

PART 1

Piecewise Deterministic Markov Processes
and Quantization

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Piecewise Deterministic Markov Processes

1.1. Introduction

In 1980, Davis [DAV 93] introduced in probability theory the class of piecewise deterministic Markov processes (PDMPs) as a general class of models suitable for formulating optimization problems in queuing and inventory systems, maintenance-replacement models, investment scheduling and in many other areas of operation research. These models are described by two variables. To the usual Euclidean variable representative of the state of the process, we add a discrete variable, called regime or mode that takes values in a finite or countable set. In this context, the state variable represents the physical parameters of the system under consideration. For example, it can be the position or the orientation of a satellite or the pressure in a tank. The mode characterizes the regimes of operation of the physical system from nominal to failure regime.

From a mathematical point of view, the notion of a PDMP is very intuitive and simple to describe. Starting from a point of the state space, the process follows a deterministic trajectory, namely a flow indexed by the mode, until the first jump time, which occurs either spontaneously in a random manner or when the trajectory hits the boundary of the state space. Between two jumps, the mode is assumed to be constant. In both cases, a new point and a new regime are selected by a random operator and the process restarts from this new point under this new mode. There exist two types of jumps. The first type is deterministic. From the mathematical point of view, it is given by the trajectory hitting the boundary of the state space. From the physical point of view, it can be seen as a modification of the mode of operation when a physical parameter

reaches a prescribed level, for example when the pressure of a tank reaches the critical value. The second type is stochastic. It models the random nature of failures or inputs that modify the mode of operation of the system.

This chapter is dedicated to the definition of PDMPs and statement of the main properties that we will use throughout the book. It is organized as follows. Section 1.2 presents general notations. In section 1.3, we give the formal definition of a PDMP. In section 1.4, we state and comment the main technical assumptions that will be required in this book. This section may be skipped by readers who are only interested in applications. In section 1.5, we define a time-augmented PDMP and establish that the properties of the original PDMP transfer to the time-augmented one. In section 1.6, we define and study a discrete-time Markov chain naturally embedded into a PDMP. This chain is at the heart of our numerical approximations. Section 1.7 provides some technical properties of the stopping times of a PDMP. This section may also be skipped at first reading. Finally, in section 1.8, we give some simple examples of PDMPs that will serve to illustrate our results throughout the book. More involved examples are detailed in Chapter 2. The readers interested in examples outside the reliability scope or further technical details on PDMPs are referred to [DAV 93].

1.2. Notation

The purpose of this section is to standardize some terminology and notation that will be used throughout the book. The set of non-negative integers is denoted by \mathbb{N} , that of positive integers by \mathbb{N}^* , \mathbb{R} is the set of real numbers and we use \mathbb{R}^+ for the non-negative real numbers. The d -dimensional Euclidean space is denoted by \mathbb{R}^d . For $a, b \in \mathbb{R}$, $a \wedge b = \min(a, b)$ (respectively, $a \vee b = \max(a, b)$) is the minimum (respectively, maximum) of a and b , and $a^+ = a \vee 0$. By convention, set $\inf \emptyset = +\infty$.

Let \mathcal{X} be a metric space with distance $d_{\mathcal{X}}$. For a subset A of \mathcal{X} , ∂A is the boundary of A , \bar{A} is its closure and A^c is its complement. We denote by $\mathfrak{B}(\mathcal{X})$ the Borel σ -field of \mathcal{X} and by $\mathbf{B}(\mathcal{X})$ the set of real-valued, bounded and measurable functions defined on \mathcal{X} . For any function $w \in \mathbf{B}(\mathcal{X})$, we write C_w for its upper bound, that is:

$$C_w = \sup_{x \in \mathcal{X}} |w(x)|.$$

Let $\mathbf{L}(\mathcal{X})$ be the subset of $\mathbf{B}(\mathcal{X})$ of Lipschitz-continuous functions, and for any function $w \in \mathbf{L}(\mathcal{X})$, denote by L_w its Lipschitz constant:

$$L_w = \sup_{x \neq x' \in \mathcal{X}} \frac{|w(x) - w(x')|}{d_{\mathcal{X}}(x, x')}.$$

for a Markov kernel P on $(\mathcal{X}, \mathfrak{B}(\mathcal{X}))$ and functions w and w' in $\mathbf{B}(\mathcal{X})$, set:

$$Pw(x) = \int_{\mathcal{X}} w(y)P(x, dy), \quad (wPw')(x) = w(x) \int_{\mathcal{X}} w'(y)P(x, dy),$$

for any $x \in \mathcal{X}$.

1.3. Definition of a PDMP

PDMPs are defined as follows. Let M be the finite set of the possible modes of the system. For all modes m in M , let E_m be an open subset in \mathbb{R}^d endowed with the usual Euclidean norm $|\cdot|$. Set:

$$E = \{(m, \zeta), m \in M, \zeta \in E_m\}.$$

Define on E the following distance, for $x = (m, \zeta)$ and $x' = (m', \zeta') \in E$:

$$|x - x'| = |\zeta - \zeta'| \mathbb{1}_{\{m=m'\}} + \infty \mathbb{1}_{\{m \neq m'\}}.$$

A PDMP on the state space E is defined by three local characteristics (Φ, λ, Q) where:

– the flow $\Phi(x, t) = (m, \Phi_m(\zeta, t))$ for all $x = (m, \zeta)$ in E and $t \geq 0$, where $\Phi_m : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ is continuous and for all $s, t \geq 0$, we have $\Phi_m(\cdot, t+s) = \Phi_m(\Phi_m(\cdot, s), t)$. It describes the deterministic trajectory of the process between jumps. Set:

$$t^*(x) = t_m^*(\zeta) = \inf\{t > 0 : \Phi_m(\zeta, t) \in \partial E_m\},$$

the time the flow takes to reach the boundary of the domain starting from position ζ in mode m ;

– the jump intensity $\lambda : \overline{E} \rightarrow \mathbb{R}^+$ is measurable and has the following integrability property: for any $x = (m, \zeta)$ in E , there exists $\epsilon > 0$ such that:

$$\int_0^\epsilon \lambda(m, \Phi_m(\zeta, t)) dt < +\infty.$$

It characterizes the frequency of jumps. For all $x = (m, \zeta)$ in E , and $t \leq t^*(x)$, set:

$$\Lambda(m, \zeta, t) = \int_0^t \lambda(m, \Phi_m(\zeta, s)) ds;$$

– the Markov kernel Q on $(\overline{E}, \mathfrak{B}(\overline{E}))$ represents the transition measure of the process and allows us to select the new location and mode after each jump. It satisfies:

$$Q(x, \{x\} \cup \partial E) = 0$$

for all x in \overline{E} , meaning that each jump is made in E and changes the location and/or the mode of the process.

