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# Building a Model for a Coupled Problem

There are numerous and varied heat and humidity exchange coupled problems in the environment, and more specifically in man's surrounding environment (comfort, habitat, clothing, etc.) and a common methodology to approach these can be established. First of all, we need to position ourselves in relation to the digital/IT tools currently offered in "the market" and which allow for a resolution of numerous physics problems. The readers may be under the impression that the difficulty resides rather in making a choice among all these tools/software. It is common for specialized research departments to use software adapted to their fields (habitat, aviation, automobile, etc.) though a layman perceives them as some sort of magic "black box". When results come out, the reliability interval is often uncertain, as the given problem was never treated for a neighboring configuration. It should be noted that solving a mathematical model numerically with elaborated software presupposes the formulation of a number of simplifying hypotheses that may be valid for a given configuration, but risky for another. To take an extreme example, outside of our field of study, media report on the progress of IPCC works concerning climate heating predictions while they highlight the uncertainty of 20-year predictions. At planetary scale, ocean/atmosphere models are particularly complex.

Let us therefore consider a "system" whose thermal and hydric behavior in particular conditions is to be determined: an individual in a room, a manned vehicle, an incubator, a piece of sportswear, etc. It is always possible to set the proper orders of magnitude for the behavior of a system under thermal constraints by scale analysis of the equations of an adapted model and by using the theoretical and experimental data in the literature. This first model can be preliminary to the use of software that is more complex but more difficult to interpret under the relative influence of input parameters. Through several simple examples, we will examine the implementation of such models.

## 1.1. Basic equations of the models (Appendix 1)

A fluid medium (humid air, liquid water, etc.) put in motion by a machine (forced ventilation, pump, etc.), wind, temperature gradients (natural convection), can be described by a number of variables depending on space and time: pressure p, temperature T, velocity $\vec{V}$, density ρ, enthalpy h, etc.

The (quite) general conservation equations given here are written in condensed notation, using a pseudo vector $\vec{\nabla}$ (nabla), or gradient, which in Cartesian coordinates x, y, z can be written: $\vec{\nabla}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)$.

Mass conservation can be written as:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\rho\vec{V} = 0 \qquad [1.1a]$$

Or if we use the differential operator in the direction of movement $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}.\vec{\nabla}$ , we then have:

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}.\vec{V} = 0 \qquad [1.1b]$$

For vapor contained in incompressible air, mass conservation is written as:

$$\frac{D\rho_v}{Dt} = \vec{\nabla}.(D_v\vec{\nabla}\rho_v) \qquad [1.2]$$

This equation is based on Fick's law of diffusion, which gives the mass diffusive flux $\vec{J_v}$ (kg/m$^2$s) of the vapor species ($\rho_v$) in the air (ρ): $\vec{J_v} = -D_v\vec{\nabla}\rho_v$, where $D_v$ is the diffusion coefficient. This law is valid for humid air with $\rho_v \ll \rho$. For gas mixtures where this order of magnitude is no longer valid, a more precise law is applied.

The momentum conservation for the "volume forces" limited to gravity, for "Newtonian" fluids with constant viscosity coefficient $\mu$ is written as:

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} p + \rho \vec{g} + \mu \nabla^2 \vec{V} \quad [1.3]$$

The first member is the inertia term and the second member contains pressure, gravity and viscosity terms.

The energy conservation or the first law of thermodynamics is written as:

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \vec{\nabla}.\left(\lambda \vec{\nabla} T\right) + \mu \Phi \quad [1.4]$$

The first member expresses enthalpy conservation, and the second one contains the "heat" corresponding to compression or expansion, an important quantity for certain machines, the conductive/convective transfer and the viscous dissipation. In certain cases, a radiation energy term can be added. Viscous dissipation, always positive, is expressed as a function of the derivatives of velocity components. At low velocities it is negligible. Therefore, the most common form of [1.4] applicable at low velocities is:

$$\rho c_p \left(\frac{\partial T}{\partial t} + \vec{V}.\vec{\nabla} T\right) = \vec{\nabla}.\left(\lambda \vec{\nabla} T\right) \quad [1.5]$$

We should note that equations [1.2] and [1.5] are similar in form, and this will have very practical analogical consequences.

We add to these the so-called equations of "state" of the fluid. For a perfect gas, which is the case of air at human environment temperature and atmospheric pressure, we have:

$$p = \rho r T \,; Dh = c_p DT \quad [1.6]$$

For a liquid at moderate pressures and temperatures we have:

$$\rho = cst \,;\ c_p = c_v = c \,; Dh = cDT + \frac{Dp}{\rho} \quad [1.7]$$

## 1.2. Boundary layers

A fluid in movement is limited by a solid wall surrounded by "boundary layers", regions that despite their low thickness play a key role in the exchanges (of momentum, heat, etc.), and where important gradients (of velocity, temperature, etc.) develop. This concept originated in the development of aviation in the 1920s. Airplanes with airfoils that generated the best lift for a wide incidence range had to be designed.

### 1.2.1. *Forced convection [SCH 60]*

The typical example treated by fluid mechanics texts is that of a heated cylinder, with diameter D, placed perpendicular to a uniform fluid flow of velocity V. The development of boundary layers (in terms of velocity and temperature) starting from the stagnation point, then a detachment that generates unsteady wake behind the cylinder can be observed (Figure 1.1). Another velocity layer accompanies the wake's vortices. The thermal boundary layer, for heat conduction "across" the stream filaments, is continuous and adapts to velocity heterogeneities. The Reynolds number Re = VD/υ, where υ is the kinematic viscosity, describes the flow regime as subcritical when the dynamic boundary layer remains laminar before detachment, or supercritical when the dynamic boundary layer transits from laminar to turbulent regime before detachment. The point of detachment moves downstream when turbulence emerges in the boundary layer. The critical Reynolds number is close to $4.5.10^5$.

