1

Modeling

We will call *modeling* the step that consists of finding a more or less accurate state representation of the system we are looking at. In general, constant parameters appear in the state equations (such as the mass or the inertial moment of a body, the coefficient of viscous friction, the capacitance of a capacitor, etc.). In these cases, an identification step may prove to be necessary. In this book, we will assume that all the parameters are known, otherwise we invite the reader to consult Eric Walter's book [WAL 14] for a broad range of identification methods. Of course, no systematic methodology exists that can be used to model a system. The goal of this chapter and of the following exercises is to present, using several varied examples, how to obtain a state representation.

1.1. Linear systems

In the continuous-time case, linear systems can be described by the following state equations:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

Linear systems are rather rare in nature. However, they are relatively easy to manipulate using linear algebra techniques and often approximate in an acceptable manner the nonlinear systems around their operating point.

1.2. Mechanical systems

The fundamental principle of dynamics allows us to easily find the state equations of mechanical systems (such as robots). The resulting calculations are relatively complicated for complex systems and the use of computer algebra systems may prove to be useful. In order to obtain the state equations of a mechanical system composed of several subsystems $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$, assumed to be rigid, we follow three steps:

1) *Obtaining the differential equations*. For each subsystem $\mathcal{S}_k$, with mass m and inertial matrix $\mathbf{J}$, the following relations must be applied:

$$\begin{array}{ll} \sum_i \mathbf{f}_i & = m\mathbf{a} \\ \sum_i \mathcal{M}_{\mathbf{f}_i} & = \mathbf{J}\dot{\omega} \end{array}$$

where the $\mathbf{f}_i$ are the forces acting on the subsystem $\mathcal{S}_k$, $\mathcal{M}_{\mathbf{f}_i}$ represents the torque created by the force $\mathbf{f}_i$ on $\mathcal{S}_k$, with respect to its center. The vector a represents the tangential acceleration of $\mathcal{S}_k$ and the vector $\dot{\omega}$ represents the angular acceleration of $\mathcal{S}_k$. After decomposing these $2m$ vectorial equations according to their components, we obtain $6m$ scalar differential equations such that some of them might be degenerate.

2) *Removing the components of the internal forces*. In differential equations there are the so-called *bonding* forces, which are internal to the whole mechanical system, even though they are external to each subsystem composing it. They represent the action of a subsystem $\mathcal{S}_k$ on another subsystem $\mathcal{S}_\ell$. Following the action–reaction principle, the

existence of such a force, denoted by $\mathbf{f}^{k,\ell}$, implies the existence of another force $\mathbf{f}^{\ell,k}$, representing the action of $\mathcal{S}_\ell$ on $\mathcal{S}_k$, such that $\mathbf{f}^{\ell,k} = -\mathbf{f}^{k,\ell}$. Through a formal manipulation of the differential equations and by taking into account the equations due to the action-reaction principle, it is possible to remove the internal forces. The resulting number of differential equations has to be reduced to the number n of degrees of freedom $q_1, \ldots, q_n$ of the system.

3) *Obtaining the state equations*. We then have to isolate the second derivative $\ddot{q}_1, \ldots, \ddot{q}_n$ from the set of n differential equations in such a manner to obtain a vectorial relation such as:

$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{u})$$

where $\mathbf{u}$ is the vector of external forces that are not derived from a potential (in other words, those which we apply to the system). The state equations are then written as:

$$\frac{d}{dt}\begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{q}} \\ \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{u}) \end{pmatrix}$$

A mechanical system whose dynamics can be described by the relation $\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{u})$ will be referred to as *holonomic*. For a holonomic system, $\mathbf{q}$ and $\dot{\mathbf{q}}$ are thus independent. If there is a so-called *non-holonomic* constraint that links the two of them (of the form $h(\mathbf{q}, \dot{\mathbf{q}}) = 0$), the system will be referred to as *non-holonomic*. Such systems may be found for instance in mobile robots with wheels [LAU 01]. Readers interested in more details on the modeling of mechanical systems may consult [KHA 07].

1.3. Servomotors

A mechanical system is controlled by forces or torques and obeys a dynamic model that depends on many poorly known coefficients. This same mechanical system represented by a kinematic model is controlled by positions, velocities or accelerations. The kinematic model depends on well-known geometric coefficients and is a lot easier to put into equations. In practice, we move from a dynamic model to its kinematic equivalent by adding servomotors. In summary, a servomotor is a direct current motor with an electrical control circuit and a sensor (of the position, velocity or acceleration). The control circuit computes the voltage u to give to the motor in order for the value measured by the sensor corresponds to the setpoint w. In practice, the signal w is generally given in the form of a square wave called *pulse-width modulation* (PWM)). There are three types of servomotors:

– the *position servo*. The sensor measures the position (or the angle) x of the motor and the control rule is expressed as $u = k(x - w)$. If k is large, we may conclude that $x \simeq w$;

– the *velocity servo*. The sensor measures the velocity (or the angular velocity) $\dot{x}$ of the motor and the control rule is expressed as $u = k(\dot{x} - w)$. If k is large, we have $\dot{x} \simeq w$;

– the *acceleration servo*. The sensor measures the acceleration (tangential or angular) $\ddot{x}$ of the motor and the control rule is expressed as $u = k(\ddot{x} - w)$. If k is large, we have $\ddot{x} \simeq w$.

1.4. Exercises

EXERCISE 1.1.– Integrator

The integrator is a linear system described by the differential equation $\dot{y} = u$. Find a state representation for this system. Give this representation in a matrix form.

EXERCISE 1.2.– Second order system

Let us consider the system with input u and output y described by the second order differential equation:

$$\ddot{y} + a_1\dot{y} + a_0 y = bu$$

Taking $\mathbf{x} = (y, \dot{y})$, find a state equation for this system. Give it in matrix form.

EXERCISE 1.3.– Mass-spring system

Let us consider a system with input u and output q_1 as shown in Figure 1.1 (u is the force applied to the second carriage, q_i is the deviation of the i^{th} carriage with respect to its point of equilibrium, k_i is the stiffness of the i^{th} spring and α is the coefficient of viscous friction).

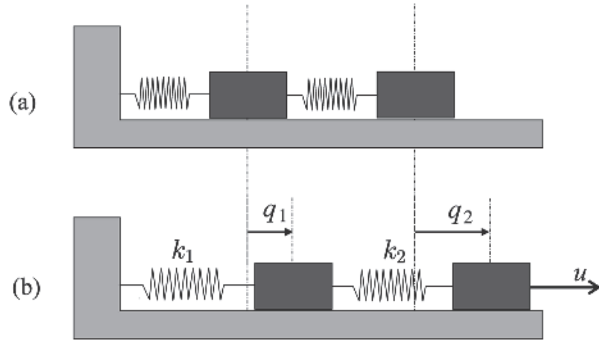

Figure 1.1. *a) Mass-spring system at rest, b) system in any state*

Let us take the state vector:

$$\mathbf{x} = (q_1, q_2, \dot{q}_1, \dot{q}_2)^{\mathrm{T}}$$

1) Find the state equations of the system.

2) Is this system linear?

EXERCISE 1.4.– Simple pendulum

Let us consider the pendulum in Figure 1.2. The input of this system is the momentum u exerted on the pendulum around its axis. The output is $y(t)$, the algebraic distance between the mass m and the vertical axis:

1) Determine the state equations of this system.

2) Express the mechanical energy E_m as a function of the state of the system. Show that the latter remains constant when the momentum u is nil.

EXERCISE 1.5.– Dynamic modeling of an inverted rod pendulum

Let us consider the so-called *inverted rod pendulum* system, which is composed of a pendulum of length ℓ placed in an unstable equilibrium on a carriage, as represented in Figure 1.3. The value u is the force exerted on the carriage of mass M, x indicates the position of the carriage, θ is the angle between the pendulum and the vertical axis and $\vec{R}$ is the force exerted by the carriage on the pendulum. At the extremity B of the pendulum, a point mass m is fixated. We may ignore the mass of the rod. Finally, A is the point of articulation between the rod and the carriage and $\vec{\Omega} = \dot{\theta}\vec{k}$ is the rotation vector associated with the rod.

1) Write the fundamental principle of dynamics as applied on the carriage and the pendulum.

2) Show that the velocity vector at point B is expressed by the relation $\mathbf{v}_B = \left(\dot{x} - \dot{\theta}\cos\theta\right)\vec{\imath} - \dot{\theta}\sin\theta\vec{\jmath}$. Calculate the acceleration $\dot{\mathbf{v}}_B$ of point B.

3) In order to model the inverted pendulum, we will take the state vector $\mathbf{x} = \left(x, \theta, \dot{x}, \dot{\theta}\right)$. Justify this choice.

4) Find the state equations for the inverted rod pendulum.

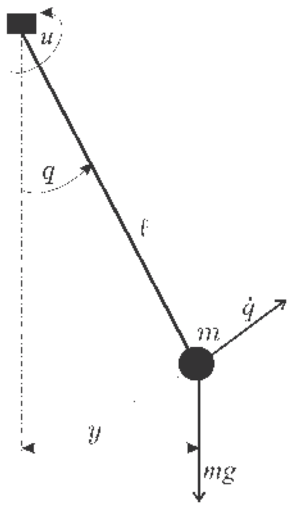

Figure 1.2. *Simple pendulum with state vector* $\mathrm{x} = (q, \dot{q})$

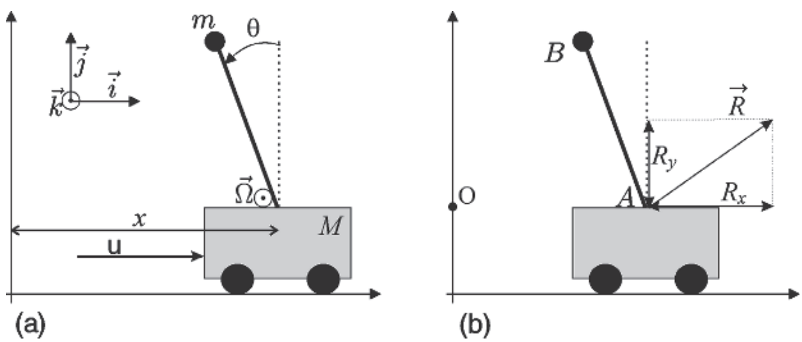

Figure 1.3. *Inverted rod pendulum*

EXERCISE 1.6.– Kinematic modeling of an inverted rod pendulum

In a kinematic model, the inputs are no longer forces or moments, but kinematic variables, in other words positions, velocities or accelerations. It is the role of servomotors to translate these kinematic variables into forces or moments.

