \quad [1.77]$$

Assuming $s_j = \mathrm{i}\omega_j$ where $\omega_j \in \mathbb{R}$ is the jth undamped natural frequency from equation [1.76], we obtain

$$-2\mathrm{i}\omega_j\mathrm{i}\frac{\partial \omega_j}{\partial p} = \frac{\partial \omega_j^2}{\partial p} = \frac{\mathbf{z}_j^T\left[\frac{\partial \mathbf{K}}{\partial p} - \omega_j^2\frac{\partial \mathbf{M}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T\mathbf{M}\mathbf{z}_j} \quad [1.78]$$

which is a well-known result.

2) *Viscously damped systems* (section 1.2): in this case, $\mathbf{G}(s) = \mathbf{C}$, a constant matrix with respect to, results

$$\begin{aligned} \mathbf{D}(s) &= s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} \\ \text{and} \quad \gamma_j &= \mathbf{z}_j^T\left[2s_j\mathbf{M} + \mathbf{C}\right]\mathbf{z}_j. \end{aligned} \quad [1.79]$$

Using these, from equation [1.76], we obtain

$$\frac{\partial s_j}{\partial p} = -\frac{\mathbf{z}_j^T\left[s_j^2\frac{\partial \mathbf{M}}{\partial p} + s_j\frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j}{\mathbf{z}_j^T\left[2s_j\mathbf{M} + \mathbf{C}\right]\mathbf{z}_j}. \quad [1.80]$$

Thus, the result obtained in equation [1.76] generalizes earlier expressions of the derivative of eigenvalues. The derivative of associated eigenvectors is considered in the next section.

1.3.2. *Sensitivity of the eigenvectors*

1.3.2.1. *Modal approach*

The various methods of calculating the derivative of eigenvectors can be divided into three main categories [MUR 88]: (1) adjoint method or modal method, (2) direct method and (3) iterative method. We adopt the modal method where the derivative of each eigenvector is expanded in the space of the complete set of eigenvectors. The main difficulty in applying available methodologies for the modal method to non-viscously damped systems is that the eigenvectors do not satisfy any familiar orthogonality relationship. We propose an approach to calculate the derivative of eigenvector without using the orthogonality relationship.

It turns out that the eigenvalue problem of the dynamic stiffness matrix (given by equation [1.68]) plays an important role. For any given $s \in \mathbb{C}$, the eigenvalue problem associated with the dynamic stiffness matrix can be expressed by equation [5.53] (Chapter 5, [ADH 14]). The eigenvalues and eigenvectors of the dynamic stiffness matrix are given by $\nu_k(s)$ and $\boldsymbol{\varphi}_k(s)$, respectively. It is assumed that all the eigenvalues are distinct for any fixed value of s. The symbols $\nu_k(s)$ and $\boldsymbol{\varphi}_k(s)$ indicate functional dependence of these quantities on the complex parameter s. Such a continuous dependence is expected whenever $\mathbf{D}(s)$ is a sufficiently smooth matrix function of s. It should be noted that because $\mathbf{D}(s)$ is an $N \times N$ complex matrix for a fixed s, the number of eigenvalues (and consequently the eigenvectors) must be N. Further, it can be shown that, for distinct eigenvalues, $\boldsymbol{\varphi}_k(s)$ also satisfies an orthogonality relationship although $\mathbf{z}_k$ does not enjoy any such simple relationship. We normalize $\boldsymbol{\varphi}_k(s)$ as in equation [5.53] (Chapter 5, [ADH 14]).

It is possible to establish the relationships between the original eigenvalue problem of the system defined by equation [1.69] and that by equation [5.53] (Chapter 5, [ADH 14]). Consider the case when the parameter s approaches any one of the system eigenvalues, say s_j. Since *all* the $\nu_k(s)$ are assumed to be distinct, for non-trivial eigenvectors, comparing equations [1.69] and [5.53] (Chapter 5, [ADH 14]), we can conclude that one and only one of the $\nu_k(s)$ must be zero when $s \to s_j$. Further discussion is given in section 5.3.1 (Chapter 5, [ADH 14]). Considering the rth set, equation [5.53] (Chapter 5, [ADH 14]) can be rewritten as

$$\mathbf{Z}_r(s)\boldsymbol{\varphi}_r(s) = \mathbf{0} \tag{1.81}$$

where

$$\mathbf{Z}_r(s) = \mathbf{D}(s) - \nu_r(s)\mathbf{I} \in \mathbb{C}^{N \times N}. \tag{1.82}$$

In view of [5.60] (Chapter 5, [ADH 14]), from the preceding equation, it is clear that

$$\lim_{s \to s_j} \mathbf{Z}_r(s) = \mathbf{D}(s)|_{s=s_j}. \tag{1.83}$$

From this equation, together with [5.53] (Chapter 5, [ADH 14]), we conclude that in the limit $s \to s_j$, the eigenvalue problem given by equation [1.81] approaches to the original eigenvalue problem given by [1.69].

Differentiating [1.81] with respect to the design parameter p, we have

$$\begin{aligned}&\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s)+\mathbf{Z}_r(s)\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p}=\mathbf{0}\\ \text{or}\quad &\mathbf{Z}_r(s)\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p}=-\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s).\end{aligned} \qquad [1.84]$$

Premultiplying the preceding equation by $\mathbf{D}^{-1}(s)$ and using [1.82], we have

$$\left[\mathbf{I}-\mathbf{D}^{-1}(s)\nu_r(s)\right]\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p}=-\mathbf{D}^{-1}(s)\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s). \qquad [1.85]$$

The derivative of eigenvector of the original system with respect to the design parameter p, that is $\frac{\partial \mathbf{z}_j}{\partial p}$, should be obtained from equation [1.85] by taking the limit $s \to s_j$. Because $\lim_{s\to s_j}\mathbf{D}(s)$ is at most of rank $(N-1)$, it is not possible to obtain $\frac{\partial \mathbf{z}_j}{\partial p}$ directly from equation [1.85]. We avoid this difficulty by expanding $\mathbf{D}^{-1}(s)$ in terms of the poles and their associated residues. From equation [5.63] given in Chapter 5 of [ADH 14], the inverse of the dynamic stiffness matrix can be expressed as

$$\mathbf{D}^{-1}(s)=\sum_{j=1}^{m}\frac{\mathbf{R}_j}{s-s_j} \qquad [1.86]$$

where

$$\mathbf{R}_j=\frac{\mathbf{z}_j\mathbf{z}_j^T}{\gamma_j}. \qquad [1.87]$$

Substituting $\mathbf{D}^{-1}(s)$ from equation [1.86] into equation [1.85], using [5.60], [ADH 14] and [1.87], and taking the limit as $s \to s_j$, we obtain

$$\begin{aligned}\frac{\partial \mathbf{z}_j}{\partial p}&=-\lim_{s\to s_j}\sum_{k=1}^{m}\frac{\mathbf{z}_k\mathbf{z}_k^T}{\gamma_j(s-s_k)}\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s)\\ &=a_{jj}\mathbf{z}_j-\sum_{\substack{k=1\\k\neq j}}^{m}\frac{\mathbf{z}_k^T\frac{\partial \mathbf{D}(s)}{\partial p}\big|_{s=s_j}\mathbf{z}_j}{\gamma_k(s_j-s_k)}\mathbf{z}_k\end{aligned} \qquad [1.88]$$

where

$$a_{jj}=-\lim_{s\to s_j}\frac{\mathbf{z}_j^T\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s)}{\gamma_j(s-s_j)}. \qquad [1.89]$$

