Part 1

Elastic Solutions to Single Crack Problems

1

Fundamentals of Plane Elasticity

The purpose of this chapter is to present the solution to plane elasticity problems, based on the use of complex-valued potentials. An isotropic linear elastic behavior is considered (except in section 1.8).

1.1. Complex representation of Airy's biharmonic stress function

Let U be an Airy stress function, from which the stress components in plane elasticity conditions are derived according to:

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}; \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}; \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \qquad [1.1]$$

Let $\Pi\boldsymbol{\sigma} = \boldsymbol{\sigma} - \sigma_{zz}\underline{e}_z \otimes \underline{e}_z$ denote the projection on the plane $(\underline{e}_x, \underline{e}_y)$ of a stress tensor $\boldsymbol{\sigma}$ defined by [1.1]. It is readily proven that $\Pi\boldsymbol{\sigma}$ is given by:

$$\Pi\boldsymbol{\sigma} = (\Delta U)\,(\mathbf{1} - \underline{e}_z \otimes \underline{e}_z) - \boldsymbol{\nabla}(\boldsymbol{\nabla} U) \qquad [1.2]$$

This expression is useful for the derivation of the components of $\boldsymbol{\sigma}$ in polar coordinates as a function of the partial derivatives of U. To do so, we recall that:

$$\Delta U = U_{,rr} + \frac{1}{r}U_{,r} + \frac{1}{r^2}U_{,\theta\theta} \qquad [1.3]$$

and

$$\begin{aligned}\boldsymbol{\nabla}(\boldsymbol{\nabla}U) = {} & U_{,rr}\underline{e}_r \otimes \underline{e}_r + (\frac{1}{r}U_{,r} + \frac{1}{r^2}U_{,\theta\theta})\underline{e}_\theta \otimes \underline{e}_\theta \\ & +(\frac{1}{r}U_{,r\theta} - \frac{1}{r^2}U_{,\theta})(\underline{e}_\theta \otimes \underline{e}_r + \underline{e}_r \otimes \underline{e}_\theta)\end{aligned} \qquad [1.4]$$

Introducing [1.3] and [1.4] into [1.2], we obtain:

$$\sigma_{,rr} = \frac{1}{r}U_{,r} + \frac{1}{r^2}U_{,\theta\theta};\ \sigma_{,r\theta} = -\frac{1}{r}U_{,r\theta} + \frac{1}{r^2}U_{,\theta};\ \sigma_{,\theta\theta} = U_{,rr} \qquad [1.5]$$

Equations [1.5] are the counterpart in polar coordinates of equations [1.1]. The compatibility condition of the strains, which reads:

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0 \qquad [1.6]$$

is ensured, in the case of an isotropic linear elastic behavior, by the condition

$$\Delta\Delta U = 0 \qquad [1.7]$$

As a matter of fact, under plane stress or strain conditions, the assumption of linear isotropy allows to write the state equations in the form:

$$\begin{aligned} &\varepsilon_{xx} = A_{11}\sigma_{xx} + A_{12}\sigma_{yy} \\ &\varepsilon_{yy} = A_{12}\sigma_{xx} + A_{22}\sigma_{yy} \\ &2\varepsilon_{xy} = A_{66}\sigma_{xy} \end{aligned} \qquad [1.8]$$

Under plane stresses, the elastic compliances A_{ij} are:

$$A_{11} = A_{22} = \frac{1}{E}; \quad A_{12} = -\frac{\nu}{E}; \quad A_{66} = \frac{2(1+\nu)}{E} \qquad [1.9]$$

Under plane strains, these relations become:

$$A_{11} = A_{22} = \frac{1-\nu^2}{E}; \quad A_{12} = -\frac{\nu(1+\nu)}{E}; \quad A_{66} = \frac{2(1+\nu)}{E} \qquad [1.10]$$

In both plane strains and plane stresses, the A_{ij} satisfy:

$$2A_{12} + A_{66} = 2A_{11} = 2A_{22} \qquad [1.11]$$

Combining [1.1] with [1.10] and using [1.11], we see that condition [1.6] reduces to [1.7]. Such a biharmonic function U is now considered. Let $P = \Delta U$. By definition, P is a harmonic function. Let Q denote the conjugate function, defined up to a constant by:

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}; \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \qquad [1.12]$$

This implies that the complex-valued function $f(x+iy) = P(x,y) + iQ(x,y)$ is holomorphic, which means that the limit (with $z = x + iy$)

$$\lim_{dz \to 0} \frac{f(z+dz) - f(z)}{dz}$$

exists. Indeed, at the first order in dx and dy:

$$f(z+dz)-f(z)=\frac{\partial P}{\partial x}dx+\frac{\partial P}{\partial y}dy+i\left(\frac{\partial Q}{\partial x}dx+\frac{\partial Q}{\partial y}dy\right) \quad [1.13]$$

Using [1.12] with [1.13] yields

$$f(z+dz)-f(z)=\left(\frac{\partial P}{\partial x}+i\frac{\partial Q}{\partial x}\right)dz \quad [1.14]$$

so:

$$f'(z)=\lim_{dz\to 0}\frac{f(z+dz)-f(z)}{dz}=\frac{\partial P}{\partial x}+i\frac{\partial Q}{\partial x} \quad [1.15]$$

Following [MUS 53], consider now a primitive $\phi(z)=p+iq$ of $f(z)/4$:

$$\phi'(z)=\frac{1}{4}f(z)$$

where p and q are two conjugate harmonic functions. Therefore, we have:

$$\frac{\partial p}{\partial x}=\frac{\partial q}{\partial y}=\frac{P}{4}; \quad P=2(\phi'(z)+\overline{\phi'(z)}) \quad [1.16]$$

We can see that

$$p_1=U-px-qy$$

is harmonic, and that

$$px+qy=\mathcal{R}e(\overline{z}\phi(z))$$

Finally, let $\chi(z)$ denote the holomorphic function whose real part is p_1:

$$p_1 = \mathcal{R}e(\chi(z)) = \frac{1}{2}\left(\chi(z) + \overline{\chi(z)}\right)$$

Following these definitions, we have:

$$U = \frac{1}{2}\left(\chi + \overline{\chi} + \overline{z}\phi(z) + z\overline{\phi(z)}\right) \qquad [1.17]$$

For future purposes, let us determine the partial derivatives of U. Observing that $\partial z/\partial x = \partial \overline{z}/\partial x = 1$, we first obtain:

$$\frac{\partial U}{\partial x} = \frac{1}{2}\left(\chi'(z) + \overline{\chi'(z)} + \phi(z) + \overline{\phi(z)} + \overline{z}\phi'(z) + z\overline{\phi'(z)}\right) \qquad [1.18]$$

In turn, $\partial z/\partial y = -\partial \overline{z}/\partial y = i$ yields:

$$\frac{\partial U}{\partial y} = \frac{i}{2}\left(\chi'(z) - \overline{\chi'(z)} + \overline{\phi(z)} - \phi(z) + \overline{z}\phi'(z) - z\overline{\phi'(z)}\right) \qquad [1.19]$$

It is convenient to summarize these results in the form:

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \qquad [1.20]$$

with the notation $\psi(z) = \chi'(z)$.