Let us take the state equations of the inverted rod pendulum established in the previous exercise:

$$\frac{d}{dt}\begin{pmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{\theta} \\ \frac{-m\sin\theta(\ell\dot{\theta}^2 - g\cos\theta)}{M+m\sin^2\theta} \\ \frac{\sin\theta((M+m)g - m\ell\dot{\theta}^2\cos\theta)}{\ell(M+m\sin^2\theta)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{M+m\sin^2\theta} \\ \frac{\cos\theta}{\ell(M+m\sin^2\theta)} \end{pmatrix} u$$

1) Instead of taking the force u on the carriage as input, let us rather take the acceleration $a = \ddot{x}$. What does the state model become?

2) Show how, by using a proportional control such as $u = K(a - \ddot{x})$ with large K, it is possible to move from a dynamic model to a kinematic model. In what way does this control recall the servomotor principle or the operational amplifier principle?

EXERCISE 1.7.– Segway

The segway represented on the left side of Figure 1.4 is a vehicle with two wheels and a single axle. It is stable since it is controlled. In the modeling step, we will of course assume that the engine is not controlled.

Its open loop behavior is very close to that of the planar unicycle represented in Figure 1.4 on the right hand side. In this figure, u represents the exerted momentum between the body and the wheel.

The link between these two elements is a pivoting pin. We will denote by B the center of gravity of the body and by A that of the wheel. C is a fix point on the disk. Let us denote by α the angle between the vector $\overrightarrow{AC}$ and the horizontal axis and by θ the angle between the body of the unicycle and the vertical axis. This system has two degrees of freedom α and θ. The state of out system is given by the vector $\mathbf{x} = (\alpha, \theta, \dot{\alpha}, \dot{\theta})^{\mathrm{T}}$.

The parameters of our system are:

– for the disk: its mass M, its radius a, its moment of inertia J_M;

– for the pendulum: its mass m, its moment of inertia J_p, the distance ℓ between its center of gravity and the center of the disk.

Find the equations of the systems.

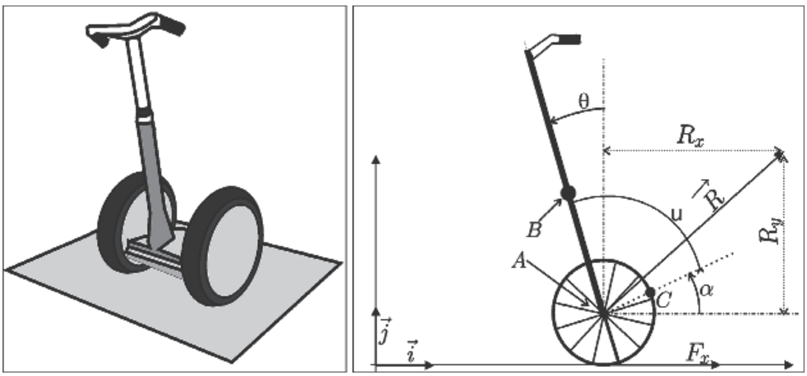

Figure 1.4. *The segway has two wheels and one axle*

EXERCISE 1.8.– Hamilton's method

Hamilton's method allows us to obtain the state equations of a conservative mechanical system (in other words, whose energy is conserved) only from the expression of a single function: its energy. For this, we define the *Hamiltonian* as the mechanical energy of the system, in other words the sum of the potential energy and the kinetic energy. The Hamiltonian can be expressed as a function $H(\mathbf{q}, \mathbf{p})$ of the degrees of freedom $\mathbf{q}$ and of the associated amount of movement (or kinetic moments in the case of a rotation) $\mathbf{p}$. The Hamilton equations are written as:

$$\begin{cases} \dot{\mathbf{q}} = \frac{\partial H(\mathbf{q},\mathbf{p})}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} = -\frac{\partial H(\mathbf{q},\mathbf{p})}{\partial \mathbf{q}} \end{cases}$$

1) Let us consider the simple pendulum shown in Figure 1.6. This pendulum has a length of ℓ and is composed of a single point mass m. Calculate the Hamiltonian of the system. Deduce the state equations from this.

2) Show that if a system is described by Hamilton equations, then the Hamiltonian is constant.

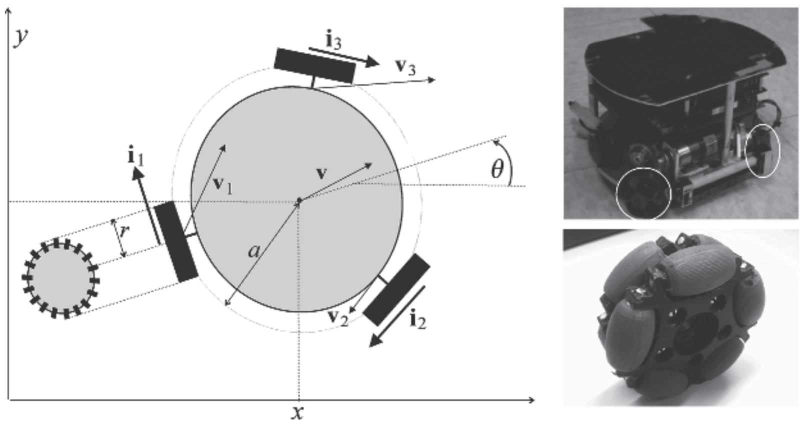

Figure 1.5. *Holonomic robot with omni wheels*

EXERCISE 1.9.– Omnidirectional robot

Let us consider the robot with three omni wheels, as shown in Figure 1.5. An omni wheel is a wheel equipped with a multitude of small rollers over its entire periphery that allow it to slide sideways (in other words perpendicularly to its nominal movement direction). Let us denote by $\mathbf{v}_i$ the velocity vector of the contact point of the i^{th} wheel. If $\mathbf{i}_i$ is the normed direction vector indicating the nominal movement direction of the wheel, then the component of $\mathbf{v}_i$ according to $\mathbf{i}_i$ corresponds to the rotation ω_i of the wheel whereas its complementary component (perpendicular to $\mathbf{i}_i$) is linked to the rotation of the peripheral rollers. If r is the radius of the wheel, then we have the relation $r\omega_i = \langle \mathbf{v}_i, \mathbf{i}_i \rangle = \|\mathbf{v}_i\| . \|\mathbf{i}_i\| . \cos\alpha_i$, where $\alpha_i = \cos(\widehat{\mathbf{v}_i, \mathbf{i}_i})$. If $\cos\alpha_i = \pm 1$, the wheel is in its nominal state, i.e. it behaves like a classical wheel. If $\cos \alpha_i = 0$, the wheel no longer turns and it is in a state of skid.

1) Give the state equations of the system. We will use the state vector $\mathbf{x} = (x, y, \theta)$ and the input vector $\omega = (\omega_1, \omega_2, \omega_3)$.

2) Propose a loop that allows us to obtain a model tank described by the following state equations:

$$\begin{cases} \dot{x} = v \cos \theta \\ \dot{y} = v \sin \theta \\ \dot{\theta} = u_1 \\ \dot{v} = u_2 \end{cases}$$

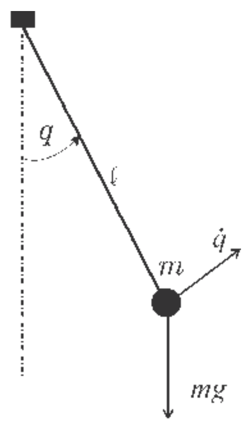

Figure 1.6. *Simple pendulum*

EXERCISE 1.10.– Modeling a tank

The robot tank in Figure 1.7 is composed of two parallel motorized crawlers (or wheels) whose accelerations (which form the inputs u_1 and u_2 of the system) are controlled by two independent motors. In the case where wheels are considered, the stability of the system is ensured by one or two idlers, not represented on the figure. The degrees of freedom of the robot are the x, y coordinates of the center of the axle and its orientation θ.

1) Why can't we choose as state vector the vector $\mathbf{x} = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})^{\mathrm{T}}$?

2) Let us denote by v_1 and v_2 the center velocities of each of the motorized wheels. Let us choose as state vector the vector $\mathbf{x} = (x, y, \theta, v_1, v_2)^{\mathrm{T}}$. What might justify such a choice? Give the state equations of the system.

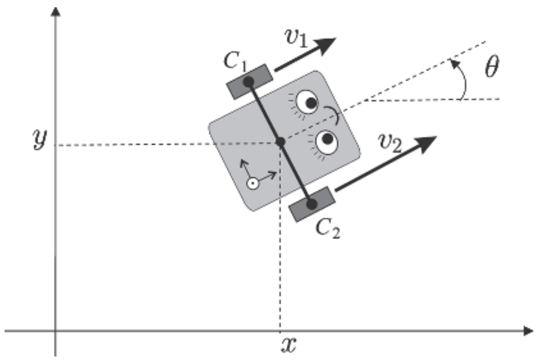

Figure 1.7. *Robot tank viewed from above*

EXERCISE 1.11.– Modeling a car

Let us consider the car as shown in Figure 1.8. The driver of the car (on the left hand side on the figure) has two controls: acceleration of the front wheels (assumed to be motorized) and rotation velocity of the steering wheel. The brakes here represent a negative acceleration. We will denote by δ the angle between the front wheels and the axis of the car, by θ the angle made by the car with respect to the horizontal axis and by (x, y) the coordinates of the middle of the rear axle. The state variables of our system are composed of:

– the position coordinates, in other words all the knowledge necessary to draw the car, more specifically the x, y

coordinates of the center of the rear axle, the orientation θ of the car, and the angle δ of the front wheels;

– the kinetic coordinate v representing the velocity of the center of the front axle (indeed, the sole knowledge of this value and the position coordinates allows to calculate all the velocities of all the other elements of the car).

