1

Framework of Discrete Mechanics

1.1. Frames of reference and uniform motions

Any change in the position of a particle defined by its position $\mathbf{x}$, at time t, depends on the frame of reference in relation to which the motion is observed. As no absolute rest state exists, it is possible to choose an inertial frame of reference, wherein a body remains at rest or animated with uniform rectilinear motion when not subjected to any external force. In view of the principle of relativity, the physical laws take the same form in all inertial frames of reference. This principle holds true for velocity values which are much lesser than the speed of light. Herein, we shall not take account of the relativistic effects, and our discussion fits into the context of mechanics at moderate velocities, far lesser than the celerity of light. We are left with the fundamental principles of restricted (special) relativity theory or of general relativity in the presence of gravitational forces, which apply for all velocity levels.

The case of a uniform rotational motion is similar in nature: an observer at rest in the rotating frame of reference is subject to a centripetal force which is equal and opposite to the centrifugal force deriving from a scalar potential $\Phi = \rho/2\ (\mathbf{\Omega} \times \mathbf{r})^2$, where $\mathbf{\Omega}$ is the constant speed of rotation of the frame of reference.

A uniform rectilinear motion "eludes" definition by the law of Mechanics; the acceleration is zero and the sum of the forces at work is

also null. In the presence of an external force, such as constant gravity, an observer is at rest in a frame of reference linked to the Earth when a different force is exerted upon him/her – in this case, the gravitational pull of the ground; the body forces, in this scenario, derive from a scalar potential Φ. The fundamental law of dynamics therefore becomes $-\nabla\Phi + \rho\,\mathbf{g} = 0$, which is the equation of static of fluids.

In both cases, the acceleration due to gravity and the centripetal acceleration are compensated by the gradient of a scalar potential. If the corresponding forces could no longer be described on the basis of true potentials, the medium would be subject to acceleration, and therefore would lose the state of relative rest. These two examples illustrate that forces which derive from a scalar potential do not give rise to motion: they simply contribute to a modification of the definition of the potential. The Hodge–Helmholtz decomposition, which separates the two components of a vector into a gradient of scalar potential and a rotational of a vector potential, suggests that for a medium at rest, the acceleration vector is null, and therefore, with the exception of the sign, the gradient of the scalar potential is equal to the rotational of the vector potential. The Hodge–Helmholtz theorem predicts, for a simply-connected domain, that a field such that $\nabla a = \nabla \times \mathbf{b}$ is a constant. This constant is, simultaneously, the gradient of a scalar potential and the rotational of a vector potential; it is a harmonic field which corresponds to a decomposition into three Hodge–Helmholtz terms. Generally speaking, the third term $\mathbf{V}_h$ is practically impossible to extract directly, and its existence probably needs to be linked to the uniform motions.

A constant introduced on the right-hand side of the motion balance equation can be interpreted as a gradient of a scalar potential, or as the rotational of a vector potential; in both cases, all that changes is the definition of the existing potentials. The directional aspect of the gradient operator suggests that the uniform rectilinear motions will be carried by a scalar potential, whilst the uniform rotational motions will be contained in a vector potential.

The Law of Dynamics formulated by Newton cannot be used to find the uniform motions – be they rectilinear or rotational. The model of

the physical phenomena constructed here in order to take account of the numerous effects observed for a continuum – e.g. viscosity, capillarity, rotation, dissipation, etc. – will therefore not take account of these uniform motions. However, if they are present, it is essential that they do not give rise to any artifact for the model; such is the case of the rotational motion, which must be prevented from engendering mechanical dissipation. Note that, at a small timescale, a uniform rotational motion can be considered to be a uniform rectilinear motion.

If we cannot find out the acceleration, why then are we interested in the position of a point of the medium as a function of time? In any case, our knowledge of the particle's absolute position at a later moment in time will be altered by errors, because we can always superimpose a uniform velocity field $\mathbf{V}'$ to calculate it using the formula $\mathbf{x} = \mathbf{x}^o + (\mathbf{V} + \mathbf{V}')\ dt$. In solid mechanics, the problem can be resolved by adopting a Lagrangian approach using a reference state. As for fluid mechanics, where we are only interested in the velocities and their variations, the question simply does not arise. A more unified approach to mechanics – both fluid and solid – would lead us to consider only the velocities; the displacements would then be deduced by an incremental process based on the evolution from one state of mechanical equilibrium to another.

Another important question merits particular attention: do we actually need a frame of reference? If we consider that the velocity is a vector $\mathbf{W}$, then it is necessary to perform elementary operations such as the scalar product, which uses the components of the vector. In this case, we introduce a frame of reference anchored to a given system of coordinates. If we now consider that the velocity is an oriented scalar, following a fixed direction Γ, it can be considered either as a new vector $\mathbf{V} = (\mathbf{W} \cdot \mathbf{t})\ \mathbf{t}$ or as a scalar $V = (\mathbf{W} \cdot \mathbf{t})$, where $\mathbf{t}$ is the unit vector over Γ. Evidently, merely knowing $\mathbf{V}$ and $\mathbf{t}$ is insufficient to find the local vector $\mathbf{W}$, but is it really necessary to do so? If we replace the scalar product with a geometric projection, and differential geometry can be used to write all the operators on the basis of the components of $\mathbf{V}$ alone, then knowing the velocity vector $\mathbf{W}$ is no longer useful, in the same way as a frame of reference is no longer needed. It is the concept of a continuum, where all the values are

defined at a single point, which creates the need for a frame of reference. Certainly, the lengths, surface areas, volumes, normals, etc. of the topologies need to be known, and therefore calculated previously, in order to be able to apply the differential operators.

Based on this observation, it is possible to do away with the notion of a vector and that of a tensor, instead using the concept of a component associated with elementary topology – that of the oriented bipoint. Hence, by simply knowing the scalars V on all of the oriented edges Γ, we are able to define a motion on a discrete topology made up of edges and points. The aim of Discrete Mechanics is to construct physical models on that basis. To find the starting point for this theory, we need to go back to the primary form of the fundamental law of dynamics formulated by Newton.

It is no easy task to reconstruct the velocity $\mathbf{W}$; it is a free vector, whose definition is linked to the chosen topology – polygonal or polyhedral – and depends on the number of sides or faces which make it up. For example, we could construct a vector associated with the barycenter of the polygon or the polyhedron, but there are other choices that could judiciously be made. Thankfully, most of the necessary operations can be performed simply with the components $\mathbf{V}$ on the oriented edges Γ. For example, the rotational over a flat surface made up of oriented edges of the vector $\mathbf{W}$ will be calculated as the circulation of its components $\mathbf{V}$ on the edge Γ using the Stokes theorem. The other operators – the gradient, the divergence, the primal and dual rotationals, etc. will be defined in a similar manner on similarly oriented flat surfaces or portions thereof.

1.2. Concept of a Discrete Medium

The "Discrete Mechanics" developed in this book offers a very different approach to that of Continuum Mechanics. It involves a differentiated consideration of different types of objects: points, oriented edges, oriented surfaces and volumes.

Figure 1.1 illustrates a control volume Ω comprising points, edges and surfaces. If it is necessary to adjust the volume Ω to tend toward

zero, then the transformation applied to all the different elements will be the same. The reduction of the volume toward zero will be homothetic, and the resulting object will be similar to the original at all intermediary scales.

