1

Background

1.1. Introduction

This chapter provides some background necessary for the remainder of this book. The common operations and functions are presented first. The common transforms required for calculations are detailed in section 1.3. Then, some background on discrete and continuous probabilities is provided in section 1.4. Finally, some elements of digital signal processing are recalled in section 1.5.

1.2. Common operations and functions

1.2.1. *Convolution*

The convolution between two signals $s(t)$ and $h(t)$ is defined as:

$$r(t) = (s * h)(t) = \int_{-\infty}^{\infty} s(u)h(t-u)du \quad [1.1]$$

We will write it as $(s * h)(t) = s(t) * h(t)$ with a notational abuse.

The convolution is linear and invarious to time. Consequently, for any continuous signals $s_1(t)$ and $s_2(t)$, any complex values α_1, α_2 and any time delay t_1, t_2, we can write:

$$\begin{aligned}(\alpha_1 s_1(t-t_1) + \alpha_2 s_2(t-t_2)) * h(t) \\ = \alpha_1 (s_1 * h)(t-t_1) + \alpha_2 (s_2 * h)(t-t_2)\end{aligned} \quad [1.2]$$

1.2.2. *Scalar product*

The scalar product between two continuous signals $s(t)$ and $r(t)$ is defined as:

$$\langle s, r\rangle = \int_{-\infty}^{\infty} s(t)r^*(t)dt \tag{1.3}$$

For any continuous signals $s_1(t)$, $s_2(t)$, $r_1(t)$ and $r_2(t)$ and any complex values α_1, α_2, the following linearity properties hold:

$$\begin{aligned} \langle \alpha_1 s_1 + \alpha_2 s_2, r\rangle &= \alpha_1 \langle s_1, r\rangle + \alpha_2 \langle s_2, r\rangle \\ \langle s, \alpha_1 r_1 + \alpha_2 r_2\rangle &= \alpha_1^* \langle s, r_1\rangle + \alpha_2^* \langle s, r_2\rangle \end{aligned} \tag{1.4}$$

For vectors of size $m \times 1$, $\mathbf{s} = [s_0, s_1, ..., s_{m-1}]^T$ and $\mathbf{r} = [r_0, r_1, ..., r_{m-1}]^T$, the scalar product is similarly defined:

$$\langle \mathbf{s}, \mathbf{r}\rangle = \sum_{k=0}^{m-1} s_k r_k^* = \mathbf{r}^H \mathbf{s} \tag{1.5}$$

1.2.3. *Dirac function, Dirac impulse and Kronecker's symbol*

The continuous Dirac function, denoted as $t \mapsto \delta(t)$, is defined in the following way: for any finite-energy signal $s(t)$ and any real value τ,

$$\int_{-\infty}^{\infty} s(t)\delta(t-\tau)dt = s(t-\tau) \tag{1.6}$$

From [1.6] and [1.1], the convolution of any function $s(t)$ by a Dirac function delayed by τ is equal to the function $s(t)$ delayed by τ:

$$s(t) * \delta(t-\tau) = s(t-\tau) \tag{1.7}$$

The Dirac impulse is defined for discrete signals as follows:

$$\delta_n = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n \neq 0 \end{cases} \tag{1.8}$$

The Kronecker's symbol is a function of two variables which is equal to 1 if both variables are equal, and to 0 elsewhere.

$$\delta_{n,n_0} = \begin{cases} 1 \text{ if } n = n_0 \\ 0 \text{ if } n \neq n_0 \end{cases} \qquad [1.9]$$

As a result, the Dirac impulse is equal to the Kronecker symbol when $n_0 = 0$.

1.2.4. *Step function*

The step function is a continuous function which is equal to 0 when the input value is negative, and equal to 1 when the input value is positive:

$$u(t) = \begin{cases} 0 \text{ if } t \leq 0 \\ 1 \text{ if } t > 0 \end{cases} \qquad [1.10]$$

1.2.5. *Rectangular function*

The rectangular function is a continuous function which is equal to 0 outside of a time interval T that starts at $t = 0$:

$$\Pi_T(t) = \begin{cases} 0 \text{ if } t \notin [0, T[\\ 1 \text{ if } t \in [0, T[\end{cases} \qquad [1.11]$$

1.3. Common transforms

1.3.1. *Fourier transform*

1.3.1.1. *Fourier transform of a continuous signal*

The Fourier transform is a particularly important tool of the field of digital communications. It allows us to study a signal no longer in the time domain, but in the frequency domain. The spectral properties of a signal are more relevant to characterize it than its time properties. For instance, due to its spectral properties, a signal can be determined as baseband or passband, its bandwidth can be characterized and so on.