In deriving equation [1.88], we have also made use of the relationships [5.61], of [ADH 14] and [1.83]. Note that the limiting value of a_{jj}, the coefficient associated with $\mathbf{z}_j$, cannot be obtained from [1.89] because the denominator approaches zero in the limit. A different approach is presented below to overcome this difficulty.

For a fixed value of s, $\boldsymbol{\varphi}_k(s), \forall\, k = 1, \cdots, N$ form a complete basis. For this reason, $\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} \in \mathbb{C}^N$ can be expanded uniquely in terms of all $\boldsymbol{\varphi}_k(s)$, that is we can write

$$\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = \sum_{k=1}^{N} \alpha_k^{(r)}(s) \boldsymbol{\varphi}_k(s) \qquad [1.90]$$

where $\alpha_k^{(r)}(s) \in \mathbb{C}$ are non-zero constants. The normalization relationship for the rth mode can be expressed from equation [5.56] given in Chapter 5 of [ADH 14] as

$$\boldsymbol{\varphi}_r^T(s)\mathbf{D}(s)\boldsymbol{\varphi}_r(s) = \nu_r(s). \qquad [1.91]$$

Differentiating this equation with respect to the design parameter p, we obtain

$$\frac{\partial \boldsymbol{\varphi}_r^T(s)}{\partial p}\mathbf{D}(s)\boldsymbol{\varphi}_r(s) + \boldsymbol{\varphi}_r^T(s)\frac{\partial \mathbf{D}(s)}{\partial p}\boldsymbol{\varphi}_r(s) + \boldsymbol{\varphi}_r^T(s)\mathbf{D}(s)\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = \frac{\partial \nu_r(s)}{\partial p}. \qquad [1.92]$$

Using the symmetry property of $\mathbf{D}(s)$ and [1.82], the above equation can be rearranged as

$$2\boldsymbol{\varphi}_r^T(s)\mathbf{D}(s)\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = -\boldsymbol{\varphi}_r^T(s)\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s). \qquad [1.93]$$

Substituting $\frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p}$ from equation [1.90] and using the orthogonality relationship given by [5.56] of [ADH 14], from the above equation we obtain

$$\alpha_r^{(r)}(s) = -\frac{\boldsymbol{\varphi}_r^T(s)\frac{\partial \mathbf{Z}_r(s)}{\partial p}\boldsymbol{\varphi}_r(s)}{2\nu_r(s)}. \qquad [1.94]$$

Now, taking the limit $s \to s_j$ on equation [1.90] and using [5.61] of [ADH 14], we have

$$\begin{aligned} \lim_{s \to s_j} \frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} &= \lim_{s \to s_j} \sum_{k=1}^{N} \alpha_k^{(r)}(s) \boldsymbol{\varphi}_k(s) \\ \text{or} \quad \frac{\partial \mathbf{z}_j}{\partial p} &= \left(\lim_{s \to s_j} \alpha_r^{(r)}(s) \right) \mathbf{z}_j + \lim_{s \to s_j} \sum_{\substack{k=1 \\ k \neq r}}^{N} \alpha_k^{(r)}(s) \boldsymbol{\varphi}_k(s). \end{aligned} \qquad [1.95]$$

Because it is assumed that all the eigenvalues are distinct, the associated eigenvectors are also distinct. Thus, $\lim_{s \to s_j} \boldsymbol{\varphi}_k(s) \neq \mathbf{z}_j, \forall k = 1, \cdots, N; \neq r$. So, comparing the coefficient of $\mathbf{z}_j$ in equations [1.88] and [1.95], it is clear that

$$\begin{aligned} a_{jj} &= \lim_{s \to s_j} \alpha_r^{(r)}(s) \\ &= - \lim_{s \to s_j} \frac{\boldsymbol{\varphi}_r^T(s) \frac{\partial \mathbf{Z}_r(s)}{\partial p} \boldsymbol{\varphi}_r(s)}{2 \nu_r(s)} \qquad \text{(from [1.94])}. \end{aligned} \qquad [1.96]$$

The above limit cannot be evaluated directly because from [5.60] Chapter 5 of [ADH 14], $\lim_{s \to s_j} \nu_r(s) = 0$. Now, differentiate equation [1.81] with respect to p to obtain

$$\frac{\partial \mathbf{Z}_r(s)}{\partial p} \boldsymbol{\varphi}_r(s) + \mathbf{Z}_r(s) \frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = \mathbf{0}. \qquad [1.97]$$

Premultiplying the above equation by $\boldsymbol{\varphi}_r^T(s)$, we obtain

$$\boldsymbol{\varphi}_r^T(s) \frac{\partial \mathbf{Z}_r(s)}{\partial p} \boldsymbol{\varphi}_r(s) + \boldsymbol{\varphi}_r^T(s) \mathbf{Z}_r(s) \frac{\partial \boldsymbol{\varphi}_r(s)}{\partial p} = 0. \qquad [1.98]$$

Taking transpose of equation [1.81] and considering the symmetry property of $\mathbf{Z}_r(s)$, it follows that the second term of the left-hand side of the above equation is zero. Thus, equation [1.98] reduces to

$$\boldsymbol{\varphi}_r^T(s) \frac{\partial \mathbf{Z}_r(s)}{\partial p} \boldsymbol{\varphi}_r(s) = 0. \qquad [1.99]$$

The above equation shows that in the limit the left-hand side of equation [1.96] has a “0 by 0” form. So, applying l’Hôspital’s rule, using [1.83], [5.61] and [5.68] in Chapter 5 of [ADH 14], from equation [1.96], we obtain

$$a_{jj} = -\frac{\mathbf{z}_j^T \frac{\partial^2[\mathbf{D}(s)]}{\partial s\,\partial p}|_{s=s_j}\mathbf{z}_j}{2\frac{\partial \nu_r(s)}{\partial s}|_{s=s_j}} = -\frac{\mathbf{z}_j^T \frac{\partial^2[\mathbf{D}(s)]}{\partial s\,\partial p}|_{s=s_j}\mathbf{z}_j}{2\left(\mathbf{z}_j^T \frac{\partial \mathbf{D}(s)}{\partial s}|_{s=s_j}\mathbf{z}_j\right)}. \qquad [1.100]$$

This expression can now be used to obtain the derivative of $\mathbf{z}_j$ in equation [1.88].