The application of this concept is subject to the same restriction as is that relating to Continuum Mechanics: the discrete conservation equations are valid only provided the length of each edge is greater than the mean free path of the molecules.

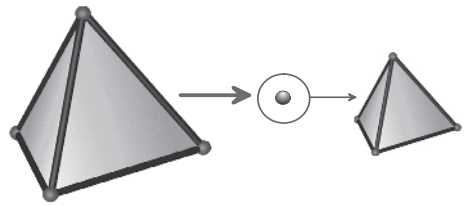

Figure 1.1. *In Continuum Mechanics, an elementary volume is reduced to a single point, and that point needs to be anchored within a fixed and absolute frame of reference; in Discrete Mechanics, the directions are also preserved when we look at the topology, regardless of the scale*

Thus, the microscopic structure of the material (in terms of molecules, atoms, etc.) can be ignored. The term "discrete" used throughout this book refers to the topology of the reference volume, made up of points, edges, surfaces and volumes. The formulation given below is formal, and in no case refers to the numerical methods used to discretize the space into elementary volumes.

Thus, let us consider the elementary topology used in Discrete Mechanics (see Figure 1.2). It is composed of two points connected by an edge, and a volume around a point, which we call the dual volume. The segment is oriented arbitrarily, and the dual volume has an outer normal. The set of edges connecting all the corners (or nodes) of the same surface is called the primal topology.

No condition is fixed to define the elements of the elementary topology: the edges may be curvilinear, and the dual surfaces themselves can have any form at all. However, if we consider that the

dimensions of the object in question tend toward zero, the elements of the primal surface are, essentially, flat surfaces. When the primal surface is made up of more than three points, it can always be subdivided into triangles. As the theorems of differential geometry are applicable for all types of surfaces, the applicability of the method presented herein is not limited by these geometric aspects. This approach theoretically formalizes the concept introduced by Harlow and Welch about the localization of the unknowns on a Cartesian primal topology [HAR 65]. The spirit of Harlow and Welch's publication is similar to the work of Tonti [TON 13], who questions the necessity of numerically discretizing the physical equations to show a direct link between the physical model and an algebraic description.

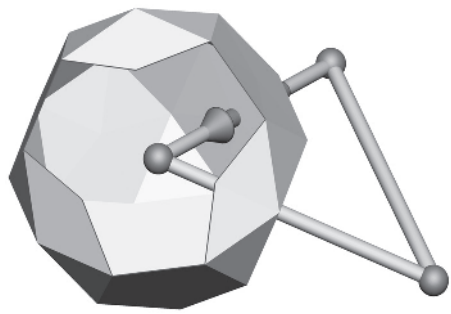

Figure 1.2. *Definition of the elementary topology: points connected by edges and a dual volume*

1.2.1. *Vectors and components*

In mathematics, the notion of a vector is clearly defined, using a base. From the physical point of view, the notion of a vector is somewhat trickier to fully comprehend, because it cannot be measured directly – for instance, we cannot measure its components in prevailing directions by using Laser Doppler Anemometry. By reconstructing a particular vector, we actually find a set, or field, of vectors, applicable in three-dimensional space.

Now, the question becomes whether knowing a vector's components alone will enable us to circumvent the process of

reconstructing it to directly access the laws of mechanics. If we let $\mathbf{W}$ represent the velocity, its component in the direction δ of unitary vector $\mathbf{t}$ will be $\mathbf{V} = (\mathbf{W} \cdot \mathbf{t})\ \mathbf{t}$ is also a vector. The whole of Discrete Mechanics is based on this concept. The velocity remains undefined; only the components are defined on each edge of the primal topology.

Figure 1.3 illustrates all of the notations which will be used later on.

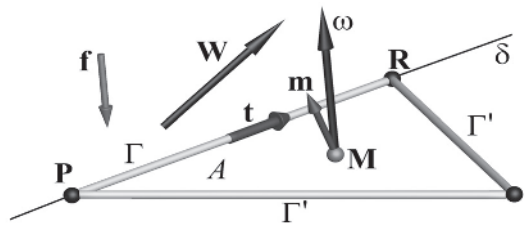

Figure 1.3. *Definition of the physical system*

The symbols and notations given in Figure 1.3 have the following meanings:

– $\mathbf{W} = u\ \mathbf{e}_1 + v\ \mathbf{e}_2 + w\ \mathbf{e}_3$ the velocity vector in an orthonormal Cartesian system;

– $\mathbf{P} = \varphi_1 \mathbf{e}_1 + \varphi_2 \mathbf{e}_2 + \varphi_3 \mathbf{e}_3$ the heat flux in an orthonormal Cartesian system;

– $\mathbf{t}$, the unit vector carried by δ;

– $\mathbf{m}$, the unit vector in M normal to the plane A;

– $\mathbf{n}$, the outward unit vector normal to the surface of the dual topology;

– $\mathbf{V} = (\mathbf{W} \cdot \mathbf{t})\ \mathbf{t}$, the velocity vector projected along the δ axis;

– $\mathbf{\Phi} = (\mathbf{P} \cdot \mathbf{t})\ \mathbf{t}$, the heat flux vector projected along the δ axis;

– $\mathbf{U} = dt\ \mathbf{V}$, the displacement vector projected along the Δ axis;

– P and R, the nodes delimiting the edges;

– $\Gamma = [P\ R]$, the edge carried by δ;

– $\mathbf{f}$, the vector representing the forces outside of the volume;

– p, the pressure on the nodes of the primal topology;

– T, the temperature on the nodes of the primal topology;

– ω, the rotational vector of the velocity vector on the primal boundary $\Gamma + \Gamma'$;

– $\boldsymbol{\omega} = \mu \nabla \times \mathbf{V}$, the vector potential of the quantity of acceleration;

– Ω, the volume of the dual topology around the point P;

– S, the surface of the dual topology around P;

– A, the surface of the primal topology with the boundary $\Gamma + \Gamma'$.

Consider a line δ in space and an oriented edge Γ, defined by its endpoints P and R - two points on the line Δ. That line may be straight, as shown in Figure 1.3, or curved. The unit vector $\mathbf{t}$ carried by δ in oriented in the same direction as Γ. It is on that edge that the equilibrium between the various mechanical actions will be expressed.

The primal topology is constructed on the basis of other curvilinear or rectilinear edges Γ' with the endpoints P and R. The boundary $\Gamma + \Gamma'$ is closed and defines a surface which is not necessarily flat A, which itself is oriented, with the normal $\mathbf{m}$, which enable us to calculate the rotational of a vector.

The dual topology, which is not represented here, around the point P defines a closed surface S whose outward normal is $\mathbf{n}$. The volume associated with R has the same portion of surface as the point P; this curvilinear or faceted surface can be used to calculate the fluxes exchanged between the points P and R.

1.2.2. *Physical meaning of the differential operators*

The primal and dual discrete topologies are constructed around a physical system, with the volume Ω and surface Σ. The choice adopted is to make the primal topology coincide with the surface of the system; this strategy enables us to simply define the scalar potential with a point on the surface. The presentation of the differential operators may be greatly different depending on the degree of formalization of the differential geometry [MAR 02]. The succinct and non-exhaustive presentation given here is based on a simple physical approach which

allows us to define the operators associated with the switch from one topology to another, on the basis of scalar or vectorial information. It should be remembered that although the classic notion of a continuum has been set aside, the material is a continuum where the directions of the edges and of the normal to the surfaces are also preserved at all scales of observation.