The Fourier transform of a continuous signal $s(t)$ is:

$$TF[s](f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt \tag{1.12}$$

It is often denoted by $S(f) = TF[s](f)$.

The inverse Fourier transform allows us to recover the time signal $s(t)$ from its Fourier transform $S(f)$:

$$TF^{-1}[S](t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}df \tag{1.13}$$

The Fourier transform is linear. In digital communications, the following property is of particular interest: the Fourier transform of a convolutional product is equal to the product of both Fourier transforms. The opposite property also stands:

$$\begin{aligned} &TF[s * h](f) = TF[s] \times TF[h] \\ &\Leftrightarrow TF[s \times h](f) = TF[s] * TF[h] \end{aligned} \tag{1.14}$$

We list here the Fourier transform of some functions that are useful in digital communications and signal processing:

– the Fourier transform of the Dirac function is a constant: $s(t) = \delta(t) \Leftrightarrow S(f) = 1$;

– the Fourier transform of the rectangular function between $-T/2$ and $T/2$ is a cardinal sine function: $s(t) = \Pi_T(t - T/2) \Leftrightarrow S(f) = T\frac{sin(\pi fT)}{\pi fT}$;

– the Fourier transform of a cosine is a sum of two Dirac functions: $s(t) = \cos(2\pi f_0 t) \Leftrightarrow S(f) = \frac{1}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right)$;

– the Fourier transform of a sine is a difference between two Dirac functions: $s(t) = \sin(2\pi f_0 t) \Leftrightarrow S(f) = \frac{1}{2}\left(\delta(f - f_0) - \delta(f + f_0)\right)$.

1.3.1.2. *Discrete Fourier transform*

Le $\mathbf{x}$ be a discrete signal composed of N samples, $\mathbf{x} = [x_0, x_1, ..., x_{N-1}]$, with sampling frequency F_e and sampling period $T_e = 1/F_e$. $x_n = x(nT_e)$ is the sample corresponding to time nT.

Its Fourier transform is defined as:

$$TF[x](f) = X(f) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n e^{-j2\pi f n T_e} \quad [1.15]$$

The discrete Fourier transform is generally determined for some discrete frequency values:

$$X\left(\frac{kF_e}{N}\right) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n e^{-j2\pi \frac{nk}{N}} \quad [1.16]$$

The N samples of the Fourier transform are denoted as $\mathbf{X} = [X_0, X_1, ..., X_{N-1}]$ with $X_k = X\left(\frac{kF_e}{N}\right)$.

The inverse discrete Fourier transform is similarly defined:

$$TF^{-1}[X](t) = x(nT_e) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X\left(\frac{kF_e}{N}\right) e^{j2\pi \frac{nk}{N}} \quad [1.17]$$

Energy is maintained between the time sample's vector $\mathbf{x}$ and the frequency sample vector $\mathbf{X}$. This is proven by Parseval's equality:

$$\begin{aligned}
\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |x(nT_e)|^2 &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(nT_e) x^*(nT_e) \\
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(nT_e) \sum_{k=0}^{N-1} X^*\left(\frac{kF_e}{N}\right) e^{-j2\pi \frac{nk}{N}} \\
&= \sum_{k=0}^{N-1} X^*\left(\frac{kF_e}{N}\right) \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(nT_e) e^{-j2\pi \frac{nk}{N}} \\
&= \sum_{k=0}^{N-1} X^*\left(\frac{kF_e}{N}\right) X\left(\frac{kF_e}{N}\right) \\
&= \sum_{k=0}^{N-1} \left| X\left(\frac{kF_e}{N}\right) \right|^2 \quad [1.18]
\end{aligned}$$

The discrete Fourier transform and its inverse can be implemented with low complexity by using the fast Fourier transform (FFT). This recursive algorithm was established by Cooley and Tuckey in 1965. For N samples, the FFT requires $N \log_2(N)$ operations, whereas a direct application of equation [1.17] would require N^2 operations.