Calculate the state equations of the system. We will assume that the two wheels have the same velocity v (even though in reality, the inner wheel during a turn is slower than the outer one). Thus, as illustrated on the right-hand side figure, everything happens as if there were only two virtual wheels situated at the middle of the axles.

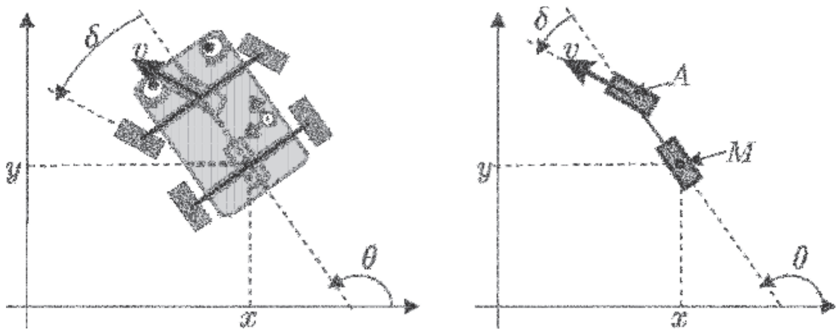

Figure 1.8. *Car moving on a plane (view from above)*

EXERCISE 1.12.– Car-trailer system

Let us consider the car studied in Exercise 1.11. Let us add a trailer to this car whose attachment point is found in the middle of the rear axle of the car, as illustrated by Figure 1.9. Find the state equations of the car-trailer system.

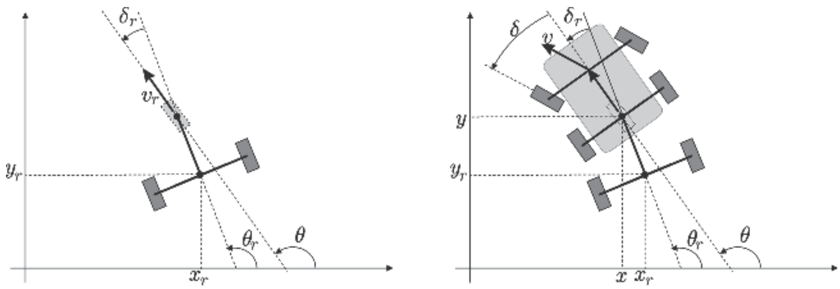

Figure 1.9. *Car with a trailer*

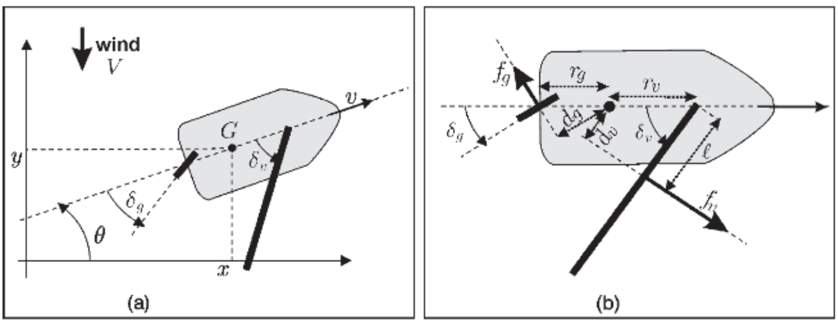

Figure 1.10. *Sailboat to be modeled*

EXERCISE 1.13.– Sailboat

Let us consider the sailboat to be modeled represented in Figure 1.10. The state vector $\mathbf{x} = (x, y, \theta, \delta_v, \delta_g, v, \omega)^{\mathrm{T}}$, of dimension 7, is composed of the coordinates x, y of the center of gravity G of the boat (the drift is found at G), of the orientation θ, the angle δ_v of the sail, the angle δ_g of the rudder, the velocity v of the center of gravity G and of the angular velocity ω of the boat. The inputs u_1 and u_2 of the system are the derivatives of the angles δ_v and δ_g. The parameters (assumed to be known and constant) are: V the velocity of the wind, r_g the distance of the rudder to G, r_v the distance of the mast to G, α_g the lift of the rudder (if the rudder is perpendicular to the vessel's navigation, the water exerts a force of $\alpha_g v$ on the rudder), α_v the lift of the sail (if the sail is stationary and perpendicular to the wind, the latter exerts a force of $\alpha_v V$), α_f the coefficient of friction of the boat on the water in the direction of navigation (the water exerts a force opposite to the direction of navigation on the boat equal to $\alpha_f v^2$), α_θ the angular coefficient of friction (the water exerts a momentum of friction on the boat equal to $-\alpha_\theta \omega$); given the form of the boat that is rather streamlined in order to maintain its course, α_θ will be large compared to α_f), J the inertial momentum of the boat, ℓ the distance between the center of pressure of the sail and the mast, β the coefficient of drift (when the sail is released, the boat tends to drift in the direction of the wind with a velocity equal to βV). The state vector is composed of the coordinates of position, i.e. the coordinates x, y of the inertial center of the boat, the orientation θ, and the angles δ_v and δ_g of the sail and the rudder and the kinetic coordinates v and ω representing, respectively, the velocity of the center of rotation G and the angular velocity of the vessel. Find the state equations for our system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, where $\mathbf{x} = (x, y, \theta, \delta_v, \delta_g, v, \omega)^{\mathrm{T}}$ and $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$.

EXERCISE 1.14.– Direct current motor

A direct current motor can be described by Figure 1.11, in which u is the supply voltage of the motor, i is the current absorbed by the motor, R is the armature resistance, L is the armature inductance, e is the electromotive force, ρ is the coefficient of friction in the motor, ω is the angular velocity of the motor and T_r is the torque exerted by the motor on the load.

Let us recall the equations of an ideal direct current motor: $e = K\Phi\omega$ and $T = K\Phi i$. In the case of an induction-independent motor, or a motor with permanent magnets, the flow Φ is constant. We are going to put ourselves in this situation.

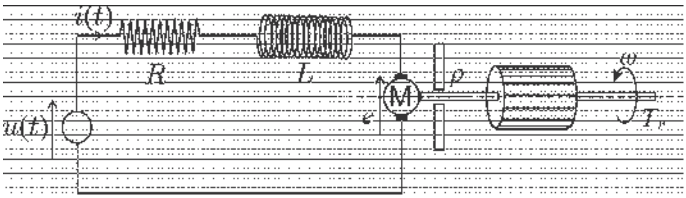

Figure 1.11. *Direct current motor*

1) We take as inputs of the system T_r and u. Find the state equations.

2) We connect a ventilator to the output of the system with a characteristic of $T_r = \alpha\omega^2$. Give the new state equations of the motor.

EXERCISE 1.15.– Electrical circuit

The electrical circuit of Figure 1.12 has the input as voltage $u(t)$ and the output as voltage $y(t)$. Find the state equations of the system. Is this a linear system?

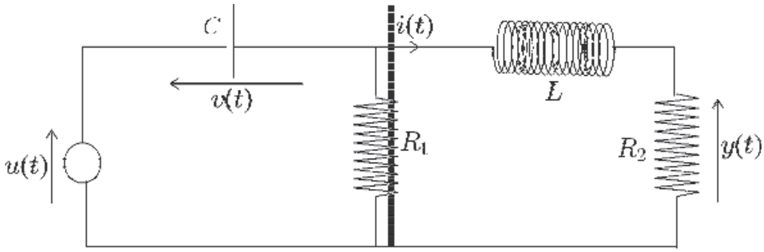

Figure 1.12. *Electrical circuit to be modeled*

EXERCISE 1.16.– The three containers

1) Let us consider two containers placed as shown in Figure 1.13.

In the left container, the water flows without friction in the direction of the right container. In the left container, the water flows in a fluid way, as opposed to the right container, where there are turbulences. These are turbulences that absorb the kinetic energy of the water and transform it into heat. Without these turbulences, we would have a perpetual back-and-forth movement of the water between the two containers. If a is the cross-section of the canal, then it shows the so-called *Torricelli*'s law that states that the water flow from the right container to the left one is equal to:

$$Q_D = a.sign\left(z_A - z_B\right)\sqrt{2g|z_A - z_B|}$$

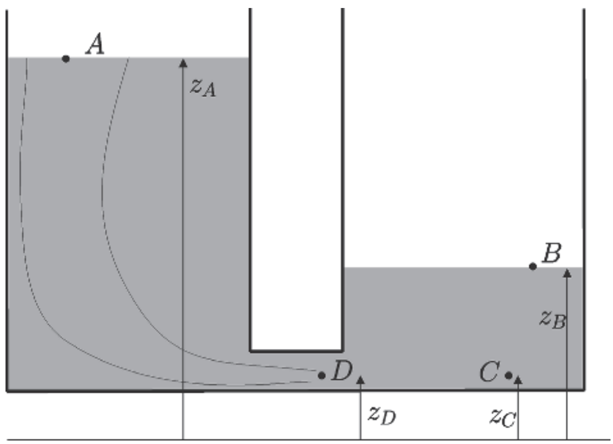

Figure 1.13. *Hydraulic system composed of two containers filled with water and connected with a canal*

2) Let us now consider the system composed of three containers as represented in Figure 1.14.