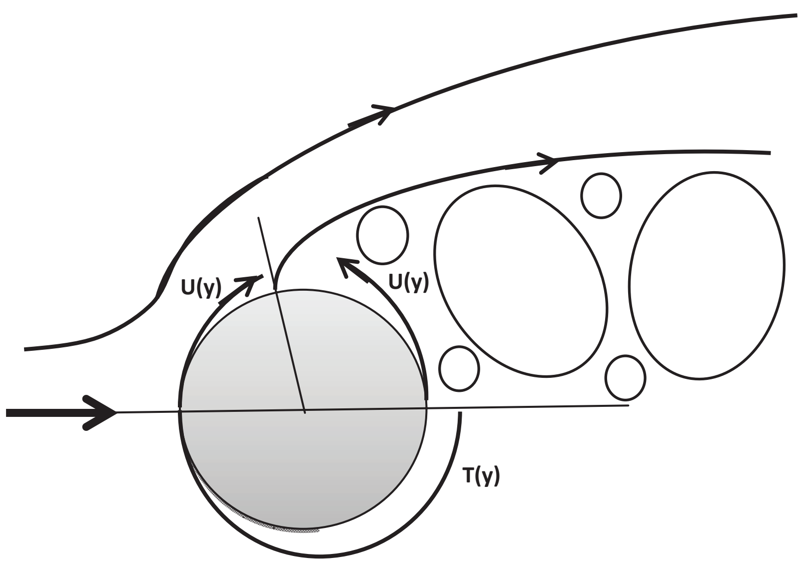


**Figure 1.1.** *Boundary layers around a cylinder in forced convection*

Let us suppose that this cylinder has a surface that is being heated at constant temperature and also kept humid (saturated) by an internal device. A third boundary in humidity ($\rho_v$) is thus established around the cylinder. The thermal boundary layers, which are key to heat exchange, are "globalized" by a heat transfer coefficient $h_{cv}$ (not to be confused with the enthalpy h) whose definition by heat flux density is the following:

$$h_{cv}\left(T_p - T_o\right) = -\lambda\left(\frac{\partial T}{\partial n}\right)_p \qquad [1.8]$$

$T_p$ and $T_o$ are, respectively, the temperature of the wall and of the fluid stream away from the wall, and the temperature gradient is defined at the wall along the exterior wall surface normal. For the cylinder example, $h_{cv}$ is a function of the curvilinear abscissa. This transfer coefficient is rendered dimensionless in the form of a Nusselt number $Nu = h_{cv}L/\lambda$, where L is a length characteristic to the problem (D for a cylinder). Integrating over the surface we get an average transfer coefficient and an average Nusselt number often denoted by $\bar{h}_{cv}, \overline{Nu}$. In the case of forced convection, experimental or theoretical data take the form Nu = f (Re, Pr) which is often expressed as $Nu \approx Re^n Pr^p$, where Pr is Prandtl number, a characteristic of the fluid written as $Pr = \upsilon/\alpha$; $\alpha = \lambda/\rho c_p$.

Similarly, in order to characterize the "vapor" boundary layer, a mass transfer coefficient denoted by k is defined by the mass density flux:

$$k\left(\rho_{vp} - \rho_{vo}\right) = -D_v\left(\frac{\partial \rho_v}{\partial n}\right)_p \qquad [1.9]$$

The dimensionless form of this coefficient is the Sherwood number $Sh = kL/D_v$, a function of Reynolds number characterizing the flow, and of Schmidt number $Sc = \upsilon/D_v$, which connects the physical properties of fluids (vapor and air in the mentioned example). In general, the data refer to surface averaged values. For a wall kept humid in such a manner that it can be considered a liquid surface, we have $\rho_{vp} = \rho_{vsat}(T_p)$, which is the condition for air to be saturated with water vapor.

In the same context of forced convection the analogy of equations [1.2], [1.5] allows us to affirm that for similar boundary conditions (constant temperature and humidity at the wall, for example) the relations Nu = f(Re, Pr) and Sh = f(Re, Sc) will be similar in form.

### 1.2.2. *Natural convection [BEJ 84, BEJ 93]*

Let us now consider the previous example heated cylinder in still air. Natural convection occurring around the cylinder under the influence of thermal body forces (warm air is lighter than cold air) generates a thermal boundary layer and a "plume"-like wake above it (Figure 1.2).

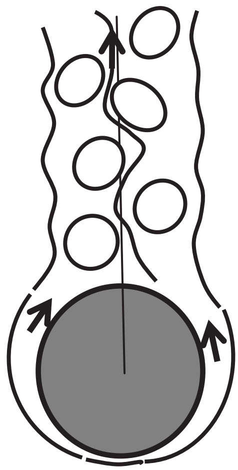

**Figure 1.2.** *Natural convection around a heated cylinder*

The heating intensity (wall temperature $T_p$) determines the nature of the flow (laminar, turbulent) and is expressed by a number, obtained by rendering the equations dimensionless, namely the Rayleigh number $Ra = \frac{g\beta(T_p - T_o)L^3}{\alpha\nu}$, where g is gravitational acceleration, $\beta = 1/T$ for a perfect gas (T Kelvin), L is a length characteristic to the problem (here, D). An alternative to Ra is the Grashof number Gr = Ra/Pr. The laminar/turbulent transition takes place around $Gr = 10^9$. The experimental correlations for various geometries of the objects on which natural convection flows occur take the form Nu = f(Ra, Pr) or Nu = f(Gr, Pr), Nusselt numbers often being averaged on the transfer surface.

Let us now complexify the previous example by supposing that the cylinder surface is not only heated, but also kept humid (a porous surface, for example). A humidity boundary layer forms ($\rho_v$) and the problem of vapor interfering with thermal body force emerges. The component of this force on the cylinder varies depending on the slope of the wall surface taken into consideration. To show the importance of the 2 terms of possible body force, let us consider the simple geometry of a vertical wall and write the heat transfer equations [1.3] and [1.5] projected on the 2 axes, a horizontal x axis and a vertical, ascending y axis. The respective components of velocity are u, v. The boundary layer hypotheses induce a number of simplifications u<<v, $\frac{\partial^2}{\partial x^2} \ll \frac{\partial^2}{\partial y^2}$ that lead to the system:

$$\begin{cases} -\frac{\partial p}{\partial x} = 0 \\ \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\rho g - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2}\right) \\ u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right) \end{cases} \quad [1.10]$$

The external pressure on the boundary layer is the hydrostatic pressure p such that $p+\rho_o gy$ = cst and the system momentum conservation equation [1.10] becomes:

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -(\rho - \rho_o)g + \mu\left(\frac{\partial^2 v}{\partial x^2}\right) \quad [1.11]$$

The density of a two-component gas mixture, such as air and water vapor, is a function $\rho = \rho(T, \rho_v)$, which for small variations around a rest position $T_o$, $\rho_o$, $\rho_{vo}$ gives:

$$\rho = \rho_o + \left(\frac{\partial \rho}{\partial T}\right)_o (T - T_o) + \left(\frac{\partial \rho}{\partial \rho_v}\right)_o (\rho_v - \rho_{vo}) \quad [1.12]$$

The derivatives of this equation are calculated starting from the humid air equations (the dry part of air will be denoted by a and the vapor part will be denoted by v):

$$\begin{cases} p = p_a + p_v \,;\, \rho = \rho_a + \rho_v \\ p_a = \rho_a \frac{R}{Mol_a} T \,;\, p_v = \rho_v \frac{R}{Mol_v} T \end{cases} \quad [1.13]$$

Mol denotes the molar mass. When $p = p_o$, equation [1.12] becomes:

$$\begin{Bmatrix} \rho = \rho_o(1 + \beta_T(T - T_o) + \beta_v(\rho_v - \rho_{vo})) \\ \beta_T = -\frac{p_o Mol_a}{RT_o^2 \rho_o} \,;\, \beta_v = \frac{1}{\rho_o}\left(1 - \frac{Mol_a}{Mol_v}\right) \end{Bmatrix} \quad [1.14]$$

The two expansive body forces terms generated by the force of gravity result from the density difference that accompanies temperature or humidity differences. Equation [1.14] shows that air density decreases when temperature increases and when humidity increases ($Mol_v/Mol_a = 0.622$). The ratio of the body forces terms indicates the relative importance of the 2 phenomena: $N = \frac{\beta_v(\rho_{vp}-\rho_{vo})}{\beta_T(T_p-T_o)}$. For humid air $N \cong 0.6\frac{\Delta\rho_v/\rho_a}{\Delta T/T_o}$ which for coupled problems results in values that are frequently $\frac{\Delta\rho_v}{\rho_a} \in (10^{-2};\ 10^{-3})$; $\frac{\Delta T}{T} \approx 10^{-1}$.