1.3.2. *The z transform*

The z transform is used in the fields of signal processing and digital communications to model filtering operations, and especially time delays.

It is applied on discrete sampled signals $\mathbf{x} = [x_0, x_1, ..., x_{N-1}]$, and is defined as follows:

$$TZ[x](z) = X(z) = \sum_{n=0}^{N-1} x_n z^{-n} \qquad [1.19]$$

We can see that the z transform is equal, up to a proportionality factor, to the discrete Fourier transform [1.19] when $z = e^{j2\pi f T_e}$.

1.4. Probability background

Probability theory is a mathematical domain that describes and models random processes. In this section, we present a summary of this theory. We recommend for further reading [PAP 02] and [DUR 10].

Let X be an experiment or an observation that can be repeated under similar circumstances several times. At each repetition, the result of this observation is an event denoted by x , which can take several possible outcomes. The set of these values is denoted by $\mathcal{A}_X$.

The result $X = x$ of this observation is not known before it takes place. X is consequently called a random variable. It is modeled by the frequency of appearance of all its outcomes.

Two classes of random variables can be distinguished:

– discrete random variables, when the set of outcomes is discrete;

– continuous random variables, when their distribution functions are continuous.

1.4.1. *Discrete random variables*

A discrete random variable X takes its values in a discrete set, called its alphabet $\mathcal{A}_X$. This alphabet may be infinite (for instance, if $\mathcal{A}_X = \mathbb{N}$) or finite with a size n if $\mathcal{A}_X = \{x_1, x_2, \ldots, x_n\}$. Each outcome is associated with a probability of occurrence $P_X = \{p_1, p_2, \ldots, p_n\}$:

$$Pr(X = x_i) = p_i \quad p_i \geq 0 \quad \text{and} \quad \sum_{x_i \in \mathcal{A}_X} p_i = 1 \qquad [1.20]$$

For discrete random variables, the probability density $f_X(x)$ is defined by:

$$f_X(x) = \sum_{x_i \in \mathcal{A}} \delta(x - x_i) p_i \qquad [1.21]$$

where $\delta(u)$ is the Dirac function.

1.4.1.1. *Joint probability*

Let X and Y be two discrete random variables with respective sets of possible outcomes $\mathcal{A}_X = \{x_1, x_2, \ldots, x_n\}$ and $\mathcal{A}_Y = \{y_1, y_2, \ldots, y_m\}$.

$Pr(X = x_i, Y = y_j)$ is called the joint probability of the events $X = x_i$ and $Y = y_j$. Of course, the following property is verified:

$$\sum_{x_i \in \mathcal{A}_x} \sum_{y_j \in \mathcal{A}_y} Pr(X = x_i, Y = y_j) = 1 \qquad [1.22]$$

1.4.1.2. *Marginal probability*

The probability $Pr(X = x_i)$ can be computed from the set of joint probabilities $Pr(X = x_i, Y = y_j)$:

$$Pr(X = x_i) = \sum_{y_j \in \mathcal{A}_y} Pr(X = x_i, Y = y_j) \qquad [1.23]$$

1.4.1.3. *Conditional probability*

$$Pr(X = x_i|Y = y_j) = \frac{Pr(X = x_i, Y = y_j)}{Pr(Y = y_j)} \qquad [1.24]$$

Similarly, we can write:

$$Pr(Y = y_j|X = x_i) = \frac{Pr(X = x_i, Y = y_j)}{Pr(X = x_i)} \qquad [1.25]$$

As a result, the following relation stands:

$$\begin{aligned} Pr(Y = y_j, X = x_i) &= Pr(X = x_i|Y = y_j)Pr(Y = y_j) \\ &= Pr(Y = y_j|X = x_i)Pr(X = x_i) \end{aligned} \qquad [1.26]$$

which can be further developed to:

$$\begin{aligned} Pr(Y = y_j|X = x_i) &= \frac{Pr(X = x_i|Y = y_j)Pr(Y = y_j)}{Pr(X = x_i)} \\ &= \frac{Pr(X = x_i|Y = y_j)Pr(Y = y_j)}{\sum_{y_k \in \mathcal{A}_y} Pr(X = x_i, Y = y_k)} \\ &= \frac{Pr(X = x_i|Y = y_j)Pr(Y = y_j)}{\sum_{y_k \in \mathcal{A}_y} Pr(X = x_i|Y = y_k)Pr(Y = y_k)} \end{aligned} \qquad [1.27]$$