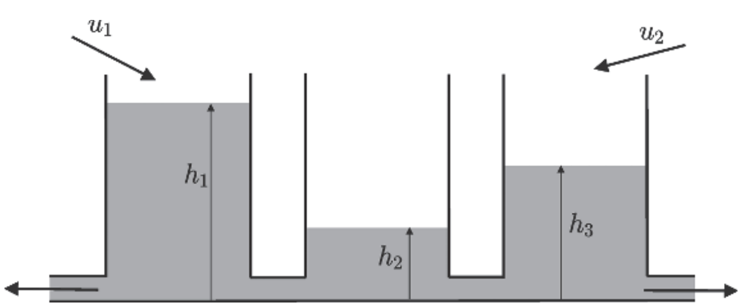

Figure 1.14. *System composed of three containers filled with water and connected with two canals*

The water from containers 1 and 3 can flow toward container 2, but also toward the outside with atmospheric pressure. The associated flow rates are given, following Toricelli's relation, by:

$$Q_{1\text{ext}} = a.\sqrt{2gh_1}$$
$$Q_{3\text{ext}} = a.\sqrt{2gh_3}$$

Similarly, the flow rate from a container i toward a container j is given by:

$$Q_{ij} = a.sign\left(h_i - h_j\right)\sqrt{2g|h_i - h_j|}$$

The state variables of this system that may be considered are the heights of the containers. In order to simplify, we will assume that the surfaces of the containers are all equal to $1\ \text{m}^2$; thus, the volume of water in a container is interlinked with its height. Find the state equations describing the dynamics of the system.

EXERCISE 1.17.– Pneumatic cylinder

Let us consider the pneumatic cylinder with return spring as shown in Figure 1.15. Such a cylinder is often referred to as single-acting since the air under pressure is only found in one of the two chambers.

The parameters of this system are the stiffness of the spring k, the surface of the piston a and the mass m at the end of the piston (the masses of all the other objects are ignored). We assume that everything happens under a constant temperature T_0. We will take as state vector $\mathbf{x} = (z, \dot{z}, p)$ where z is the position of the cylinder, $\dot{z}$ its velocity and p the pressure inside the chamber. The input of the system is the volumetric flow rate u of the air toward the chamber of the cylinder. In order to simplify, we will assume that there is vacuum in the spring chamber and that when

$z = 0$ (the cylinder is in the left hand limit) the spring is in equilibrium. Find the state equations of the pneumatic cylinder.

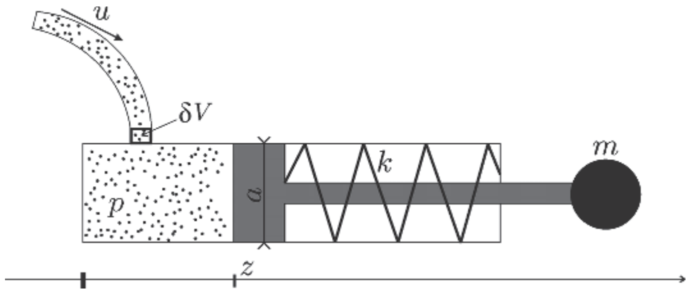

Figure 1.15. *Single-acting pneumatic cylinder*

EXERCISE 1.18.– Fibonacci sequence

We will now study the evolution of the number $y(k)$ of rabbit couples on a farm as a function of the year k. At year 0, there is only a single couple of newborn rabbits on the farm (and thus $y(0) = 1$). The rabbits only become fertile a year after their birth. It follows that at year 1, there is still a single couple of rabbits, but this couple is fertile (and thus $y(1) = 1$). A fertile couple gives birth, each year, to another couple of rabbits. Thus, at year 2, there is a fertile couple of rabbits and a newborn couple. This evolution can be described in Table 1.1, where N means *newborn* and A means *adult*.

$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
N	A	A	A	A
		N	A	A
			N	A
				N
				N

Table 1.1. *Evolution of the number of rabbits*

Let us denote by $x_1(k)$ the number of newborn couples, by $x_2(k)$ the number of fertile couples and by $y(k)$ the total number of couples.

1) Give the state equations that govern the system.

2) Give the recurrence relation satisfied by $y(k)$.

EXERCISE 1.19.– Bus network

We consider a public transport system of buses with 4 lines and 4 buses. There are only two stations where travelers can change lines. This system can be represented by a Petri net (see Figure 1.16). Each token corresponds to a bus. The places p_1, p_2, p_3, p_4 represent the lines. These places are composed of a number that corresponds to the minimum amount of time that the token must remain in its place (this corresponds to the transit time). The transitions t_1, t_2 ensure synchronization. They are only crossed when each upstream place of the transition has at least one token that has waited sufficiently long. In this case, the upstream places lose a token and the downstream places gain one. This structure ensures that the correspondence will be performed systematically and that the buses will leave in pairs.

1) Let us assume that at time $t = 0$, the transitions t_1 and t_2 are crossed for the first time and that we are in the configuration of Figure 1.16 (this corresponds to the initialization). Give the crossing times for each of the transitions.

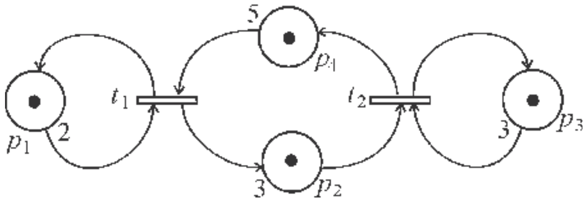

Figure 1.16. *Petri net of the bus system*

2) Let us denote by $x_i(k)$ the time when the transition t_i is crossed for the kth time. Show that the dynamics of the model can be written using states:

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$$

where $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$ is the state vector. Remember that here k is not the time, but an event number.

3) Let us now attempt to reformulate elementary algebra by redefining the addition and multiplication operators (see [BAC 92]) as follows:

$$\begin{cases} a \oplus b = \max(a, b) \\ a \otimes b = a + b \end{cases}$$

Thus, $2 \oplus 3 = 3$, whereas $2 \otimes 3 = 5$. Show that in this new algebra (called max-plus), the previous system is linear.

4) Why do you think that matrix calculus (such as we know it) could not be used easily in this new algebra?

1.5. Solutions

Solution to Exercise 1.1 (integrator)

One possible state representation for the integrator is the following:

$$\begin{cases} \dot{x}(t) = u(t) \\ y(t) = x(t) \end{cases}$$

The matrices associated with this system are $\mathbf{A} = (0)$, $\mathbf{B} = (1)$, $\mathbf{C} = (1)$ and $\mathbf{D} = (0)$. They all have dimension 1×1.

Solution to Exercise 1.2 (second-order system)

By taking $\mathbf{x} = (y, \dot{y})$, this differential equation can be written in the form:

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -a_1 x_2 - a_0 x_1 + bu \end{pmatrix} \\ \quad y \quad = \quad x_1 \end{cases}$$

or, in a standard form using state matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$:

$$\begin{cases} \dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ b \end{pmatrix} u \\ y = \begin{pmatrix} 1 \, 0 \end{pmatrix} \mathbf{x} \end{cases}$$

Solution to Exercise 1.3 (mass-spring system)

1) The fundamental principle of dynamics applied to carriage 1 then to carriage 2 gives:

$$\begin{cases} -k_1 q_1 - \alpha \dot{q}_1 + k_2 (q_2 - q_1) = m_1 \ddot{q}_1 \\ u - \alpha \dot{q}_2 - k_2 (q_2 - q_1) \qquad = m_2 \ddot{q}_2 \end{cases}$$

In other words:

$$\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{m_1} \left(- \left(k_1 + k_2 \right) q_1 + k_2 q_2 - \alpha \dot{q}_1 \right) \right) \\ \frac{1}{m_2} \left(k_2 q_1 - k_2 q_2 - \alpha \dot{q}_2 + u \right) \end{pmatrix}$$

We thus have the following state representation:

$$\begin{cases} \dot{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{\alpha}{m_1} & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & -\frac{\alpha}{m_2} \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{pmatrix} u \\ q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \mathbf{x} \end{cases}$$

2) Yes, this system is linear since it can be written as:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

Solution to Exercise 1.4 (simple pendulum)

1) Following the fundamental principle of dynamics, we have:

$$-\ell m g \sin q + u = J\ddot{q}$$

where ℓ is the length of the pendulum. However, for our example, $J = m\ell^2$, therefore:

$$\ddot{q} = \frac{u - \ell m g \sin q}{m\ell^2}$$

Let us take as state vector $\mathbf{x} = (q, \dot{q})$. The state equations of the system are then written as:

$$\begin{array}{ccc} \frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} & = & \begin{pmatrix} \dot{q} \\ \frac{u-\ell m g \sin q}{m\ell^2} \end{pmatrix} \\ y & = & \ell \sin q \end{array}$$

or:

$$\begin{array}{ccc} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} & = & \begin{pmatrix} x_2 \\ \frac{u-\ell m g \sin x_1}{m\ell^2} \end{pmatrix} \\ y & = & \ell \sin x_1 \end{array}$$

2) The mechanical energy of the pendulum is given by:

$$E_m = \underbrace{\frac{1}{2} m\ell^2 \dot{q}^2}_{\text{kinetic energy}} + \underbrace{mg\ell\,(1 - \cos q)}_{\text{potential energy}}$$

When the torque u is nil, we have:

$$\begin{aligned} \frac{dE_m}{dt} &= \frac{1}{2} m\ell^2 \left(2\dot{q}\ddot{q}\right) + mg\ell\dot{q}\sin q \\ &= m\ell^2 \left(\dot{q}\frac{-\ell mg \sin q}{m\ell^2}\right) + mg\ell\dot{q}\sin q = 0 \end{aligned}$$

The mechanical energy of the pendulum therefore remains constant, which is coherent with the fact that the pendulum without friction is a conservative system.

Solution to Exercise 1.5 (dynamic modeling of an inverted rod pendulum)

1) The fundamental principle of dynamics applied to the carriage and the pendulum gives us:

$$\begin{array}{rll} (u - R_x)\,\vec{i} & = M\ddot{x}\vec{i} & \text{(carriage in translation)} \\ R_x\vec{i} + R_y\vec{j} - mg\vec{j} & = m\dot{\mathbf{v}}_B & \text{(pendulum in translation)} \\ R_x\ell\cos\theta + R_y\ell\sin\theta & = \ 0\ddot{\theta} & \text{(pendulum in rotation)} \end{array}$$

where $\mathbf{v}_B$ is the velocity vector of point B. For the third equation, the inertial momentum of the pendulum was defined nil.