The humid body force term is usually negligible compared to the thermal one. Let us note that $\beta_T$ reduces to 1/T for purely thermal natural convection.

For natural convection problems that are mainly thermal, but imply mass transfer, the similarity of equations in temperature and humidity, associated with similar boundary conditions leads to deducing the correlations Sh = f(Ra, Sc) from those of Nu = f(Ra, Pr), similarly to forced convection.

What happens to these systems of equations when, above a certain Reynolds or Rayleigh number, a part of the boundary layers, in either forced or natural convection, go through a transition to a turbulent regime? Turbulence is characterized by local velocity, pressure, temperature… fluctuations over a characteristic time that allows for a definition of average values over this period of time. The averaging procedures for the previously introduced equations lead to additional quadratic terms that we can "interpret" as friction or turbulent conduction.

A flow classification vocabulary was thus introduced: forced or natural convection, laminar or turbulent regime, to which we can add steady state (wind tunnel experiment) or unsteady state (wind on a building), internal flow (through a pipe) or external flow (around an obstacle).

## 1.3. Heat balance for a "system" and boundary conditions [FOH 10]

The heat balance equation for a "system" under study (a human body, a room, etc.) is the first law of thermodynamics or the energy conservation law equation, generally without the term corresponding to mechanical power exchange with a machine. For an "open" system, defined on a domain $\mathcal{D}$ that allows for incoming or outgoing mass fluxes through sections $\sigma_i$, the equation is written:

$$\begin{cases} \frac{dE}{dt} + h_i \dot{m}_i = \dot{Q} \\ E = \int_{\mathcal{D}} \rho e d\mathcal{V};\ e = u + \frac{V^2}{2} + gz \\ h = e + p/\rho \end{cases} \qquad [1.15]$$

E is the energy of the system that "receives" a heat flux $\dot{Q}$ at its boundary (transferred through radiation, convection on the wall) and eventually an internal flux of combustion heat or radiation. The notations e, u, h, ρ are relative to mass unit (or volume unit for ρ) with u as internal energy, h as enthalpy. Pressure is denoted by p, potential energy by ρgz, $\dot{m}$ is the outgoing mass flow rate from a domain, and which in forced convection is defined by $\int_{\sigma_i} \rho\vec{V}.\vec{n}_{ext} d\sigma$ and in diffusion through a boundary layer is written $\int_{\sigma_i} -D_v \frac{\partial \rho_v}{\partial n_{ext}} d\sigma$, where $\vec{n}_{ext}$ is the external normal to the boundary of the domain $\mathcal{D}$. For a perfect gas du = $c_v$ dT, dh = $c_p$ dT. For a liquid within a range of normal temperature and pressure $c_p = c_v$ and du = dh. Depending on the given problem, certain hypotheses will be needed: incompressibility, negligible kinetic energy, secondary gravity effect, etc.

The radiative flux at a solid surface in a gas medium, which is limited here to air, is quite complex and therefore its calculation requires more than a general approach: shape factors, surface radiative properties (emission, reflections and absorption) depending on wavelength. Within the frame of the environment described here, radiative problems are simplified as follows: possible solar radiation is calculated by several equations that take into account directional geometry and meteorological data. Long wavelength radiation (infrared) is comprehended by simplified linearization of exchanges between gray surfaces. Let $\dot{q}_{ray} d\sigma$ be the net radiative flux (emitted minus incident) from a surface dσ. Conductive heat density flux in a solid medium is generally written $\vec{\dot{q}} = -\lambda\, \vec{\nabla} T$. Forced or natural movements driven in a fluid medium by temperature gradients generate convection boundary layers. At the solid-fluid boundary, where normal $\vec{n}$ is directed toward fluid, flux transferred on an elementary surface dσ will be expressed as $\vec{\dot{q}}.\vec{n}d\sigma = -\lambda \frac{\partial T}{\partial n} d\sigma$. Given the $\dot{Q}$ definition, the following can be written:

$$\dot{Q} = -\int_{\sigma} \vec{\dot{q}}.\vec{n}d\sigma - \int_{\sigma} \dot{q}_{ray}d\sigma - \int_{vol} \dot{q}_{ray}dv$$

For the calculation of radiative flux for a volume element (see Chapter 4) an adequate method, different from the one for the flux on a surface element, will be used.

The case of liquid boundaries subjected to evaporation or condensation should be carefully considered. It is a generalization of the expression written for the boundary condition on a surface. Let an elementary free surface thin layer be an evaporation ($\dot{m}_v > 0$) or condensation ($\dot{m}_v < 0$) site for a mass flux $\dot{m}_v \vec{n}_{ext} d\sigma$ undergoing phase change compensated by a transfer of thermal energy per unit of mass $\Delta h_v = h_{vap} - h_{liq}$ at a given temperature, namely that of the free surface (Figure 1.3).

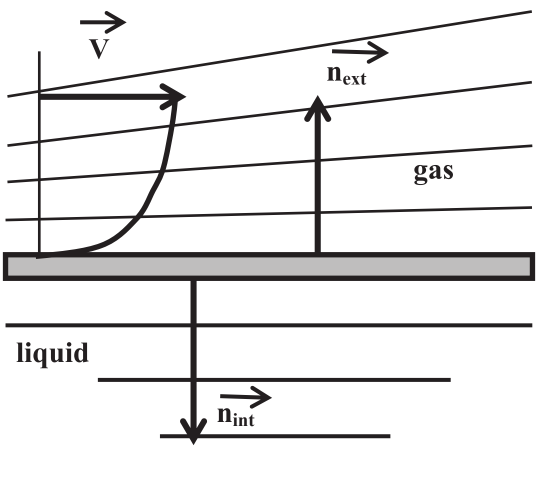


**Figure 1.3.** *Exchanges at a free surface*

Let $\overrightarrow{q_n}.\vec{n}d\sigma$ be the radiation heat flux along direction $\vec{n}$ from an elementary surface dσ. On each side of the surface fluids transmit energy either by conduction alone (no movement) or by convection (conduction across a boundary layer) or by radiation in a gas medium. Conduction flux density can be written as: $\dot{q_n} = -\lambda_n \frac{\partial T}{\partial n}$, where $\lambda_n$ is the conductivity of medium n. The hot surface effect is denoted by $\dot{q_n} > 0$ and the cold surface effect by $\dot{q_n} < 0$.