From these characteristics, it can be shown that there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ on which a process $(X_t)_{t \geq 0}$ can be defined as a strong Markov process. The process $(X_t)_{t \geq 0}$ has two components $X_t = (m_t, \zeta_t)$ where the first component m_t is usually called the mode or the regime and the second component ζ_t is the so-called Euclidean variable. The motion of this process can then be defined iteratively. Starting at an initial point $X_0 = (m_0, \zeta_0)$ with $m_0 \in M$ and $\zeta_0 \in E_{m_0}$, the first jump time T_1 is determined by the following survival function:

$$\mathbb{P}_{(m_0, \zeta_0)}(T_1 > t) = e^{-\Lambda(m_0, \zeta_0, t)} \mathbb{1}_{\{t < t^*(m_0, \zeta_0)\}}. \quad [1.1]$$

On the interval $[0, T_1)$, the process $(X_t)_{t \geq 0}$ follows the deterministic trajectory given by $\Phi_{m_0}(\zeta_0, t)$ and the regime m_t is constant and equal to m_0 . At the random time T_1 , a jump occurs. A jump can produce either a discontinuity in the Euclidean variable ζ_t and/or a change of mode. The process restarts at a new mode and/or position $X_{T_1} = (m_{T_1}, \zeta_{T_1})$, according to the distribution $Q(m_0, \Phi_{m_0}(\zeta_0, T_1), \cdot)$. An inter-jump time $T_2 - T_1$ is then selected in a similar way to equation [1.1], and on the interval $[T_1, T_2)$ the process follows the path $m_t = m_{T_1}$ and $\zeta_t = \Phi_{m_{T_1}}(\zeta_{T_1}, t - T_1)$. The process $(X_t)_{t \geq 0}$ thus defined is a PDMP on the state space E (see section 1.6 for an

explicit construction). An example of such a trajectory is given in Figure 1.1. The first jump is random, but the second jump is forced (or deterministic) since the process hits the boundary of the domain.

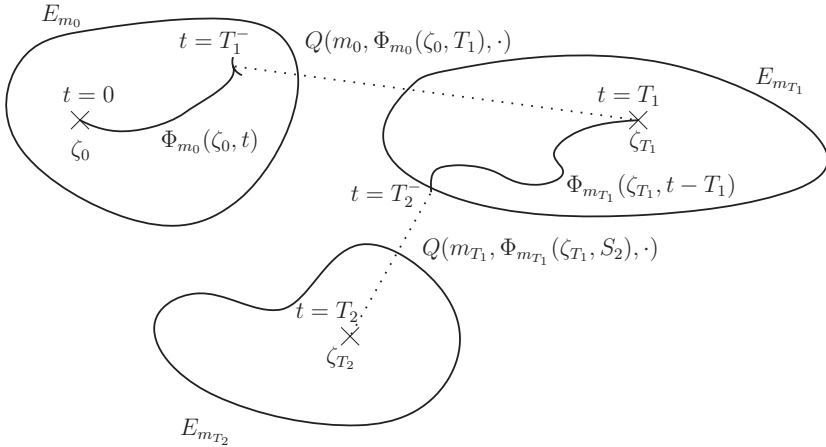


Figure 1.1. A sample trajectory of a generic PDMP

In order to avoid any technical problems due to the possible explosion of the process, we make the following standard assumption (see section 24 in [DAV 93]).

ASSUMPTION 1.1.– For all $(x, t) \in E \times \mathbb{R}^+$, $\mathbb{E}_x \left[\sum_{n=1}^{\infty} \mathbb{1}_{\{T_n < t\}} \right] < +\infty$.

We will also assume throughout the book that the deterministic time to reach the boundary of the state space is bounded.

ASSUMPTION 1.2.– The exit time t^* is in $\mathbf{B}(E)$ and bounded by C_{t^*} .

In most practical applications, the physical properties of the system ensure that either t^* is bounded, or the problem has a natural deterministic time horizon T . In the latter case, there is no loss of generality in considering that t^* is bounded by this deterministic time horizon. This leads to replacing C_{t^*} by T (see also section 2.2.2).

For notational convenience, we set $T_0 = 0$. The sequence $(Z_n)_{n \in \mathbb{N}}$ with $Z_n = X_{T_n}$ describes the post jump locations of the process $(X_t)_{t \geq 0}$ and

$(S_n)_{n \in \mathbb{N}}$ with $S_n = T_n - T_{n-1}$ for $n \geq 1$ and $S_0 = 0$ gives the times between two consecutive jumps.

1.4. Regularity assumptions

Here we introduce some technical regularity assumptions on the local characteristics of the PDMP that will be useful in the following chapters in order to prove the convergence of our approximation procedures; this section may be skipped at first reading. We start by introducing new sets of functions that have some regularity property along the flow.

1.4.1. Lipschitz continuity along the flow

Denote by $\mathbf{L}_\Phi(E)$ the set of functions $w \in \mathbf{B}(E)$ that are Lipschitz continuous along the flow, i.e. the real-valued, bounded, measurable functions defined on E and satisfying the following conditions:

– for all $x \in E$, the map $w(\Phi(x, \cdot)): [0, t^*(x)] \rightarrow \mathbb{R}$ is continuous, the limit $\lim_{t \rightarrow t^*(x)} w(\Phi(x, t))$ exists and is denoted by $w(\Phi(x, t^*(x)))$;

– there exists $[w]_1^E \in \mathbb{R}^+$ such that for $x, y \in E$ and $t \in [0, t^*(x) \wedge t^*(y)]$, we have

$$|w(\Phi(x, t)) - w(\Phi(y, t))| \leq [w]_1^E |x - y|;$$

– there exists $[w]_2^E \in \mathbb{R}^+$ such that for all $x \in E$ and $t, t' \in [0, t^*(x)]$, we have:

$$|w(\Phi(x, t)) - w(\Phi(x, t'))| \leq [w]_2^E |t - t'|;$$

– there exists $[w]_*^E \in \mathbb{R}^+$ such that for all $x, y \in E$, we have:

$$|w(\Phi(x, t^*(x))) - w(\Phi(y, t^*(y)))| \leq [w]_*^E |x - y|.$$

Denote by $\mathbf{L}_\Phi(\partial E)$ the subset of $\mathbf{B}(\partial E)$ of real-valued, bounded, measurable functions defined on ∂E for which there exists $[w]_*^{\partial E} \in \mathbb{R}^+$ such that for $x, y \in E$, we have:

$$|w(\Phi(x, t^*(x))) - w(\Phi(y, t^*(y)))| \leq [w]_*^{\partial E} |x - y|.$$

Whenever there is no ambiguity, we will use the simpler notation $[w]_i$ instead of $[w]_i^E$ for $i \in \{1, 2, *\}$ and $[w]_*$ instead of $[w]_*^E$.