The denominator in the above equation can be related to the normalization constant γ_j given by equation [1.71]. The term $\frac{\partial^2[\mathbf{D}(s)]}{\partial s\,\partial p}|_{s=s_j}$ appearing in the numerator may be obtained by differentiating equation [1.68] as

$$\frac{\partial^2[\mathbf{D}(s)]}{\partial s\,\partial p}|_{s=s_j} = 2s_j\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{G}(s)}{\partial p}|_{s=s_j} + s_j\frac{\partial^2[\mathbf{G}(s)]}{\partial s\,\partial p}|_{s=s_j}. \qquad [1.101]$$

From equations [1.88] and [1.101], the derivative of $\mathbf{z}_j$ is obtained as

$$\frac{\partial \mathbf{z}_j}{\partial p} = -\frac{1}{2\gamma_j}\left(\mathbf{z}_j^T \frac{\partial^2[\mathbf{D}(s)]}{\partial s\,\partial p}|_{s=s_j}\mathbf{z}_j\right)\mathbf{z}_j - \sum_{\substack{k=1\\k\neq j}}^{m}\frac{\mathbf{z}_k^T \frac{\partial \mathbf{D}(s)}{\partial p}|_{s=s_j}\mathbf{z}_j}{\gamma_k(s_j - s_k)}\mathbf{z}_k. \qquad [1.102]$$

This is the most general expression for the derivative of eigenvectors of linear dynamic systems. Equation [1.102] can be applied directly to derive the derivative of eigenvectors for various special cases:

1) *Undamped systems* (section 1.1): in this case, $\mathbf{G}(s) = 0$ results in the order of the characteristic polynomial $m = 2N$; s_j is purely imaginary so that $s_j = \mathrm{i}\omega_j$. Using [1.77], equation [1.101] results

$$\frac{\partial^2[\mathbf{D}(s)]}{\partial s\,\partial p}|_{s=s_j} = 2s_j\frac{\partial \mathbf{M}}{\partial p}. \qquad [1.103]$$

Recalling that the eigenvalues appear in complex conjugate pairs and all $\mathbf{z}_j$ are real, from [1.102], we obtain

$$\begin{aligned}\frac{\partial \mathbf{z}_j}{\partial p} &= -\frac{1}{2}\frac{\left(2\mathrm{i}\omega_j\mathbf{z}_j^T\frac{\partial \mathbf{M}}{\partial p}\mathbf{z}_j\right)}{2\mathrm{i}\omega_j\left(\mathbf{z}_j^T\mathbf{M}\mathbf{z}_j\right)}\mathbf{z}_j \\ &- \sum_{\substack{k=1\\k\neq j}}^{N}\frac{\mathbf{z}_k^T\left[\frac{\partial \mathbf{K}}{\partial p} - \omega_j^2\frac{\partial \mathbf{M}}{\partial p}\right]\mathbf{z}_j}{2\mathrm{i}\omega_k\left(\mathbf{z}_k^T\mathbf{M}\mathbf{z}_k\right)}\left[\frac{1}{\mathrm{i}\omega_j - \mathrm{i}\omega_k} - \frac{1}{\mathrm{i}\omega_j + \mathrm{i}\omega_k}\right]\mathbf{z}_k.\end{aligned} \qquad [1.104]$$

Considering the unity mass normalization, that is $\mathbf{z}_k^T \mathbf{M} \mathbf{z}_k = 1, \forall k = 1, \cdots, N$, the preceding equation can be rewritten as

$$\frac{\partial \mathbf{z}_j}{\partial p} = -\frac{1}{2}\left(\mathbf{z}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{z}_j\right) \mathbf{z}_j + \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{\mathbf{z}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \omega_j^2 \frac{\partial \mathbf{M}}{\partial p}\right] \mathbf{z}_j}{(\omega_j^2 - \omega_k^2)} \mathbf{z}_k \qquad [1.105]$$

which is a well-known result.

2) *Viscously damped systems* (section 1.2): in this case, $\mathbf{G}(s) = \mathbf{C}$, a constant matrix with respect to s and $m = 2N$. Using [1.79], equation [1.101] results

$$\frac{\partial^2 [\mathbf{D}(s)]}{\partial s \, \partial p}\Big|_{s=s_j} = 2 s_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}. \qquad [1.106]$$

Recalling that the eigenvalues and eigenvectors appear in complex conjugate pairs, from [1.102], we obtain

$$\begin{aligned} \frac{\partial \mathbf{z}_j}{\partial p} &= -\frac{1}{2\gamma_j}\left(\mathbf{z}_j^T \left[2 s_j \frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{C}}{\partial p}\right] \mathbf{z}_j\right) \mathbf{z}_j \\ &\quad - \frac{1}{\gamma_j^* 2\mathrm{i}\Im(s_j)}\left(\mathbf{z}_j^{*T} \frac{\partial \mathbf{D}(s)}{\partial p}\Big|_{s=s_j} \mathbf{z}_j\right) \mathbf{z}_j^* \\ &\quad - \sum_{\substack{k=1 \\ k \neq j}}^{N} \left[\frac{\mathbf{z}_k^T \frac{\partial \mathbf{D}(s)}{\partial p}\big|_{s=s_j} \mathbf{z}_j}{\gamma_k (s_j - s_k)} \mathbf{z}_k + \frac{\mathbf{z}_k^{*T} \frac{\partial \mathbf{D}(s)}{\partial p}\big|_{s=s_j} \mathbf{z}_j}{\gamma_k^* (s_j - s_k^*)} \mathbf{z}_k^*\right]. \end{aligned} \qquad [1.107]$$

Thus, the result obtained in equation [1.102] generalizes earlier expressions of the derivative of eigenvectors.