Gradient of a scalar

The gradient operator applied to a scalar p, ∇p, represents the difference of that scalar over a distance δ in a given direction. Unlike with the concept of Continuum Mechanics, the gradient vector defined here has only one component, assigned as a scalar to the edge Γ. The gradient of a scalar in the space has no meaning – it is an illegal operation in Discrete Mechanics. The gradient is calculated solely on a bipoint PR linked by an edge.

Primal rotational of a vector

The primal rotational of a vector $\mathbf{W}$, $\nabla_p \times \mathbf{W}$, is associated with the circulation of its components $\mathbf{V}$ over all the edges Γ constituting the primal boundary. It is represented by a vector $\mathbf{n}$ orthogonal to the primal surface. This surface is considered to be flat. This apparent restriction disappears as the surface area ds tends toward zero; however, it will remain a condition for the application of the theorems of differential geometry in particular context discussed here.

Divergence of a vector

The divergence represents the flux of a vector $\mathbf{W}$, $\nabla \cdot \mathbf{W}$, across all the facets of the dual surface. The scalar obtained as assigned to the single point inside the dual volume. The flux is calculated on the basis of the components $\mathbf{V}$ on the edges Γ of the vector itself. If the vector $\mathbf{W}$ is a rotational, calculated as the circulation of another vector on each primal boundary, then the divergence will be strictly null.

Dual rotational of a vector

The dual rotational of a vector $\mathbf{W}$, $\nabla_d \times \mathbf{W}$, physically represents the flux of the vector $\mathbf{W}$ across that portion of the dual surface associated with the edge Γ. This flux is calculated using the circulation of the vector, or rather, of its components, on the boundary delimiting the dual portion which, in general, is not flat. The result of this operation is assigned to the edge Γ as a vector or an oriented scalar, if necessary.

1.2.3. *Use of the theorems of differential geometry*

The velocity $\mathbf{W}$ is defined in relation to an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ composed by unit vectors in each of the spatial directions. The essential point in the description of the mechanical motion given in the coming chapters is that it is not necessary to consider the velocity vector $\mathbf{W}$ in itself, and therefore it is possible to discount the orthonormal Cartesian frame of reference needed to define it. Only its component $\mathbf{V} = (\mathbf{W} \cdot \mathbf{t})\ \mathbf{t}$ on the edge Γ will be used to describe the motion. On the basis of the various components, it is possible to determine the vector $\mathbf{W}$ in an appropriate frame of reference.

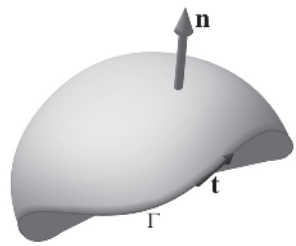

Figure 1.4. *Stokes theorem on the edge* Γ

Indeed, the operators of differential geometry ∇, $\nabla\cdot$, $\nabla\times$ are invariant when the base changes. It is therefore possible to carry out a certain number of transformations based on knowledge of the components only. Take, for example, the calculation of the rotational of

the vector $\mathbf{W}$ based on Stokes' theorem (equation [1.1]):

$$\iint_A \nabla \times \mathbf{W} \cdot \mathbf{m} \; ds = \int_\Gamma \mathbf{W} \cdot \mathbf{t} \; dl = \int_\Gamma \mathbf{V} \cdot \mathbf{t} \; dl$$
$$= \iint_A \nabla \times \mathbf{V} \cdot \mathbf{m} \; ds \qquad [1.1]$$

The vectors $\mathbf{W}$ and $\mathbf{V}$ have the same rotational $\omega = \nabla \times \mathbf{W} = \nabla \times \mathbf{V}$. The discrete rotational is defined by:

$$\nabla \times \mathbf{V} = \frac{1}{[A]} \int_\Gamma \mathbf{V} \cdot \mathbf{t} \; dl \qquad [1.2]$$

where $[A]$ is the measurement of the curved surface A.

Similarly, the divergence theorem is based on the projection of the vector $\mathbf{W}$ along the normal to the oriented surface S:

$$\iiint_V \nabla \cdot \mathbf{W} \; dv = \iint_S \mathbf{W} \cdot \mathbf{n} \; ds \qquad [1.3]$$

and the discrete divergence is defined as

$$\nabla \cdot \mathbf{W} = \frac{1}{[V]} \iint_S \mathbf{W} \cdot \mathbf{n} \; ds \qquad [1.4]$$

where $[V]$ is the measurement of the dual volume V. In this case, the flux of the vector $\mathbf{W}$ will be written on the basis of the components $\mathbf{V}$ on all of the portions of the dual surface S associated with each segment Γ around the point P.

Finally, the discrete gradient of a scalar value p is calculated very simply over the edge Γ by way of the relation:

$$\nabla p = \frac{1}{[L]} \int_\Gamma \nabla p \; dl \qquad [1.5]$$

where $[L]$ is the distance PR. Hereafter, this gradient will be assumed to be constant along the entire length of the edge Γ, giving us:

$$(p_R - p_P) = \frac{1}{L}\int_P^R d\,p\,dl = \int_\Gamma \nabla p \cdot \mathbf{t}\,dl \qquad [1.6]$$

There is an important difference which exists in the use of the Stokes and Green-Ostrogradski theorems; in the current context, the gradient- and rotational operators are the only ones capable of describing mechanical equilibrium. The divergence operator is secondary in this context, and can be used to calculate the fluxes so as to then return the values of the scalar variables. This distinction is already present in the Hodge–Helmholtz decomposition, because that formula can be used to decompose any vector into a gradient and a rotational.

It will be necessary to define the momentum vector $\rho\mathbf{V}$ along the edge. We shall suppose that the density along the edge is a constant $\overline{\rho}$:

$$\overline{\rho} = \frac{1}{[L]}\int_L \rho\ dl \qquad [1.7]$$

and that the component $\mathbf{V}$ is also constant along the edge; the momentum can be written interchangeably as $\rho\mathbf{V} = \overline{\rho\mathbf{V}} = \overline{\rho}\ \overline{\mathbf{V}}$. It is possible to theoretically calculate the exact value of $\overline{\rho}$ for a conservation of momentum in the context of a non-conservative formulation of the motion balance equations.

The body forces are represented by their volumetric density $\mathbf{f}$. Their effects on the motion are perceived by way of the projection $\mathbf{f}\cdot\mathbf{t}$. These forces can be associated with a scalar potential such as $\mathbf{g} = \nabla\phi$, as with gravity for a constant density, for instance, or indeed with a vector potential; generally speaking, they will have both components.

1.2.4. *Two essential properties*

The circulation of a vector on a boundary or the flux over a surface are concepts which have been extended to apply to the material point.

Hence, the properties $\nabla \times \nabla p = 0$ and $\nabla \cdot \nabla \times \mathbf{V} = 0$ are supposed to be satisfied in the continuum formulation. However, a contour or a surface cannot be reduced to a point, and it is only by extension that these concepts can be used in the context of a continuum. In Discrete Mechanics, the topologies can be reduced but, whatever the scale, they remain homothetic to the original geometries, and the local rotational can be defined, if necessary, as the extension of the discrete rotational. The same is true for the flux density over the dual topology.