We also need a notion of local Lipschitz-continuity along the flow up to a given time horizon that may be smaller than the exit time t^* . For all $u \geq 0$, denote by $\mathbf{L}_\Phi^u(E)$ the set of functions $w \in \mathbf{B}(E)$ that are Lipschitz continuous along the flow until time u , i.e. the real-valued, bounded, measurable functions defined on E and satisfying the following conditions:

- for all $x \in E$, the map $w(\Phi(x, \cdot)): [0, t^*(x) \wedge u] \rightarrow \mathbb{R}$ is continuous and $\lim_{t \rightarrow t^*(x) \wedge u} w(\Phi(x, t))$ exists and is denoted by $w(\Phi(x, t^*(x) \wedge u))$;
- there exists $[w]_1^{E,u} \in \mathbb{R}^+$ such that for all $x, y \in E$ and $t \in [0, t^*(x) \wedge t^*(y) \wedge u]$, we have:

$$|w(\Phi(x, t)) - w(\Phi(y, t))| \leq [w]_1^{E,u} |x - y|;$$

- there exists $[w]_2^{E,u} \in \mathbb{R}^+$ such that for $x \in E$ and $t, t' \in [0, t^*(x) \wedge u]$, we have:

$$|w(\Phi(x, t)) - w(\Phi(x, t'))| \leq [w]_2^{E,u} |t - t'|;$$

- there exists $[w]_*^{E,u} \in \mathbb{R}^+$ such that for all $x, y \in E$, if $t^*(x) \leq u$ and $t^*(y) \leq u$, we have:

$$|w(\Phi(x, t^*(x))) - w(\Phi(y, t^*(y)))| \leq [w]_*^{E,u} |x - y|.$$

We will also drop the superscript E in the Lipschitz constants whenever there is no ambiguity regarding the state space. Note that for all $u \leq u'$, we have $\mathbf{L}_\Phi^{u'}(E) \subset \mathbf{L}_\Phi^u(E)$ with $[w]_i^{E,u} \leq [w]_i^{E,u'}$, $i \in \{1, 2, *\}$. As the exit time t^* is bounded by C_{t^*} , then $\mathbf{L}_\Phi^u(E) = \mathbf{L}_\Phi(E)$ for all $u \geq C_{t^*}$.

Note that any function w in $\mathbf{L}_\Phi(E)$ is also in $\mathbf{L}(E)$ with $L_w \leq [w]_1$ as for all $x \in E$, $x = \Phi(x, 0)$. Conversely, if the flow Φ and the exit time t^* are bounded and Lipschitz continuous: $\Phi \in \mathbf{L}(E \times \mathbb{R}^+)$ and $t^* \in \mathbf{L}(E)$, then any function $w \in \mathbf{L}(E)$ is in $\mathbf{L}_\Phi(E)$ with $[w]_1 \leq L_w L_\Phi$, $[w]_2 \leq L_w L_\Phi$ and $[w]_* \leq L_w L_\Phi (L_{t^*} + 1)$.

1.4.2. Regularity assumptions on the local characteristics

We gather here the assumptions that are going to be needed in the following chapters. They may not be all required at the same time. We make

no special assumption on the flow as we will rather assume that the functions of interest are Lipschitz continuous along the flow. However, we may need some regularity for the exit time.

ASSUMPTION 1.3.– The exit time is Lipschitz continuous: $t^* \in \mathbf{L}(E)$.

We require some boundedness and weaker form of Lipschitz continuity along the flow for the jump intensity.

ASSUMPTION 1.4.– The jump intensity λ is in $\mathbf{B}(E)$ and bounded by C_λ .

ASSUMPTION 1.5.– There exists $[\lambda]_1 \in \mathbb{R}^+$ such that for all $x, y \in E$ and $t \in [0, t^*(x) \wedge t^*(y)]$, we have:

$$|\lambda(\Phi(x, t)) - \lambda(\Phi(y, t))| \leq [\lambda]_1 |x - y|.$$

Finally, we need some global and local Lipschitz continuity properties for the Markov kernel Q .

ASSUMPTION 1.6.– The Markov kernel Q is Lipschitz in the following sense: there exists $[Q] \in \mathbb{R}^+$ such that for all functions $w \in \mathbf{L}_\Phi(E)$:

1) for all $x, y \in E$ and $t \in [0, t^*(x) \wedge t^*(y)]$, we have

$$|Qw(\Phi(x, t)) - Qw(\Phi(y, t))| \leq [Q][w]_1 |x - y|;$$

2) for all $x, y \in E$, we have

$$|Qw(\Phi(x, t^*(x))) - Qw(\Phi(y, t^*(y)))| \leq [Q][w]_* |x - y|.$$

ASSUMPTION 1.7.– The Markov kernel Q is locally Lipschitz in the following sense: there exists $[Q] \in \mathbb{R}^+$ such that for all $u \geq 0$ and for all functions $w \in \mathbf{L}_\Phi^u(E)$:

1) for all $x, y \in E$ and $t \in [0, t^*(x) \wedge t^*(y) \wedge u]$, we have

$$|Qw(\Phi(x, t)) - Qw(\Phi(y, t))| \leq [Q][w]_1^u |x - y|;$$

2) for all $x, y \in E$ such that $t^*(x) \vee t^*(y) \leq u$, we have

$$|Qw(\Phi(x, t^*(x))) - Qw(\Phi(y, t^*(y)))| \leq [Q]([w]_*^u + [w]_1^u) |x - y|.$$

The necessity for the weaker notion of local Lipschitz continuity will be made clear in the next section. Our last assumption is often satisfied in applications. It states that the jumps cannot send the process too close to the boundary of E .

ASSUMPTION 1.8.– There exists $\epsilon > 0$ such that for all x in E , $Q(x, B_\epsilon) = 1$, where:

$$B_\epsilon = \{x \in E : t^*(x) \geq \epsilon\}.$$

1.5. Time-augmented process

In some applications, it may be useful to consider the time-augmented process $(X_t, t)_{t \geq 0}$, for instance to compute expectations of time-dependent functionals, see section 4.5. The formal definition of this process comes from [DAV 93] as well as the proof that it is still a PDMP. The new state space is:

$$\tilde{E} = E \times \mathbb{R}^+,$$

equipped with the distance defined, for all $\xi = (x, t), \xi' = (x', t') \in \tilde{E}$, by:

$$|\xi - \xi'| = |x - x'| + |t - t'|.$$

On this state space, we define the process:

$$\tilde{X}_t = (X_t, t),$$

for $t \geq 0$. The local characteristics of $(\tilde{X}_t)_{t \geq 0}$, denoted by $(\tilde{\Phi}, \tilde{\lambda}, \tilde{Q})$, are given for all $\xi = (x, t) \in \tilde{E}$, $s \leq t^*(x)$ and $A \in \mathfrak{B}(E)$ by:

$$\begin{aligned} \tilde{\Phi}(\xi, s) &= (\Phi(x, s), t + s), \\ \tilde{\lambda}(\xi) &= \lambda(x), \\ \tilde{Q}(\xi, A \times \{t\}) &= Q(x, A), \\ \tilde{t}^*(\xi) &= \inf\{t > 0 : \tilde{\Phi}(\xi, s) \in \partial\tilde{E}\} = t^*(x). \end{aligned}$$

As both processes $(X_t)_{t \geq 0}$ and $(\tilde{X}_t)_{t \geq 0}$ have the same jump times, assumption 1.1 clearly holds for $(\tilde{X}_t)_{t \geq 0}$. The interesting question is whether the Lipschitz properties on the local characteristics of the original process

$(X_t)_{t \geq 0}$ transfer to that of $(\tilde{X}_t)_{t \geq 0}$ or not. As $\tilde{t}^*(\xi) = t^*(x)$ and $\tilde{\lambda}(\xi) = \lambda(x)$, the answer is yes for t^* and λ .