1.3.2.2. *Numerical example: a two degree-of-freedom system*

We consider a two degree-of-freedom system shown in Figure 1.8 to illustrate a possible use of the expressions derived so far. The system considered here is similar to the system used in section 1.2.2.3 except that the dissipative element connected between the two masses is not a simple viscous dashpot but a non-viscous damper. The equation of motion describing the free vibration of the system can be expressed by [1.66], with

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \qquad [1.108]$$

and

$$\boldsymbol{\mathcal{G}}(t) = g(t)\hat{\mathbf{I}}, \quad \text{where} \quad \hat{\mathbf{I}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \qquad [1.109]$$

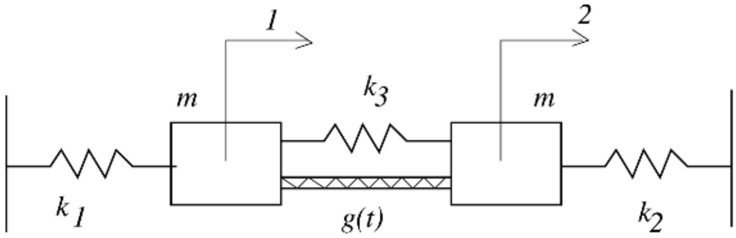

Figure 1.8. *A two degree-of-freedom spring-mass system with non-viscous damping,* $m = 1$ *Kg,* $k_1 = 1{,}000$ *N/m,* $k_3 = 100$ *N/m,* $g(t) = c\left(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}\right)$*,* $c = 4.0$ *Ns/m,* $\mu_1 = 10.0\, s^{-1}$*,* $\mu_2 = 2.0\, s^{-1}$

The damping function $g(t)$ is assumed to be the Golla–Hughes–McTavish (GHM) model [GOL 85, MCT 93] so that

$$g(t) = c\left(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}\right); \quad c, \mu_1, \mu_2 \geq 0, \tag{1.110}$$

where c is a constant and μ_1 and μ_2 are known as the relaxation parameters. In equation [1.110], if the function associated with c was a delta function, c would serve the purpose of the familiar viscous damping constant. Taking the Laplace transform of [1.109], we obtain

$$\begin{aligned} &\mathbf{G}(s) = G(s)\hat{\mathbf{I}} \\ &\quad \text{where} \quad G(s) = \mathcal{L}\left[g(t)\right] = c\left(\frac{\mu_1}{s+\mu_1} + \frac{\mu_2}{s+\mu_2}\right). \end{aligned} \tag{1.111}$$

Substituting [1.108] and [1.111] in equation [1.70], it may be shown that the system has six eigenvalues – four of which correspond to the two elastic modes (together with corresponding complex conjugate pairs) and the remaining two correspond to two non-viscous modes. For convenience, arrange the eigenvalues as

$$s_{e_1}, s_{e_2}, s_{e_1}^*, s_{e_2}^*, s_{nv_1}, s_{nv_1} \tag{1.112}$$

where $(\bullet)_e$ denotes elastic modes and $(\bullet)_{nv}$ denotes non-viscous modes.

We consider the derivative of eigenvalues with respect to the relaxation parameter μ_1. The derivative of the system matrices with respect to this parameter may be obtained as

$$\frac{\partial \mathbf{M}}{\partial \mu_1} = \mathbf{O}, \quad \frac{\partial \mathbf{G}(s)}{\partial \mu_1} = \hat{\mathbf{I}}\frac{c\,s}{\left(s+\mu_1\right)^2} \quad \text{and} \quad \frac{\partial \mathbf{K}}{\partial \mu_1} = \mathbf{O}. \tag{1.113}$$

Further, from equation [1.111], we also obtain

$$\begin{aligned}\frac{\partial \mathbf{G}(s)}{\partial s} &= -\hat{\mathbf{I}}c\left\{\frac{\mu_1}{(s+\mu_1)^2}+\frac{\mu_2}{(s+\mu_2)^2}\right\}\\ \frac{\partial^2[\mathbf{G}(s)]}{\partial s\,\partial \mu_1} &= -\hat{\mathbf{I}}c\frac{s-\mu_1}{(s+\mu_1)^3}.\end{aligned} \qquad [1.114]$$

Using equations [1.113] and [1.114], the terms γ_j, $\frac{\partial \mathbf{D}(s)}{\partial p}$ and $\frac{\partial^2[\mathbf{D}(s)]}{\partial s\,\partial \mu_1}$ appearing in equations [1.76] and [1.102] can be evaluated.

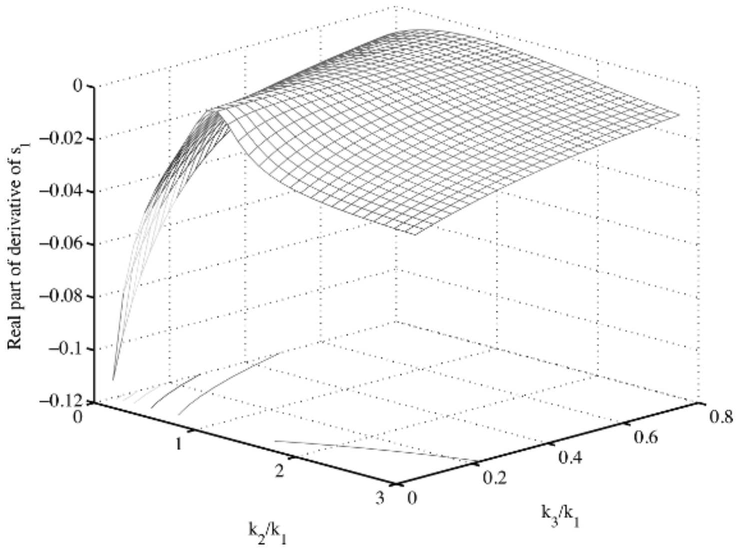

Figure 1.9. *Real part of the derivative of the first eigenvalue with respect to the relaxation parameter* μ_1

Figures 1.9 and 1.10 show the real part of the derivative of first and second eigenvalues with respect to μ_1 over a parameter variation of k_2 and k_3. These results are obtained by direct application of equation [1.76]. The system considered here shows the so-called "veering" [DU 11, BOI 09] when the eigenvalues are plotted against a system parameter. In the veering range (that is when $k_2 \approx k_1$ and $k_3 \approx 0$), rapid changes take place in the eigensolutions. From Figures 1.9 and 1.10, it may be noted that around the veering range the first eigenvalue is not very sensitive to μ_1 while the second eigenvalue is very sensitive in this region. In the first mode, both the blocks move in the same direction and consequently the damper is not stretched, resulting in insensitiveness to the relaxation parameter μ_1. In the second mode, the blocks move away from each other. This results in stretching of the damping block and increases sensitiveness to the relaxation parameter μ_1.