1.2. Force acting on a curve or an element of arc

Let us consider a curve oriented by the tangent unit vector $\underline{t}$:

$$\underline{t} = \frac{dx}{ds}\underline{e}_x + \frac{dy}{ds}\underline{e}_y$$

where s denotes the curvilinear abscissa. The positive direction of the normal unit $\underline{n}$ is defined such that $(\underline{n}, \underline{t})$ is oriented like $(\underline{e}_x, \underline{e}_y)$. This being the case, we have:

$$\underline{n} = \underline{t} \wedge \underline{e}_z = \frac{dy}{ds}\underline{e}_x - \frac{dx}{ds}\underline{e}_y$$

Using [1.1], the components of the stress vector $\underline{T} = \boldsymbol{\sigma} \cdot \underline{n}$ read:

$$T_x = \underline{e}_x \cdot \boldsymbol{\sigma} \cdot \underline{n} = \sigma_{xx}n_x + \sigma_{xy}n_y = \frac{d}{ds}\left(\frac{\partial U}{\partial y}\right)$$
$$T_y = \underline{e}_y \cdot \boldsymbol{\sigma} \cdot \underline{n} = \sigma_{yx}n_x + \sigma_{yy}n_y = -\frac{d}{ds}\left(\frac{\partial U}{\partial x}\right) \quad [1.21]$$

The elementary force $\underline{T}ds$ acting on ds is represented by a complex dF with real and imaginary parts $T_x ds$ and $T_y ds$. Using [1.21], this yields:

$$dF = (T_x + iT_y)ds = -i\,d\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right) \quad [1.22]$$

By integration, we obtain the resultant force $\underline{F}$ acting on a given arc oriented from A to B. Introducing [1.20] into [1.22], the components F_x and F_y are given by:

$$F_x + iF_y = -i[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}]_{s_A}^{s_B} \quad [1.23]$$

The boundary conditions on a loaded arc are an important application of this result. In the following, let $f(z)$ be defined as:

$$f(z) = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} = i\int_{AB_z}(T_x + iT_y)ds + \text{Const} \quad [1.24]$$

where the point A is fixed and z denotes the affix of point B_z. $f(z)$ is a complex representation of the resultant force acting between A and B_z on the considered arc. $f(z)$ is defined up to constant.

For instance, consider a uniform pressure acting on the loaded arc:

$$T_x = -pn_x = -p\frac{dy}{ds}; \quad T_y = -pn_y = p\frac{dx}{ds}$$

or

$$(T_x + iT_y)ds = p(-dy + idx) = ip(dx + idy) = ip\, dz$$

Introducing this result into [1.24], we obtain:

$$df = -p\, dz; \quad f = -pz + \text{Const} \qquad [1.25]$$

1.3. Derivation of stresses

Consider the choice $ds = dy$ in [1.22], for which $\underline{t}$ is equal to $\underline{e}_y$ so that $\underline{n}$ is equal to $\underline{e}_x$. This implies that $T_x = \sigma_{xx}$ and $T_y = \sigma_{xy}$:

$$\begin{aligned}\sigma_{xx} + i\sigma_{xy} &= -i\frac{\partial}{\partial y}\left(\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right)\\ &= \phi'(z) + \overline{\phi'(z)} - z\overline{\phi''(z)} - \overline{\psi'(z)}\end{aligned} \qquad [1.26]$$

In turn, if $ds = -dx$, $\underline{t}$ is along $-\underline{e}_x$, so that $\underline{n} = \underline{e}_y$. Hence, we have $T_x = \sigma_{xy}$ and $T_y = \sigma_{yy}$:

$$\begin{aligned}\sigma_{yy} - i\sigma_{xy} &= \frac{\partial}{\partial x}\left(\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right)\\ &= \phi'(z) + \overline{\phi'(z)} + z\overline{\phi''(z)} + \overline{\psi'(z)}\end{aligned} \qquad [1.27]$$

Combinations of these relations successively yield:

$$\sigma_{xx} + \sigma_{yy} = 2\left(\phi'(z) + \overline{\phi'(z)}\right) = P \qquad [1.28]$$

where [1.16] has been used, and

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left(\psi'(z) + \overline{z}\phi"(z)\right) \qquad [1.29]$$

The stress components in cartesian and polar coordinates being related by:

$$\begin{aligned} &\sigma_{rr} + \sigma_{\theta\theta} = \sigma_{xx} + \sigma_{yy} \\ &\sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{r\theta} = e^{2i\theta}(\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}) \end{aligned} \qquad [1.30]$$

it is readily seen from [1.28] and [1.29] that:

$$\begin{aligned} &\sigma_{rr} + \sigma_{\theta\theta} = 2\left(\phi'(z) + \overline{\phi'(z)}\right) \\ &\sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{r\theta} = 2e^{2i\theta}\left(\psi'(z) + \overline{z}\phi"(z)\right) \end{aligned} \qquad [1.31]$$

The stresses are not modified if $\phi(z)$ is replaced by $\phi(z) + iCz + \gamma$ and if $\psi(z)$ is replaced by $\psi(z) + \gamma'$, where γ and γ' are complex-valued constants and C is a real-valued constant. Let us assume that the origin $z = 0$ is part of the domain of study. If the boundary conditions prescribe stresses only, the arbitrariness of the definition of $\phi(z)$ and $\psi(z)$ allows us to choose them in such a way that:

$$\phi(0) = 0; \quad \psi(0) = 0; \quad \mathrm{Im}\,\phi'(0) = 0 \qquad [1.32]$$