Equation [1.27] is called the Bayes law. From this equation, we can see that $Pr(X = x_i|Y = y_j)$ is the *a posteriori* probability, whereas $Pr(Y = y_i)$ is the *a priori* probability.

1.4.1.4. *Independence*

If two discrete random variables X and Y are independent, then:

$$Pr(X, Y) = Pr(X)P(Y) \qquad [1.28]$$

and

$$Pr(X|Y) = Pr(X) \qquad [1.29]$$

1.4.2. *Continuous random variables*

The random variable X is continuous if its cumulative distribution function $F_X(x)$ is continuous. $F_X(x)$ is related to the probability density in the following way:

$$f_X(x) = \frac{dF_X(x)}{dx} \quad \Leftrightarrow \quad F_X(x) = \int_{-\infty}^{x} f_X(u)du \qquad [1.30]$$

The random variable mean is defined as:

$$m_X = E\big[X\big] = \int_{-\infty}^{\infty} x f_X(x) dx \qquad [1.31]$$

Its N^{th} moment is equal to:

$$E\big[X^N\big] = \int_{-\infty}^{\infty} x^N f_X(x) dx \qquad [1.32]$$

1.4.3. *Jensen's inequality*

Let us first recall that a function $f(x)$ is convex if, for any x, y and $0 < \lambda < 1$, the following inequality stands:

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda)y) \qquad [1.33]$$

Let f be a convex function, $[x_1, \ldots, x_n]$ a real n-tuple belonging to the definition set of f and $[p_1, \ldots, p_n]$ a real positive n-tuple such that $\sum_{i=1}^{n} p_i = 1$. Thus:

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f\left(x_i\right) \qquad [1.34]$$

Jensen's inequality is obtained by interpreting the p_i terms as probabilities: if $f(x)$ is convex for any real discrete random variable X, then:

$$f\left(E[X]\right) \leq E\left[f(X)\right] \qquad [1.35]$$

1.4.4. *Random signals*

The signals used in digital communications depend on time t.

Signal $x(t)$ is deterministic if the function $t \mapsto x(t)$ is perfectly known. If, on the contrary, the values taken by $x(t)$ are unknown, the signal follows a random process. At time t, the random variable is denoted by $X(t)$, and an outcome of this random variable is denoted as $x(t)$. The set of all signal values $x(t)$, for any t in the definition domain, is a given outcome of the random process X.

A random process is defined by its probability density and statistical moments. The probability density is equal to:

$$f_X(x,t) = \lim_{\Delta x \to 0} \frac{Pr\left(x \leq X(t) \leq x + \Delta x\right)}{\Delta x} \qquad [1.36]$$

The random process is stationary if its probability density is independent of the time: $f_X(x,t) = f_X(x)\ \forall t$. As a result, all of its statistical properties are independent of t. Its probability density can thus be obtained from equation [1.30] in the following way:

$$f_X(x) = \lim_{\Delta X \to 0} \frac{F_{X+\Delta X}(x) - F_X(x)}{\Delta X} \qquad [1.37]$$

$m_x(t)$, the mean of the random variable $x(t)$ from the random process X, is defined as:

$$m_x(t) = E\big[x(t)\big] \qquad [1.38]$$

The autocorrelation function $R_{xx}(\tau)$ of a random variable is:

$$R_{xx}(t_1, t_2) = E\big[x(t_1)x^*(t_2)\big] \qquad [1.39]$$

The random process X is second-order stationary or wide-sense stationary if, for any random signal $x(t)$:

– its mean $m_x(t)$ is independent of t;

– its autocorrelation function verifies $R_{xx}(t_1, t_2) = R_{xx}(t_1 + t, t_2 + t)\ \forall t$.