LEMMA 1.1.— If assumption 1.2 (1.3, 1.4 and 1.5, respectively) holds for $(X_t)_{t \geq 0}$, then it holds for $(\tilde{X}_t)_{t \geq 0}$ and $C_{\tilde{t}^*} = C_{t^*}$ ($L_{\tilde{t}^*} = L_{t^*}$, $C_{\tilde{\lambda}} = C_{\lambda}$ and $[\tilde{\lambda}]_1 = [\lambda]_1$ respectively).

In order to investigate continuity properties for \tilde{Q} , we need to be able to compare functions continuous along the flow Φ of the original PDMP with functions continuous along the flow $\tilde{\Phi}$ of the time-augmented process.

LEMMA 1.2.— Set $u, t \geq 0$ and $w \in \mathbf{L}_{\tilde{\Phi}}^u(\tilde{E})$. Denote by w_t the function of $\mathbf{B}(E)$ defined by $w_t(x) = w(x, t)$ for all $x \in E$. Under assumption 1.3, w_t is in $\mathbf{L}_{\Phi}^{u \wedge t}(E)$ with:

$$\begin{aligned} [w_t]_1^{E, t \wedge u} &\leq [w]_1^{\tilde{E}, u}, & [w_t]_2^{E, t \wedge u} &\leq [w]_1^{\tilde{E}, u} + [w]_2^{\tilde{E}, u}, \\ [w_t]_*^{E, t \wedge u} &\leq (1 + L_{t^*})[w]_*^{\tilde{E}, u}. \end{aligned}$$

PROOF.— For $x, x' \in E$ and $s \leq t^*(x) \wedge t^*(x') \wedge t \wedge u$, we have:

$$\begin{aligned} |w_t(\Phi(x, s)) - w_t(\Phi(x', s))| &= |w(\tilde{\Phi}((x, t-s), s)) \\ &\quad - w(\tilde{\Phi}((x', t-s), s))|. \end{aligned}$$

We now use the fact that $w \in \mathbf{L}_{\tilde{\Phi}}^u(\tilde{E})$ which yields since $s \leq u$:

$$\begin{aligned} |w_t(\Phi(x, s)) - w_t(\Phi(x', s))| &\leq [w]_1^{\tilde{E}, u} |(x, t-s) - (x', t-s)| \\ &= [w]_1^{\tilde{E}, u} |x - x'|. \end{aligned}$$

Hence, $[w_t]_1^{E, t \wedge u} \leq [w]_1^{\tilde{E}, u}$. Similarly, for $s, s' \leq t^*(x) \wedge t^*(x') \wedge t \wedge u$, we have:

$$\begin{aligned} |w_t(\Phi(x, s)) - w_t(\Phi(x, s'))| &= |w(\tilde{\Phi}((x, t-s), s)) - w(\tilde{\Phi}((x, t-s'), s'))| \\ &\leq [w]_1^{\tilde{E}, u} |s - s'| + [w]_2^{\tilde{E}, u} |s - s'|. \end{aligned}$$

Finally, for $x, x' \in E$ such that $t^*(x) \vee t^*(x') \leq t \wedge u$, we have:

$$\begin{aligned}
 & |w_t(\Phi(x, t^*(x))) - w_t(\Phi(x', t^*(x')))| \\
 &= |w(\tilde{\Phi}((x, t - t^*(x)), t^*(x))) - w(\tilde{\Phi}((x', t - t^*(x')), t^*(x')))| \\
 &= |w(\tilde{\Phi}((x, t - t^*(x)), \tilde{t}^*(x, t - t^*(x)))) \\
 &\quad - w(\tilde{\Phi}((x', t - t^*(x')), \tilde{t}^*(x', t - t^*(x'))))| \\
 &\leq [w]_{*}^{\tilde{E}, u} |(x, t - t^*(x)) - (x', t - t^*(x'))| \\
 &\leq [w]_{*}^{\tilde{E}, u} (1 + L_{t^*}) |x - x'|,
 \end{aligned}$$

as $w \in \mathbf{L}_{\tilde{\Phi}}^u(\tilde{E})$ and $\tilde{t}^*(x, t - t^*(x)) \vee \tilde{t}^*(x', t - t^*(x')) \leq u$ and using the Lipschitz continuity assumption 1.3 on t^* . \square

Because of the dependence in t , the global Lipschitz continuity along the flow $\tilde{\Phi}$ for w does not in general imply the global Lipschitz continuity of w_t along the flow Φ . As a result, in general, if the Markov kernel Q satisfies the global Lipschitz assumption 1.6, then \tilde{Q} may not satisfy it. This is why we introduced the weaker notion of local Lipschitz continuity. This one remains valid for the time-augmented process, provided that the Lipschitz continuity assumption on t^* also holds.

LEMMA 1.3.– If assumptions 1.7 and 1.3 hold for $(X_t)_{t \geq 0}$, then assumption 1.7 holds for $(\tilde{X}_t)_{t \geq 0}$ and:

$$[\tilde{Q}] \leq ([Q] \vee 1)(1 + L_{t^*}).$$

PROOF.– For $\xi = (x, t) \in \tilde{E}$ and $w \in \mathbf{L}_{\tilde{\Phi}}^u(\tilde{E})$, we have, by definition of \tilde{Q} :

$$\tilde{Q}w(\xi) = \int_{\xi' \in \tilde{E}} w(\xi') \tilde{Q}((x, t), d\xi') = \int_{z \in E} w(z, t) Q(x, dz) = Qw_t(x).$$

Let $\xi = (x, t)$ and $\xi' = (x', t') \in \tilde{E}$. Let $s \in [0; \tilde{t}^*(\xi) \wedge \tilde{t}^*(\xi') \wedge u]$. We have:

$$\begin{aligned}
 & |\tilde{Q}w(\tilde{\Phi}(\xi, s)) - \tilde{Q}w(\tilde{\Phi}(\xi', s))| = |\tilde{Q}w(\Phi(x, s), t + s) \\
 &\quad - \tilde{Q}w(\Phi(x', s), t' + s)| \\
 &= |Qw_{t+s}(\Phi(x, s)) - Qw_{t'+s}(\Phi(x', s))|.
 \end{aligned}$$

We split it into the sum of two differences:

$$\begin{aligned} & |Qw_{t+s}(\Phi(x, s)) - Qw_{t'+s}(\Phi(x', s))| \\ & \leq |Qw_{t+s}(\Phi(x, s)) - Qw_{t+s}(\Phi(x', s))| + |Q(w_{t+s} - w_{t'+s})(\Phi(x', s))|. \end{aligned}$$

Since $s \leq (t + s) \wedge u$, we may use lemma 1.2 and the Lipschitz continuity assumption 1.7 on Q to obtain:

$$\begin{aligned} |Qw_{t+s}(\Phi(x, s)) - Qw_{t+s}(\Phi(x', s))| & \leq [Q][w_{t+s}]_1^{E, (t+s) \wedge u} |x - x'| \\ & \leq [Q][w]_1^{\tilde{E}, u} |x - x'|. \end{aligned}$$