The energy balance can thus be written (equation [1.15], E = 0):

$$\dot{m_v} \Delta h_v = -\dot{q}_{ray} - \dot{q}_{n,ext} - \dot{q}_{n,int} \quad [1.16]$$

Evaporation or condensation takes place starting from a boundary layer of vapor density $\rho_v$. The mass flux density is expressed as $\dot{m_v} = -D_v \frac{\partial \rho_v}{\partial n_{ext}}$ and [1.16] and can be developed as:

$$-D_v \frac{\partial \rho_v}{\partial n_{ext}} \Delta h_v = -\dot{q}_{ray} + \lambda_{ext} \frac{\partial T}{\partial n_{ext}} + \lambda_{int} \frac{\partial T}{\partial n_{int}} \quad [1.17]$$

This is a general relation applicable to both evaporation and condensation. Let us note that for each free surface problem it is possible to correctly write the boundary condition for a domain $\mathcal{D}$ precisely defined with the free surface boundary crossing either the liquid (output enthalpy $\dot{m_v} h_{liq}$, heat flux from the liquid), or the gas (output enthalpy $\dot{m_v} h_{vap}$, heat flux from the gas). Let us now apply this approach to the following example.

## 1.4. On the problem of cooling of a cup of tea

This example illustrates how, starting from physical reality, a model can be set up by formulating hypotheses and searching the specialist literature for transfer coefficients drawing on physical analogies.

A tea enthusiast takes the time to wait for the liquid to reach the proper temperature for tasting it. Let us consider a cup of tea filled with this drink at 90°C. We propose to predict its cooling. Let the geometry of a cylinder cup be the following: height H = 10 cm, internal diameter D = 6 cm, wall thickness ep = 5mm. The wall is made of sandstone with the following thermal characteristics: conductivity $\lambda$ = 1.5 W/mK; density $\rho$ = 2,300 kg/m$^3$; specific heat $c_p$ = 710J/ kgK; emissivity $\varepsilon$ = 0.95. The cup is set on a table in a room with a calm atmosphere, at temperature $T_a$ = 20°C and absolute humidity $W_a = 5.10^{-3}$.

The inventory of thermal transfers around the cup is as follows (Figure 1.4): natural convection and radiation on the vertical side around the cup, natural convection, radiation and evaporation on the horizontal liquid surface, conduction in the cup's thickness and on the table.

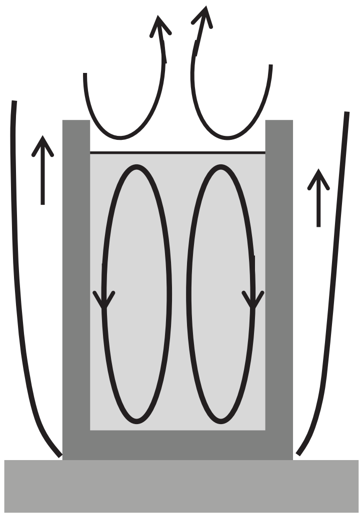

**Figure 1.4.** *Heat transfers around a cup of tea*

### 1.4.1. *Balance equations*

The heat balance equation or the first law of thermodynamics can be written on the whole for the "system" cup + liquid, the domain's boundary including the full free surface (outgoing vapor):

$$\frac{dE}{dt} + h_{vap}\dot{m}_{vap} = -\dot{Q}_{conv} - \dot{Q}_{ray} - \dot{Q}_{cond} \quad [1.18]$$

E is the total internal energy of the filled cup. The second member is the sum of heat fluxes dissipated by convection, radiation and conduction through the table. These fluxes are algebraic quantities, but bear a negative sign because they transfer energy outside ("outgoing" fluxes). Conduction through the cup is not represented, as it is internal to the system. E decomposition can be written as:

$$E = (Mc_p\overline{T})_t + (Mc_p\overline{T})_{liq} \quad [1.19]$$

Indices t and liq refer to the solid object cup and to the liquid. $\overline{T}$ is a spatial average temperature and $M_{liq}$ is a function of time, as for temperatures. The derivation of E leads to:

$$(Mc_p)_t \frac{d\overline{T}_t}{dt} + (Mc_p)_{liq} \frac{d\overline{T}_{liq}}{dt} + (c_p\overline{T})_{liq} \frac{dM_{liq}}{dt} + h_{vap}\dot{m}_{evap} = -\dot{Q}_{conv} - \dot{Q}_{ray} - \dot{Q}_{cond} \quad [1.20]$$

Let us introduce mass conservation:

$$\frac{dM_{liq}}{dt} = -\dot{m}_{liq};\ \dot{m}_{liq} = \dot{m}_{evap} \quad [1.21]$$

and the differential evaporation enthalpy $\Delta h_v = h_{vap} - h_{liq}$; $h_{liq} = (c_pT)_{liq}$. We obtain:

$$(Mc_p)_t \frac{d\overline{T}_t}{dt} + (Mc_p)_{liq} \frac{d\overline{T}_{liq}}{dt} = -\dot{Q}_{conv} - \dot{Q}_{ray} - \dot{Q}_{cond} - \dot{Q}_{evap} \quad [1.22]$$

where $\dot{Q}_{evap} = \dot{m}_{evap}\Delta h_v(T_{surf})$; $T_{surf}$ is the free surface temperature, $\dot{m}_{evap} > 0$.

This equation [1.22] can be formulated directly, somewhat intuitively. We already have three temperatures $\overline{T}_t, \overline{T}_{liq}, T_{surf}$ that were introduced for one equation. It is now time to formulate the physical hypotheses and to explain the terms.

The energy balance on the free surface (equation [1.17]) gives a second equation:

$$\dot{Q}_{liq} = \dot{Q}_{conv,surf} + \dot{Q}_{ray,surf} + \dot{Q}_{evap} \quad [1.23]$$

which expresses that the heat flux provided by the liquid mass to the surface, to be determined by internal transfers, is dissipated by evaporation, convection and radiation.