When the domain of study is infinite, another possibility is to define $\phi(z)$ and $\psi(z)$ by conditions at infinity of the form:

$$\phi(\infty) = 0; \quad \psi(\infty) = 0; \quad \mathrm{Im}\,\phi'(\infty) = 0 \qquad [1.33]$$

1.4. Derivation of displacements

In plane strains, the isotropic linear elastic constitutive equation reads:

$$2\mu\frac{\partial \xi_x}{\partial x} = \sigma_{xx} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)$$
$$2\mu\frac{\partial \xi_y}{\partial y} = \sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{yy}\right) \qquad [1.34]$$

Observing that $\sigma_{xx} = \partial^2 U/\partial y^2 = P - \partial^2 U/\partial x^2$, and using [1.16] together with [1.28], we obtain:

$$2\mu\frac{\partial \xi_x}{\partial x} = P(1-\nu) - \frac{\partial^2 U}{\partial x^2} \qquad [1.35]$$

which can be integrated in the form (see [1.16]):

$$2\mu\xi_x = 4(1-\nu)p - \frac{\partial U}{\partial x} \qquad [1.36]$$

We recall that the partial derivatives of U have been determined previously (see equations [1.18] and [1.19]).

Similarly, note that $\sigma_{yy} = \partial^2 U/\partial x^2 = P - \partial^2 U/\partial y^2$. Again, we use [1.16] and [1.28], which yields:

$$2\mu\frac{\partial \xi_y}{\partial y} = P(1-\nu) - \frac{\partial^2 U}{\partial y^2} \qquad [1.37]$$

A primitive of [1.37] reads:

$$2\mu\xi_y = 4(1-\nu)q - \frac{\partial U}{\partial y} \qquad [1.38]$$

Equations [1.36] and [1.38] define the displacement up to a rigid body motion. Finally, a combination of these equations together with [1.20] gives:

$$2\mu(\xi_x + i\xi_y) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \qquad [1.39]$$

where $\kappa = 3 - 4\nu$.

1.5. General form of the potentials ϕ and ψ

Considering applications, the domain of study S is the complex plane, except a bounded region with closed contour L. Therefore, the studied domain is non-simply connected. We aim to determine the general form of the complex-valued functions ϕ and ψ. Without loss of generality, it can be assumed that the point $z = 0$ is located within the region bounded by L, that is $z = 0 \notin S$.

Owing to [1.28], we first note that the real part of $\phi'(z)$ is single-valued, but this is possibly not the case for the imaginary part. Therefore, the integral of $\phi'(z)$ on a closed contour surrounding L is a priori not 0 and denoted by $2i\pi A$ ($A \in \mathbb{R}$). There exists a single-valued holomorphic function $F(z)$ defined on S such that:

$$\phi'(z) = A\log(z) + F(z)$$

By integration, we obtain:

$$\phi(z) = A(z\log(z) - z) + \mathcal{F}(z) \quad \text{with} \quad \mathcal{F}(z) = \int_{z_o}^{z} F(u)\,du$$

where z_o is some fixed point in S. Again, if $\mathcal{F}(z)$ is not single-valued, there exists a complex-valued constant B such that $\mathcal{F}(z) - B\,\log(z)$ is single-valued:

$$\phi(z) = Az\log(z) + B\log(z) + \phi^*(z) \qquad [1.40]$$

where $\phi^*(z)$ is a single-valued holomorphic function defined on S. A similar reasoning starting from [1.29] shows that there exists a complex-valued constant C such that:

$$\psi(z) = C\,\log(z) + \psi^*(z) \qquad [1.41]$$

where $\psi^*(z)$ is a single-valued holomorphic function defined on S.

We now recall [1.39], and take advantage of the fact that the displacement is single-valued. An anticlockwise integration around L yields:

$$2\mu[\xi_x + i\xi_y]_L = 2i\pi\left(Az(\kappa+1) + B\kappa + \overline{C}\right)$$

from which the following identities are derived:

$$A = 0; \quad B\kappa + \overline{C} = 0 \qquad [1.42]$$

We now apply [1.23] to the whole contour L:

$$F_x + iF_y = -i[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}]_L \qquad [1.43]$$

where F_x and F_y denote the components of the resultant force acting on the contour. In order for the unit normal $\underline{n}$ to point outward with respect to S, note that the contour must be oriented clockwise. Using [1.40], [1.41] and [1.42], we find that:

$$F_x + iF_y = 2\pi\left(\overline{C} - B\right)$$

Eventually, combining this result with [1.42], $\phi(z)$ and $\psi(z)$ take the form:

$$\phi(z) = -\frac{F_x + iF_y}{2\pi(1+\kappa)}\log(z) + \phi^*(z)$$
$$\psi(z) = \frac{\kappa(F_x - iF_y)}{2\pi(1+\kappa)}\log(z) + \psi^*(z) \qquad [1.44]$$

Let us finally add the assumption that the stresses are bounded at infinity. This being the case, consider the Laurent series expansions of $\phi^*(z)$ and $\psi^*(z)$ in S:

$$\phi^*(z) = \sum_{-\infty}^{+\infty} a_n z^n; \quad \psi^*(z) = \sum_{-\infty}^{+\infty} b_n z^n$$

We can easily see that [1.28] requires $a_n = 0$ for $n \geq 2$. In the same line of reasoning, [1.29] requires $b_n = 0$ for $n \geq 2$. It is therefore possible to put $\phi(z)$ and $\psi(z)$ in the form:

$$\phi(z) = -\frac{F_x + iF_y}{2\pi(1+\kappa)}\log(z) + \Gamma z + \phi_o(z)$$
$$\psi(z) = \frac{\kappa(F_x - iF_y)}{2\pi(1+\kappa)}\log(z) + \Gamma' z + \psi_o(z) \qquad [1.45]$$

where $\Gamma = \alpha + i\beta$ and $\Gamma' = \alpha' + i\beta'$ are complex-valued constants, and $\phi_o(z)$ and $\psi_o(z)$ being single-valued holomorphic (including the point at infinity) functions defined on S. This means that they can be put in the form (no strictly positive power in the series expansion):

$$\phi_o(z) = \sum_{-\infty}^{0} a_n z^n; \quad \psi_o(z) = \sum_{-\infty}^{0} b_n z^n \qquad [1.46]$$

In the case of stress boundary conditions, [1.33] allows us to choose $\beta = 0$ as well as $a_o = b_o = 0$, so that:

$$\phi_o(z) = \sum_{-\infty}^{-1} a_n z^n; \quad \psi_o(z) = \sum_{-\infty}^{-1} b_n z^n \qquad [1.47]$$

We still have to interpret $\Gamma = \alpha$ and $\Gamma' = \alpha' + i\beta'$. Introducing [1.45] into [1.28] and [1.29], and considering the limit $|z| \to \infty$, we obtain:

$$\sigma_{xx}^{\infty} = 2\alpha - \alpha'; \quad \sigma_{yy}^{\infty} = 2\alpha + \alpha'; \quad \sigma_{xy}^{\infty} = \beta' \qquad [1.48]$$

or:

$$\Gamma = \frac{1}{4}\left(\sigma_{xx}^{\infty} + \sigma_{yy}^{\infty}\right); \quad \Gamma' = \frac{1}{2}\left(\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty}\right) + i\sigma_{xy}^{\infty} \qquad [1.49]$$

1.6. Examples

For illustrative purpose, two examples are now briefly presented.