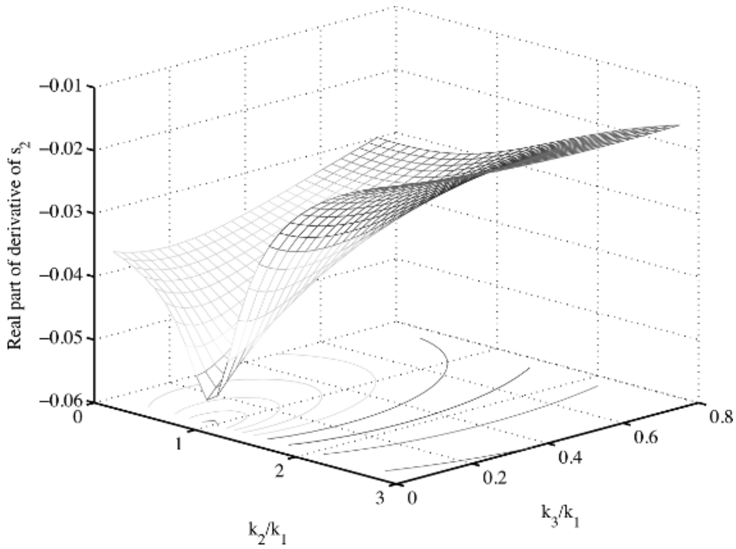

Figure 1.10. *Real part of the derivative of the second eigenvalue with respect to the relaxation parameter* μ_1

It is useful to understand the effect of different parameters on the eigenvalues. Figures 1.11 and 1.12 show the imaginary part of the derivative of first and second eigenvalues with respect to the damping parameters c, μ_1 and μ_2 over a parameter variation of k_2. The value of k_3 is fixed at $k_3 = 100$. These plots show that the damping parameters not only affect the real part of the eigenvalues but also affect the imaginary part. Again, observe that in the veering range, the first eigenvalue is insensitive to the damping parameters while the second eigenvalue is sensitive to them.

Now, we turn our attention to the derivative of eigenvectors. Figures 1.13 and 1.14 show the real part of the derivative of first and second eigenvectors with respect to k_2 over a parameter variation of k_2. It is useful to compare these results with the corresponding results by considering the damping mechanism to be viscous, i.e. when $g(t)$ given by equation [1.110] has the form $g(t) = c\delta(t)$. In Figures 1.13 and 1.14, the derivative of both eigenvectors for the corresponding viscously damped system is also plotted. Observe that around the veering range, the derivatives of both eigenvectors are different for viscously and non-viscously damped systems. This illustrates that the nature of damping affects the parameter sensitivity of the real part of complex modes.

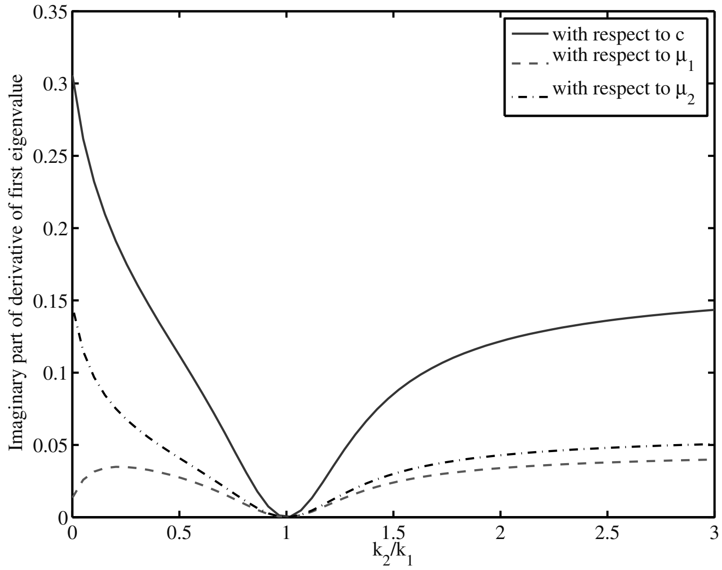

Figure 1.11. *Imaginary part of the derivative of the first eigenvalue with respect to the damping parameters* c*,* μ_1 *and* μ_2

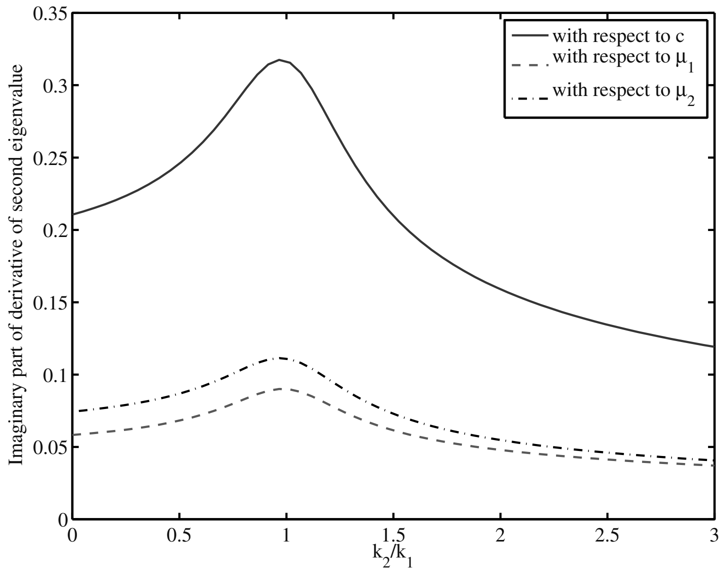

Figure 1.12. *Imaginary part of the derivative of the second eigenvalue with respect to the damping parameters* c*,* μ_1 *and* μ_2

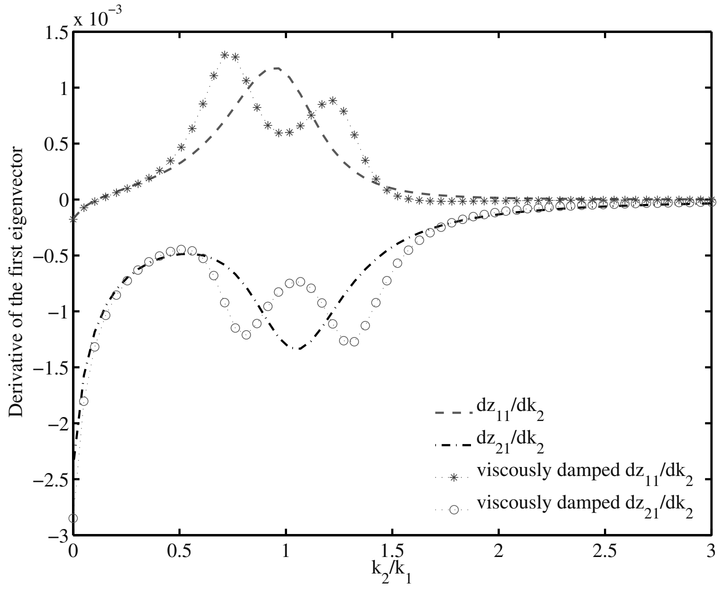

Figure 1.13. *Real part of the derivative of the first eigenvector with respect to* k_2