### 1.4.2. *Research of transfer correlations*

The temperature field is indeed dependant on the natural convection flows. Careful observation of the liquid surface leads to detecting natural convection cells that are slowly changing. These cells are related to Bénard cells, observed between two horizontal plates, the lower one being heated and the higher one cooled. We will use the data of this configuration. The vertical walls may disturb the development of convection cells, but "physical intuition" of the problem validates this choice. The Rayleigh number that determines the convective movement depends on the height H and on ΔT, the temperature difference between the plates. Literature indicates a correlation that may serve as an order of magnitude (Appendix 1):

$$\begin{cases} \overline{Nu}_H = 0.069 Ra_H^{1/3} Pr_{liq}^{0.074} \ ; \ Ra_H = \left(\frac{g\beta}{\alpha\upsilon}\right)_{liq} \Delta T_{liq} H^3 \\ \overline{Nu}_H = \frac{\bar{q} H}{\lambda_{liq} \Delta T_{liq}} \ ; 3.10^5 < Ra_H < 7.10^9 \end{cases} \quad [1.24]$$

This correlation is established for a permanent exchange and we will use it to get a good order of magnitude for the slow cooling of the liquid mass. The passage from flux density on the free surface to flux on the whole surface is expressed as $\dot{Q}_{liq} = \bar{q}\sigma_{surf}$. We set an uniform temperature gradient along the cup's vertical axis and thus an average mass temperature $\overline{T}_{liq} = T_{surf} + \frac{\Delta T_{liq}}{2}$. Let us note that [1.24] shows that the flux $\bar{q}$ does not depend on height H.

The convective flux on the free surface can be written as:

$$\dot{Q}_{conv,surf} = h_{cv,surf}(T_{surf} - T_o)\sigma_{surf} \quad [1.25]$$

The radiative flux on the free surface is denoted:

$$\dot{Q}_{ray,surf} = h_{ray,surf}(T_{surf} - T_o)\sigma_{surf} \quad [1.26]$$

The evaporative flux on the free surface is expressed as:

$$\dot{Q}_{evap} = k\left(\rho_{vsat}\left(T_{surf}\right) - \rho_{vo}\right)\Delta h_v \sigma_{surf} \quad [1.27a]$$

Or in terms of humid air data (Appendix 2), $W = \frac{\rho_v}{\rho_a}$:

$$\dot{Q}_{evap} \cong k\overline{\rho_a}\left(W_{sat}\left(T_{surf}\right) - W_o\right)\Delta h_v \sigma_{surf} \quad [1.27b]$$

The mass transfer coefficient k “summarizes” the vapor boundary layer to the free surface, basis of the convection plume in the air. This coefficient is evaluated by analogy with heat transfer by natural convection above a hot plate. For this configuration we obtain (Appendix 1):

$$\begin{cases} \overline{Nu_L} = 0.54 Ra_L^{1/4};\ Ra_L = \left(\frac{g\beta}{\alpha\upsilon}\right)_{air} \Delta T_{air} L^3 \\ \overline{Nu_L} = \frac{h_{cv,surf} L}{\lambda_{air}};\ 10^4 < Ra_L < 10^7 \end{cases} \qquad [1.28]$$

The reference length L is equal to surface area divided by perimeter, which for a disk of diameter D, gives L = D/4. The temperature difference is $\Delta T_{air} = T_{surf} - T_o$. Correlation [1.28] does not depend on the Prandtl number, and it leads to:

$$\overline{Sh_L} = \overline{Nu_L} \Rightarrow \frac{\bar{k}L}{D_v} = \frac{\bar{h}_{cv,surf}}{\lambda} L \Rightarrow \bar{k} = D_v \frac{\bar{h}_{cv,surf}}{\lambda}$$

For a good order of magnitude: $D_v = 2.6.10^{-5}$ m$^2$/s; $\lambda = 2.6.10^{-2}$ W/mK, namely k(m/s) = $10^{-3} h_{cv,surf}$ (W/m$^2$s).

### 1.4.3. *Surface temperature as a function of average temperature of the liquid*

Equation [1.23] further developed becomes:

$$\begin{cases} 0.069 \lambda_{liq} \Delta T_{liq}^{4/3} \left(\frac{g\beta}{\alpha\upsilon}\right)_{liq}^{1/3} Pr_{liq}^{0.074} = k\bar{\rho}_a \left(W_{sat}(T_{surf}) - W_o\right)\Delta h_v + \left(h_{cv,surf} + h_{ray}\right)\Delta T_{air} \\ k = h_{cv,surf} \frac{D_v}{\lambda_{air}};\ h_{cv,surf} = 0.54 \lambda_{air} \left(\frac{g\beta}{\alpha\upsilon}\right)_{air}^{1/4} \left(\frac{\Delta T_{air}}{L}\right)^{1/4} \end{cases} \qquad [1.29]$$

The coefficient $h_{ray}$ results from a linearization of a radiative flux between a “clear” surface and surrounding walls at temperature $T_o$:

$$\dot{q} = \epsilon\sigma\left(T_p^4 - T_o^4\right) \cong 4\epsilon\sigma\overline{T}^3\left(T_p - T_o\right) = h_{ray}\left(T_p - T_o\right) \qquad [1.30]$$

where ε is the emissivity of the gray surface (close to unit) and σ denotes the Stefan–Boltzmann constant, with a value of $5.67.10^{-8}$ W/m$^2$K. $\overline{T}$ is an average temperature (in Kelvin) between $T_p$ and $T_o$. For example, $\overline{T}$ = 293 K, ε = 0.95 give $h_{ray}$ = 5.4 W/m$^2$K.

Equation [1.29] is in fact a function $T_{surf}(\overline{T}_{liq})$. The various physical constants introduced have to be evaluated for an average value between the 2 temperatures. Solving this equation with a little program or on a spreadsheet readily allows for physics related remarks (Figure 1.5).

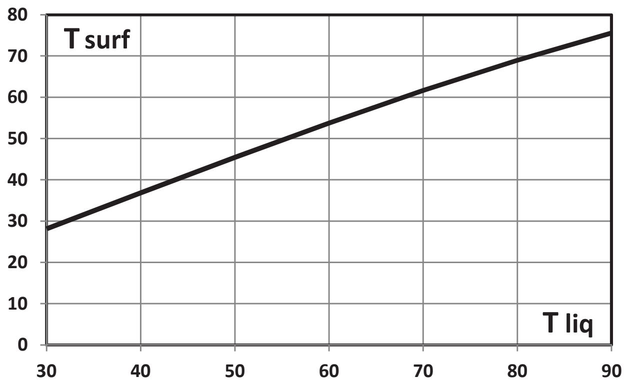


**Figure 1.5.** *Surface temperature as a function of the average liquid temperature*

The difference $\Delta T_{liq}$ follows a quasi-linear decrease between 30 and 90°C. The heat flux generated by the liquid mass and redistributed across the surface drastically decreases with temperature: 61.6 W when $\overline{T}_{liq}$ = 90°C and 2.4 W when $\overline{T}_{liq}$ = 30°C. The relative importance of various heat fluxes across the surface reinforces the physics intuition of phenomena.

| $T_{liq}$ | $\dot{Q}_{liq}$ | $\dot{Q}_{evap}/\dot{Q}_{liq}$ | $\dot{Q}_{cv,surf}/\dot{Q}_{liq}$ | $\dot{Q}_{ray,surf}/\dot{Q}_{liq}$ |
|---|---|---|---|---|
| 90°C | 61.1 W | 0.94 | 0.05 | 0.01 |
| 30°C | 2.4 W | 0.84 | 0.11 | 0.05 |

**Table 1.1.** *Magnitude of heat fluxes across the liquid surface*

Surface convection and radiation come second to evaporation.