1.6.1. *Circular cavity under pressure*

Consider an infinite domain with a circular cavity (radius R) subjected to a uniform internal pressure p. The stresses at infinity are equal to 0. Since the resulting force of the stresses acting on the cavity wall is 0, [1.45] takes on the form:

$$\phi(z) = \sum_{-\infty}^{-1} a_n z^n; \quad \psi(z) = \sum_{-\infty}^{-1} b_n z^n$$

Combining [1.24] and [1.25] yields:

$$|z| = R: \qquad \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} = -pz$$

which also reads:

$$|z| = R: \quad pz + \sum_{-\infty}^{-1} a_n z^n + \sum_{-\infty}^{-1} n\overline{a}_n R^{2(n-1)} z^{2-n} + \sum_{-\infty}^{-1} \overline{b}_n R^{2n} z^{-n} = 0$$

In which we have replaced $\overline{z}$ by R^2/z for the points on the circle with radius R. This implies that $\overline{b_{-1}}R^{-2} = -p$. All the other coefficients (a_n or b_n) are equal to 0:

$$\phi(z) = 0; \quad \psi(z) = -p\frac{R^2}{z}$$

In polar coordinates, the stresses are given by:

$$\sigma_{rr} + \sigma_{\theta\theta} = 0$$

$$\sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{r\theta} = 2e^{2i\theta} p\frac{R^2}{z^2}$$

which yields:

$$\sigma_{\theta\theta} = -\sigma_{rr} = p\frac{R^2}{r^2}; \quad \sigma_{r\theta} = 0$$

1.6.2. *Circular cavity in a plane subjected to uniaxial traction at infinity*

As in the previous section, the domain S is infinite with a circular cavity centered at the origin (radius R). The stresses at infinity are defined by the tensor

$$\boldsymbol{\sigma}^{\infty} = p\underline{e}_y \otimes \underline{e}_y$$

The cavity wall is free of stress. Using [1.48], we obtain:

$$2\alpha - \alpha' = 0; \quad 2\alpha + \alpha' = p; \quad \beta' = 0$$

which yields:

$$\Gamma = \frac{p}{4}; \quad \Gamma' = \frac{p}{2}$$

$$\phi(z) = \frac{p}{4}z + \sum_{-\infty}^{-1} a_n z^n; \quad \psi(z) = \frac{p}{2}z + \sum_{-\infty}^{-1} b_n z^n$$

These expressions are introduced in the boundary condition [1.24]:

$$|z| = R: \quad \frac{p}{2}z + \sum_{-\infty}^{-1} a_n z^n + \sum_{-\infty}^{-1} n\bar{a}_n R^{2(n-1)} z^{2-n} + \frac{p}{2}\frac{R^2}{z} + \sum_{-\infty}^{-1} \bar{b}_n R^{2n} z^{-n} = 0$$

As in the previous example, the coefficients of the above power series must be equal to 0. This implies that:

$$a_{-1} = \bar{b}_{-1} = -\frac{p}{2}R^2; \quad \bar{b}_{-3} = a_{-1}R^2$$

All the other a_n and b_n are equal to 0. The corresponding functions $\phi(z)$ and $\psi(z)$ are:

$$\phi(z) = \frac{p}{4}z - \frac{p}{2}\frac{R^2}{z}; \quad \psi(z) = \frac{p}{2}z - \frac{p}{2}\frac{R^2}{z} - \frac{p}{2}\frac{R^4}{z^3}$$

from which the stress field can be derived.

1.7. Conformal mapping

Conformal mapping (see, e.g. [MUS 53]) is introduced with a view to solve the problem of the elliptic hole, which corresponds to the so-called inhomogeneity model of a crack in the plane strain framework.

1.7.1. *Application of conformal mapping to plane elasticity problems*

Consider a function ω defined on the domain Σ of the complex plane $\mathbb{C}$, valued in the codomain $S \subset \mathbb{C}$:

$$\begin{aligned} \omega :\ & \Sigma \to S \\ & \zeta \to z = \omega(\zeta) \end{aligned}$$

It is assumed that the function $\omega(\zeta)$ is holomorphic[1] on Σ and that $\omega'(\zeta) \neq 0$. This function is said to be a conformal map in the sense that it preserves the angles. More precisely, consider the complex numbers $d\zeta_1$ and $d\zeta_2$ representing two elementary vectors with origin ζ. Their images dz_1 and dz_2 are:

$$dz_i = \omega'(\zeta) d\zeta_i; \quad \mathbf{Arg}(dz_i) = \mathbf{Arg}(\omega'(\zeta)) + \mathbf{Arg}(d\zeta_i)$$

It follows that

$$\mathbf{Arg}(dz_2) - \mathbf{Arg}(dz_1) = \mathbf{Arg}(d\zeta_2) - \mathbf{Arg}(d\zeta_1)$$

If the domain Σ is bounded while the codomain S is infinite, it implies that $z = \omega(\zeta)$ approaches infinity in the neigborhood

1 Except possibly at a pole in the case of an infinite codomain S.

of some point of Σ. For instance, $z = \infty$ is the image of $\zeta = 0$ when $\omega(\zeta)$ is of the form:

$$\omega(\zeta) = \frac{C}{\zeta} + \text{holomorphic function on } \Sigma \qquad [1.50]$$

Accordingly, $\omega(\zeta) = 1/\zeta$ provides a conformal transform from the circular domain Σ, defined by $|\zeta| < 1$ centered at $\zeta = 0$ with radius 1 on the infinite codomain S defined by $|\zeta| > 1$.