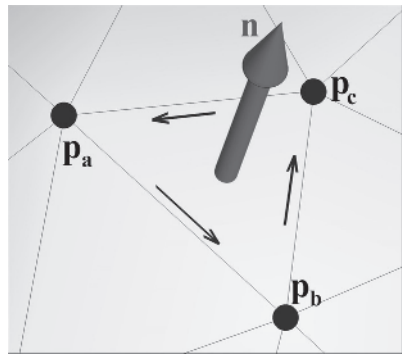

Figure 1.5. *Property 1: the rotational of a gradient is null on the primal topology. This property is verified on all types of topologies: polygons and polyhedrons with any number of faces*

Figure 1.5 shows a primal topology, over which it is possible to calculate the circulation of the velocity vector $\mathbf{W}$, which is also the circulation of its components $\mathbf{V}$ along the boundary Γ.

$$\begin{cases} \displaystyle\int_a^b \nabla p \cdot \mathbf{t}\, dl = p_b - p_a \\ \displaystyle\int_\Gamma \nabla p \cdot \mathbf{t}\, dl = 0 \\ \displaystyle\iint_{\mathcal{S}} \nabla \times \big(\nabla p\big) \cdot \mathbf{n}\, ds = 0 \\ \nabla_h \times \big(\nabla_h\, p\big) = 0 \end{cases} \qquad [1.8]$$

The calculation of the circulation of the vector ∇p along a closed path Γ is given by the system of equations [1.8]. Here, p is a simple scalar, and irrespective of the function chosen, the result is the same.

Regardless of the primal topology that is chosen, the rotational of the vector ∇p is always equal to zero. On the closed path, each contribution of p to the circulation is used twice, with opposite signs.

The second property $\nabla \cdot \nabla \times \mathbf{V} = 0$, applicable over the whole of the dual topology, is illustrated by Figure 1.6. The global flux is calculated on all of the facets:

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \Gamma_i = \sum_{i=1}^{n} \iint_s \nabla \times \mathbf{V} \cdot \mathbf{n}\, ds = 0 \\ \displaystyle\iint_{\mathcal{S}} \big(\nabla \times \mathbf{V}\big) \cdot \mathbf{n}\, ds = 0 \\ \displaystyle\iiint_{\mathcal{V}} \nabla \cdot \big(\nabla \times \mathbf{V}\big)\, ds = 0 \\ \nabla_h \cdot \big(\nabla_h \times \mathbf{V}\big) = 0 \end{cases} \qquad [1.9]$$

As each edge is used twice with opposite directions of travel, its net contribution is null, as is shown by the relations in system [1.9].

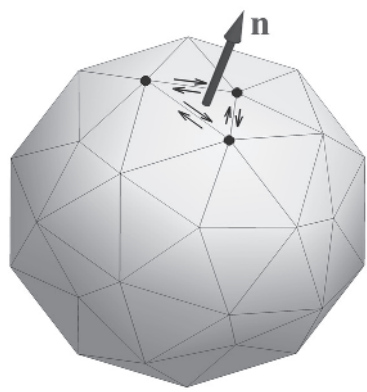

Figure 1.6. *Property 2: the divergence of a rotational is null over the dual topology. This property is verified on all types of topologies: polygons and polyhedrons with any number of faces*

The two rotationals of the relations $\nabla_p \times \nabla p = 0$ and $\nabla \cdot \nabla_d \times \mathbf{V} = 0$ are not defined on the same topology. For the rotational applied to the

gradient, it is the primal rotational; for the second, it corresponds to the dual rotational applied to the flux of the primal rotational.

These properties [1.8] and [1.9] are obtained by construction of the operators which apply to values that are localized in a unique manner: the scalars at the points and the components of the vectors on the edges connecting two points. The gradient operator is, naturally, calculated using values at the extremities of the bipoint, and assigned to the corresponding edge. The primal rotational is defined on the faces which are supposed to be flat, so the rotational vector is represented by a vector which is normal to the face in question. The dual rotational defined on the edge is calculated on the basis of the fluxes over the facets of the primal topology. The divergence of the vector $\mathbf{V}$ calculated on the dual topology is assigned to the point P.

The example given in Figure 1.7 clearly illustrates the advantage to using a formulation which respects these two properties. It is a planar, rectilinear, uniform motion which cannot give rise to any gradient of a scalar potential p or any rotational of a vector potential $\boldsymbol{\omega}$. Here, the topology is based on a pattern of regular triangles. The initial solution corresponds to a state of rest where all the components $\mathbf{V}$ are zero. A constant flow rate is imposed on the left-hand wall of the domain. As the medium is considered to be incompressible ($\nabla \cdot \mathbf{V} = 0$), the solution is obtained instantly in the form of values for the components such that $\mathbf{V} = V_0\ \mathbf{t}$, where V_0 represents the imposed velocity and $\mathbf{t}$ the unit vector carried by each edge. The scalar potential p is null, as is the rotational $\nabla \times \mathbf{V}$. Any reconstruction of the velocity vector $\mathbf{W}$ would give $\mathbf{W} = V_0\ \mathbf{e}_x$; the motion is indeed rectilinear and uniform, everywhere. This result can be obtained with any topology made up of polygons or polyhedra.

The stream lines shown in Figure 1.7 do correspond to the motion we are looking for. It is easy to see that the superposition of that motion on any other motion will have no effect whatsoever on the values of the potential fields of the latter. If a field deriving from a scalar potential were added, the motion would not be altered.

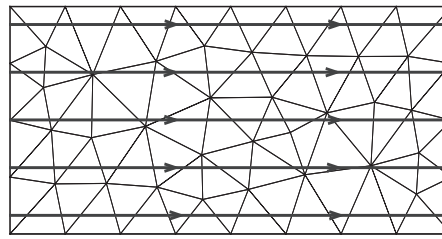

Figure 1.7. *Uniform rectilinear plane flow: stream lines in a triangle-based primal topology: the components of the velocity along each edge are such that* $\mathbf{V} = V_0\,\mathbf{t}$*, exactly*

Similarly, a motion of uniform rotation superposed upon any other will not modify the latter, provided an appropriate potential is added to the existing scalar potential. Consider the case of the primal topology shown in Figure 1.8, where a medium which is initially at rest is gradually caused to rotate, maintaining the velocity of the surface of the outer circle at $V_\theta = \Omega_0\,R$, where V_θ is the imposed velocity and R the radius of the circle. The momentum transfer due to the viscous effects leads the system to reach a state of uniform rotation.

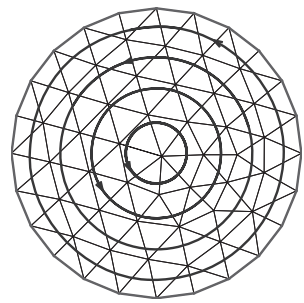

Figure 1.8. *Uniform rotational flow: stream lines in a triangle-based primal topology: the velocity components on each edge are such that* $\mathbf{V} = \Omega_0\,r\,\mathbf{t}$

The steady-state solution to the problem obtained on the basis of the discrete equations corresponds to a polar velocity equal to $\mathbf{V} = \Omega_0\,r\,\mathbf{t}$, and constant rotational $\nabla \times \mathbf{V} = 2\,\Omega_0$ and a pressure field defined by the potential $\phi = -\rho\,\left(\mathbf{\Omega} \times \mathbf{r}\right)^2/2$, such that $\nabla(p + \phi)$ is zero. In the

moving frame of reference whose axis is orthogonal to the plane of the cylinder, the system would be in a state of rest.