For the second term, we have, from the definition of Q ,:

$$|Q(w_{t+s} - w_{t'+s})(\Phi(x', s))| \leq [w]_1^{\tilde{E}, u} |t - t'|,$$

hence, the first result:

$$|\tilde{Q}w(\tilde{\Phi}(\xi, s)) - \tilde{Q}w(\tilde{\Phi}(\xi', s))| \leq ([Q] \vee 1)[w]_1^{\tilde{E}, u} |\xi - \xi'|.$$

We now reason similarly to bound $|\tilde{Q}w(\tilde{\Phi}(\xi, \tilde{t}^*(\xi))) - \tilde{Q}w(\tilde{\Phi}(\xi', \tilde{t}^*(\xi')))|$ where $\xi = (x, t)$ and $\xi' = (x', t') \in \tilde{E}$ are such that $\tilde{t}^*(\xi) \vee \tilde{t}^*(\xi') \leq u$. Assuming, without loss of generality, that $\tilde{t}^*(x) \geq \tilde{t}^*(x')$, we have:

$$\begin{aligned} & |\tilde{Q}w(\tilde{\Phi}(\xi, \tilde{t}^*(\xi))) - \tilde{Q}w(\tilde{\Phi}(\xi', \tilde{t}^*(\xi')))| \\ & = |Qw_{t+\tilde{t}^*(x)}(\Phi(x, \tilde{t}^*(x))) - Qw_{t'+\tilde{t}^*(x')}(\Phi(x', \tilde{t}^*(x')))| \\ & \leq |Qw_{t+\tilde{t}^*(x)}(\Phi(x, \tilde{t}^*(x))) - Qw_{t+\tilde{t}^*(x)}(\Phi(x', \tilde{t}^*(x')))| \\ & \quad + |Q(w_{t+\tilde{t}^*(x)} - Qw_{t'+\tilde{t}^*(x')})(\Phi(x', \tilde{t}^*(x')))| \\ & \leq [Q]((1 + L_{t_*})[w]_*^{\tilde{E}, u} + [w]_1^{\tilde{E}, u})|x - x'| + [w]_1^{\tilde{E}, u}(|t - t'| \\ & \quad + L_{t_*}|x - x'|), \end{aligned}$$

hence, the result. □

1.6. Embedded Markov chain

Associated with a PDMP $(X_t)_{t \geq 0}$, there exists a natural embedded discrete-time Markov chain $(\Theta_n)_{n \in \mathbb{N}}$ with $\Theta_n = (Z_n, S_n)$, the sequence of post-jump locations and inter-arrival times of the process. This Markov chain contains all the information about the random part of the process $(X_t)_{t \geq 0}$. Indeed, if we know the jump times and the positions after each jump, then we can reconstruct the deterministic part of the trajectory between jumps. As we will see in Chapter 3, $(\Theta_n)_{n \in \mathbb{N}}$ will be the cornerstone of our approach in order to develop numerical techniques for PDMPs.

This section gathers technical results about this Markov chain. The first result comes directly from the definition of the dynamics of the PDMP. It specifies the law of (Z_1, S_1) given $Z_0 = X_0 = x$ for all x in E .

LEMMA 1.4.– For all w in $\mathbf{B}(E)$, w' in $\mathbf{B}(\mathbb{R}^+)$ and x in E , we have:

$$\begin{aligned} \mathbb{E}_x[w(Z_1)w'(S_1)] &= \int_0^{t^*(x)} \lambda Qw(\Phi(x, s))e^{-\Lambda(x, s)}w'(s)ds \\ &\quad + Qw(\Phi(x, t^*(x)))e^{-\Lambda(x, t^*(x))}w'(t^*(x)). \end{aligned}$$

In particular, note that the law of (Z_{n+1}, S_{n+1}) only depends on Z_n .

We now study the distribution of the jump times $(T_n)_{n \geq 1}$. For this, we need an explicit path-wise construction of the trajectories of the PDMP. Suppose, a sequence $(U_n)_{n \in \mathbb{N}}$ of independent random variables with uniform distribution on $[0; 1]$ is defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $x \in E, \omega \in \Omega$ and $t \geq 0$ and let us focus on the construction of the trajectory $(X_t(\omega))_{t \geq 0}$ of the process starting from point x . Let:

$$F(x, t) = \exp(-\Lambda(x, t))\mathbb{1}_{\{t < t^*(x)\}},$$

be the survival function of the first jump time T_1 . Define its generalized inverse as:

$$F^{\leftarrow}(x, u) = \inf\{t \geq 0 : F(x, t) \leq u\}.$$

Let then $S_1(\omega) = T_1(\omega) = F^{\leftarrow}(x, U_1(\omega))$ and for all $t < T_1(\omega)$, set:

$$X_t(\omega) = \Phi(x, t).$$

If $T_1(\omega) < +\infty$, choose X_{T_1} with distribution $Q(\cdot, \Phi(x, T_1(\omega)))$. Assume that the trajectory is constructed until time $T_n(\omega)$. If $T_n(\omega) < +\infty$, set:

$$S_{n+1}(\omega) = F^{\leftarrow}(X_{T_n}, U_n(\omega)), \quad T_{n+1}(\omega) = T_n(\omega) + S_{n+1}(\omega).$$

If $T_{n+1}(\omega) < +\infty$, choose $X_{T_{n+1}}$ with distribution $Q(\cdot, \Phi(X_{T_n}, S_{n+1}))$. The trajectory is finally constructed by induction. With this construction, we can bound the probability that the jump times are smaller than a given threshold.

LEMMA 1.5.– Let H be a survival function such that for all $t \in \mathbb{R}^+$ and for all $x \in E$, $H(t) \leq F(x, t)$. Then, there exists a sequence of independent random variables $(\tilde{S}_n)_{n \in \mathbb{N}}$ with distribution H and such that for all $T > 0$ and $N \in \mathbb{N}$:

$$\mathbb{P}(T_N < T) \leq \mathbb{P}(\tilde{T}_N < T),$$

where $\tilde{T}_N = \sum_{n=0}^N \tilde{S}_n$.

PROOF.– Let H be such a survival function and let \tilde{F}^{\leftarrow} be its generalized inverse, i.e.:

$$\tilde{F}^{\leftarrow}(u) = \inf\{t \geq 0 : H(t) \leq u\}.$$

The assumption made on H yields for all $x \in E$, $\tilde{F}^{\leftarrow}(u) \leq F^{\leftarrow}(x, u)$. For all $n \in \mathbb{N}$ and for all $\omega \in \Omega$, set:

$$\tilde{S}_n(\omega) = \tilde{F}^{\leftarrow}(U_n(\omega)).$$

Notice that we are using the same U_n as in the definition of S_n , allowing us to write that $\tilde{S}_n \leq S_n$ a.s. and therefore $\tilde{T}_n \leq T_n$ a.s. The result follows. \square

See the crack propagation model in section 1.8.4 for an example of a practical construction of H and an explicit bound for $\mathbb{P}(T_N < T)$.

1.7. Stopping times

For the sake of completeness, we present now some technical results related to the special structure of stopping times of PDMPs. This section can be skipped by readers mostly interested in applications.

For $n \in \mathbb{N}$, let \mathcal{M}_n be the family of all (\mathcal{F}_t) -stopping times which are dominated by T_n and for $n < p$, let $\mathcal{M}_{n,p}$ be the family of all (\mathcal{F}_t) -stopping times ν satisfying $T_n \leq \nu \leq T_p$.