Let S be the domain on which the mechanical problem is defined. The location in the complex plane is z. The preimage Σ of S by the conformal transformation $z = \omega(\zeta)$ is defined on the ζ-plane and valued in the z-plane. Let $\phi_1(z)$ and $\psi_1(z)$ denote the complex potentials of the solution sought in the z-plane. Let the functions $\phi(\zeta)$ and $\psi(\zeta)$ be defined on the ζ-plane by the change in variable $z = \omega(\zeta)$:

$$\phi(\zeta) = \phi_1(\omega(\zeta)); \quad \psi(\zeta) = \psi_1(\omega(\zeta)) \qquad [1.51]$$

Differentiating the definition [1.51] of ϕ, we obtain:

$$\phi_1'(z) = \frac{\phi'(\zeta)}{\omega'(\zeta)} \qquad [1.52]$$

It is assumed that the problem is defined on the z-plane by a loading of the type [1.24] on a contour L_z, in which $f(z)$ is given. L_ζ denoting the preimage $\omega^{-1}(L_z)$, the change in variables yields:

$$\left(\forall\zeta \in L_\zeta = \omega^{-1}(L_z)\right) \quad f(\omega(\zeta)) = \phi(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi'(\zeta)} + \overline{\psi(\zeta)} \qquad [1.53]$$

The unknowns are the functions $\phi(\zeta)$ and $\psi(\zeta)$ that should be determined from the boundary condition [1.53]. The potentials of the initial problem are then retrieved from [1.51].

In particular, an important a case occurs when the domain of study S is the complex plane with a hole bounded by the closed contour L_z. This is the subject of the following.

1.7.2. *The domain* Σ *is the unit disc* $|\zeta| \leq 1$

From now on, Σ^+ is the unit disc $|\zeta| \leq 1$. Let us assume that we know a conformal transformation $z = \omega(\zeta)$ defined on Σ^+ such that $S = \omega(\Sigma^+)$ (Figure 1.1). We have seen that ω is of the form [1.50]. The contour L_ζ is the circle γ ($|\zeta| = 1$). In this section, we aim to show that [1.53] provides a functional equation for the unknown $\phi(\zeta)$. The other unknown $\psi(\zeta)$ can then be explicitly derived from the solution to the latter. The loading considered herein is defined by stress boundary conditions. Owing to [1.33] written for $\phi_1(z)$ and $\psi_1(z)$ at infinity, and observing that $\zeta = 0$ is the preimage of $z = \infty$ by a transformation ω of the form [1.50], it is convenient to introduce the corresponding conditions at $\zeta = 0$:

$$\phi(0) = 0; \quad \psi(0) = 0 \qquad [1.54]$$

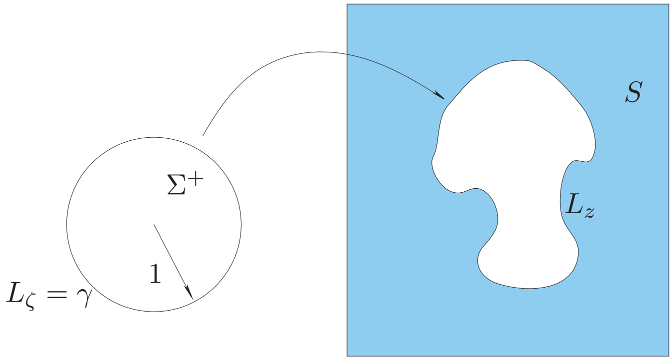

Figure 1.1. *The domain* S *mapped on the disc* $|\zeta| < 1$

To begin with, the resultant force acting on L_z is equal to 0 while the stresses tend to 0 at infinity ($\boldsymbol{\sigma}^\infty = 0$). It follows that functions $\phi_1(z)$ and $\psi_1(z)$ are holomorphic on S, including the point $z = \infty$ [2]. This implies that the functions $\phi(\zeta)$ and $\psi(\zeta)$ are holomorphic in Σ^+.

Let us start with the determination of $\psi(\zeta)$ in the domain $|\zeta| < 1$ under the assumption that $\phi(\zeta)$ has been determined. Since $\psi(\zeta)$ is holomorphic in the domain $|\zeta| \leq 1$, Cauchy's formula reads:

$$|\zeta| < 1 \qquad \psi(\zeta) = \frac{1}{2i\pi} \int_\gamma \frac{\psi(\sigma)}{\sigma - \zeta}\, d\sigma \qquad [1.55]$$

Introducing the expression of $\psi(\sigma)$ derived from [1.53] into [1.55] yields:

$$|\zeta| < 1 \quad \psi(\zeta) = \frac{1}{2i\pi} \int_\gamma \frac{\overline{f} d\sigma}{\sigma - \zeta} - \frac{1}{2i\pi} \int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \frac{\phi'(\sigma) d\sigma}{\sigma - \zeta} \qquad [1.56]$$

where formula [1.93] of Appendix 1.9 has been used as well as the condition $\overline{\phi(0)} = 0$ (see equation [1.54]); [1.56] states that $\psi(\zeta)$ can be determined in Σ^+ provided that $\phi'(\sigma)$ is known on the edge γ of the disc.

Let us now move to the determination of $\phi(\zeta)$ within the unit disc. The starting point is the theorem [1.93] written for $\psi(\zeta)$ and combined with [1.54]. We obtain:

$$|\zeta| < 1: \qquad \frac{1}{2i\pi} \int_\gamma \frac{\overline{\psi(\sigma)}}{\sigma - \zeta}\, d\sigma = \overline{\psi(0)} = 0 \qquad [1.57]$$

2 This means that these functions can be expanded in a power series of the form $\sum_{k \geq 0} a_k z^{-k}$ in the neighborhood of infinity.

Then, the value $\psi(\sigma)$ of $\psi(\zeta)$ on the edge γ of the disc is taken from [1.53]:

$$|\sigma| = 1 \qquad \overline{\psi(\sigma)} = f(\omega(\sigma)) - \phi(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\phi'(\sigma)} \tag{1.58}$$

Introducing this value into [1.57] yields:

$$\forall\zeta \text{ such that } |\zeta| < 1: \quad \phi(\zeta) + \frac{1}{2i\pi}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\phi'(\sigma)}}{\sigma - \zeta}\,d\sigma = \frac{1}{2i\pi}\int_\gamma \frac{f\,d\sigma}{\sigma - \zeta} \tag{1.59}$$

where Cauchy's formula:

$$|\zeta| < 1 \qquad \phi(\zeta) = \frac{1}{2i\pi}\int_\gamma \frac{\phi(\sigma)}{\sigma - \zeta}\,d\sigma \tag{1.60}$$

has been used; [1.59] is a functional equation with respect to the unknown $\phi(\zeta)$.