Thus, it can simply be denoted as:

$$R_{xx}(\tau) = E\big[x(t)x^*(t-\tau)\big] \qquad [1.40]$$

In this case, the power spectrum density $\gamma_{xx}(f)$ is obtained by applying the Fourier transform on the autocorrelation function:

$$\begin{aligned}\gamma_{xx}(f) &= TF\left[R_{xx}\right](f)\\ &= \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-j2\pi f\tau}d\tau\end{aligned} \qquad [1.41]$$

Reciprocally, the autocorrelation function $R_{xx}(\tau)$ is determined from the power spectrum density as follows:

$$\begin{aligned}R_{xx}(\tau) &= TF^{-1}\left[\gamma_{xx}\right](\tau)\\ &= \int_{-\infty}^{+\infty} \gamma_{xx}(f)e^{+j2\pi f\tau}df\end{aligned} \qquad [1.42]$$

Generally, the mean and autocorrelation function of a stationary random process are estimated from a set of outcomes of the signal $X(t)$. When the mean over time tends to the random process's mean, the random process is ergodic. Only one time set of outcomes of the random process X is required to evaluate its mean and autocorrelation function. Most random processes that are considered in digital communications are second-order stationary and ergodic.

For discrete signals (for instance, signals that have been sampled from a continuous random signal $x(t)$ at frequency $\frac{1}{T_e}$) $x_n = x(nT_e)$, the autocorrelation function $R_{xx}(\tau)$ is only defined at discrete times $\tau = nT_e$, and the power spectrum density becomes:

$$\begin{aligned}\gamma_{xx}(f) &= TF\left[R_{xx}\right](f)\\ &= \sum_{n=-\infty}^{+\infty} R_{xx}(nT_e)e^{-j2\pi f nT_e}\end{aligned} \qquad [1.43]$$

The power spectrum density can be estimated with the periodogram. When N samples are available, it is equal to:

$$\widetilde{\gamma}_{xx}(f) = \frac{1}{NT_e} \left| \sum_{n=0}^{N-1} x(nT_e) e^{-j2\pi nT_e f} \right|^2 \quad [1.44]$$

1.4.4.1. *Power*

The power of $x(t)$ is defined as:

$$\begin{aligned} P &= \int_{-\infty}^{+\infty} \gamma_{xx}(f) df \\ &= R_{xx}(0) \\ &= E\left[|x(t)|^2 \right] \end{aligned} \quad [1.45]$$

For discrete signals, it is equal to:

$$P = E\left[|x_n|^2 \right] \quad [1.46]$$

1.4.4.2. *Energy*

The energy of a random signal $x(t)$ with finite energy is:

$$\mathcal{E} = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt \quad [1.47]$$

For discrete signals, it is equal to:

$$\mathcal{E} = \sum_{n=-\infty}^{+\infty} |x_n|^2 \quad [1.48]$$

1.4.4.3. *Characteristic function*

The characteristic function of a random variable is:

$$\Phi_x(u) = E\left[e^{iuX}\right] = \int_{-\infty}^{\infty} e^{jux} f_X(x) dx \quad [1.49]$$

1.4.4.4. *Cyclostationary signals*

Cyclostationary signals are not stationary. As a result, their autocorrelation function depends on both variables t_1 and t_2. However, the signals are wide-sense cyclostationary with period T_c if:

$$R_{xx}(t_1, t_2) = R_{xx}(t_1 + T_c, t_2 + T_c) \quad \forall (t_1, t_2) \qquad [1.50]$$

1.5. Background on digital signal processing

This section presents some digital signal processing background that is useful for digital communications. For more information on this subject, the readers can refer to [OPP 96].

1.5.1. *Sampling*

1.5.1.1. *Sampling theorem*

An analog signal $x(t)$ is expressed by a set of continuous values of t. Sampling at period T_e consists of expressing the signal x only at time multiples in T_e. The resulting sequence of samples is a digital signal, whose samples are denoted as $x_n = x(nT_e)$, for $n \in \mathbb{Z}$.