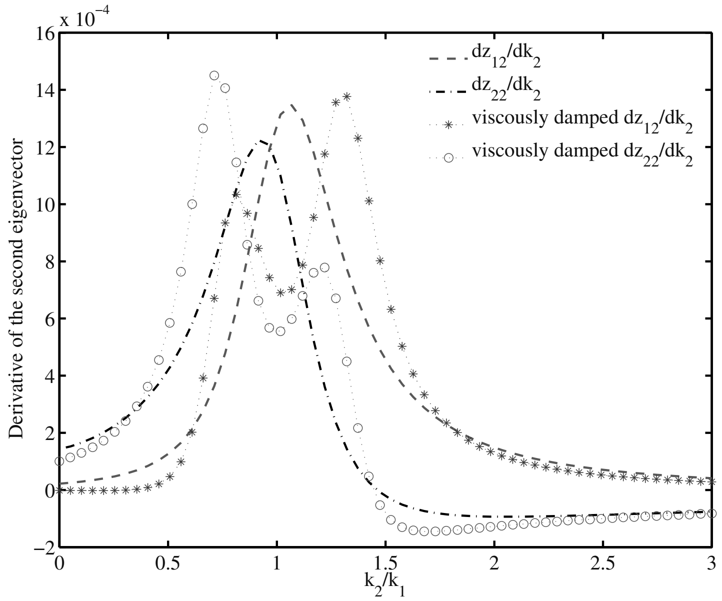

Figure 1.14. *Real part of the derivative of the second eigenvalue with respect to* k_2

1.3.2.3. *Nelson's method*

For large-scale structures with non-viscous damping, obtaining all of the eigenvectors is a computationally expensive task because the number of eigenvectors of a non-viscously damped system is much larger, in general, than the number for a viscously damped system. This motivates the extension of Nelson's method to calculate the derivatives of eigenvectors of non-viscously damped systems.

Differentiating equation [1.69] with respect to the design parameter p, we have

$$\mathbf{D}(s_j)\frac{\partial \mathbf{z}_j}{\partial p} = \mathbf{h}_j \qquad [1.115]$$

where

$$\mathbf{h}_j = -\frac{\partial \mathbf{D}(s_j)}{\partial p}\mathbf{z}_j = -\left[2s_j\frac{\partial s_j}{\partial p}\mathbf{M} + s_j^2\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial s_j}{\partial p}\mathbf{G}(s_j) + s_j\frac{\partial\left[\mathbf{G}(s_j)\right]}{\partial p} + \frac{\partial \mathbf{K}}{\partial p}\right]\mathbf{z}_j \qquad [1.116]$$

is known. For unique results, we need to normalize the eigenvectors. There are many approaches to the normalization of the eigenvectors. A convenient approach (see section 5.6.2 of [ADH 14]) is to normalize $\mathbf{z}_j$ such that

$$\mathbf{z}_j^T\frac{\partial \mathbf{D}(s)}{\partial s}|_{s=s_j}\mathbf{z}_j = \gamma_j \qquad [1.117]$$

$$\text{or}\quad \mathbf{z}_j^T\mathbf{D}'(s_j)\mathbf{z}_j = \gamma_j, \quad \forall j = 1, \cdots, m \qquad [1.118]$$

where

$$\mathbf{D}'(s) = \frac{\partial \mathbf{D}(s)}{\partial s} = \left[2s\mathbf{M} + \mathbf{G}(s) + s\mathbf{G}'(s)\right] \in \mathbb{C}^{N\times N} \qquad [1.119]$$

and $\gamma_j \in \mathbb{C}$ is some non-zero constant.

Equation [1.115] cannot be solved to obtain the eigenvector derivative because the matrix is singular. For distinct eigenvalues, this matrix has a null space of dimension 1. Following Nelson's approach, the eigenvector derivative is written as

$$\frac{\partial \mathbf{z}_j}{\partial p} = \mathbf{v}_j + d_j\mathbf{z}_j \qquad [1.120]$$

where $\mathbf{v}_j$ and d_j have to be determined. These quantities are not unique since any multiple of the eigenvector may be added to $\mathbf{v}_j$. A convenient choice is to identify the element of maximum magnitude in $\mathbf{z}_j$ and make the corresponding element in $\mathbf{v}_j$ equal to zero. Although other elements of $\mathbf{v}_j$ could be set to zero, this choice is most likely to produce a numerically well-conditioned problem. Because $\mathbf{D}(s_j)\mathbf{z}_j = \mathbf{0}$ due to equation [1.69], substituting equation [1.120] into equation [1.115], gives

$$\mathbf{D}_j\mathbf{v}_j = \mathbf{h}_j \qquad [1.121]$$

where

$$\mathbf{D}_j = \mathbf{D}(s_j) = \left[s_j^2\mathbf{M} + s_j\,\mathbf{G}(s_j) + \mathbf{K}\right] \ \in \mathbb{C}^{N\times N}. \qquad [1.122]$$

This may be solved, including the constraint on the zero element of $\mathbf{v}_j$, by solving the equivalent problem

$$\begin{bmatrix} \mathbf{D}_{j11} & \mathbf{0} & \mathbf{D}_{j31} \\ 0 & 1 & 0 \\ \mathbf{D}_{j31} & \mathbf{0} & \mathbf{D}_{j33} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_{j1} \\ x_{j2}\,(=0) \\ \mathbf{v}_{j3} \end{Bmatrix} = \begin{Bmatrix} \mathbf{h}_{j1} \\ 0 \\ \mathbf{h}_{j3} \end{Bmatrix} \qquad [1.123]$$

where the $\mathbf{D}_j$ is defined in equation [1.122], and has the row and column corresponding to the zeroed element of $\mathbf{v}_j$ replaced with the corresponding row and column of the identity matrix. This approach maintains the banded nature of the structural matrices, and hence is computationally efficient.