Let us now consider the situation when the asymptotic stress state $\boldsymbol{\sigma}^\infty$ and the resultant force acting on the contour L_z (with components F_x and F_y) are possibly not equal to 0. The general form of functions $\phi_1(z)$ and $\psi_1(z)$ is provided by [1.45]. Recalling the form [1.50] of the conformal transformation defined on Σ^+, this implies that $\phi(\zeta)$ and $\psi(\zeta)$ are of the following type:

$$\begin{aligned} \phi(\zeta) &= \frac{F_x + iF_y}{2\pi(1+\kappa)}\log(\zeta) + \frac{\Gamma C}{\zeta} + \phi_o(\zeta) \\ \psi(\zeta) &= -\frac{\kappa(F_x - iF_y)}{2\pi(1+\kappa)}\log(\zeta) + \frac{\Gamma' C}{\zeta} + \psi_o(\zeta) \end{aligned} \tag{1.61}$$

where $\phi_o(\zeta)$ and $\psi_o(\zeta)$ are holomorphic on Σ^+. However, since $\phi(\zeta)$ and $\psi(\zeta)$ are not holomorphic in $\zeta = 0$, the identities [1.56] and [1.59] are no longer valid and the reasoning which has led to them must be modified. The idea consists of introducing [1.61] into [1.58], which now takes the form:

$$|\sigma| = 1 \qquad \overline{\psi_o(\sigma)} = f_o - \phi_o(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\phi_o'(\sigma)} \qquad [1.62]$$

in which f was replaced by f_o, defined by:

$$f_o = f - \frac{F_x + iF_y}{2\pi}\log\sigma - \frac{\Gamma C}{\sigma} - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\left(\frac{F_x - iF_y}{2\pi(1+\kappa)}\sigma - \Gamma\overline{C}\sigma^2\right) - \overline{\Gamma}'\,\overline{C}\sigma \qquad [1.63]$$

Since the functions ϕ_o and ψ_o are holomorphic, they can be derived from [1.56] and [1.59] in which the following changes have to be made:

$$f \to f_o, \quad \phi \to \phi_o, \quad \psi \to \psi_o$$

1.7.3. *The domain Σ is the complement Σ^- of the unit disc*

From now on, Σ^- denotes the complement of the unit disc. Hence, $\zeta \in \Sigma^-$ is equivalent to $|\zeta| \geq 1$. We assume that S is mapped by a conformal transformation $z = \omega(\zeta)$ defined on Σ^-. Furthermore, the edge γ of the unit disc is the preimage of the contour L_z ($L_\zeta = \gamma$). The general form of the transformation $\omega(\zeta)$ is

$$\omega(\zeta) = R\zeta + \omega_o(\zeta) \qquad [1.64]$$

where $R \neq 0$ is a constant and $\omega_o(\zeta)$ is holomorphic in Σ^- including at infinity.

We aim to show that [1.53] provides a functional equation for the unknown $\phi(\zeta)$. The other unknown $\psi(\zeta)$ can then be explicitly derived from the solution to the latter. As discussed in section 1.7.2, we restrict to stress boundary conditions, and we start with a loading in which $F_x = F_y = \Gamma = \Gamma' = 0$ (see equation [1.45]). In other words, the functions $\phi_1(z)$ and $\psi_1(z)$ are holomorphic, including the point $z = \infty$. This holds true for the functions $\phi(\zeta)$ and $\psi(\zeta)$, which are also holomorphic, including $\zeta = \infty$.

Let us apply theorem [1.96] of Appendix 1.9 to the function $\psi(\zeta)$ in [1.53]:

$$\big(\forall\sigma \in \gamma = \omega^{-1}(L_z)\big) \quad \overline{\psi(\sigma)} = f(\omega(\sigma)) - \phi(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\phi'(\sigma)} \quad [1.65]$$

This identity is first divided by $\sigma - \zeta$ and then integrated along γ. This yields:

$$|\zeta| > 1: \quad \frac{1}{2i\pi}\int_\gamma \frac{f\,d\sigma}{\sigma - \zeta} + \phi(\zeta) - \frac{1}{2i\pi}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\phi'(\sigma)}}{\sigma - \zeta}\,d\sigma = 0 \quad [1.66]$$

where theorem [1.94] was applied to $\phi(\zeta)$ and the relation $\phi(\infty) = 0$ was used (see equation [1.33]); [1.66] constitutes the functional equation with respect to the unknown $\phi(\zeta)$. It is the counterpart of [1.59] for the domain $\Sigma = \Sigma^-$.

We still have to determine $\psi(\zeta)$. To do so, theorem [1.94] is applied together with the condition $\psi(\infty) = 0$:

$$|\zeta| > 1: \qquad \psi(\zeta) = -\frac{1}{2i\pi}\int_\gamma \frac{\psi(\sigma)}{\sigma - \zeta}\,d\sigma \quad [1.67]$$

Hence, the value of $\psi(\sigma)$ follows from [1.65]:

$$\big(\forall\sigma \in \gamma = \omega^{-1}(L_z)\big) \quad \psi(\sigma) = \overline{f(\omega(\sigma))} - \overline{\phi(\sigma)} - \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\phi'(\sigma) \quad [1.68]$$

Again using theorem [1.96], applied to $\phi(\zeta)$, it appears that:

$$|\zeta| > 1 : \ \psi(\zeta) = -\frac{1}{2i\pi}\int_\gamma \frac{\overline{f}}{\sigma-\zeta}\,d\sigma + \frac{1}{2i\pi}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\phi'(\sigma)}{\sigma-\zeta}\,d\sigma \quad [1.69]$$

We now consider the situation when the resultant force acting on $\mathcal{L}_z$ and the asymptotic stress state are possibly not equal to 0.

