Part 1

Integration of One-Variable Functions

1

Functions without Second-kind Discontinuities

DEFINITION 1.1.– *A function* $f : I \to \mathbb{R}$*, where* $I \subset \mathbb{R}$ *is an interval, is said to be continuous at a point* $x_0 \in I$ *if* $\lim_{x \to x_0} f(x) = f(x_0)$*. Otherwise,* f *is said to be discontinuous (or has a jump) at* x_0 *or that* x_0 *is a discontinuity point of* f*.*

A discontinuity point x_0 *of a function* f *is called:*

1) a first-kind (or simple) discontinuity point if there exist finite one-sided limits

$$f(x_0 - 0) := \lim_{x \to x_0 - 0} f(x) \quad \textit{and} \quad f(x_0 + 0) := \lim_{x \to x_0 + 0} f(x);$$

2) a second-kind discontinuity point in the remaining cases, where at least one of the limits $f(x_0 \pm 0)$ *is infinite (infinite discontinuity) or does not exist (oscillating discontinuity).*

The functions $f : I \to \mathbb{R}$ *that do not have second-kind discontinuities in* I *are called regular. We denote the set of all such functions by* $D(I)$ *(the letter* D *from "discontinuous"). In the case of a concrete interval, we will omit the parentheses:* $D[a,b] := D([a,b])$*,* $D[a,b) := D([a,b))$*, etc. It is obvious that* $C(I) \subset D(I)$ *(where, as usual,* $C(I)$ *denotes the set of all continuous functions* $f : I \to \mathbb{R}$*).*

REMARK 1.1.– Because of its generality, the term *regular* is not perfect for functions without second-order discontinuities. As we will see further, mathematically, a more appropriate term is *uniformly measurable*; however, it is rather long and inconvenient.

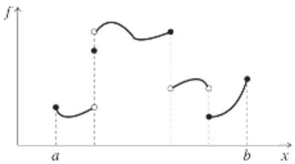

Figure 1.1. *A regular function* $f \in D[a,b]$

DEFINITION 1.2.– *A function* $\varphi\colon [a,b] \to \mathbb{R}$ *is called a step function if the interval* $[a,b]$ *can be partitioned into finitely many intervals* I_k, $k = 1,2,\ldots,m$ *(*$\bigcup_{k=1}^{m} I_k = [a,b]$*) so that, in each interval* I_k, φ *is constant (i.e.* $\exists y_k \in \mathbb{R}$, $k = 1,2,\ldots,m : \varphi(x) = y_k$ *for* $x \in I_k$*). We denote the set of all such functions by* $S[a,b]$.

Note that here we consider one-point sets $\{c\}$ *as closed intervals* $[c,c]$.

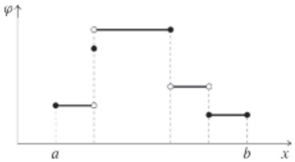

Figure 1.2. *A step function* $\varphi \in S[a,b]$

PROPOSITION 1.1.– Let $f \in C[a,b]$. Define the sequence of step functions $\{\varphi_n\}$ by

$$\varphi_n(x) := \begin{cases} f(x_k^n) \; for \; x \in [x_k^n, x_{k+1}^n), \; k = 0,1,\ldots,n-1, \\ f(b) \quad for \; x = x_n^n = b; \end{cases}$$

$$x_k^n := a + \frac{b-a}{n} k, \quad k = 0,1,2,\ldots,n.$$

Then $\varphi_n \rightrightarrows f$ in the interval $[a,b]$, that is, the sequence $\{\varphi_n\}$ converges to f uniformly in $[a,b]$.

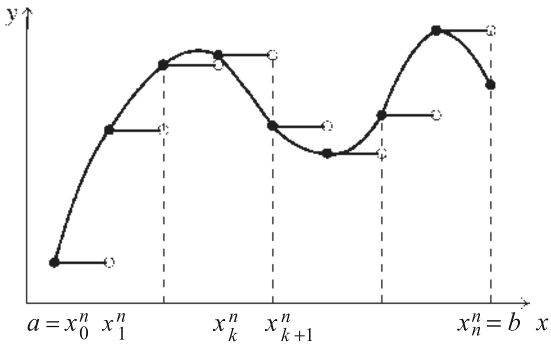

Figure 1.3. *Approximation of a continuous function by step functions*

PROOF.– Take arbitrary $\varepsilon > 0$. Since f is uniformly continuous in $[a,b]$ (Cantor theorem), there exists $\delta > 0$ such that

$$\left|f(x) - f(y)\right| < \varepsilon \quad \text{if} \quad |x - y| < \delta,\ x, y \in [a,b].$$