### 1.4.4. *Liquid temperature as a function of time*

Equation [1.22] will be solved as a differential equation in $\overline{T}_{liq}(t)$ by formulating several additional hypotheses. Conduction in the cup's thickness can be written as a one-dimensional conduction equation (thickness is small compared to diameter):

$$\rho c_p \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2} \qquad [1.31a]$$

The "time constant" deduced from orders of magnitude suggested by this equation is:

$$\tau = \frac{e_p}{\alpha}\ ;\ \alpha = \frac{\lambda}{\varrho c_p} \qquad [1.31b]$$

From the numerical data for earthenware mentioned above ($\lambda$ = 1.5 W/mK; $\rho$ = 2,300 kg/m$^3$, $c_p$ = 710 J/kgK) and for a thickness $e_p$ = 5 mm, we have: $\tau$ = 27 s. Considering that a full cup takes much longer to cool, let us formulate the hypothesis of a quasi-steady linear profile of temperature within the cup's thickness. This will allow us to write the equality of the conductive flux in the cup's thickness and the convective and radiative fluxes in the air around the cup:

$$\left(h_{cv,p} + h_{ray,p}\right)\left(T_p - T_o\right) = \frac{\lambda}{e_p}\left(\overline{T}_{liq} - T_p\right) \qquad [1.32]$$

Let us examine the literature in search of a natural correlation on a vertical wall, in the laminar case. In effect, a rapid calculation of the Rayleigh number indicates this boundary layer is laminar.

$$\overline{Nu} = 0.51 Ra^{1/4}\ ;\ \overline{Nu} = \frac{h_{cv,p} H}{\lambda_a}\ ; Ra = \left(\frac{g\beta}{\alpha\nu}\right)_{air} \Delta T_p H^3 \qquad [1.33]$$

where $\Delta T_p = T_p - T_o$, $T_p$ is the temperature of the cup's external side surface. Considering [1.32] the temperature difference can be written:

$$T_p - T_o = a_p\left(\overline{T}_{liq} - T_o\right);\ a_p = \frac{\lambda_t/e_p}{h_{cv,p} + h_{ray,p} + \lambda_t/e_p} \qquad [1.34]$$

We will not take into consideration the conductive flux on the horizontal support plane, which might be notable in the case of a metal table, but negligible on a wooden plate. Equation [1.22] can then be written:

$$\left[\left(Mc_p\right)_t \frac{(1+a_p)}{2} + \left(Mc_p\right)_{liq}\right] \frac{d\overline{T}_{liq}}{dt} = -\dot{Q}_{liq} - (h_{cv,p} + h_{ray,p})(T_p - T_o)\sigma_p \qquad [1.35]$$

To solve this last equation it is convenient to calculate numerically its second member as a function of $\overline{T}_{liq}$, and to deduce an adequate representation under the form of a second degree equation $\overline{T}_{liq} = -ax^2 + b$. In effect, coefficients $\lambda, \frac{g\beta}{\alpha\upsilon}, Pr$ are functions of temperature. $M_{liq}$ will be considered constant in the first member, $a_p$ practically equal to unit, $h_{ray} = 5.4$ W/m$^2$K.

The integration of $A\frac{dx}{dt} = -ax^2 + b$ gives: $t = \frac{A}{2\sqrt{ab}} Ln\left(\frac{x\sqrt{a}+\sqrt{b}}{x\sqrt{a}-\sqrt{b}}\right) + cst$ from which we deduce x(t), that is to say $\overline{T}_{liq}(t)$ (Figure 1.6).

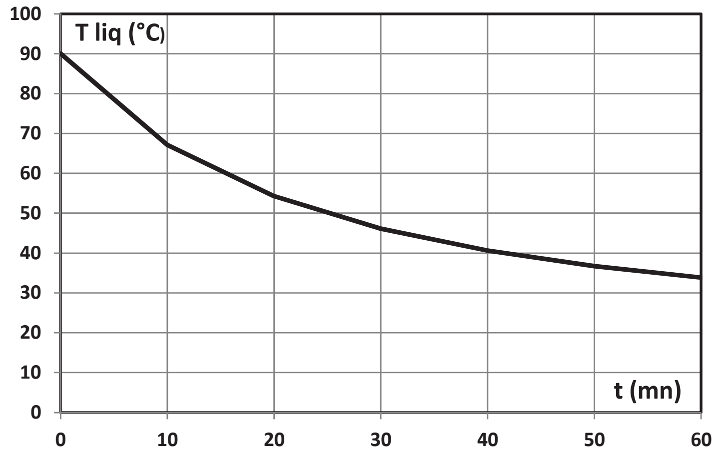


**Figure 1.6.** *Cooling of liquid in a cup*

We deduce from this the importance of the flux convected/radiated by the cup's vertical wall in the total flux provided by the liquid at the free surface, namely $\left(\dot{Q}_{cv,p} + \dot{Q}_{ray,p}\right)/\dot{Q}_{liq}$ which is 0.29 at 90°C and 0.88 at 30°C.

To finish the problem, let us investigate what happens when the cup reaches $T_o$, the external temperature. A quick examination of the subject may lead us to erroneously conclude that it reaches a state of equilibrium. Indeed, natural convection of mass takes over, as is shown by the linearization of humid air density:

$$\varrho = \varrho_o(1 + \beta_v(\varrho - \varrho_o)) \; ; \; \beta_v = \frac{1}{\varrho_o}\left(1 - \frac{Mol_a}{Mol_v}\right) \qquad [1.36]$$

The analogy between mass and heat transfer justifies the correspondences:

$$\Delta T \rightarrow \Delta\rho_v \; ; \; \beta\Delta T \rightarrow \beta_v\Delta\rho_v \; ; \; \alpha \rightarrow D_v \; ; \; \lambda \rightarrow D_v \; ; Ra \rightarrow Ra_m \; ; Pr \rightarrow Sc \; ; Nu \rightarrow Sh$$

We can thus transform Nu(Ra) in Sh($Ra_m$) correlations. The case of same order thermal and mass body forces is more complicated and is still poorly studied. The evaporation flux requires a heat flux coming out of water or air. As soon as water temperature goes below ambient temperature by several degrees, thermal body force becomes predominant again. The N number, a ratio of the body forces terms, is equal to unit when liquid temperature is approximately 1.5°C below the ambient temperature. Equations [1.22] and [1.23] remain valid but we now have to search for correlations for the case of a free surface that is colder than the environment. Instead of [1.28] we obtain: $\overline{Nu_L} = 0.27Ra^{1/4}$ with the same definitions of Nu and Ra but $\Delta T = T_o - T_{surf}$. The same calculation method as previously, slightly simplified for "thermal equilibrium", leads to $\overline{T}_{liq} \cong 17.5°C$. In this pseudo-equilibrium case the ambient air provides a heat flux convected/radiated by the free surface and side walls. We did not take into consideration the mass reduction but it is common knowledge that a thin water layer on a plate will eventually disappear due to evaporation.