The two uniform motions, rectilinear and rotational, do not lead to precisely the same conclusions; whilst for the rectilinear motion, the scalar potential is null, in the case of uniform rotation we find a scalar potential, created by the centrifugal effects. However, the viscous effects are null in both cases. This suggests that we can add the gradient of any function without giving rise to permanent motion.

It could be said that the two properties $\nabla_p \times \nabla p = 0$ and $\nabla \cdot \nabla_d \times \mathbf{V} = 0$ mimic the properties applied to continua. More seriously, we can say that the properties assigned to the continuum are the result of reaching the system's limit; for example, the rotational only has meaning if we are able to previously calculate the circulation in a plane along a path of finite length.

1.2.5. *Tensorial values*

The notion of a tensor is needed here, when we wish to combine the variations of the same property into a single point in the direction of observation. In that the direction of observation Δ is fixed, the various values, mechanical properties, stresses, etc. will be simple scalars or vectors oriented along Γ. Certain materials have tensorial properties such as heat conductivity, permeability, certain mechanical properties, etc., such as the components of the heat flux, of the displacement or the velocity, the tensorial values will be carried by the edge Γ by projection.

Consider, for example, the case of diffusion of the heat flux $\mathbf{\Phi}$ in an anisotropic material. The matrix $\mathbf{\Lambda}$ representing the heat conductivity tensor can be diagonalized, and its eigenvectors define the principal directions of the tensor $\mathbf{\Lambda}$; in its principal basis, the matrix is diagonal $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$.

The flux carried by the edge $\mathbf{t}$ can, formally, be written $\mathbf{\Phi} \cdot \mathbf{t} = -k\,(T_R - T_P)/L$, where k is the scalar representing the conductivity on

the edge and $(T_R - T_P)/L$ the discrete gradient. Hence, we can identify the heat flux in direction $\mathbf{t}$ and calculate the value of k:

$$k = \boldsymbol{\lambda} \cdot \mathbf{t} \qquad [1.10]$$

where in this case, $\mathbf{t}$ is represented on the principal basis. This property will be constant along the entire length of the edge.

It should be noted that any value taken by the oriented edge of unit vector $\mathbf{t}$, can interchangeably be written as a scalar which is constant along the whole length of the edge, or as an oriented vector.

The behavioral laws, the constitutive laws, the state laws, etc., are only necessary to describe the behavior of a medium, be it fluid or solid, as a function of certain scalar or vectorial variables such as the temperature, pressure, mechanical stress, etc. If the properties of the medium are variable as a function of the direction, as is the case with anisotropic media, it may be advantageous to describe its behavior using a symmetrical tensor to simulate that direction, and enable us to calculate the stress in any given direction. However, the tensor is defined by a base, which is generally orthonormal, and it is there that the problem lies. The very general nature of this approach also introduces a certain number of difficulties which need to be overcome in other ways – notably by way of the principle of material frame indifference, for example. These tensorial properties are then introduced into the conservation laws.

The six independent coefficients of the Cauchy stress tensor, defined at a single point, are expressed on the basis of the velocity or the displacement to give the 81 coefficients of the elasticity tensor corresponding to the stress/strain relation given by Hooke's law. For an isotropic medium, these are reduced to two coefficients: the Lamé coefficients, which do not have the same degree of representativeness in fluids and solids.

This strong link between the constitutive laws and the conservation laws can be broken without adversely affecting the representativeness of the model constructed on the basis of the fundamental law of

dynamics. The anisotropy of the medium, just like its inhomogeneity, has no direct influence on the model. The physical properties, including the viscosity attributed to the primal surface and the compressibility coefficient defined at the endpoints of each edge, are dependent on scalar variables such as pressure, temperature or density, or of vectorial variables such as the vector potential. The way in which we deal with anisotropy is similar to the approach for inhomogeneity: the viscosity is defined per plane and compressibility per point, and their values are variable in space and, of course, over time.

1.2.6. *The scalar and vectorial potentials*

The physical values adopted to describe the evolutions of the physical system are:

– the vectorial variables located on the direction edge of unit vector $\mathbf{t}$; for example, the velocity $\mathbf{V}$, the displacement $\mathbf{U}$, the heat flux $\mathbf{\Phi}$, etc. are defined at the midpoint of the edge $[PR]$. These values will be constant along the whole length of the edge;

– the scalar potentials of the stresses linked to the velocities, displacements, fluxes, etc. located at the vertices of the primal topology. These values are constant throughout the whole of the dual volume, but are assigned to the point. The scalar values are entirely separate variables such as the density ρ, the pressure p or the absolute temperature T. They are defined at P and depend directly on the vectorial variables which feed into them – the velocity $\mathbf{V}$ and the heat flux $\mathbf{\Phi}$. When the mechanical effects associated with solid behaviors are taken into account, we need to introduce the accumulator linked to the shear $\boldsymbol{\omega}$.

The scalar potential p contains two terms: the accumulation potential p^o corresponding to equilibrium, and its deviator, which represents all the contributions liable to alter the pressure, the velocity, the flux, etc. Similarly, the vector potential $\boldsymbol{\omega}$ is the sum of an accumulator $\boldsymbol{\omega}^o$ and a deviator, which is a function of the rotational and the velocity. All these values will be defined below. It should be noted that the accumulation of shear stresses is negligible in fluid media. However, by using the term

$\boldsymbol{\omega}^o$, we are able to represent the fluid and solid behaviors in a continuous manner, and describe complex rheological behaviors.

Depending on the physics we are looking at, many other variables can be defined, but for reasons of clarity, our presentation here will be limited to the mechanical effects of viscosity and elasticity. The pressure used in these examples will be the thermodynamic pressure corresponding to mechanical equilibrium, which is written as p^o. In order to describe the effects of compression it will be necessary to take account of the thermodynamic and thermal evolution of the system. These aspects are described in this author's previous publication on Discrete Mechanics [CAL 13a].

The values which we shall call potentials $(\rho, p, T, \boldsymbol{\omega})$ will be updated as a function of the fluxes of matter and heat. For example, the density of a fluid contained in a non-deformable envelope will increase over time as a function of the degree of momentum $\rho\mathbf{V}$ injected. These values therefore appear in the form of accumulators, fed back by the divergence of the influx and outflux into/out of the dual volume across the dual surface S. The vectorial accumulator $\boldsymbol{\omega}$ is updated on the basis of the rotational of the displacement $\nabla \times \mathbf{U} = dt\, \nabla \times \mathbf{V}$.

These physical properties defined at point P are variables, taken from the conservation balances, but cannot be found by solving equations. The scalar conservation laws merely serve to feed back these properties into the values necessary to express mechanical equilibrium. In the Lagrangian approach adopted here, we consider that the initial equilibrium at time t, defined by the values $(\rho^o, p^o, T^o, \boldsymbol{\omega}^o)$ and described by the vectorial equations on the fluxes, can be broken by a number of causes (sources, boundary conditions, evolution over time, etc.) and the system will evolve to reach a different state of equilibrium at time $t + dt$.