We start with the expressions $\phi_1(z)$ and $\psi_1(z)$ given in [1.45]. Owing to [1.51] and recalling the expression [1.64] of $\omega(\zeta)$, the potentials $\phi(\zeta)$ and $\psi(\zeta)$ defined on Σ^- are found in the form:

$$\begin{aligned} \phi(\zeta) &= -\frac{F_x+iF_y}{2\pi(1+\kappa)}\log(\zeta) + \Gamma R\zeta + \phi_o(\zeta) \\ \psi(\zeta) &= \frac{\kappa(F_x-iF_y)}{2\pi(1+\kappa)}\log(\zeta) + \Gamma' R\zeta + \psi_o(\zeta) \end{aligned} \quad [1.70]$$

where $\phi_o(\zeta)$ and $\psi_o(\zeta)$ are holomorphic on Σ^- (including the point at infinity).

We now revisit the previous reasoning starting from equation [1.65]: this equation is written in terms of $\phi_o(\zeta)$ and $\psi_o(\zeta)$, and replaces f by a new function f_o defined on γ by:

$$\begin{aligned} f_o = f - \frac{\overline{\Gamma'}R}{\sigma} - \Gamma R\left(\sigma + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\right) \\ + \frac{F_x-iF_y}{2\pi(1+\kappa)}\sigma\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} + \frac{F_x+iF_y}{2\pi}\log\sigma \end{aligned} \quad [1.71]$$

It is now possible to determine $\phi_o(\zeta)$ and $\psi_o(\zeta)$ from [1.66]:

$$|\zeta| > 1\text{: } \frac{1}{2i\pi}\int_\gamma \frac{f_o d\sigma}{\sigma - \zeta} + \phi_o(\zeta) - \frac{1}{2i\pi}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\phi'_o(\sigma)}}{\sigma - \zeta} d\sigma = 0 \quad [1.72]$$

and [1.69]:

$$|\zeta| > 1\text{: } \psi_o(\zeta) = -\frac{1}{2i\pi}\int_\gamma \frac{\overline{f}_o}{\sigma - \zeta} d\sigma + \frac{1}{2i\pi}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\phi'_o(\sigma)}{\sigma - \zeta} d\sigma \quad [1.73]$$

1.8. The anisotropic case

1.8.1. *General features*

We now consider the anisotropic case, in plane strains or plane stress conditions. It is assumed that the plane of study is a plane of material symmetry. The linear elastic constitutive equations therefore read:

$$\begin{aligned} \varepsilon_{xx} &= A_{11}\sigma_{xx} + A_{12}\sigma_{yy} + A_{16}\sigma_{xy} \\ \varepsilon_{yy} &= A_{12}\sigma_{xx} + A_{22}\sigma_{yy} + A_{26}\sigma_{xy} \\ 2\varepsilon_{xy} &= A_{16}\sigma_{xx} + A_{26}\sigma_{yy} + A_{66}\sigma_{xy} \end{aligned} \quad [1.74]$$

In the following, a_{ij} (respectively, b_{ij}) will denote the coefficients A_{ij} in plane stress (respectively, plane strain) conditions. Coefficients a_{ij} are directly related to the coefficients of the tensor of compliance:

$$\begin{aligned} &a_{11} = S_{1111}; \quad a_{22} = S_{2222}; \quad a_{12} = S_{1122} \\ &a_{16} = 2S_{1112}; \quad a_{26} = 2S_{2212}; \quad a_{66} = 4S_{1212} \end{aligned} \quad [1.75]$$

The plane strain coefficients b_{ij} are related to a_{ij} according to $(i, j = 1, 2, 6)$:

$$b_{ij} = a_{ij} - a_{i3}\frac{a_{3j}}{a_{33}} \quad [1.76]$$

with

$$a_{31} = S_{1133}, \quad a_{32} = S_{2233}, \quad a_{33} = S_{3333}, \quad a_{36} = 2S_{3312}$$

In the isotropic case, the conditions of geometrical compatibility [1.6] imply that the Airy function U (see equation [1.1]) is biharmonic $\Delta\Delta U = 0$ (see section 1.1). Owing to [1.74], they now take the form:

$$A_{22}\frac{\partial^4 U}{\partial x^4} - 2A_{26}\frac{\partial^4 U}{\partial x^3 \partial y} + (2A_{12} + A_{66})\frac{\partial^4 U}{\partial x^2 \partial y^2} - 2A_{16}\frac{\partial^4 U}{\partial x \partial y^3} + A_{11}\frac{\partial^4 U}{\partial y^4} = 0 \qquad [1.77]$$

Observing that all terms involve a fourth–order derivative, we find solutions in the form $f(x + \mu y)$, where μ is a complex constant and $f(z)$ is a differentiable function of the (*a priori*) complex variable $z = x + \mu y$. It appears that [1.77] will be satisfied provided that μ is a root of the polynomial equation:

$$A_{22} - 2A_{26}\mu + (2A_{12} + A_{66})\mu^2 - 2A_{16}\mu^3 + A_{11}\mu^4 = 0 \qquad [1.78]$$

The latter has two pairs of conjugate roots, respectively, μ_1, $\overline{\mu}_1$ and μ_2, $\overline{\mu}_2$. By convention, the imaginary parts of μ_1 and μ_2 are positive. Among other classical relations between coefficients of [1.78], let us note in particular that:

$$\frac{A_{22}}{A_{11}} = |\mu_1|^2|\mu_2|^2 \qquad [1.79]$$

Since $U(x, y)$ is a real-valued function, we find the solution to [1.77] in the form:

$$U(x, y) = 2\mathcal{R}e\left(f_1(x + \mu_1 y) + f_2(x + \mu_2 y)\right) \qquad [1.80]$$

where $f_1(z_1)$ and $f_2(z_2)$ are two differentiable functions of the two distinct complex variables:

$$z_1 = x + \mu_1 y; \quad z_2 = x + \mu_2 y \qquad [1.81]$$

1.8.2. *Stresses, displacements and boundary conditions*

Following [LEK 63] (see also [SIH 65]), let us introduce the complex potentials $\phi_1(z_1) = f_1'(z_1)$ and $\phi_2(z_2) = f_2'(z_2)$. From the definition [1.1] of the Airy function, we obtain:

$$\begin{aligned} \sigma_{xx} &= 2\mathcal{R}e\left(\mu_1^2\phi_1' + \mu_2^2\phi_2'\right) \\ \sigma_{yy} &= 2\mathcal{R}e\left(\phi_1' + \phi_2'\right) \\ \sigma_{xy} &= -2\mathcal{R}e\left(\mu_1\phi_1' + \mu_2\phi_2'\right) \end{aligned} \qquad [1.82]$$