## 1.5. Bather on a beach

Human thermal behavior models will be examined in Chapter 3. For now, let us try to quantify the orders of magnitude of the exchanges between a naked individual on a beach and his environment, which we can compare to his own heat production (his metabolism) amounting to approximately 100 W. The bather or "beachgoer" exchanges heat through natural convection (without wind), forced convection (with sea breeze), solar radiation and infrared (long wavelength) radiation and evaporation through transpiration.

Let us evaluate convection and for this purpose let us set a simple geometrical model of a standing individual as a cylinder of height H = 1.80 m and side surface $\sigma_p$ = 1.80 m², which is the skin surface area of a representative individual. We will not take into consideration in this context the exchanges through the upper base of the cylinder. The diameter of the cylinder is deduced from the above as D = 0.318 m.

Let us suppose a sea breeze with a speed V = 0.5 m/s. For the flow around the cylinder, Reynolds number Re = VD/υ is very close to $10^4$ (υ = $1.6.10^{-5}$ m²/s). Among numerous correlations in the literature let us choose the following:

$$\overline{Nu} = 0.26Re^{0.6}Pr^{0.37};\ 10^3 < Re < 2.10^5 \qquad [1.37]$$

It is an average Nusselt number integrated over the surface of the cylinder. The calculation gives (Pr = 0.72, λ = 0.026 W/mK) $\overline{Nu}$ = 57.6 and a transfer coefficient $\overline{h}_{cv} = \overline{Nu}.\frac{\lambda}{D} = 4.7\ W/m^2\ K$. The flux lost by the skin through convection can be expressed: $\dot{Q}_{cv} = \overline{h}_{cv}(T_p - T_a)\sigma_p$. Let us fix the air temperature at $T_a$ = 25°C. The skin temperature $T_p$, regulated by blood flow, is in the range 31–40°C. At the latter temperature the body is near heatstroke. For $T_p$ = 34°C we get $\dot{Q}_{cv} = 76\ W$.

Let us now suppose a calm air, with no wind, and let us consider convective exchange by natural convection. This will be determined by the Rayleigh number $Ra = \frac{g\beta}{\alpha v}\Delta T H^3$.

For values ΔT = $T_p$-$T_a$; $T_p$ = 34°C, $T_a$ = 25°C, gβ/αυ = $90.7.10^6$ m$^3$K, we have Ra = $4.7.10^9$, which would classify this flow as turbulent. Lienhard (Appendix 1) proposes the correlation:

$$\overline{Nu} = 0.52Ra^{1/4} \qquad [1.38]$$

The calculation then gives $\overline{Nu}$ = 136; $\overline{h}_{cv}$= 6.4 W/m²K; $\dot{Q}_{cv}$= 103 W. We note that other correlations, for example that of Sparrow for a vertical cylinder with diameter equal to the height $\overline{Nu} = 0.775Ra^{0.208}$ would give a transfer coefficient below 30% which is a common uncertainty when we manipulate literature data that were obtained in a particular context and applied to another context.

These small convection transfer calculations result in orders of magnitude that are close to those for forced and natural convection. It is a configuration of combined natural/forced convection for which few data are available.

Obviously, we could suppose that the flux exchanged will be the maximum of the two fluxes thus calculated.

The exchanges by long wavelength (infrared) radiation between skin and environment can be evaluated by [1.30] with an emissivity of skin $\varepsilon = 0.95$ or a value of the equivalent transfer coefficient $h_{ray} = 6.2$ W/m²K and an emitted flux $\dot{Q}$ of the order of 100 W.

Let us now consider the solar (short wavelengths) flux which is likely to "warm up" the skin of the "beachgoer". Everyone has experienced the extreme variability of this flux. The cylindrical geometry of the human model offers the advantage of easier calculation. Let us refer to the expression of solar radiation (Appendix 3). The question is here to evaluate a global flux received as thermal power and which depends on local meteorological data, namely global and diffuse radiation on a horizontal surface, as continuously recorded data in a weather station. Weather data are published as monthly values, but going back to the source of data allows us to get the proper orders of magnitude of clear sky radiation during summer months. Let us consider here a global solar radiation at ground level $G_o$ between 500 and 1,000 W/m² and diffuse solar radiation $D_o$, which is highly dependent on the cloud cover of 30% of global radiation.

Let us suppose a surface element inclined at an angle i to the horizontal and in azimuth direction $a_{surf}$ ($a_{surf} = 0$ to the south, positive to the west, negative to the east), the global flux received is the sum of direct and diffuse fluxes:

$$\begin{cases} G_o(i,a) = B_o R_b + D(i,a);\ B_o = G_o - D_o \\ R_b = \frac{cosh.sini.\cos(a_{sol} - a_{surf}) + sinh.cosi}{sinh} \\ D(i,a) = D_o\left(\frac{1+cosi}{2}\right) + \rho_{sol} G_o\left(\frac{1-cosi}{2}\right) \end{cases} \qquad [1.39]$$

where h is the angular height of sun and $a_{sol}$ its azimuth at the given moment of time, $\rho_{sol}$ is the ground albedo, that is the reflection coefficient for solar radiation. Let us consider here $\rho_{sol} = 0.4$ which is a high value, compatible with a sand and a water surface nearby. It is important to understand that diffuse radiation comes partly from the ground and partly from the sky, solid angles being in brackets in the expression for D.

The integration over the surface of "human cylinder" is quite easy. On the whole, azimuths have no influence, and the integration gives:

$$\int_{\sigma_p} G(i,a)d\sigma = B_o \frac{2RH}{tgh} + \frac{(D_o+\rho_{sol}G_o)\sigma_p}{2};\ \sigma_p = \pi DH \qquad [1.40]$$

Let us deduce several typical values. Let us consider the angular height of sun h = 45° in the middle of summer, during hot, rush hours on the beach. During a beautiful day values can easily reach $G_o$ = 1,000 W/m², $D_o$ = 300 W/m² or a flux integrated over the cylinder of 1,030 W. This is 10 times the metabolic production. Obviously, skin does not absorb all the incident solar flux, a part of it being reflected (20–30%). How does the human body react to this level of heat? Skin temperature increases and transpiration starts with a flux that, for complete skin wetness (see [1.27b]), can be expressed as:

$$\dot{Q}_{evap} = k\rho_a\left(W_{sat}\left(T_p\right) - W_a\right)\Delta h_v \sigma_p \qquad [1.41]$$

What would be the maximum value of this flux in the given context?