Sampling is equivalent to multiplying the analog time-domain signal $x(nT_e) = x(t)w_{T_e}(t)$ by a Dirac comb function, composed of Dirac functions at period nT_e with $n \in \mathbb{Z}$, defined by:

$$w_{T_e}(t) = \sum_{-\infty}^{\infty} \delta(t - nT_e) = \begin{cases} 1 \text{ if } t = nT_e, n \in \mathbb{Z} \\ 0 \text{ elsewhere} \end{cases} \qquad [1.51]$$

The Fourier transform of a Dirac comb with time period T_e is another Dirac comb, with frequency period $F_e = 1/T_e$:

$$TF[w_{T_e}](f) = F_e w_{F_e}(f) \qquad [1.52]$$

Since the Fourier transform of the product $x(t)w_{T_e}(t)$ is the convolution of the Fourier transforms, we get:

$$
\begin{aligned}
TF[x(t)w_{T_e}(t)](f) &= X(f) * w_{F_e}(f) \\
&= F_e \sum_{-\infty}^{\infty} X(f - nF_e)
\end{aligned}
\qquad [1.53]
$$

where $X(f)$ is the Fourier of $x(t)$.

As a result, the spectrum of the time-domain signal sampled at period T_e is equal to the spectrum of the frequency-domain signal $X(t)$ with replications at every frequency multiple of F_e.

Let us first assume that spectrum $X(f)$ is passband and band-limited, with maximum frequency f_{max}. Equation [1.53] indicates that if $F_e/2 < f_{\text{max}}$, the replications of $X(f)$ will overlap.

If, on the contrary $F_e/2 \geq f_{\text{max}}$, then the pasted versions are separated in the frequency domain. In order to recover the analog signal from the digital signal, it is necessary to apply on the digital signal spectrum [1.53] a lowpass filter with values different from zero in the interval $[-F_e/2, F_e/2]$. This allows us to extract $F_eX(f)$ only, and consequently to recover the spectrum of the original analog signal. $x(t)$ is finally obtained lossily by applying an inverse Fourier transform to $X(f)$.

This result is called Shannon's sampling theorem: a signal that does not possess any spectral component higher than f_{max} can be fully determined by the sequence of its values regularly spaced by $T_e = 1/F_e$, if and only if $F_e/2 \geq f_{\text{max}}$.

1.5.2. *Discrete, linear and time-invariant systems*

When signals are transmitted on a channel, they may be degraded by it. Passing through a channel can mathematically be written as a filtering operation, which may be continuous in the time-domain if the transmitted signal is continuous. In this case, the receiver will digitize the filtered signal. The filtering operation may also be considered in the digital domain. The channel can then be characterized by its impulse response, which corresponds to its response to a Dirac impulse at sampling times nT_e. In this section, we present a summary of the most common digital filters.

A digital filter is a discrete, linear and time-invariant system.

A system is discrete if to any input discrete sequence $\mathbf{x} = [x_n]_{\{-\infty \leq n \leq \infty\}}$ corresponds an output discrete sequence $\mathbf{y}$:

$$S[\mathbf{x}] = \mathbf{y} \quad [1.54]$$

A system is linear if for any input couple of sequences $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $S[\mathbf{x}_1] = \mathbf{y}_1$ and $S[\mathbf{x}_2] = \mathbf{y}_2$, and for any real or complex scalar a:

$$S[\mathbf{x}_1 + a\mathbf{x}_2] = \mathbf{y}_1 + a\mathbf{y}_2 \quad [1.55]$$

A system is time-invariant if to any input sequence shifted by m samples $\tilde{\mathbf{x}}$ such that $\tilde{x}_n = x_{n-m}\ \forall n \in \mathbb{Z}$ corresponds the output sequence shifted by m samples $\tilde{\mathbf{y}}$, such that $\tilde{y}_n = y_{n-m}\ \forall n \in \mathbb{Z}$:

$$S[\tilde{\mathbf{x}}] = \tilde{\mathbf{y}}\ \forall m \in \mathbb{Z} \quad [1.56]$$

A digital filtering operation possesses these three characteristics. It is identified by its impulse response $[h_n]_{n=-\infty}^{\infty}$:

$$\begin{aligned} S[\mathbf{x}]_n = y_n &= \sum_{m=-\infty}^{\infty} x_m h_{n-m}\ \forall n \in \mathbb{Z} \\ &= \sum_{m=-\infty}^{\infty} x_{n-m} h_m\ \forall n \in \mathbb{Z} \end{aligned} \quad [1.57]$$