1.3. The physical characteristics

The continuum hypothesis is associated with the notion of assignment to the point P of any and all variables – be they scalar,

vectorial or tensorial – defined in a finite control volume surrounding the point in question. For instance, the mean density is the volumetric mean, calculated for the volume:

$$\overline{\rho} = \frac{1}{[V]} \iiint_{\Omega} \rho \ dv \qquad [1.11]$$

The continuum hypothesis reduces the volume to a single point, by adopting $\rho = \overline{\rho}$. Whilst this hypothesis poses no difficulty for a scalar, its application to vectorial or tensorial values eliminates all concept of direction. To define a vector in the space, therefore, it will be necessary to create a base upon which to express its three components. Worse still, in order to preserve the vectors over an oriented surface, it is necessary to resort to a tensorial form to express the scalar product and obtain the vector we seek for a given direction.

In linear elasticity, the Lamé mechanical coefficients – λ, the compression viscosity coefficient, and μ, the shear viscosity coefficient – are linked by simple laws to measurable values: Young's modulus E and the Poisson coefficient σ. Whilst the coefficient λ has a definite physical meaning in a solid material, the same is not true for fluids. Measuring λ (generally by ultrasounds) yields results which are highly disparate, and dependent on the working method. Discrete Mechanics [CAL 13a] shows that the single physical coefficient, linked to the effects of undeniable compression, is the compressibility coefficient, which is easily measurable both in fluids and solids.

In Discrete Mechanics, the physical properties of the media are simple coefficients whose values need to be known. Unlike in Continuum Mechanics, there is no particular formalization of these coefficients that can prove a constitutive law. The state laws, the phenomenological relations, the rheology of the fluid or solid media may yield specific formalizations, but these do not enter directly into the establishment of the conservation equations.

The scalar values will be defined at the point P, which is one of the endpoints of the edge Γ. They could include the thermodynamic properties such as the compressibility coefficient of the material χ_T, its coefficient of thermal expansion β, its specific heats, at constant

pressure or constant volume, c_p or c_v, etc. Other values are defined on the edge Γ, including the heat conductivity k, the permeability K, etc. They will be considered constant along the whole length of the edge. Finally, the viscosity of the medium μ makes sense only when it is associated with the primal surface, and is constant on that surface.

These properties do not depend primarily on the vectorial variables, the velocity or the flux; they may, of course, depend on the scalar values, the pressure, the temperature, etc. These physical characteristics are known locally, and obtained independently by any means (measurements, tables, laws, etc.).

1.4. Equilibrium stress state

Before giving an expression of the stress state, it is advisable to define mechanical equilibrium. Mechanical equilibrium is obtained when the law of dynamics is satisfied, $\rho\,\boldsymbol{\gamma} - \mathbf{f} = 0$ if $\mathbf{f}$ represents all the volumetric forces applied to the system studied at time t. This choice of the concept of equilibrium precludes all motions where acceleration is null, the uniform rectilinear motion and the block rotational motion. For the second case, the constant centrifugal acceleration counterbalances the centripetal force; we shall see that the case poses various problems for non-connected domains, which we shall look at later on.

Any disturbance to this state of equilibrium due to modifications made to one of the source terms, the boundary conditions, etc., will lead to a change in the variable used – i.e. velocity or displacement – which leads the system to a different state of equilibrium at time $t + dt$, for which we shall also have $\rho\,\boldsymbol{\gamma} - \mathbf{f} = 0$. What changes between the two states of equilibrium is the residual stress state manifested by two potentials – the equilibrium pressure p^o and the vector potential of the acceleration $\boldsymbol{\omega}^o$. Hence, for a state of mechanical equilibrium, the sum of the relative contributions to these two values is null and the acceleration is also null; the motion is limited to the cases discussed above.

The question that now arises is of the description of the residual state of equilibrium $(p^o, \boldsymbol{\omega}^o)$; the dimension of $\boldsymbol{\omega}^o$ is also that of the pressure, and we shall write the equilibrium stresses as $(p^o, \boldsymbol{\omega}^o)$.

Consider a medium at equilibrium Ω and divide it, by introducing the surface $\mathcal{S}$ (see Figure 1.9), without the upper part, which is removed. The problem is to define the stress state enabling us to maintain the remaining part also in a state of equilibrium. The unit vector $\mathbf{n}$ is the support for the forces of pressure applied at point P to the elementary surface ds, i.e. $d\mathbf{F} = -pds$. The separation between the pressure effects and the viscous effects will be discussed later on, but we shall introduce a formal separation of these two effects, which will be defined by the two independent potentials.

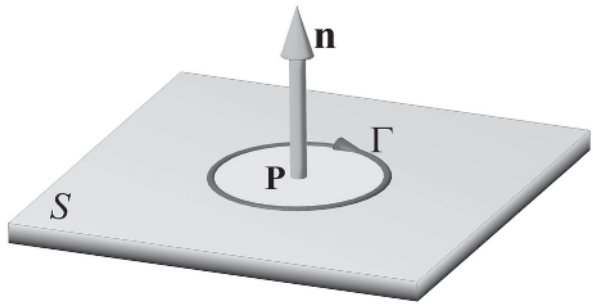

Figure 1.9. *Stress state at equilibrium for a medium separated from its upper part*

The shear stress applies to the surface $\mathcal{S}$ orthogonal to the vector $\mathbf{n}$. This does not mean that that support of the applied force is in the plane of that surface. Such is the case for a rotation force which is exerted on the medium in the plane and whose support is the normal to that plane $\mathbf{n}$. We then introduce the rotational operator $\nabla \times \mathbf{V}$ to describe the local rotation of the medium. That rotation can be calculated on the basis of the circulation of $\mathbf{V}$ along the boundary Γ surrounding the surface ds. It is still necessary that the boundary be able to be defined, which requires that its length be finite, along with the surface ds, in order to be able to calculate the rotational.

Here, we can see what could be a difficulty linked to the multidimensional nature of the operator $\nabla \times \mathbf{V}$: it only exists in one direction. Thus, the concept of a rotational for a material point is meaningless; this operator is always attached to a surface whose normal is clearly defined.

The notion of the shear stress of the medium in only introduced by the spatial variation over $\mathcal{S}$ of the local rotational. It is no longer necessary to refer to a fixed system of coordinates – the direction of the normal to the plane is enough to take care of the forces of pressure and shear.

The rotation/shear stress will be calculated on the basis of the rotational of the velocity and a coefficient which will depend on the nature of the medium; it will be assumed to be constant over the surface ds. This coefficient, which will be written as μ, will be called the rotational viscosity, and its existence is linked to the first Lamé coefficient of the medium.

The pressure stress $\mathbf{T}_p$ and the rotation/shear stress written as $\mathbf{T}_v$, local and instantaneous, will therefore be written as:

$$\begin{cases} \mathbf{T}_p = -p\,\mathbf{n} \\ \mathbf{T}_v = -\mu\,\nabla \times \mathbf{V} \end{cases} \qquad [1.12]$$

It should be noted that p^o and $\boldsymbol{\omega}^o$ are the stresses are time t, where all the forces applied before that instant are "remembered". The formalism presented here enables us to take account of the entire history of the medium, i.e. its evolution over time from an initial neutral state. For a given instantaneous state of strain, there may be multiple paths by which that state can be reached, and $(p^o, \boldsymbol{\omega}^o)$ will, alone, contain the whole of the medium's history. It is not helpful to know the local and instantaneous stresses, in that these two potentials will have accumulated stresses over time. These potential can therefore be used to take account of the behavior of media with continuous memory.