Let us consider the possible data: $T_a$ = 25°C, $W_a = 7.10^{-3}$, $T_p$ = 40°C, $W_{sat}(T_p) = 48.8.10^{-3}$, $\rho_a$ = 1.18 kg/m$^3$, $\Delta h_v = 2.5.10^6$ J/kgK. Because of the analogy between heat and vapor transfers in boundary layers k(m/s) = $10^{-3}h_{cv}$(W/m²s) and $h_{cv}$ = 6W/m²K. For the "human cylinder" the calculation gives an evaporated power $\dot{Q}_{evap}$ = 1,330 W.

Heat balance is therefore often possible, but not certain, because heatstroke is also possible. On the contrary, this small calculation can explain why a bather coming out of water under a clouded sky feels cold. For example, let us consider the following fixed values: $G_o$ = 500 W/m², $D_o$ = 200 W/m², h = 45°, $T_p$ = 34°C, $T_a$ = 25°C, $W_a = 7.10^{-3}$, $W_{sat}(T_p)$ = $34.5.10^{-3}$, $\rho_a$ = 1.18 kg/m$^3$, k(m/s) = $10^{-3}$ $h_{cv}$(W/m²s), $h_{cv}$ = 6 W/m²K. We get $\dot{Q}_{evap}$ = 876 W and the received solar radiation is 666 W. When evaporation, convection and long wavelength radiation are added, metabolism cannot compensate. What does the bather do? He dries in order to limit evaporation.

Going to the beach means bathing, and cooling in the water after a "grill" under the sun is a great pleasure. What is the heat loss due to sea bathing? Let us go back to the vertical motionless "cylinder/human body" model, head above water and immersed height H of 1.5 m. The order of magnitude of heat loss in water can be obtained through Lienhard correlation [1.38] provided that skin temperature is known. Immersion in water corresponds to a "thermal shock" that will make skin go from its "air" temperature (33–40°C) to a temperature close to that of water $T_{eau}$ that we fix at 20°C. It takes several seconds for this physiological adaptation to take place, and it implies internal regulation: vasoconstriction that diverts blood from external flow (skin) to internal flow. The muscles and body core are sufficiently irrigated to distribute heat produced by metabolism, while the skin and related fat layers are "left" to their own devices, meaning that they undergo conductive heat transfers. A steady conductive regime is established in this skin/fat layer with thickness $e_p$ within a time expressed by [1.31a], [1.31b] $\tau = e_p^2/\alpha$. Let us take the thermal data of a typical human body: $\lambda_c$ = 0.5 W/mK, $\rho_c$ = 860 kg/m$^3$, $c_{pc}$ = 3,500 J/kg and thus $\alpha_c = (\lambda/\rho c_p)_c$ = 1.66.10$^{-7}$m²/s and if we estimate $e_p$ between 0.5 and 1 cm, $\tau$ is approximately between 2.5 and 10 min. Beyond this period of time needed for the adaptation of the skin/fat layer we can presume a quasi-steady state of this layer and a slow cooling of the body according to the following energy equation:

$$\begin{cases} Mc_p \frac{d\overline{T}_n}{dt} = \dot{Q}_{met} - h_{cv}\left(T_p - T_{eau}\right)\sigma_p \\ \quad h_{cv}\left(T_p - T_{eau}\right) = \frac{\lambda_c}{e_p}\left(T_n - T_p\right) \end{cases} \qquad [1.42]$$

$\dot{Q}_{met}$ is the metabolic energy supplied by the body at the level of its central core supposedly at temperature (rendered uniform by blood flow) $T_n$. The transfer coefficient $h_{cv}$ is obtained from [1.38]:

$$\overline{N}u = 0.52Ra^{1/4}; \overline{N}u = \frac{h_{cv}H}{\lambda}; Ra = \frac{g\beta}{\alpha\upsilon}\Delta T H^3;\ \Delta T = T_p - T_{eau} \qquad [1.43]$$

Let us consider $T_{eau}$ = 20°C, $T_n(t = 0)$ = 37°C, $e_p$ = 1 cm, H = 1.5 m, D = 0.318 m, $\sigma_p = \pi DH$, M = 68.3 kg. We will not take into account the exchanges between the head above water and the air. Water physical constants are taken at an average temperature water/skin of 25°C: $\lambda$ = 0.60 W/mK, $g\beta/\alpha\upsilon$ = 1.98.10$^{10}$ m$^{-3}$K$^{-1}$.

Equations [1.42] and [1.43] are solved by iterative calculation, for example on a spreadsheet. At the beginning of cooling we get $h_{cv}$ = 151.5 W/m²K, a convective flux lost in water $\dot{Q}_{conv}$ = 957.8 W for a motionless person metabolic flux of 100 W. A person weighing 80 kg who stands motionless for 15 min in water at 20°C would lower his body temperature by approximately 3°C.

Let us now consider the fetal position, in which losses are minimized, and assimilate it with a sphere with diameter D = 0.61 m and a volume equal to the one of the previously discussed immersed cylinder. Churchill correlation with D as characteristic length is indicated for the loss calculation made using the same method as previously:

$$\overline{Nu} = 2 + 0.589Ra^{1/4}\left[1 + (\frac{0.469}{Pr})^{9/16}\right]^{-4/9} ; 1 < Ra < 10^{11} \qquad [1.44]$$

For water at this temperature scale, Pr = 6.21. We get $h_{cv}$ = 189.5 W/m²K and a convective flux lost in water $\dot{Q}_{conv}$ = 787 W.

Let us now consider the swimmer is moving at a speed V = 1 m/s. Careful observation of the flow around the swimmer shows that stream filaments around the body cover a distance L of approximately 50 cm before being ejected sideways. It seems possible to use for an order of magnitude a "flat plate" of laminar or turbulent flow correlation, and the flow characteristic Reynolds number is here Re = VL/υ which is $5.6.10^5$, which indicates the beginning of a laminar/turbulent transition and Nusselt number is then a solution of:

$$\overline{Nu} = \frac{h_{cv}L}{\upsilon} = 0.037\left(Re^{\frac{4}{5}} - 23{,}550\right)Pr^{\frac{1}{3}}; Re > 5.10^5 \qquad [1.45]$$

We then get: $h_{cv}$ = 1,315 W/m²K and $\dot{Q}_{conv}$ = 1,227 W. The metabolic flux that results from physical effort is here far higher than 100 W. A possible flux for an energetic swim at 600 W leads to a cooling of the said swimmer body by 2° within 15 min.

This highly simplified human body behavior model gives orders of magnitude which are quite satisfactory. In Chapter 3, we will take a look at more elaborate models.