Let τ be an $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ -stopping time. Let us recall the important result from [DAV 93].

THEOREM 1.1.– There exists a sequence $(R_n)_{n \in \mathbb{N}^*}$ of non-negative random variables such that R_n is $\mathcal{F}_{T_{n-1}}$ -measurable and $\tau \wedge T_{n+1} = (T_n + R_{n+1}) \wedge T_{n+1}$ on $\{\tau \geq T_n\}$.

In this section, important sharp properties of stopping times for PDMPs are established and discussed. They will be used in sections 7.6 and 8.2.

LEMMA 1.6.– Define $\bar{R}_1 = R_1$, and $\bar{R}_k = R_k \mathbb{1}_{\{S_{k-1} \leq \bar{R}_{k-1}\}}$. Then, we have:

$$\tau = \sum_{n=1}^{\infty} \bar{R}_n \wedge S_n.$$

PROOF.– Clearly, on $\{T_k \leq \tau < T_{k+1}\}$, we have $R_j \geq S_j$ and $R_{k+1} < S_{k+1}$ for all $j \leq k$. Consequently, by definition $\bar{R}_j = R_j$ for all $j \leq k+1$, whence:

$$\begin{aligned} \sum_{n=1}^{\infty} \bar{R}_n \wedge S_n &= \sum_{n=1}^k \bar{R}_n \wedge S_n + \{\bar{R}_{k+1} \wedge S_{k+1}\} + \sum_{n=k+2}^{\infty} \bar{R}_n \wedge S_n \\ &= T_k + R_{k+1} + \sum_{n=k+2}^{\infty} \bar{R}_n \wedge S_n. \end{aligned}$$

Since $\bar{R}_{k+1} = R_{k+1} < S_{k+1}$, we have $\bar{R}_j = 0$ for all $j \geq k+2$. Therefore, $\sum_{n=1}^{\infty} \bar{R}_n \wedge S_n = T_k + R_{k+1} = \tau$, showing the result. \square

There exists a sequence of measurable mappings $(r_k)_{k \in \mathbb{N}_*}$ defined on $E \times (\mathbb{R}_+ \times E)^{k-1}$ with value in \mathbb{R}_+ satisfying:

$$\begin{aligned} R_1 &= r_1(Z_0), \\ R_k &= r_k(Z_0, \Gamma_{k-1}), \end{aligned}$$

where $\Gamma_k = (S_1, Z_1, \dots, S_k, Z_k)$.

DEFINITION 1.1.– Consider $p \in \mathbb{N}_*$. Let $(\widehat{R}_{p,k})_{k \in \mathbb{N}_*}$ be a sequence of mappings defined on $E \times (\mathbb{R}_+ \times E)^p \times \Omega$ with value in \mathbb{R}_+ defined by:

$$\widehat{R}_{p,1}(y, \gamma, \omega) = r_{p+1}(y, \gamma),$$

and for $k \geq 2$:

$$\widehat{R}_{p,k}(y, \gamma, \omega) = r_{p+k}(y, \gamma, \Gamma_{k-1}(\omega)) \mathbf{1}_{\{S_{k-1} \leq \widehat{R}_{p,k-1}\}}(y, \gamma, \omega).$$

PROPOSITION 1.1.– Assume that $\tau \leq T_N$. Then, we have:

$$\tau = T_p + \widehat{\tau}(Z_0, \Gamma_p, \theta_{T_p}),$$

on $\{T_p \leq \tau\}$ where $\widehat{\tau}: E \times (\mathbb{R}_+ \times E)^p \times \Omega \rightarrow \mathbb{R}_+$ is defined by:

$$\widehat{\tau}(y, \gamma, \omega) = \sum_{n=1}^{N-p} \widehat{R}_{p,n}(y, \gamma, \omega) \wedge S_n(\omega). \quad [1.2]$$

PROOF.– First, let us prove by induction that for $k \in \mathbb{N}_*$, we have:

$$\widehat{R}_{p,k}(Z_0, \Gamma_p, \theta_{T_p}) = \overline{R}_{p+k}. \quad [1.3]$$

Indeed, $\widehat{R}_{p,1}(Z_0, \Gamma_p, \theta_{T_p}) = R_{p+1}$, and on the set $\{\tau \geq T_p\}$, $R_{p+1} = \overline{R}_{p+1}$. Consequently, $\widehat{R}_{p,1}(Z_0, \Gamma_p) = \overline{R}_{p+1}$. Now, assume that $\widehat{R}_{p,k}(Z_0, \Gamma_p, \theta_{T_p}) = \overline{R}_{p+k}$. Then, we have:

$$\begin{aligned} \widehat{R}_{p,k+1}(Z_0(\omega), \Gamma_p(\omega), \theta_{T_p}(\omega)) &= r_{p+k+1}(Z_0(\omega), \Gamma_p(\omega), \Gamma_k(\theta_{T_p}(\omega))) \\ &\quad \times \mathbf{1}_{\{S_k \leq \widehat{R}_{p,k}\}}(Z_0(\omega), \Gamma_p(\omega), \theta_{T_p}(\omega)). \end{aligned}$$

By definition, we have $\Gamma_k(\theta_{T_p}(\omega)) = (S_{p+1}(\omega), Z_{p+1}(\omega), \dots, S_{p+k}(\omega), Z_{p+k}(\omega))$ and $\mathbb{L}_{\{S_k \leq \widehat{R}_{p,k}\}}(Z_0(\omega), \Gamma_p(\omega), \theta_{T_p}(\omega)) = \mathbb{L}_{\{S_{p+k} \leq \overline{R}_{p+k}\}}(\omega)$ by the induction hypothesis. Therefore, we get $\widehat{R}_{p,k+1}(Z_0, \Gamma_p, \theta_{T_p}) = \overline{R}_{p+k+1}$, showing [1.3]. Combining equations [1.2] and [1.3] yields:

$$\widehat{\tau}(Z_0, \Gamma_p, \theta_{T_p}) = \sum_{n=1}^{N-n} \overline{R}_{p+n} \wedge S_{p+n}. \quad [1.4]$$

However, we have already seen that on the set $\{T \geq T_p\}$, we have $R_k = \overline{R}_k \geq S_k$, for $k \leq p$. Consequently, using equation [1.4], we obtain:

$$T_p + \widehat{\tau}(Z_0, \Gamma_p, \theta_{T_p}) = \sum_{k=1}^p S_k + \sum_{k=p+1}^N \overline{R}_k \wedge S_k = \sum_{k=1}^N \overline{R}_k \wedge S_k.$$

Since $\tau \leq T_N$, we obtain from lemma 1.6 and its proof that $\tau = \sum_{n=1}^N \bar{R}_n \wedge S_n$, showing the result. \square

PROPOSITION 1.2.– Let $(U_n)_{n \in \mathbb{N}^*}$ be a sequence of non-negative random variables such that U_n is $\mathcal{F}_{T_{n-1}}$ -measurable and $U_{n+1} = 0$ on $\{S_n > U_n\}$, for all $n \in \mathbb{N}^*$. Set:

$$U = \sum_{n=1}^{\infty} U_n \wedge S_n.$$

Then, U is an $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ -stopping time.