It is not necessary to define a total stress $\mathbf{T}^o$ including the effects of pressure and those of shear/rotation. However, if it were necessary to do so, we would write:

$$\mathbf{T}^o = -p^o\,\mathbf{n} + \boldsymbol{\omega}^o \qquad [1.13]$$

The influences of pressure and of shear stress are thus represented by forces with the same carrier: the normal to the plane $\mathcal{S}$. It is therefore no longer essential to use a tensorial value – the Cauchy stress tensor used in Continuum Mechanics – to describe a local stress applied in the direction normal to a face in a given direction.

1.4.1. *Two examples of mechanical equilibrium*

Consider an example of a steady-state flow (see Figure 1.10) engendered by the animation of two parallel horizontal walls at velocity V_o. The pressure is constant throughout the domain in question. The motion corresponds to a constant shear stress, and can be characterized as a row of co-rotating vortices throughout the thickness of the fluid layer.

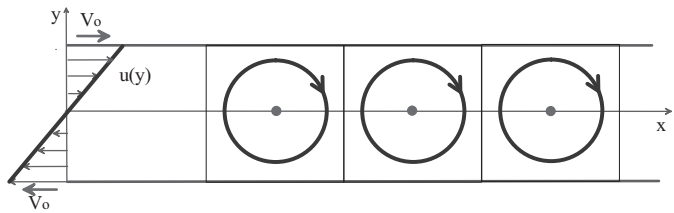

Figure 1.10. *Couette flow engendered by a row of vortices animated by a stress applied in the direction orthogonal to the plane*

The combination of these vortices gives rise to a so-called Couette flow, where the vertical velocity component is zero and its axial component is equal to $u(y) \propto y$. The vector potential is constant and equal to $\boldsymbol{\omega}^o = \mu\,\nabla \times \mathbf{V}$.

The case of a Poiseuille flow (see Figure 1.11) can also be interpreted as the superposition of vortices caused by rotational

stresses along the axes orthogonal to the plane (x, y) of the domain in question. The intensity of the vortices is a linear function of y.

In this example, the pressure stress is not constant, and we can show that mechanical equilibrium represents one solution in terms of the linear pressure $p(x) \propto x$.

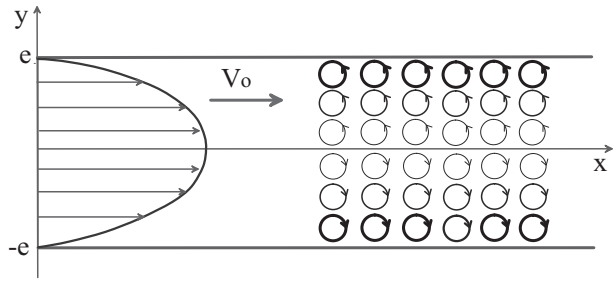

Figure 1.11. *Poiseuille flow defined by a set of rotational stresses whose intensities vary with the y value*

As these two examples illustrate, the variation of the local rotational stress may give rise to a shear stress; thus, the solution to the problem at hand needs to take account of the imposed boundary conditions. The derivation of the motion balance equation is established on the basis of the concept of local stress, pressure and viscosity. The physical meaning of the rotational operator associated with the viscosity μ suggests the possibility of representing the effects of viscosity in the motion balance equation for any given motions.

1.5. Thermodynamic non-equilibrium

In order to prevent the confusion which results from the typical jargon employed in each discipline, it must be understood that local non-equilibrium as defined here is not the same as the local non-equilibrium encountered in molecular physics ([CHA 99]). For our purposes, the material is always considered to be a continuum, so that we can make use of the concepts of pressure, density and local temperature. The length of the edge d of Γ is greater than the mean free

path of the molecules l, $d >> l$. Hence, we can see the advantage inherent in looking at edges instead of points, as is done in Continuum Mechanics.

Local Thermodynamic Equilibrium (LTE) is generally used to describe the relationship which exists between the traditional thermodynamic variables, such as pressure, temperature or density, for a divariant fluid. This relation is called the state law, and if we limit the variables to those already given, it can be expressed in the form $f(\rho, p, T) = 0$, where p and T are the thermodynamic pressure and the thermodynamic temperature.

Although no principle exists which explicitly links these potentials, the state laws are commonly used for simplicity's sake. Only the thermodynamic coefficients have an undeniable reality – the compressibility coefficient χ_T, the dilatation coefficient β, etc. There is no reason why they could not easily be deduced using a law, but under no circumstances can these laws constitute a closure of the conservation equations. As we shall see, each conservation equation has its own potential, and the state law is not needed to bring closure to the system. Indeed, direct use of the state law can even negate the conservation of the various values – particularly the conservation of mass.

Hence, when establishing the discrete equations for Discrete Mechanics, we shall not invoke this hypothesis; only the thermophysical property values which have an incontestable meaning will be used.

Discrete Mechanics is based on the fact that the equations found using this approach describe mechanical equilibrium. We can define the main vectorial values, which are the velocity $\mathbf{V}$ and the heat flux vector $\mathbf{\Phi}$. These fluxes enter into the domain across its surface, with an outward normal $\mathbf{n}$. Any equation stemming from this theory describes a state of equilibrium associated with the values of the equilibrium potentials $\rho^o, p^o, T^o, \boldsymbol{\omega}^o$. If the equilibrium is broken because of the alteration of one of the boundary conditions, one of the source terms, etc. then the system of equations can re-establish mechanical

equilibrium by an appropriate adaptation of the velocity, the flux, etc. Symbolically, if an equation is written as

$$\mathbf{L}(\rho^o, p^o, T^o, \boldsymbol{\omega}^o, \mathbf{V}^o, \boldsymbol{\Phi}^o) = 0 \qquad [1.14]$$

then it is a state of mechanical equilibrium.

The equilibrium potentials $(\rho^o, p^o, T^o, \boldsymbol{\omega}^o)$ are strictly independent, and therefore cannot be connected by any state law. This assertion is, of course, accepted when the time-constants of the phenomena are very small (as is the case with shockwaves, explosions, etc.) but is generally adopted here.

Figure 1.12 shows the diagram illustrating the Lagrangian evolution of a system at equilibrium at time t and evolving toward a different state of equilibrium at time $t + dt$. The choice of the symbol dt to represent the time difference may be surprising for some, but it is neither a characteristic time (which could have been written as τ in a physical approach to denote changes of scale) nor a numerically-based time-lag (which would be written as Δt), used for simulations as an increment of time. Thus, it is the time taken to evolve from one state of equilibrium to another, and the symbol employed here is that which is used to write the material derivative. The discrete approach developed here can be extended to the notion of time; only the state of equilibrium at time t^o and the instantaneous state t can be used to quantify the variables involved in the system. Between these two times, the values of these variables are unknown, as are the physical properties of the medium. This is, at once, a handicap (because this incremental process is linked to the observation time-constant dt) and an advantage, in that the history of the medium's evolution is contained in the potentials p^o and $\boldsymbol{\omega}^o$. They express the stresses undergone by the medium throughout its entire history of evolution, or in other words, they contain the medium's memory.