It only remains to compute the scalar constant, d_j, to obtain the eigenvector derivative. For this, the normalization equation [1.118] must be used. Differentiating equation [1.118] and using the symmetry property of $\mathbf{D}'(s)$, we have

$$\mathbf{z}_j^T\frac{\partial\mathbf{D}'(s_j)}{\partial p}\mathbf{z}_j + 2\mathbf{z}_j^T\mathbf{D}'(s_j)\frac{\partial\mathbf{z}_j}{\partial p} = 0. \qquad [1.124]$$

Substituting $\frac{\partial\mathbf{z}_j}{\partial p}$ from equation [1.120], we have

$$\frac{1}{2}\mathbf{z}_j^T\frac{\partial\mathbf{D}'(s_j)}{\partial p}\mathbf{z}_j + \mathbf{v}_j^T\mathbf{D}'(s_j)\mathbf{z}_j + d_j\mathbf{z}_j^T\mathbf{D}'(s_j)\mathbf{z}_j = 0. \qquad [1.125]$$

Noting that the coefficient associated with d_j is the normalization constant given by equation [1.118], we have

$$d_j = -\frac{1}{\gamma_j}\left\{\frac{1}{2}\mathbf{z}_j^T\frac{\partial\mathbf{D}'(s_j)}{\partial p}\mathbf{z}_j + \mathbf{z}_j^T\mathbf{D}'(s_j)\mathbf{v}_j\right\}. \qquad [1.126]$$

The first term on the right-hand side can be obtained by substituting $s = s_j$ into equation [1.119] and differentiating

$$\frac{\partial \mathbf{D}'(s_j)}{\partial p} = 2\frac{\partial s_j}{\partial p}\mathbf{M} + 2s_j\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial\left[\mathbf{G}(s_j)\right]}{\partial p} + \frac{\partial s_j}{\partial p}\mathbf{G}'(s_j) + s_j\frac{\partial\left[\mathbf{G}'(s_j)\right]}{\partial p} \qquad [1.127]$$

where $\frac{\partial[\mathbf{G}(s_j)]}{\partial p}$ is given in equation [1.73] and

$$\begin{aligned}\frac{\partial\left[\mathbf{G}'(s_j)\right]}{\partial p} &= \frac{\partial s_j}{\partial p}\frac{\partial \mathbf{G}'(s)}{\partial s}|_{s=s_j} + \frac{\partial \mathbf{G}'(s)}{\partial p}|_{s=s_j} \\ &= \frac{\partial s_j}{\partial p}\frac{\partial^2 \mathbf{G}(s)}{\partial s^2}|_{s=s_j} + \frac{\partial^2 \mathbf{G}(s)}{\partial p\,\partial s}|_{s=s_j}.\end{aligned} \qquad [1.128]$$

Equation [1.120], combined with $\mathbf{v}_j$ obtained by solving equation [1.123] and d_j obtained from equation [1.126], completely defines the derivative of the eigenvectors.

1.3.2.4. *Numerical example*

We consider a two degree-of-freedom system shown in Figure 1.8 to illustrate the use of the expressions derived here. Here, the dissipative element connected between the two masses is not a simple viscous dashpot but a non-viscous damper. The equation of motion describing the free vibration of the system can be expressed by equation [1.66], with

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \qquad [1.129]$$

and

$$\boldsymbol{\mathcal{G}}(t) = g(t)\hat{\mathbf{I}}, \quad \text{where} \quad \hat{\mathbf{I}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \qquad [1.130]$$

The damping function $g(t)$ is assumed to be a "double exponential model", with

$$g(t) = c\left(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}\right); \quad c, \mu_1, \mu_2 \geq 0 \qquad [1.131]$$

where c is a constant, and μ_1 and μ_2 are known as the relaxation parameters. In equation [1.131], if the function associated with c was a delta function, c would be the familiar viscous damping constant. Taking the Laplace transform of equation [1.130], we obtain

$$\mathbf{G}(s) = c\hat{\mathbf{I}}\left\{(1 + s/\mu_1)^{-1} + (1 + s/\mu_2)^{-1}\right\}. \qquad [1.132]$$

Substituting equations [1.129] and [1.132] into equation [1.70] shows that the system has six eigenvalues – four of which occur in complex conjugate pairs and correspond to the two elastic modes. The other two eigenvalues are real and negative and they correspond to the two non-viscous modes. The eigenvalues and the eigenvectors of the system are shown in Table 1.1. The normalization constants γ_j are selected such that $\gamma_j = 2s_j$ for the elastic modes and $\gamma_j = 1$ for the non-viscous modes.

Quantity	Elastic mode 1	Elastic mode 2	Non-viscous mode 1	Non-viscous mode 2
s_j	$-0.0387 \pm 38.3232\mathrm{i}$	$-1.5450 \pm 97.5639\mathrm{i}$	-2.8403	-5.9923
$\mathbf{z}_j$	$\begin{Bmatrix} -0.7500 \pm 0.0043\mathrm{i} \\ -0.6616 \mp 0.0041\mathrm{i} \end{Bmatrix}$	$\begin{Bmatrix} 0.6622 \mp 0.0035\mathrm{i} \\ -0.7501 \pm 0.0075\mathrm{i} \end{Bmatrix}$	$\begin{Bmatrix} -0.0165 \\ 0.0083 \end{Bmatrix}$	$\begin{Bmatrix} 0.0055 \\ -0.0028 \end{Bmatrix}$

Table 1.1. *Eigenvalues and eigenvectors for the example*

We consider the derivative of eigenvalues with respect to the stiffness parameter k_1 and the relaxation parameter μ_1. The derivative of the relevant system matrices with respect to k_1 may be obtained as

$$\frac{\partial \mathbf{M}}{\partial k_1} = \mathbf{O}, \quad \frac{\partial \mathbf{G}(s)}{\partial k_1}|_{s=s_j} = \mathbf{O}, \quad \frac{\partial \mathbf{K}}{\partial k_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad [1.133]$$

$$\begin{aligned} \frac{\partial \left[\mathbf{D}(s_j)\right]}{\partial k_1} &= \left(2s_j\mathbf{M} + \mathbf{G}(s_j) - cs_j\hat{\mathbf{I}} \left\{\mu_1^{-1}\left(1 + s_j/\mu_1\right)^{-2} + \mu_2^{-1}\left(1 + s_j/\mu_2\right)^{-2}\right\}\right) \\ &\quad \times \frac{\partial s_j}{\partial k_1} + \frac{\partial \mathbf{K}}{\partial k_1} \end{aligned} \qquad [1.134]$$

and

$$\frac{\partial \left[\mathbf{D}'(s_j)\right]}{\partial k_1} = \left(2\mathbf{M} - 2c\hat{\mathbf{I}}\left\{\mu_1^{-1}\left(1 + s_j/\mu_1\right)^{-3} + \mu_2^{-1}\left(1 + s_j/\mu_2\right)^{-3}\right\}\right)\frac{\partial s_j}{\partial k_1}. \qquad [1.135]$$

Using these expressions, the derivative of the eigenvalues is eigenvectors is obtained from equations [1.80] and [1.120] shown in Table 1.2.