Equation [1.57] shows that filtering is a convolution in the digital domain:

$$(\mathbf{x} * \mathbf{h})_n = \sum_{m=-\infty}^{\infty} x_{n-m} h_m\ \forall n \in \mathbb{Z} \quad [1.58]$$

The filter coefficients $[h_m]_{-\infty \leq m \infty}$ are generally causal: $h_m = 0\ \forall\ m < 0$. The filter's impulse response is equal to the sequence $[h_m]$. It is obtained by selecting a Dirac function on sample m as the system's input, since in that case:

$$(\delta_m * \mathbf{h})_n = \sum_{m'=-\infty}^{\infty} \delta_{m',m} h_{n-m'} \; \forall n \in \mathbb{Z}$$
$$= \begin{cases} h_m \text{ if } m = n \\ 0 \text{ elsewhere} \end{cases} \quad [1.59]$$

For any integer m, the system's impulse response coefficient h_m is obtained by providing as input a Dirac δ_m.

A discrete, linear and time-invariant system is stable if to any bounded input sequence corresponds a bounded output sequence. The necessary and sufficient stability condition is:

$$\sum_{n=-\infty}^{\infty} |h_n| < +\infty \quad [1.60]$$

The frequency response of a filter with impulse response $[h_n]_{n=-\infty}^{\infty}$, denoted as $H(f)$, is the Fourier transform of its impulse response:

$$H(f) = \sum_{n=-\infty}^{\infty} h_n e^{-j2pifnT_e} \quad [1.61]$$

Its transfer function is defined as the z transform of its impulse response:

$$H(z) = \sum_{n=-\infty}^{\infty} h_n z^{-n} \quad [1.62]$$

Since filtering is a convolution in time-domain (see [1.58]), by applying the Fourier transform, it becomes a product in frequency domain:

$$y_n = h_n * x_n \Leftrightarrow Y(f) = H(f)X(f) \quad [1.63]$$

As the z transform is equivalent to the discrete Fourier transform, the filtering operation can also be written as $Y(z) = H(z)X(z)$.

Consequently, the transfer function of the filter is equal to the ratio of the z transform of the output samples and the z transform of the input samples:

$$H(z) = \frac{Y(z)}{X(z)} \tag{1.64}$$

1.5.3. *Finite impulse response filters*

A filter has a finite impulse response (FIR) if its transfer function is a polynomial. It can, therefore, be expressed as:

$$H(z) = \sum_{i=0}^{N} a_i z^{-i} \tag{1.65}$$

The output samples y_n corresponding to the input samples x_n only depend on the previous x samples:

$$y_n = \sum_{i=0}^{N} a_i x_{n-i} \tag{1.66}$$

Finite impulse response filters are always stable.

1.5.4. *Infinite impulse response filters*

A filter has an infinite impulse response (IIR) if its transfer function is an algebraic fraction whose denominator is a polynomial:

$$H(z) = \frac{\sum_{i=0}^{N} a_i z^{-i}}{1 - \sum_{j=1}^{M} b_j z^{-j}} \tag{1.67}$$

The current output sample not only depends on the previous input samples, but also on the previous output samples:

$$y_n = \sum_{i=0}^{N} a_i x_{n-i} + \sum_{j=1}^{M} b_j y_{n-j} \tag{1.68}$$

Any value z_i such that $H(z_i) = 0$ is called a zero of the filter. Any p_j such that $H(p_j) = \infty$ is a pole of the filter. The transfer function can be written as a function of the zeros and poles of H:

$$H(z) = \frac{a_0 \prod_{i=0}^{N}(z - z_i)}{\prod_{j=1}^{M}(z - p_j)} \quad [1.69]$$

In order to study the stability of IIR filters, the poles must be determined. A filter is stable if and only if all its poles are strictly inside the unit circle: $|P_j| < 1 \, \forall j \in \{1, ..., M\}$.

Moreover, representing the filter by its poles and zeros allows us to determine if the filter is lowpass, passband, etc. by studying the relative locations of the poles and zeros. We can finally notice that a real filter necessarily possesses either real or complex conjugate roots.