Taking $N \in \mathbb{N}$ such that $\frac{b-a}{N} < \delta$, we get that, for all $n > N$,

$$|\varphi_n(x) - f(x)| < \varepsilon, \quad x \in [a,b].$$

Indeed, let $n > N$. For any $x \in [a,b)$, let $[x_k^n, x_{k+1}^n)$ be the interval to which the point x belongs. Then $|x_k^n - x| < \delta$, and therefore, $|\varphi_n(x) - f(x)| = |f(x_k^n) - f(x)| < \varepsilon$. Finally, for $x = b$, we always have $\varphi_n(b) - f(b) = 0 < \varepsilon$. This means that, as required, $\varphi_n \rightrightarrows f$ in $[a,b]$ (the choice of N depends only on $\varepsilon > 0$ and not on $x \in [a,b]$). □

REMARK 1.2.– From the proof of proposition 1.1, we can easily see that the uniform convergence $\varphi_n \rightrightarrows f$ in the interval $[a,b]$ remains true if (see Figure 1.4):

1) instead of the points $x_k^n = a + \frac{b-a}{n}$ that partition the interval $[a,b]$ into n intervals $[x_k^n, x_{k+1}^n]$, $k = 0, 1, \ldots, n-1$, of equal size $\frac{b-a}{n}$, we take arbitrary points $a = x_0^n < x_1^n < \cdots < x_{k_n}^n = b$, $n \in \mathbb{N}$, satisfying the condition

$$\max_{0 \leqslant k \leqslant k_n - 1} \left|x_{k+1}^n - x_k^n\right| \to 0 \quad \text{as } n \to 0;$$

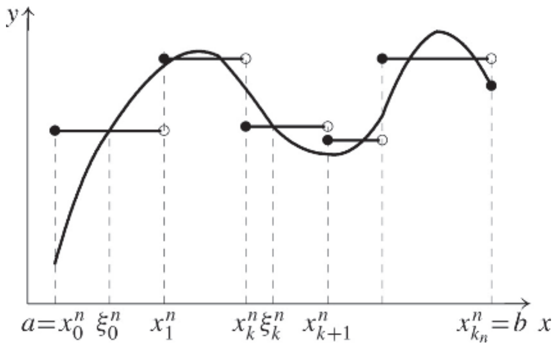

Figure 1.4. *A more general approximation of a continuous function by step functions*

2) we define the values of the step functions φ_n as the values of f taken not at the *left* points of the partition interval $[x_k^n, x_{k+1}^n]$, but rather at *arbitrary* points $\xi_k^n \in [x_k^n, x_{k+1}^n]$, that is,

$$\varphi_n(x) := \begin{cases} f(\xi_k^n) \text{ for } x \in [x_k^n, x_{k+1}^n),\ k = 0, 1, \ldots, k_n - 1, \\ f(b) \quad \text{for } x = x_{k_n}^n = b. \end{cases}$$

THEOREM 1.1.–

1) For any function $f \in D[a,b]$, there exist a sequence of step functions $\{\varphi_n\} \subset S\,[a,b]$ converging to f uniformly in the interval $[a,b]$.

2) The set of discontinuity points of any $f \in D[a,b]$ is finite or countable.

3) Every function $f \in D[a,b]$ is bounded.

REMARK 1.3.– The converse of the first statement is also true: the limit of a uniformly converging sequence of step functions is a function without second-kind discontinuities. Therefore, the class $D[a,b]$ can be characterized as the functions that are uniform limits of sequences of step functions in $[a,b]$.

PROOF.– [of theorem 1.1]

1) Let $f \in D[a,b]$. Take arbitrary $\varepsilon > 0$. Then, for every $x \in [a,b]$, there exists $\delta_x > 0$ such that

$$\left|f(t) - f(x+0)\right| < \frac{\varepsilon}{2} \quad \text{for } t \in (x, x+\delta_x) \cap [a,b]$$

and

$$\left|f(t) - f(x-0)\right| < \frac{\varepsilon}{2} \quad \text{for } t \in (x-\delta_x, x) \cap [a,b].$$

Since

$$\bigcup_{x\in[a,b]} U_{\delta_x}(x) = \bigcup_{x\in[a,b]} (x-\delta_x, x+\delta_x) \supset [a,b],$$

by the compactness of the interval $[a,b]$ (finite subcovering property), there exist $x_1, x_2, \ldots, x_n \in [a,b]$ such that

$$U_{\delta_{x_1}}(x_1) \cup U_{\delta_{x_2}}(x_2) \cup \cdots \cup U_{\delta_{x_n}}(x_n) \supset [a,b].$$

Let $\{a_0, a_1, \ldots, a_k\}$ be the set consisting of the numbers $a, b, x_i, x_i - \delta_{x_i}$, $x_i + \delta_{x_i}$ belonging to the interval $[a,b]$ and written in the increasing order: $a = a_0 < a_1 < \cdots < a_k = b$. Define the function $\varphi \in S[a,b]$ by

$$\varphi(a_i) := f(a_i), \quad i = 0, 1, \ldots, k;$$
$$\varphi(x) := f(c_j) \quad \text{for } x \in (a_{j-1}, a_j),$$

with arbitrary $c_j \in (a_{j-1}, a_j)$ (say, $c_j = (a_{j-1} + a_j)/2$).