2) Since:

$$\overrightarrow{OB} = (x - \ell\sin\theta)\,\vec{i} + \ell\cos\theta\vec{j}$$

we have:

$$\mathbf{v}_B = \left(\dot{x} - \ell\dot{\theta}\cos\theta\right)\vec{i} - \ell\dot{\theta}\sin\theta\vec{j}$$

Therefore, the acceleration of point B is given by:

$$\dot{\mathbf{v}}_B = \left(\ddot{x} - \ell\ddot{\theta}\cos\theta + \ell\dot{\theta}^2\sin\theta\right)\vec{i} - \left(\ell\ddot{\theta}\sin\theta + \ell\dot{\theta}^2\cos\theta\right)\vec{j}$$

3) It is the vector of the degrees of freedom and of their derivatives. There are no non-holonomic constraints.

4) After scalar decomposition of the dynamics equations given above, we obtain:

$$\begin{cases} M\ddot{x} & = & u - R_x & \text{(i)} \\ R_x & = m\left(\ddot{x} - \ell\ddot{\theta}\cos\theta + \ell\dot{\theta}^2\sin\theta\right) & & \text{(ii)} \\ R_y - mg & = & -m\left(\ell\ddot{\theta}\sin\theta + \ell\dot{\theta}^2\cos\theta\right) & \text{(iii)} \\ R_x\cos\theta + R_y\sin\theta = & & 0 & \text{(iv)} \end{cases}$$

These four equations describe, respectively, (i) the carriage in translation, (ii) the pendulum in translation following $\vec{i}$, (iii) the pendulum in translation following $\vec{j}$ and (iv) the pendulum in rotation. We thus verify that the number of degrees of freedom (here x and θ) added to the number of internal forces (here R_x and R_y) is equal to the number of equations. In a matrix form, these equations become:

$$\begin{pmatrix} M & 0 & 1 & 0 \\ -m & m\ell\cos\theta & 1 & 0 \\ 0 & m\ell\sin\theta & 0 & 1 \\ 0 & 0 & \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{\theta} \\ R_x \\ R_y \end{pmatrix} = \begin{pmatrix} u \\ m\ell\dot{\theta}^2\sin\theta \\ mg - m\ell\dot{\theta}^2\cos\theta \\ 0 \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} M & 0 & 1 & 0 \\ -m & m\ell\cos\theta & 1 & 0 \\ 0 & m\ell\sin\theta & 0 & 1 \\ 0 & 0 & \cos\theta & \sin\theta \end{pmatrix}^{-1} \cdot \begin{pmatrix} u \\ m\ell\dot{\theta}^2\sin\theta \\ mg - m\ell\dot{\theta}^2\cos\theta \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-m\sin\theta(\ell\dot{\theta}^2 - g\cos\theta) + u}{M + m\sin^2\theta} \\ \frac{\sin\theta((M+m)g - m\ell\dot{\theta}^2\cos\theta) + \cos\theta u}{\ell(M + m\sin^2\theta)} \end{pmatrix}$$

The state equations are therefore written as:

$$\frac{d}{dt}\begin{pmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{\theta} \\ \frac{-m\sin\theta(\ell\dot{\theta}^2 - g\cos\theta)}{M + m\sin^2\theta} \\ \frac{(\sin\theta)((M+m)g - m\ell\dot{\theta}^2\cos\theta)}{\ell(M + m\sin^2\theta)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{M + m\sin^2\theta} \\ \frac{\cos\theta}{\ell(M + m\sin^2\theta)} \end{pmatrix} u$$

or equivalently:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ \frac{-m \sin x_2 (\ell x_4^2 - g \cos x_2)}{M + m \sin^2 x_2} \\ \frac{(\sin x_2)((M+m)g - m\ell x_4^2 \cos x_2)}{\ell(M + m \sin^2 x_2)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{M + m \sin^2 x_2} \\ \frac{\cos x_2}{\ell(M + m \sin^2 x_2)} \end{pmatrix} u$$

Solution to Exercise 1.6 (kinematic modeling of an inverted rod pendulum)

1) Following the dynamic model of the inverted rod pendulum, we obtain:

$$a = \frac{1}{M + m \sin^2 \theta} \left(-m \sin \theta (\ell \dot{\theta}^2 - g \cos \theta) + u \right)$$

After isolating u, we get:

$$u = m \sin \theta \left(\ell \dot{\theta}^2 - g \cos \theta \right)) + \left(M + m \sin^2 \theta \right) a$$

Therefore:

$$\begin{aligned} \ddot{\theta} &= \frac{\sin \theta ((M+m)g - m\ell\dot{\theta}^2 \cos \theta)}{\ell(M+m \sin^2 \theta)} + \frac{\cos \theta}{\ell(M+m \sin^2 \theta)} u \\ &= \frac{\sin \theta ((M+m)g - m\ell\dot{\theta}^2 \cos \theta)}{\ell(M+m \sin^2 \theta)} + \frac{\cos \theta}{\ell(M+m \sin^2 \theta)} \left(m \sin \theta \left(\ell\dot{\theta}^2 - g \cos \theta \right) + \left(M + m \sin^2 \theta \right) a \right) \\ &= \frac{1}{\ell(M+m \sin^2 \theta)} \left((M+m) g \sin \theta - gm \sin \theta \cos^2 \theta + \left(M + m \sin^2 \theta \right) \cos \theta a \right) \\ &= \frac{g \sin \theta}{\ell} + \frac{\cos \theta}{\ell} a \end{aligned}$$

Let us note that this relation could have been obtained directly by noticing that:

$$\ell \ddot{\theta} = \underbrace{a . \cos \theta}_{\text{acceleration of } A \text{ that contributes to the rotation}} + \underbrace{g . \sin \theta}_{\text{acceleration of } B \text{ from the viewpoint of } A}$$

REMARK.– In order to obtain this relation in a more rigorous manner and without the use of the dynamic model, we would need to write the temporal derivative of the formula of the composition of the velocities (or *Varignon*'s formula), in other words:

$$\dot{\mathbf{v}}_A = \dot{\mathbf{v}}_B + \overrightarrow{AB} \wedge \overrightarrow{\dot{\omega}}$$

and write this formula in the frame of the pendulum. We obtain:

$$\begin{pmatrix} a\cos\theta \\ -a\sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -g\sin\theta \\ n \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \ell \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ \dot{\omega} \end{pmatrix}$$

where n corresponds to the normal acceleration of the mass m. We thus obtain, in addition to the desired relation, the normal acceleration $n = -a\sin\theta$ that will not be used.

The model therefore becomes:

$$\frac{d}{dt}\begin{pmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{\theta} \\ 0 \\ \frac{g\sin\theta}{\ell} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{\cos\theta}{\ell} \end{pmatrix} a$$

This model, referred to as *kinetic*, only involves positions, velocities and accelerations. It is much more simple than the dynamic model and contains less coefficients. On the other hand, it corresponds less to reality since the real input is a force and not an acceleration.

2) In the case of the inverted rod pendulum, we can move from the dynamic model with input u to a kinetic model with input a by generating u with a *proportional control* of type *high gain* of the form:

$$u = K(a - \ddot{x})$$

with K very large and where a is a new input (see Figure 1.17).

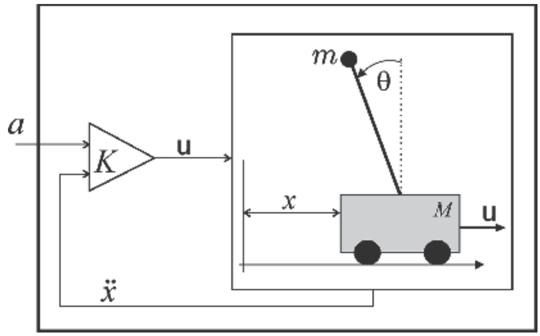

Figure 1.17. *The inverted rod pendulum, looped by a high gain K, behaves like a kinematic model*

The acceleration $\ddot{x}$ can be measured by an accelerometer. If K is sufficiently large, we will have a control u that will create the desired acceleration a, in other words we will be able to say that $\ddot{x} = a$. Thus, the system could be described by state equations of the kinematic model that do not involve any of the inertial parameters of the system. A controller that will be designed over the kinematic model will therefore be more robust than that designed over the dynamic model since the controller will work no matter what the masses (m, M), the inertial moments, the frictions, etc. are. This high gain-type control is very close to the op-amp principle. In addition to being more robust, such an approach also allows to have a simpler model that is easier to obtain.