Over the course of the system's evolution between time t and time $t + dt$, the potentials will evolve from ρ to $\rho + d\rho$, from p to $p + dp$, from T to $T + dT$ and from $\boldsymbol{\omega}$ to $\boldsymbol{\omega} + d\boldsymbol{\omega}$. The new state will be a state of mechanical equilibrium if the equation is exactly satisfied.

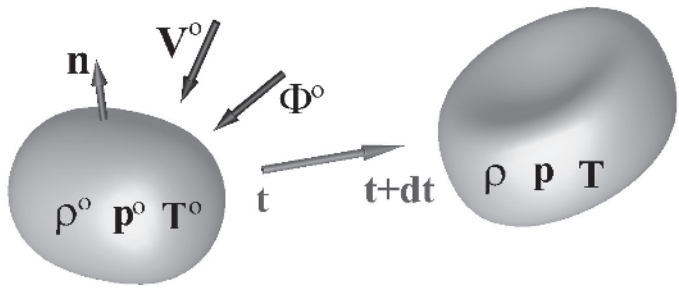

Figure 1.12. *Definition of the physical system and observation of that system between two states of mechanical equilibrium t and $t + dt$*

At no time will the state law be invoked; there is no reason for thermodynamic equilibrium to be verified at all times, locally. The system is closed by the density, which is recovered for the conservation of mass equation, whilst the pressure and the vector potential $\boldsymbol{\omega}$ are raised by the momentum conservation equation and the temperature by the heat flux conservation equation.

It should be noted that whilst the properties of the materials, the coefficients and the transport properties, all influence the solution, they have no implicit structural link to one another. They simply need to be worked out on the basis of the potentials.

1.5.1. *Forces and fluxes*

Consider the system (Figure 1.12) at time t; this exhibits a state of instantaneous equilibrium defined by the relation [1.14]. $\mathbf{V}_o$ and $\mathbf{\Phi_o}$ are the velocity flux and heat flux across the surface S of Ω. Within the system, these fluxes are written as $\mathbf{V}^o$ and $\mathbf{\Phi^o}$. The potentials at that moment in time are (ρ^o, p^o, T^o) – density, pressure and temperature.

The material system will be followed in its motion between times t and $t + dt$. The system will evolve as a function of the fluxes $\mathbf{V}_o$ and $\mathbf{\Phi_o}$ on the surface, but also because of the forces exerted within the system itself: forces of pressure, of friction, body force density or any other sources within the system. The system will evolve to reach a

different state of equilibrium, which is also defined by the relation [1.14] where the new potentials will be $(\rho, p, T, \boldsymbol{\omega})$. The fluxes of matter and heat will also have evolved toward the values $\mathbf{V}$ and $\boldsymbol{\Phi}$ calculated from the material derivatives. To work back from this to find the temporal evolutions of the potentials and fluxes, it will be necessary to perform an Eulerian description, expressing the partial derivatives.

Traditionally, the thermodynamic variables used are scalar values: temperature T, enthalpy h, internal energy e, entropy s, etc. There are as many forms of conservation of energy as there are thermodynamic potentials. The approach adopted here is different: it is the conservation of heat flux $\boldsymbol{\Phi}$ which will be preserved; the other scalars are merely accumulators of energy. These scalar values will be a function of the fluxes of matter and heat defined by $\mathbf{V}$ and $\boldsymbol{\Phi}$. Out of all these scalar values, we shall choose the temperature T to quantify the evolutions of the system's energy.

The evolution of the system's temperature is a function of the divergence of the fluxes, $\nabla \cdot \mathbf{V}$ and $\nabla \cdot \boldsymbol{\Phi}$ and of the various heat sources (production) or mechanical sources (dissipation) introduced into the domain.

Hence, for phenomena limited solely to the mechanical and thermal effects, the only variables used are vectorial ($\mathbf{V}$ and $\boldsymbol{\Phi}$) and the only accumulators are the potentials $(\rho, p, T, \boldsymbol{\omega})$. Finally, the associated thermophysical characteristics $(\chi_T, \beta, c_v, c_p)$ and the transfer coefficients (μ, k) may or may not depend solely on the potentials.

1.6. Conservation of mass

The conservation of mass for a closed domain which is tracked as it moves yields a formula identical to the conservation law when we consider a constant volume, also tracked, where we introduce the flux of mass across its surface.

Here, we consider that the dual volume is constant over time (Figure 1.12) and that the flux of mass across the boundary can be

used to calculate the evolution of the mass for an open system:

$$\frac{d}{dt}\iiint_{\Omega}\rho\,dv=-\iint_{\Sigma}\rho\,\mathbf{V}\cdot\mathbf{n}\,ds \qquad [1.15]$$

so

$$\iiint_{\Omega}\frac{\partial\rho}{\partial t}\,dv+\iint_{\Sigma}\rho\,\mathbf{V}\cdot\mathbf{n}\,ds=0 \qquad [1.16]$$

We then find

$$\iiint_{\Omega}\left(\frac{d\rho}{dt}+\rho\,\nabla\cdot\mathbf{V}\right)\,dv=0 \qquad [1.17]$$

Consider the local conservation of mass law:

$$\frac{d\rho}{dt}=-\rho\,\nabla\cdot\mathbf{V} \qquad [1.18]$$

This relation expresses that the local evolution of the density, when we track the motion of a material point, is due only to the local compressions or decompressions of any origin (be it mechanical, thermal, mass, etc.).

Although the model is limited to the case of a pure fluid or a homogeneous mixture or pseudo-fluid, it is possible to extend it to apply to a multi-component mixture (see section 5.8). In this case, it is necessary to know the velocities of each component $\mathbf{V}_i$, so there are as many motion balance equations as there are components in the mixture. The reading is then taken on the basis of each partial density ρ_i and the conservation of the total mass of the mixture is thus ensured perfectly.

The density ρ is a potential accumulator of the flux of matter within the dual volume; a positive fluid flow rate for the system across the surface leads to a negative divergence of the velocity, and therefore an increase in density.

If ρ^o is the density in the equilibrium state at time t, we can calculate the equilibrium density at time $t + dt$ by integrating the above law:

$$\rho = \rho^o \, e^{-dt \, \nabla \cdot \mathbf{V}} \qquad [1.19]$$

considering $\nabla \cdot \mathbf{V}$ to be constant throughout the time period dt. We can view the calculation of ρ simply as the updating of the accumulator:

$$\rho = \rho^o - dt \, \rho^o \, \nabla \cdot \mathbf{V} \qquad [1.20]$$

As we shall see, the conservation of mass law serves only to calculate the Lagrangian evolution of the density; not to formulate a pseudo-equation for the pressure. In no case can the density be evaluated by way of a state equation: it depends solely on the velocity divergence operator, and its variations with other variables – particularly the temperature – are taken into account only through the velocity.

In addition, the conservation of mass cannot be considered to be a law associated with the conservation of motion: it is an integral part of the conservation of motion, in the strongest possible since. Thermodynamics has shown us the equivalence of the mechanical energy and the heat, and relativistic mechanics introduced the link between the mass and the mechanical energy. In the next chapter, on conservation of momentum, we shall integrate the conservation of mass law into the conservation of momentum law, whilst remaining in the context of Newtonian mechanics.