PROOF.– Assumption 1.1 yields:

$$\begin{aligned} \{U \leq t\} = \bigcup_{n=0}^{\infty} \left[\left(\{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{t < T_{n+1}\} \right) \right. \\ \left. \cup \left(\{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{T_{n+1} \leq t\} \right) \right]. \quad [1.5] \end{aligned}$$

From the definition of U_n , we have $\{U \geq T_n\} = \{U_n \geq S_n\}$, hence we have:

$$\begin{aligned} \{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{t < T_{n+1}\} = \{S_n \leq U_n\} \cap \{T_n + U_{n+1} \leq t\} \\ \cap \{T_n \leq t\} \cap \{t < T_{n+1}\}. \end{aligned}$$

On the one hand, $\{S_n \leq U_n\} \cap \{T_n + U_{n+1} \leq t\} \cap \{T_n \leq t\} \in \mathcal{F}_t$ by theorem 2.10 *ii*) in [ELL 82], thus we have:

$$\{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{t < T_{n+1}\} \in \mathcal{F}_t. \quad [1.6]$$

On the other hand, we have:

$$\begin{aligned} \{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{T_{n+1} \leq t\} \\ = \{S_n \leq U_n\} \cap \{U_{n+1} < S_{n+1}\} \cap \{T_{n+1} \leq t\}. \end{aligned}$$

Hence, theorem 2.10 *ii*) in [ELL 82] again yields:

$$\{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{T_{n+1} \leq t\} \in \mathcal{F}_t. \quad [1.7]$$

Combining equations [1.5], [1.6] and [1.7], we obtain the result. \square

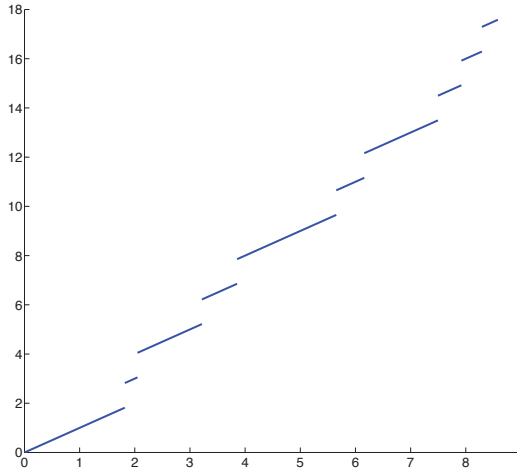


Figure 1.2. *A trajectory of the Poisson process with trend up to the 10-th jump time*

1.8. Examples of PDMPs

In this section, we give a few examples of simple PDMPs that will serve to illustrate our numerical results in the following chapters. For more involved examples in the field of reliability, the readers are referred to Chapter 2.

1.8.1. Poisson process with trend

Pure jump processes are among the simplest examples of PDMPs. Let $(N_t)_{t \geq 0}$ be a Poisson process with intensity $\lambda = 1$ and set $X_t = t + N_t$. The process $(X_t)_{t \geq 0}$ is a PDMP with state space $E = \mathbb{R}^+$ (with no boundary) and intensity $\lambda = 1$ as the inter-jump times are independent and identically distributed with exponential distribution of parameter λ . The flow is defined on $(\mathbb{R}^+)^2$ by $\Phi(x, t) = x + t$. The jump kernel satisfies $Q(x, x + 1) = 1$. An example of trajectory of the process is represented in Figure 1.2. This example is further discussed in section 5.5.

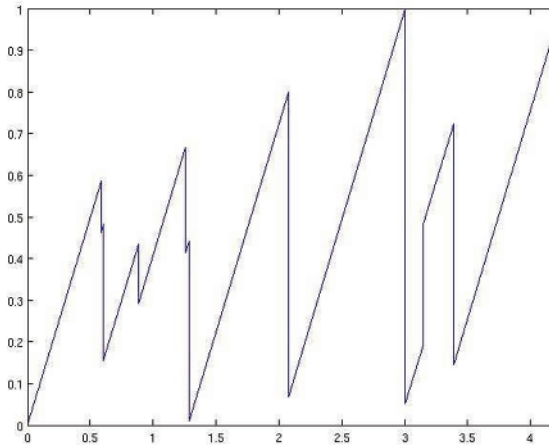


Figure 1.3. *One trajectory of the TCP process up to the 10-th jump*

1.8.2. TCP

The Transmission Control Protocol (TCP) process appears in the modeling of the famous transmission control protocol used for data transmission over the Internet. It has been designed to adapt to the various traffic conditions of the actual network. The state variable is the number of packets to be transmitted. It increases linearly at rate v until the congestion level is reached. Then, the next packet size to be transmitted is reduced at least by half.

We consider here that the state space $E = [0, 1)$ and the congestion level is $\partial E = \{1\}$. The flow is $\Phi(x, t) = x + vt$ for some positive v . We also allow random jumps with rate $\lambda(x) = \beta x^\alpha$ for some $\beta > 0$ and $\alpha \geq 1$, so that the probability to jump is higher as the boundary is closer. When a jump occurs, either randomly or at the boundary, the new position is selected according to $Q(x, \cdot)$ that is uniformly distributed on $[0, 1/2]$.

Figure 1.3 shows one sample trajectory of this process for $X_0 = 0$, $v = \alpha = 1$ and $\beta = 3$ and up to the 10-th jump. This example will be further discussed in sections 7.7 and 10.5 in relation with optimization problems.

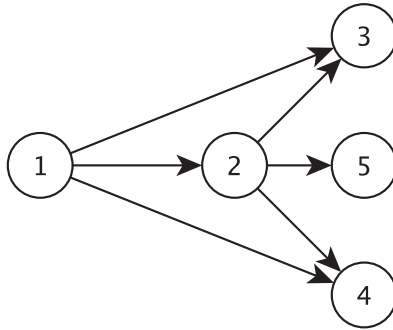


Figure 1.4. Possible transitions for the air conditioning process

1.8.3. Air conditioning unit

This example was provided by Thales Optronique SAS. We consider an air conditioning unit that can be in 5 different states:

- 1) stable;
- 2) degraded ball bearing;
- 3) failed electrovalve;
- 4) electronic failure;
- 5) ball bearing failure.

State 1 is the nominal state, state 2 is a functioning though degraded state and states 3 to 5 are failure states where the air conditioning unit no longer works. They are distinguished as they lead to different repair costs. The possible transitions between the states are given in Figure 1.4. The possible modes are naturally the 5 possible states of the system. The Euclidean variable is simply the running time in modes 1 and 2 and a cemetery state Δ in the other modes. The failure rates are constant for the ball bearing and electronic failures, but time-dependent according to a Weibull distribution for the electrovalve failure and the degradation of the ball bearing, leading to a time-dependent intensity. Note that this process has at most two jumps before reaching the cemetery state. The equipment has a lifetime of 10^5 h. Thus, the state space is $E = \{1, 2\} \times [0, 10^5) \cup \{1, 2, 3, 4, 5\} \times \{\Delta\}$. The flow in modes $m \in \{1, 2\}$ is $\Phi_m(\zeta, t) = \zeta + t$, 0 otherwise. The jump intensity in

modes 1 and 2 is:

$$\begin{aligned}\lambda(1, \zeta) &= \lambda_2(\zeta) + \lambda_3(\zeta) + \lambda_4, \\ \lambda(2, \zeta) &= \lambda_3(\zeta) + \lambda_4 + \lambda_5,\end{aligned}$$

where λ_2 and λ_3 correspond to Weibull distributions. The values of the parameters are confidential. The Markov jump kernel can be deduced from the intensities and the graph in Figure 1.4. This example will be further discussed in Chapter 9 where we study a problem of maintenance for this equipment.