Solution to Exercise 1.7 (segway)

In order to find the state equations, we apply the fundamental principle of dynamics on each subsystem, more precisely the wheel and the body. We have:

$$\left\{\begin{array}{rl} -R_x + F_x = -Ma\ddot{\alpha} & \text{(wheel in translation)} \\ F_x a + u = J_M \ddot{\alpha} & \text{(wheel in rotation)} \\ R_x \vec{i} + R_y \vec{j} - mg\vec{j} = m\dot{\mathbf{v}}_B & \text{(body in translation)} \\ R_x \ell \cos\theta + R_y \ell \sin\theta - u = J_p \ddot{\theta} & \text{(body in rotation)} \end{array}\right.$$

where $\mathbf{v}_B$ is the velocity vector of point B. Since:

$$\overrightarrow{OB} = (-a\alpha - \ell\sin\theta)\,\vec{i} + (\ell\cos\theta + a)\,\vec{j}$$

using differentiation, we obtain:

$$\mathbf{v}_B = \left(-a\dot{\alpha} - \ell\dot{\theta}\cos\theta\right)\vec{i} - \ell\dot{\theta}\sin\theta\vec{j}$$

or:

$$\dot{\mathbf{v}}_B = \left(-a\ddot{\alpha} - \ell\ddot{\theta}\cos\theta + \ell\dot{\theta}^2\sin\theta\right)\vec{i} - \left(\ell\ddot{\theta}\sin\theta + \ell\dot{\theta}^2\cos\theta\right)\vec{j}$$

Thus, after scalar decomposition, the dynamics equations become:

$$\begin{cases} -R_x + F_x = -Ma\ddot{\alpha} \\ F_x a + u = J_M\ddot{\alpha} \\ R_x = m(-a\ddot{\alpha} - \ell\ddot{\theta}\cos\theta + \ell\dot{\theta}^2\sin\theta) \\ R_y - mg = -m(\ell\ddot{\theta}\sin\theta + \ell\dot{\theta}^2\cos\theta) \\ R_x\ell\cos\theta + R_y\ell\sin\theta - u = J_p\ddot{\theta} \end{cases}$$

We verify that the number of degrees of freedom (here α and θ) added to the number of components of the internal forces (here R_x, R_y and F_x) is equal to the number of equations. In matrix form, these equations become:

$$\begin{pmatrix} Ma & 0 & -1 & 0 & 1 \\ J_M & 0 & 0 & 0 & -a \\ ma & m\ell\cos\theta & 1 & 0 & 0 \\ 0 & m\ell\sin\theta & 0 & 1 & 0 \\ 0 & -J_p & \ell\cos\theta & \ell\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \ddot{\alpha} \\ \ddot{\theta} \\ R_x \\ R_y \\ F_x \end{pmatrix} = \begin{pmatrix} 0 \\ u \\ m\ell\dot{\theta}^2\sin\theta \\ mg - m\ell\dot{\theta}^2\cos\theta \\ u \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} \ddot{\alpha} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 1\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0 \end{pmatrix} \cdot \begin{pmatrix} Ma & 0 & -1 & 0 & 1 \\ J_M & 0 & 0 & 0 & -a \\ ma & m\ell\cos\theta & 1 & 0 & 0 \\ 0 & m\ell\sin\theta & 0 & 1 & 0 \\ 0 & -J_p & \ell\cos\theta & \ell\sin\theta & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ u \\ m\ell\dot{\theta}^2\sin\theta \\ mg - m\ell\dot{\theta}^2\cos\theta \\ u \end{pmatrix}$$

In other words:

$$\begin{cases} \ddot{\alpha} = \frac{\mu_3\left(\mu_2\dot{\theta}^2-\mu_g\cos\theta\right)\sin\theta+(\mu_2+\mu_3\cos\theta)u}{\mu_1\mu_2-\mu_3^2\cos^2\theta} \\ \ddot{\theta} = \frac{\left(\mu_1\mu_g-\mu_3^2\dot{\theta}^2\cos\theta\right)\sin\theta-(\mu_1+\mu_3\cos\theta)u}{\mu_1\mu_2-\mu_3^2\cos^2\theta} \end{cases}$$

with:

$$\begin{array}{cc} \mu_1 = J_M + a^2\,(m+M)\,, & \mu_2 = J_p + m\ell^2, \\ \mu_3 = am\ell, & \mu_g = g\ell m \end{array}$$

The state equations are therefore written as:

$$\frac{d}{dt}\begin{pmatrix} \alpha \\ \theta \\ \dot{\alpha} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{\alpha} \\ \dot{\theta} \\ \frac{\mu_3\left(\mu_2\dot{\theta}^2-\mu_g\cos\theta\right)\sin\theta+(\mu_2+\mu_3\cos\theta)u}{\mu_1\mu_2-\mu_3^2\cos^2\theta} \\ \frac{\left(\mu_1\mu_g-\mu_3^2\dot{\theta}^2\cos\theta\right)\sin\theta-(\mu_1+\mu_3\cos\theta)u}{\mu_1\mu_2-\mu_3^2\cos^2\theta} \end{pmatrix}$$

By taking $\mathbf{x} = \left(\alpha, \theta, \dot{\alpha}, \dot{\theta}\right)^{\mathrm{T}}$, these equations become:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ \frac{\mu_3\left(\mu_2^2x_4-\mu_g\cos x_2\right)\sin x_2+(\mu_2+\mu_3\cos x_2)u}{\mu_1\mu_2-\mu_3^2\cos^2 x_2} \\ \frac{\left(\mu_1\mu_g-\mu_3^2x_4^2\cos x_2\right)\sin x_2-(\mu_1+\mu_3\cos x_2)u}{\mu_1\mu_2-\mu_3^2\cos^2 x_2} \end{pmatrix}$$

Solution to Exercise 1.8 (Hamilton's method)

1) The Hamiltonian is written as:

$$\begin{aligned} H\,(q,p) &= \underbrace{\frac{1}{2}m\,(\ell\dot{q})^2}_{\text{Kinetic energy}} + \underbrace{mg\ell\,(1-\cos q)}_{\text{Potential energy}} \\ &= \frac{1}{2}\frac{p^2}{m\ell^2} + mg\ell\,(1-\cos q) \end{aligned}$$

since the amount of movement of the pendulum (or rather the kinetic moment in this case) is $p = J\dot{q} = m\ell^2\dot{q}$ and thus $m\left(\ell\dot{q}\right)^2 = m\left(\ell\frac{p}{m\ell^2}\right)^2 = \frac{p^2}{m\ell^2}$. The state equations of the pendulum are therefore:

$$\begin{cases} \dot{q} = \frac{\partial H(q,p)}{\partial p} = \frac{p}{m\ell^2} \\ \dot{p} = -\frac{\partial H(q,p)}{\partial q} = -mg\ell\sin q \end{cases}$$

where the state vector here is $\mathbf{x} = (q,p)^{\mathrm{T}}$. Let us note that:

$$\ddot{q} = \frac{\dot{p}}{m\ell^2} = \frac{-mg\ell\sin q}{m\ell^2} = \frac{-g\sin q}{\ell}$$

and we come back to the differential equation of the pendulum.

2) We have:

$$\dot{H} = \frac{\partial H(\mathbf{q},\mathbf{p})}{\partial \mathbf{p}}\,\dot{\mathbf{p}} + \frac{\partial H(\mathbf{q},\mathbf{p})}{\partial \mathbf{q}}\,\dot{\mathbf{q}} = 0$$

The Hamiltonian (or, equivalently, the mechanical energy) is thus constant.

Solution to Exercise 1.9 (omnidirectional robot)

1) We have $r\omega_i = \langle \mathbf{v}_i, \mathbf{i}_i \rangle$. However, following the velocity composition formula (Varignon's formula), $\mathbf{v}_i = \mathbf{v} - a\dot{\theta}\mathbf{i}_i$. Therefore:

$$r\omega_i = < \mathbf{v} - a\dot{\theta}.\mathbf{i}_i, \mathbf{i}_i > = \langle \mathbf{v}, \mathbf{i}_i \rangle - a\dot{\theta}$$

or:

$$\mathbf{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix},\ \mathbf{i}_1 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},\ \mathbf{i}_2 = \begin{pmatrix} -\sin\left(\theta - \frac{\pi}{3}\right) \\ \cos\left(\theta - \frac{\pi}{3}\right) \end{pmatrix},$$

$$\mathbf{i}_3 = \begin{pmatrix} -\sin\left(\theta + \frac{\pi}{3}\right) \\ \cos\left(\theta + \frac{\pi}{3}\right) \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \underbrace{\frac{1}{r} \begin{pmatrix} -\sin\theta & \cos\theta & -a \\ -\sin\left(\theta - \frac{\pi}{3}\right) & \cos\left(\theta - \frac{\pi}{3}\right) & -a \\ -\sin\left(\theta + \frac{\pi}{3}\right) & \cos\left(\theta + \frac{\pi}{3}\right) & -a \end{pmatrix}}_{\mathbf{A}(\theta)} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}$$

The state equations are therefore:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \mathbf{A}^{-1}(\theta).\omega$$

2) For this we need to design an input controller $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ and an output controller $\omega = (\omega_1, \omega_2, \omega_3)^{\mathrm{T}}$. The new inputs u_1,u_2 correspond to the desired angular velocity and acceleration. Let us choose the controller:

$$\begin{cases} \dot{v} = u_2 \\ \omega = \mathbf{A}(\theta) . \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ u_1 \end{pmatrix} \end{cases}$$

The loop is represented in Figure 1.18.

The loop system is then written as:

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \quad u_1 \\ \dot{v} = \quad u_2 \end{cases}$$

Solution to Exercise 1.10 (modeling a tank)

1) The state vector cannot be chosen to be equal to $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})^{\mathrm{T}}$, which would seem natural with respect to

Lagrangian theory. Indeed, if this was our choice, some states would have no physical meaning. For instance the state:

$$\left(x = 0, y = 0, \theta = 0, \dot{x} = 1, \dot{y} = 1, \dot{\theta} = 0\right)$$

has no meaning since the tank is not allowed to skid. This phenomenon is due to the existence of wheels that creates constraints between the natural state variables. Here, we necessarily have the so-called *non-holonomic* constraint:

$$\dot{y} = \dot{x} \tan \theta$$

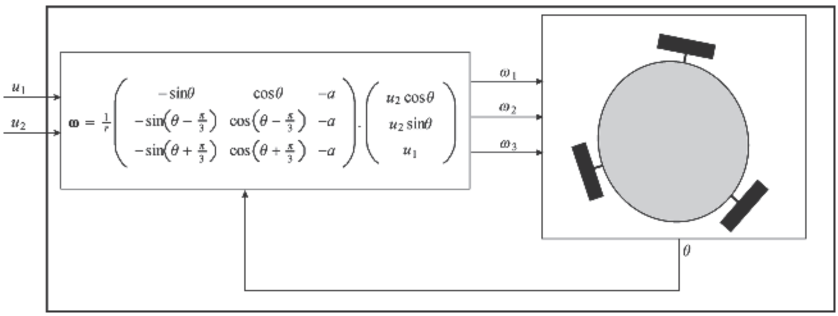

Figure 1.18. *The robot with omni wheels looped in this manner behaves like a tank*

Mechanical systems for which there are such constraints on the equality of natural state variables (by natural state variables we mean the vector $(\mathbf{q}, \dot{\mathbf{q}})$ where $\mathbf{q}$ is the vector of the degrees of liberty of our system) are said to be *non-holonomic*. When such a situation arises, it is useful to use these constraints in order to reduce the number of state variables and this, until no more constraints are left between the state variables.