The derivatives of the eigensolutions with respect to the relaxation parameter μ_1 may be obtained using similar manner. The derivative of the relevant system matrices with respect to μ_1 may be obtained as

$$\frac{\partial \mathbf{M}}{\partial \mu_1} = \mathbf{O}, \quad \frac{\partial \mathbf{K}}{\partial \mu_1} = \mathbf{O}, \quad \frac{\partial \mathbf{G}(s)}{\partial \mu_1}|_{s=s_j} = c\hat{\mathbf{I}}\, s_j\mu_1^{-2}\left(1 + s_j/\mu_1\right)^{-2}, \qquad [1.136]$$

$$\frac{\partial\left[\mathbf{D}(s_j)\right]}{\partial\mu_1} = \left(2s_j\mathbf{M} + \mathbf{G}(s_j) - cs_j\hat{\mathbf{I}}\ \left\{\mu_1^{-1}\left(1+s_j/\mu_1\right)^{-2} + \mu_2^{-1}\left(1+s_j/\mu_2\right)^{-2}\right\}\right) \times\frac{\partial s_j}{\partial\mu_1} + c\hat{\mathbf{I}}s_j^2\mu_1^{-2}\left(1+s_j/\mu_1\right)^{-2} \qquad [1.137]$$

and

$$\frac{\partial\left[\mathbf{D}'(s_j)\right]}{\partial\mu_1} = \left(2\mathbf{M} - 2c\hat{\mathbf{I}}\left\{\mu_1^{-1}\left(1+s_j/\mu_1\right)^{-3} + \mu_2^{-1}\left(1+s_j/\mu_2\right)^{-3}\right\}\right)\frac{\partial s_j}{\partial\mu_1} + 2c\hat{\mathbf{I}}\,s_j\mu_1^{-2}\left(1+s_j/\mu_1\right)^{-3}. \qquad [1.138]$$

Quantity	Elastic mode 1	Elastic mode 2	Non-viscous mode 1	Non-viscous mode 2
$\frac{\partial s_j}{\partial k_1}$	$0.0001 \pm 0.0073\mathrm{i}$	$0.0001 \pm 0.0022\mathrm{i}$	-2.7106×10^{-4}	-2.9837×10^{-5}
$\frac{\partial \mathbf{z}_j}{\partial k_1} \times 10^3$	$\begin{Bmatrix} 0.1130 \mp 0.0066\mathrm{i} \\ 0.0169 \pm 0.0041\mathrm{i} \end{Bmatrix}$	$\begin{Bmatrix} 0.0385 \mp 0.0015\mathrm{i} \\ 0.0494 \mp 0.0026\mathrm{i} \end{Bmatrix}$	$\begin{Bmatrix} 0.0072 \\ 0.0046 \end{Bmatrix}$	$\begin{Bmatrix} -0.0018 \\ -0.0018 \end{Bmatrix}$

Table 1.2. *Derivative of eigenvalues and eigenvectors with respect to the stiffness parameter* k_1

Using these expressions, the derivative of the eigenvalues and eigenvectors is obtained from equations [1.80] and [1.120] shown in Table 1.3.

Quantity	Elastic mode 1	Elastic mode 2	Non-viscous mode 1	Non-viscous mode 2
$\frac{\partial s_j}{\partial \mu_1}$	$-0.0034 \pm 0.0196\mathrm{i}$	$-0.2279 \pm 2.0255\mathrm{i}$	-0.0570	-0.4804
$\frac{\partial \mathbf{z}_j}{\partial \mu_1} \times 10^3$	$\begin{Bmatrix} 0.0022 \pm 0.0004\mathrm{i} \\ -0.0021 \mp 0.0003\mathrm{i} \end{Bmatrix}$	$\begin{Bmatrix} -0.0045 \mp 0.0012\mathrm{i} \\ 0.0098 \pm 0.0015\mathrm{i} \end{Bmatrix}$	$\begin{Bmatrix} -0.0002 \\ 0.0001 \end{Bmatrix}$	$\begin{Bmatrix} 0.0022 \\ -0.0011 \end{Bmatrix}$

Table 1.3. *Derivative of eigenvalues and eigenvectors with respect to the relaxation parameter* μ_1

1.4. Summary

Sensitivity of the eigenvalues and eigenvectors of linear damped discrete systems with respect to the system parameters has been derived. In the presence of general non-proportional viscous damping, the eigenvalues and eigenvectors of the system become complex. The results are presented in terms of changes in the mass, damping, stiffness matrices and complex eigensolutions of the second-order system so that the state-space representation of the equation of motion can be avoided. The expressions derived hereby generalize earlier results on derivatives of eigenvalues and eigenvectors of undamped systems to the damped systems. It was shown through an example problem that the use of the expression for the derivative of undamped modes can give rise to incorrect results even when the modal damping is quite low. For non-classically damped systems, the expressions for the sensitivity of eigenvalues and eigenvectors developed in this chapter should be used. These complex eigensolution derivatives can be useful in various application areas, for example finite element model updating, damage detection, design optimization and system stochasticity analysis, relaxing the present restriction to use the real undamped modes only.

In general, structural systems are expected to be non-viscously damped. The derivative of eigenvalues and eigenvectors of non-viscously damped discrete linear systems has been derived. The assumed non-viscous damping forces depend on the past history of velocities via convolution integrals over suitable kernel functions. The familiar viscous damping model is a special case corresponding to a "memory-less" kernel. It has been assumed that, in general, the mass and the stiffness matrices as well as the matrix of the kernel functions cannot be simultaneously diagonalized by any linear transformation. The analysis is, however, restricted to systems with non-repetitive eigenvalues and non-singular mass matrices. Eigenvectors of linear dynamic systems with general non-viscous damping do not satisfy any kind of orthogonality relationship (not even in the usual state space). For this reason, none of the established methodologies for determination of the derivative of eigenvectors are applicable to non-viscously damped systems. An approach is shown that utilizes the eigenvalue problem of the associated complex dynamic stiffness matrix. The original eigenvalue problem is a limiting case of this eigenvalue problem. The expressions derived for the derivative of eigenvalues and eigenvectors (equations [1.76] and [1.102]) are very general and also valid for undamped and viscously damped systems. This analysis opens up the possibility of extending the conventional modal updating and parameter estimation techniques to non-viscously damped systems.

So far, in this book, we have discussed dynamics of damped systems with known parameters. In the next two chapters, we will show how the damping parameters can be identified from structural dynamic experiments.