1.8.4. Crack propagation model

We give here a simple version of the process studied by Chiquet and Linnios in [CHI 08], which models a crack propagation. Let $(Y_t)_{t \geq 0}$ be a real-valued process representing the crack size satisfying $Y_0 > 0$ and:

$$\frac{dY_t}{dt} = A_t Y_t,$$

for all $t \geq 0$, where $(A_t)_{t \geq 0}$ is a pure jump Markov process with state space $\{\alpha, \beta\}$ where $0 < \alpha \leq \beta$ and infinitesimal generator $\mathcal{A} = (a_{i,j}, i, j \in \{\alpha, \beta\})$. Consider the PDMP $(X_t)_{t \geq 0}$ defined by $X_t = (A_t, Y_t)$, where A_t represents the mode at time t . The state space is $E = \{\alpha, \beta\} \times (0, +\infty)$ (with no boundary). The flow is:

$$\Phi_m(\zeta, t) = (\zeta e^{mt}),$$

for $\{\alpha, \beta\}$, $\zeta > 0$ and $t \geq 0$. The intensity in mode m is constant equal to $-a_{mm}$, thus assumption 1.4 holds. The jump kernel is:

$$Q((m, \zeta), (m', \zeta')) = \mathbb{1}_{\zeta=\zeta'} \mathbb{1}_{m \neq m'}.$$

Let us compute a bound for the probability that the jump times are smaller than a given threshold T . Set $H(t) = e^{-C_\lambda t}$, for $t \geq 0$. The survival function H represents, generally speaking, the worst distribution of the inter-jump times S_n in the sense that it is the one that implies the most frequent jumps. Lemma 1.5 provides a random variable $\tilde{T}_N = \sum_{n=0}^N \tilde{S}_n$ where (\tilde{S}_n) are independent and have survival function H such that:

$$\mathbb{P}(T_N < T) \leq \mathbb{P}(\tilde{T}_N < T).$$

We now bound $\mathbb{P}(\tilde{T}_N < T)$. Standard computations with exponential laws yield $\mathbb{E}[\tilde{T}_N] = Nm$ and $\text{var}[\tilde{T}_N] = N\sigma^2$ where:

$$m = \mathbb{E}[\tilde{S}_1] = C_\lambda^{-1}, \quad \sigma^2 = \text{var}[\tilde{S}_1] = C_\lambda^{-2}.$$

Assume now that N is chosen such that $Nm > T$ and note that:

$$\mathbb{P}(\tilde{T}_N < T) \leq \mathbb{P}(|\tilde{T}_N - \mathbb{E}[\tilde{T}_N]| > \mathbb{E}[\tilde{T}_N] - T).$$

Chebychev inequality then yields:

$$\mathbb{P}(\tilde{T}_N < T) \leq N\sigma^2(Nm - T)^{-2}.$$

and that the right-hand side term goes to zero when N goes to infinity.

1.8.4. Repair workshop model

This repair workshop model is adapted from [DAV 93, section 2.1]. In a factory, a machine produces goods whose daily value is $r(x)$ where $x \in [0; 1]$ represents a parameter of evolution of the machine, a setting chosen by the operator. For instance, x may be some load or some pace imposed on the machine. This machine, initially working, may break down with age-dependent hazard rate $\lambda(t)$ and is then sent to the workshop for reparation. Moreover, the direction of the factory has decided that, whenever the machine has worked for a whole year without requiring reparation, it is sent to the workshop for maintenance. We assume that after a reparation or a maintenance, that both last a fixed time s , the machine is totally repaired and is not worn down.

We, therefore, consider three modes: the machine is working ($m = 1$), being repaired ($m = 2$) or undergoing maintenance ($m = 3$). The state of the process at time t will be denoted by $X_t = (m_t, \zeta_t)$ where ζ_t is the time since the last change of mode. This component is required since the hazard rate is age-dependent. The state space is:

$$E = (\{1\} \times [0; 365]) \cup (\{2\} \times [0; s]) \cup (\{3\} \times [0; s]).$$

In each mode m , the flow is $\Phi_m(\zeta, t) = \zeta + t$, so that we have:

$$t^*((1, \zeta)) = 365 - \zeta, \quad t^*((2, \zeta)) = t^*((3, \zeta)) = s - \zeta.$$

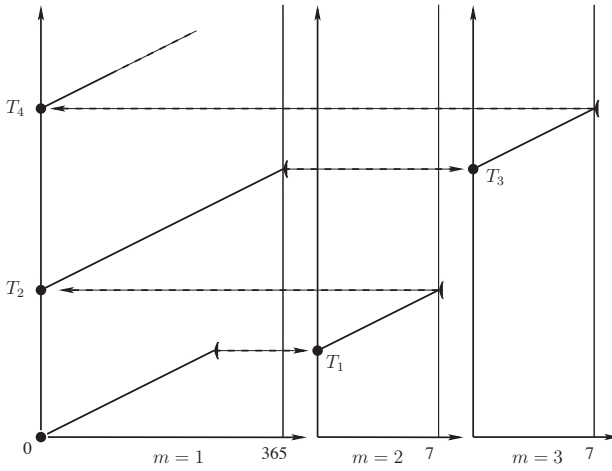


Figure 1.5. Trajectory of the repair workshop model

Therefore, the deterministic time to reach the boundary is bounded by $365 \vee s$. Concerning the transition kernel, from the point $(1, \zeta)$, the process can jump to the point $(2, 0)$ if $\zeta < 365$ and the jump is forced to $(3, 0)$ if $\zeta = 365$, thus we have for $\zeta < 365$:

$$\begin{aligned} Q((1, \zeta), (2, 0)) &= 1, & Q((1, 365), (3, 0)) &= 1, \\ Q((2, s), (1, 0)) &= 1, & Q((3, s), (1, 0)) &= 1. \end{aligned}$$

Figure 1.5 presents a sample trajectory of the process. The abscissa is the time since the last change of mode ζ_t and the coordinate is the total time elapsed since the beginning. This model will be further studied in section 4.7 where we solve an optimization problem to select parameter x so as to maximize some reward function.

We fully discuss more involved examples of PDMPs related to reliability problems in Chapter 2. Nowadays, another important field developing applications of PDMPs is biology. Although we will not focus on this field in this book, some of the ideas and techniques presented here may also be applied in that context. The interested readers may consult, for instance, the following works and references therein (not exhaustive) [AZA 14, DOU 15, FRI 14, PAK 10, RIE 12, RIE 13, ROB 14].