2) This choice of state variables is easily understood in the sense that these variables allow us to draw the tank (x, y, θ) and the knowledge of v_1,v_2 allows us to calculate the variables $\dot{x}$, $\dot{y}$, $\dot{\theta}$. Moreover, every arbitrary choice of vector

(x, y, θ, v_1, v_2) corresponds to a physically possible situation. The state equations of the system are:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} \frac{v_1+v_2}{2}\cos\theta \\ \frac{v_1+v_2}{2}\sin\theta \\ \frac{v_2-v_1}{\ell} \\ Ru_1 \\ Ru_2 \end{pmatrix}$$

where ℓ is the distance between the two wheels. The third relation on $\dot{\theta}$ is obtained by the velocity composition rule (Varignon's formula). Indeed, we have:

$$\mathbf{v}_2 = \mathbf{v}_1 + \overrightarrow{C_2C_1} \wedge \overrightarrow{\omega}$$

where $\overrightarrow{\omega}$ is the instantaneous rotation vector of the tank and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocity vectors of the centers of the wheels. Let us note that this relation is a vectorial relation that depends on the observer but that is independent of the frame. Let us express this in the frame of the tank, represented on the figure. We must be careful not to confuse the observer fixed on the ground with the frame in which the relation is expressed. This equation is written as:

$$\underbrace{\begin{pmatrix} v_2 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{v}_2} = \underbrace{\begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{v}_1} + \underbrace{\begin{pmatrix} 0 \\ \ell \\ 0 \end{pmatrix}}_{\overrightarrow{C_2C_1}} \wedge \underbrace{\begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}}_{\overrightarrow{\omega}}$$

We thus obtain $v_2 = v_1 + \ell\dot{\theta}$ or $\dot{\theta} = \frac{v_2-v_1}{\ell}$.

Solution to Exercise 1.11 (modeling a car)

Let us consider an observer fixed with respect to the ground. Following the velocity composition rule (Varignon's formula), we have:

$$\mathbf{v}_A = \mathbf{v}_M + \overrightarrow{AM} \wedge \overrightarrow{\omega}$$

where $\overrightarrow{\omega}$ is the instantaneous rotation vector of the car. Let us express this vectorial relation in the frame of the car, which is represented in the figure:

$$\begin{pmatrix} v\cos\delta \\ v\sin\delta \\ 0 \end{pmatrix} = \begin{pmatrix} v_M \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -L \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}$$

where L is the distance between the front and rear axles. Therefore:

$$\begin{pmatrix} v\cos\delta \\ v\sin\delta \end{pmatrix} = \begin{pmatrix} v_M \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ L\dot{\theta} \end{pmatrix}$$

Thus:

$$\dot{\theta} = \frac{v\sin\delta}{L}$$

and:

$$\begin{cases} \dot{x} = v_M\cos\theta = v\cos\delta\cos\theta \\ \dot{y} = v_M\sin\theta = v\cos\delta\sin\theta \end{cases}$$

The evolution equation of the car is therefore written as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v} \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} v\cos\delta\cos\theta \\ v\cos\delta\sin\theta \\ \frac{v\sin\delta}{L} \\ u_1 \\ u_2 \end{pmatrix}$$

Solution to Exercise 1.12 (car-trailer system)

By looking at the figure and using the state equations of the car, we have:

$$\dot{\theta}_r = \frac{v_r\sin\delta_r}{L_r}$$

with:

$$v_r = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{(v\cos\delta\cos\theta)^2 + (v\cos\delta\sin\theta)^2} = v\cos\delta$$

$$\delta_r = \theta - \theta_r$$

The parameter L_r represents the distance between the attachment point and the middle of the axle of the trailer. However, only θ_r has to be added as state variable to those of the car. Indeed, it is clear that x_r, y_r, v_r, δ_r, $\dot{x}_r$, $\dot{y}_r$ ... can be obtained analytically from the sole knowledge of the state of the car and the angle θ_r. Thus, the state equations of the cat-trailer system are given by:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\theta}_r \\ \dot{v} \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} v\cos\delta\cos\theta \\ v\cos\delta\sin\theta \\ \frac{v\sin\delta}{L} \\ \frac{v\cos\delta\sin(\theta-\theta_r)}{L_r} \\ u_1 \\ u_2 \end{pmatrix}$$

Solution to Exercise 1.13 (sailboat)

The modeling that we will propose here is inspired from the article [JAU 04]. Even though it is simplistic, the obtained model remains relatively consistent with reality and is used for the simulation of robotic sailboats (such as Vaimos) in order to test the behavior of the controllers. In order to perform this modeling, we will use the fundamental principle of dynamics in translation (in order to obtain an expression of the tangential acceleration $\dot{v}$) and then in rotation (in order to obtain an expression of the angular acceleration $\dot{\omega}$).

TANGENTIAL ACCELERATION $\dot{v}$.– The wind exerts an orthogonal force on the sail with an intensity equal to:

$$f_v = \alpha_v \left(V\cos(\theta + \delta_v) - v\sin\delta_v\right)$$

Concerning the water, it exerts a force on the rudder that is equal to:

$$f_g = \alpha_g v \sin \delta_g$$

orthogonal to the rudder. The friction force exerted by it on the boat is assumed to be proportional to the square of the boat's velocity. The fundamental equation of dynamics, projected following the axis of the boat gives:

$$m\dot{v} = \sin \delta_v f_v - \sin \delta_g f_g - \alpha_f v^2$$

The radial acceleration can be considered as nil if we assume that the drift is perfect.

ANGULAR ACCELERATION $\dot{\omega}$.– Among the forces that act on the rotation of the boat, we can find the forces f_v and f_g exerted by the sail and the rudder, but also a force of angular friction which we assume to be viscous. The fundamental equation of dynamics gives us:

$$J\dot{\omega} = d_v f_v - d_g f_g - \alpha_\theta \omega$$

where:

$$\begin{cases} d_v = \ell - r_v \cos \delta_v \\ d_g = \quad r_g \cos \delta_g \end{cases}$$

The state equations of the boat are therefore written as:

$$\begin{cases} \dot{x} &= v \cos \theta & \text{(i)} \\ \dot{y} &= v \sin \theta - \beta V & \text{(ii)} \\ \dot{\theta} &= \omega & \text{(iii)} \\ \dot{\delta}_v &= u_1 & \text{(iv)} \\ \dot{\delta}_g &= u_2 & \text{(v)} \\ \dot{v} &= \frac{f_v \sin \delta_v - f_g \sin \delta_g - \alpha_f v^2}{m} & \text{(vi)} \\ \dot{\omega} &= \frac{(\ell - r_v \cos \delta_v) f_v - r_g \cos \delta_g f_g - \alpha_\theta \omega}{J} & \text{(vii)} \\ f_v &= \alpha_v \left(V \cos(\theta + \delta_v) - v \sin \delta_v\right) & \text{(viii)} \\ f_g &= \alpha_g v \sin \delta_g & \text{(ix)} \end{cases}$$

Let us note that these previous equations are not differential but algebraic ones. In order to be perfectly consistent with a state equation, we would need to remove these two equations as well as the two internal forces f_v and f_g that appear in equations (vi) and (vii).

Solution to Exercise 1.14 (direct current motor)

1) The equations governing the system are given by:

$$\begin{array}{ll} u \; = Ri + L\frac{di}{dt} + e & \text{(electrical part)} \\ J\dot{\omega} = T - \rho\omega - T_r & \text{(mechanical part)} \end{array}$$

However, the equations of a direct current machine are $e = K\Phi\omega$ and $T = K\Phi i$. Therefore:

$$\begin{array}{l} u \; = Ri + L\frac{di}{dt} + K\Phi\omega \\ J\dot{\omega} = \; K\Phi i - \rho\omega - T_r \end{array}$$

We then have the following linear state equations:

$$\begin{cases} \frac{di}{dt} = -\frac{R}{L}i - \frac{K\Phi}{L}\omega + \frac{u}{L} \\ \dot{\omega} = \frac{K\Phi}{J}i - \frac{\rho}{J}\omega - \frac{T_r}{J} \end{cases}$$

where the inputs are u and T_r and the state variables are i and ω. These equations can be written in matrix form:

$$\frac{d}{dt}\begin{pmatrix} i \\ \omega \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{K\Phi}{L} \\ \frac{K\Phi}{J} & -\frac{\rho}{J} \end{pmatrix}\begin{pmatrix} i \\ \omega \end{pmatrix} + \begin{pmatrix} \frac{1}{L} & 0 \\ 0 & -\frac{1}{J} \end{pmatrix}\begin{pmatrix} u \\ T_r \end{pmatrix}$$

2) When the motor is running, the torque T_r is being imposed. The following table gives some mechanical characteristics, in continuous output, of the pair (T_r, ω):

$$\begin{array}{ll} T_r = C^{te} & \text{motor used for lifting} \\ T_r = \alpha\omega & \text{locomotive, mixer, pump} \\ T_r = \frac{\alpha}{\omega} & \text{machine tool (lathe, milling machine)} \\ T_r = \alpha\omega^2 & \text{ventilator, fast car} \end{array}$$

In our case, $T_r = \alpha\omega^2$. The motor now only has a single input that is the armature voltage $u(t)$. We have:

$$\begin{cases} \frac{di}{dt} = -\frac{R}{L}i - \frac{K\Phi}{L}\omega + \frac{u}{L} \\ \dot{\omega} = \frac{K\Phi}{J}i - \frac{\rho}{J}\omega - \frac{\alpha\omega^2}{J} \end{cases}$$

This is a nonlinear system. It now only has a single input $u(t)$.