Then $|\varphi(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$. Indeed, let $x \in [a,b]$. If $x = a_j$ for some j, then $|\varphi(x) - f(x)| = 0 < \varepsilon$. If $x \in (a_{j-1}, a_j)$, then there is i such that

$$(a_{j-1}, a_j) \subset (x_i, x_i + \delta_{x_i})$$

or

$$(a_{j-1}, a_j) \subset (x_i - \delta_{x_i}, x_i).$$

Suppose, for example, that case (1) is true. Then,

$$\begin{aligned}
\left|\varphi(x)-f(x)\right| = \left|f(c_j)-f(x)\right| \\
&\leqslant \left|f(c_j)-f(x_i+0)\right|+\left|f(x_i+0)-f(x)\right| \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}$$

since $0 < c_j - x_i < a_j - x_i < \delta_{x_i}$ and $0 < x - x_i < \delta_{x_i}$.

Thus, for arbitrary $\varepsilon > 0$, we have a function $\varphi \in S[a,b]$ such that

$$|\varphi(x)-f(x)| < \varepsilon \quad \text{for all } x \in [a,b].$$

By taking $\varepsilon = \frac{1}{n}$, $n = 1,2,\dots$, we get a sequence $\{\varphi_n\} \subset S[a,b]$ such that

$$|\varphi_n(x)-f(x)| < \frac{1}{n}, \quad x \in [a,b],\ n \in \mathbb{N}.$$

Clearly, $\varphi_n \rightrightarrows f$ in $[a,b]$.

2) Let $f \in D[a,b]$, and let $\{\varphi_n\} \subset S[a,b]$ be a sequence constructed in the previous item ($\varphi_n \rightrightarrows f$). The set T_n of discontinuity points of each φ_n is finite (since it consists of the ends of constancy intervals of φ_n). All the functions φ_n are continuous on the set $A = [a,b]\backslash(\bigcup_{n=1}^{\infty} T_n)$. Therefore, f is continuous on the set A.[1] This means that f may have discontinuities only at the points of the set $T = \bigcup_{n=1}^{\infty} T_n$. Since the set T is finite or countable (as a countable union of finite sets), the statement is proved.

3). For any $f \in D[a,b]$, take $\varphi \in S[a,b]$ such that

$$\left|f(x)-\varphi(x)\right| \leqslant 1, \quad x \in [a,b].$$

1 If all the functions in a uniformly converging sequence are continuous at some point, so is the limit function.

Denoting $M := \max_{x\in[a,b]} |\varphi(x)|$ ($\varphi \in S[a,b]$ can take only finitely many distinct values), we get

$$\left|f(x)\right| \leqslant \left|\varphi(x)\right| + 1 \leqslant M + 1, \quad x \in [a,b]. \qquad \square$$

REMARK 1.4 (for a lecturer).– If the latter proof seems to be too difficult for the first-year students, we may omit it (and the whole chapter as well) and later (in Chapter 3) define the integral for functions that are uniform limits of sequences of step functions (that is, in fact, for all functions from $D[a,b]$), without proving that such functions exhaust the whole class $D[a,b]$. Of course, in such a case, we need to prove that at least all continuous functions are integrable (proposition 1.1) and, as an easy consequence, so are the functions with finitely many first-kind discontinuities. Such functions are sufficient for developing practical integration skills.

P.1. Problems

PROBLEM 1.1.– Give an example of a function $f \in D[a,b]$ with an infinite (countable) set of discontinuity points.

PROBLEM 1.2.– May the set of discontinuity points of a function $f \in D[a,b]$ be the rationals of the interval $[a,b]$?

PROBLEM 1.3.– Does the Dini lemma hold in the class $D[a,b]$, that is, does the monotone convergence $D[a,b] \ni f_n \downarrow 0$ imply that $f_n \rightrightarrows 0$?

PROBLEM 1.4.– Show that if $f \in D[a,b]$ is right continuous (that is, $f(x+0) = f(x)$, $x \in [a,b)$), then the sequence$\{\varphi_n\}$ in theorem 1.1 can be chosen to be of right-continuous functions.