In the case of plane strains, it is sufficient to determine the components ξ_x and ξ_y of the displacement. To do so, let us put the strains given by [1.74] (with $A_{ij} = b_{ij}$) in the form:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial \xi_x}{\partial x} = 2\mathcal{R}e\left(p_1\phi_1' + p_2\phi_2'\right) \\ \varepsilon_{yy} &= \frac{\partial \xi_y}{\partial y} = 2\mathcal{R}e\left(q_1\mu_1\phi_1' + q_2\mu_2\phi_2'\right) \end{aligned} \qquad [1.83]$$

with the following notations ($j = 1,\ 2$):

$$p_j = b_{11}\mu_j^2 + b_{12} - b_{16}\mu_j; \quad q_j = b_{12}\mu_j + \frac{b_{22}}{\mu_j} - b_{26} \qquad [1.84]$$

Integration gives (up to a rigid body motion):

$$\begin{aligned} \xi_x &= 2\mathcal{R}e\left(p_1\phi_1 + p_2\phi_2\right) \\ \xi_y &= 2\mathcal{R}e\left(q_1\phi_1 + q_2\phi_2\right) \end{aligned} \qquad [1.85]$$

Moreover, the derivation of the resultant force acting on an oriented arc element – such as that presented in section 1.2 – can be directly used. [1.22] provides the resultant components along $\underline{e}_x$ and $\underline{e}_y$ of this force as a function of the curvilinear abscissa:

$$\begin{aligned} F_x(s) &= \int_0^s T_x\, ds \;=\; \frac{\partial U}{\partial y} + \text{Const.} \\ F_y(s) &= \int_0^s T_y\, ds \;=\; -\frac{\partial U}{\partial x} + \text{Const.} \end{aligned} \qquad [1.86]$$

Using [1.80] again, the conditions on a loaded edge are put in the form:

$$\begin{aligned} &2\mathcal{R}e\left(\mu_1\phi_1(z_1) + \mu_2\phi_2(z_2)\right) = F_x + \text{ Const.} \\ &2\mathcal{R}e\left(\phi_1(z_1) + \phi_2(z_2)\right) = -F_y + \text{ Const.} \end{aligned} \qquad [1.87]$$

1.9. Appendix: mathematical tools

As mentioned, γ denotes the unit circle ($|\sigma| = 1$). Cauchy's classical formula states that:

$$|\zeta| > 1 : \quad \int_\gamma \frac{d\sigma}{\sigma - \zeta} = 0 \qquad [1.88]$$

whereas

$$|\zeta| < 1 : \quad \int_\gamma \frac{d\sigma}{\sigma - \zeta} = 2i\pi \qquad [1.89]$$

Furthermore, a partial fraction decomposition reads:

$$k \geq 1 : \quad \frac{1}{\sigma^k(\sigma - \zeta)} = -\sum_{j=1}^{k} \frac{1}{\zeta^j}\frac{1}{\sigma^{k+1-j}} + \frac{1}{(\sigma - \zeta)}\frac{1}{\zeta^k} \qquad [1.90]$$

It follows that:

$$k \geq 1, \ |\zeta| > 1 : \quad \frac{1}{2i\pi} \int_{\gamma} \frac{d\sigma}{\sigma^k(\sigma - \zeta)} = -\frac{1}{\zeta^k} \qquad [1.91]$$

whereas

$$k \geq 1, \ |\zeta| < 1 : \quad \frac{1}{2i\pi} \int_{\gamma} \frac{d\sigma}{\sigma^k(\sigma - \zeta)} = 0 \qquad [1.92]$$

These identities are helpful to establish the following results.

1.9.1. *Theorem 1*

If $f(\zeta)$ is a holomorphic function on the unit disc Σ^+, then:

$$|\zeta| < 1 : \qquad \frac{1}{2i\pi} \int_{\gamma} \frac{\overline{f(\sigma)}}{\sigma - \zeta} \, d\sigma = \overline{f(0)} \qquad [1.93]$$

This result is readily established by means of a power series expansion $\sum_{k \geq 0} a_k \zeta^k$ of $f(\zeta)$. First, considering the boundary γ of the disc, we have:

$$|\sigma| = 1 : \qquad \overline{f(\sigma)} = \sum_{k \geq 0} \overline{a}_k \sigma^{-k}$$

which is introduced into the left-hand side of [1.93]. The theorem then immediately follows from [1.89] and [1.92].

1.9.2. *Theorem 2*

The matter of the present theorem is Cauchy's formula for an infinite domain: If $f(z)$ is holomorphic in Σ^- ($|\zeta| \geq 1$) (including the point at infinity), then:

$$|\zeta| > 1: \qquad \frac{1}{2i\pi}\int_\gamma \frac{f(\sigma)}{\sigma - \zeta}\, d\sigma = f(\infty) - f(\zeta) \qquad [1.94]$$

This can be proved from a Laurent power series of $f(\zeta)$ (with no positive power since $f(\zeta)$ is holomorphic at infinity):

$$f(\zeta) = \sum_{k=-\infty}^{0} a_k \zeta^k \qquad [1.95]$$

A direct application of [1.91] and [1.88] yields

$$|\zeta| > 1: \quad \frac{1}{2i\pi}\int_\gamma \frac{f(\sigma)}{\sigma - \zeta}\, d\sigma = -\sum_{k=-\infty}^{-1} a_k \zeta^k$$

and the theorem [1.94] is established.

1.9.3. *Theorem 3*

The counterpart of theorem [1.93] for a holomorphic function in Σ^- (including the point at infinity) reads:

$$|\zeta| > 1: \qquad \frac{1}{2i\pi}\int_\gamma \frac{\overline{f(\sigma)}}{\sigma - \zeta}\, d\sigma = 0 \qquad [1.96]$$

Again, the proof is based on a power series expansion of $f(\zeta)$ in the form [1.95]. Considering in particular the unit circle γ ($\overline{\sigma} = 1/\sigma$):

$$\overline{f(\sigma)} = \sum_{-\infty}^{0} \overline{a}_k \sigma^{-k}$$

Then, observing that the function σ^{-k} ($k \leq 0$) is holomorphic on the unit disc, it is readily seen that

$$k \leq 0,\ |\zeta| > 1: \qquad \int_{\gamma} \frac{\sigma^{-k} d\sigma}{\sigma - \zeta} = 0$$

The theorem [1.96] immediately follows from the two last equations.