Solution to Exercise 1.15 (RLC circuit)

Let i_1 be the electrical current in the resistance R_1 (top to bottom). Following the node and mesh rules, we have:

$$\begin{cases} u(t) - v(t) - R_1 i_1(t) = 0 \text{ (mesh rule)} \\ L\frac{di}{dt} + R_2 i(t) - R_1 i_1(t) = 0 \text{ (mesh rule)} \\ i(t) + i_1(t) - C\frac{dv}{dt} = 0 \text{ (node rule)} \end{cases}$$

Intuitively, we can understand that the memory of the system corresponds to the capacitor charge and the electromagnetic flow in the coil. Indeed, if these values are known at time $t = 0$, for a known input, the future of the system is determined in a unique manner. Thus, the possible state variables are given by the values $i(t)$ (proportional to the flow) and $v(t)$ (proportional to the charge). We obtain the state equations by removing the i_1 in the previous equations and by isolating $\frac{di}{dt}$ and $\frac{dv}{dt}$. Of course, one equation must be removed. We obtain:

$$\begin{cases} \frac{dv}{dt} = -\frac{1}{CR_1}v(t) + \frac{1}{C}i(t) + \frac{1}{CR_1}u(t) \\ \frac{di}{dt} = -\frac{1}{L}v(t) - \frac{R_2}{L}i(t) + \frac{1}{L}u(t) \end{cases}$$

Note that the output is given by $y(t) = R_2 i(t)$. Finally, we reach the state representation of a linear system given by:

$$\begin{array}{rcl} \frac{d}{dt}\begin{pmatrix} v(t) \\ i(t) \end{pmatrix} & = & \begin{pmatrix} -\frac{1}{CR_1} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_2}{L} \end{pmatrix} \begin{pmatrix} v(t) \\ i(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{CR_1} \\ \frac{1}{L} \end{pmatrix} u(t) \\ y(t) & = & \begin{pmatrix} 0 & R_2 \end{pmatrix} \begin{pmatrix} v(t) \\ i(t) \end{pmatrix} \end{array}$$

Solution to Exercise 1.16 (the three containers)

1) With the aim of applying Bernoulli's relation of the left container, let us consider a flow tube, in other words a virtual tube (see figure) in which the water has a fluid movement and does not cross the walls. Bernoulli's relation tells us that in this tube, at every point:

$$P + \rho \frac{v^2}{2} + \rho g z = \text{constant}$$

where P is the pressure at the considered point, z its height and v the velocity of the water at this point. The coefficient ρ is the bulk density of the water and g is the gravitational constant. Following Bernouilli's relation, we have:

$$P_D + \rho \frac{v_D^2}{2} + \rho g z_D = P_A + \rho \frac{v_A^2}{2} + \rho g z_A$$

in other words:

$$P_D = P_A + \rho g \left(z_A - z_D \right) - \rho \frac{v_D^2}{2}$$

Moreover, we may assume that C is far from the turbulence zone and that the water is not moving. Therefore, we have the following Bernoulli relation:

$$P_C + \rho g z_C = P_B + \rho g z_B$$

in other words:

$$P_C = P_B + \rho g \left(z_B - z_C \right)$$

Let us note that, in this turbulence zone, the water is slowed down, but we can assume that the pressure does not change, i.e. $P_C = P_D$. Thus, we have:

$$\underbrace{P_B}_{P_{\text{atm}}} + \rho g \left(z_B - z_C \right) = \underbrace{P_A}_{P_{\text{atm}}} + \rho g \left(z_A - z_D \right) - \rho \frac{v_D^2}{2}$$

As $P_A = P_B = P_{\text{atm}}$, and that $z_C = z_D$, this equation becomes:

$$\rho g \left(z_A - z_B \right) = \rho \frac{v_D^2}{2}$$

or:

$$v_D = \sqrt{2g \left(z_A - z_B \right)}$$

In the case where the level of the right container is higher than that of the left one, a similar study gives us:

$$v_D = -\sqrt{2g \left(z_B - z_A \right)}$$

The minus sign of the expression indicates that the flow now moves from the right container toward the left one. Thus the general relation for the velocity of the water in the canal is:

$$v_D = sign \left(z_A - z_B \right) \sqrt{2g |z_A - z_B|}$$

If a is the cross section of the canal, the water flow from the right container to the left one is:

$$Q_D = a.sign \left(z_A - z_B \right) \sqrt{2g |z_A - z_B|}$$

This is the so-called *Torricelli* law.

REMARK.– Initially, this law was proven in a simpler context where the water flows into emptiness. The total energy of a fluid element of mass m is conserved if we consider this latter to be falling freely in the tube flow. Thus, for the two points A and B, where A is at the surface and B in the tube, we have:

$$mgh_A + \underbrace{\frac{1}{2}mv_A^2}_{=0} = mgh_B + \frac{1}{2}mv_B^2$$

and therefore:

$$v_B = \sqrt{2g\left(h_A - h_B\right)}$$

We can then deduce Toricelli's relation. This reasoning is only possible in the case of a perfect fluid, where, given the absence of friction, we assume that the forces of tangential pressure are nil.

2) The state equations are obtained by writing that the volume of water in a container is equal to the sum of the incoming flows minus the sum of the outgoing flows, in other words:

$$\begin{aligned}\dot{h}_1 &= -Q_{1\text{ext}} - Q_{12} + u_1 \\ \dot{h}_2 &= Q_{12} - Q_{23} \\ \dot{h}_3 &= -Q_{3\text{ext}} + Q_{23} + u_2\end{aligned}$$

or:

$$\begin{aligned}\dot{h}_1 &= -a.\sqrt{2gh_1} - a.sign\left(h_1 - h_2\right)\sqrt{2g|h_1 - h_2|} + u_1 \\ \dot{h}_2 &= a.sign\left(h_1 - h_2\right)\sqrt{2g|h_1 - h_2|} \\ &\quad -a.sign\left(h_2 - h_3\right)\sqrt{2g|h_2 - h_3|} \\ \dot{h}_3 &= -a.\sqrt{2gh_3} + a.sign\left(h_2 - h_3\right)\sqrt{2g|h_2 - h_3|} + u_2\end{aligned}$$

Solution to Exercise 1.17 (pneumatic cylinder)

The input of the system is the volumetric flow rate u of the air toward the cylinder chamber. We then have:

$$u = \left(\frac{V}{n}\right)\dot{n}$$

where n is the number of gas in the chamber and V is the volume of the chamber. The fundamental principle of dynamics gives us $pa - kz = m\ddot{z}$. Therefore, the first two state equations are:

$$\begin{cases} \dot{z} = \dot{z}, \\ \ddot{z} = \frac{ap-kz}{m} \end{cases}$$

The ideal gas law ($pV = nRT$) is given by $pza = nRT$. By differentiating, we obtain:

$$a\left(\dot{p}z + p\dot{z}\right) = R\left(\dot{n}T + n\dot{T}\right)$$

By assuming an isothermal evolution, this relation becomes:

$$a\left(\dot{p}z + p\dot{z}\right) = R\dot{n}T = R\frac{nu}{V}T = pu$$

By isolating $\dot{p}$, we obtain the third state equation of our system, which is:

$$\dot{p} = \frac{p}{z}\left(\frac{u}{a} - \dot{z}\right)$$

The state equations of the system are therefore:

$$\begin{cases} \dot{z} = \dot{z} \\ \ddot{z} = \frac{ap-kz}{m} \\ \dot{p} = \frac{p}{z}\left(\frac{u}{a} - \dot{z}\right) \end{cases}$$

or, since $\mathbf{x} = (z, \dot{z}, p)$:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{ax_3 - kx_1}{m} \\ \dot{x}_3 = -\frac{x_3}{x_1}\left(x_2 - \frac{u}{a}\right) \end{cases}$$

Solution to Exercise 1.18 (Fibonacci sequence)

1) The state equations are given by:

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = x_1(k) + x_2(k) \\ \quad\quad y(k) = x_1(k) + x_2(k) \end{cases}$$

where $x_1(0) = 1$ and $x_2(0) = 0$ as the initial conditions. This system is called a *Fibonacci* system.

2) Let us now look for the recurrence relation associated with this system. For this we need to express $y(k)$, $y(k+1)$ and $y(k+2)$ as a function of $x_1(k)$ and $x_2(k)$. The resulting calculations are the following:

$$\begin{aligned} y(k) &= x_1(k) + x_2(k) \\ y(k+1) &= x_1(k+1) + x_2(k+1) = x_1(k) + 2x_2(k) \\ y(k+2) &= x_1(k+2) + x_2(k+2) = x_1(k+1) + 2x_2(k+1) \\ &= 2x_1(k) + 3x_2(k) \end{aligned}$$

In other words:

$$\begin{pmatrix} y(k) \\ y(k+1) \\ y(k+2) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}$$

By removing $x_1(k)$ and $x_2(k)$ from this system of three linear equations, we obtain a single equation given by:

$$y(k+2) - y(k+1) - y(k) = 0$$

The initial conditions are $y(0) = y(1) = 1$. It is in general in this form that the Fibonacci system is described.

Solution to Exercise 1.19 (bus network)

1) The timetable is given below.

k	1	2	3	4	5
$x_1(k)$	0	5	8	13	16
$x_2(k)$	0	3	8	11	16

Table 1.2. *Timetable for the buses*

We note a periodicity of 2 in the sequence progression.

2) The state equations are:

$$\begin{cases} x_1(k+1) = \max(x_1(k)+2, x_2(k)+5) \\ x_2(k+1) = \max(x_1(k)+3, x_2(k)+3) \end{cases}$$

3) In matrix form, we have:

$$\mathbf{x}(k+1) = \begin{pmatrix} 2\ 5 \\ 3\ 3 \end{pmatrix} \otimes \mathbf{x}(k)$$

4) The problem comes from the fact that $(\mathbb{R}, \oplus)$ is only a monoid and not a group (since the image does not exist). Therefore $(\mathbb{R}, \oplus, \otimes)$ is not a ring (as it is not necessary for matrix calculus) but a dioid.