1

Autonomous Petri Nets

1.1. Unmarked Petri nets

1.1.1. *Definitions*

A unmarked PN is a bipartite oriented 1-graph[1] provided with a mapping ϖ from the set of arcs to the positive integer set $\mathbb{N}^+$:

$$Q = \langle P, T, A, \alpha, \beta, \varpi \rangle$$

– P and T are two disjointed subsets of nodes: $P \cap T = \varnothing$:
- P is the Place subset with a finite cardinal p;
- T is the Transition subset with a finite cardinal t.

– A is the set of Arcs, α and β are the mappings associating with each arc, its origin and its goal nodes, respectively, so that:

$\forall a \in A, \quad$ if $\alpha(a) \in T$ then $\beta(a) \in P$
if $\alpha(a) \in P$ then $\beta(a) \in T$

– ϖ is a mapping or weighting function associating an integer with each arc, $\varpi : A \rightarrow \mathbb{N}^+$.

If $\mathbb{N}$ is reduced to $\{1\}$, the PN is of ordinary type (or state transition graph), otherwise the PN is of generalized type.

1 An oriented 1-graph is such that only one arc may be associated with a couple {origin node and destination node}.

Practically, rather than this formalism directly issued from the graph theory, we will use a definition where A does not explicitly appear. As it is an 1-graph, it consists of considering all the couples (P_i, T_j) or (T_i, P_j) and two applications w^- and w^+.

The PN is then defined as:

DEFINITION 1.1.– An unmarked PN or Place/Transition (P/T) net is a 4-uple $Q = \langle P, T, w^-, w^+ \rangle$ [DAV 89] where:

– P is the set of places (finite cardinal p);

– T is the set of transitions (finite cardinal t);

– $w^-(P_i, T_j) : P \times T \rightarrow \mathbb{N}$ is the backward transition function;

– $w^+(P_i, T_j) : T \times P \rightarrow \mathbb{N}$ is the forward transition function.

The value "0" associated with the couple (P_i, T_j) by w^- or w^+ means that there is no arc between P_i and T_j or T_j and P_i. If the value $k \in \mathbb{N}^+$ is associated with this couple by w^-, respectively w^+, then one arc oriented from P_i to T_j, respectively from T_j to P_i, exists between these nodes with the valuation k.

REMARK 1.1.– Another possibility is to define a PN as an n-graph (n arcs may exist between two nodes), an arc of weight n being replaced by n arcs each of them having the weight one.

1.1.2. *Drawing*

In the drawing of a PN, places and transitions are, respectively, represented by circles and streaks (or filled or empty rectangles) and the arcs are arrows to which the weights are attached. Figure 1.1 shows an example of PN with three places and two transitions respectively named as P_1, P_2, P_3, T_1, T_2. From this figure, we can write $w^-(P_1, T_1) = 1$, $w^-(P_3, T_2) = 2$, $w^-(P_2, T_1) = 2$, $w^+(P_3, T_1) = 2$, $w^+(P_1, T_1) = 1$, $w^+(P_2, T_2) = 3$.

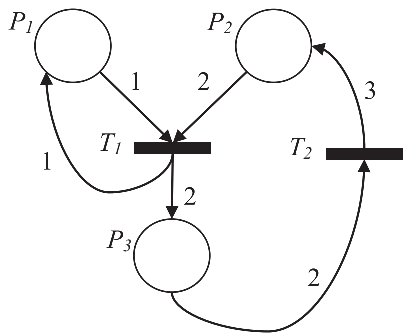

Figure 1.1. *The drawing of a PN*

1.1.3. *Other definitions*

Some of other definitions concerning particular cases of PN are summarized in the Appendix, section A.1.

1.2. Marking of a PN

Notations [DAV 89]:

– $I(T_j) = \{P_i \in P | w^-(P_i, T_j) > 0\}$ is the set of the input places of T_j;

– $O(T_j) = \{P_i \in P | w^+(P_i, T_j) > 0\}$ is the set of the output places of T_j;

– $I(P_i) = \{T_j \in T | w^+(P_i, T_j) > 0\}$ is the set of the input transitions of P_i;

– $O(P_i) = \{T_j \in T | w^-(P_i, T_j) > 0\}$ is the set of the output transitions of P_i.

For example, in Figure 1.1, $I(T_1) = \{P_1, P_2\}$ and $O(T_1) = \{P_3, P_1\}$.

The marking is a notion resulting from the association of tokens with the places of the PN. The position in the places of these tokens will evolve to represent the dynamics of the described system. This evolution is performed according to a set of rules described in section 1.3.

DEFINITION 1.2.– A marked PN is a couple $R = \langle Q, M_0 \rangle$ where Q is an unmarked PN and M_0 is an initial marking.

The marking M of a PN at a given instant is a p-sized columnar vector of integers (p is the place number of the PN), each of its component being the marking (or charge) of the place P_i that is to say the number of tokens inside P_i at the considered time instant:

$$M \in \mathbb{N}^p \qquad M^T = [M(P_1), M(P_2), \ldots, M(P_p)]$$

The initial marking M_0 is the marking at time $t = 0$.

Figure 1.2 shows the initial marking of the PN of Figure 1.1 with: $M_0^T = [1, 2, 0]$.

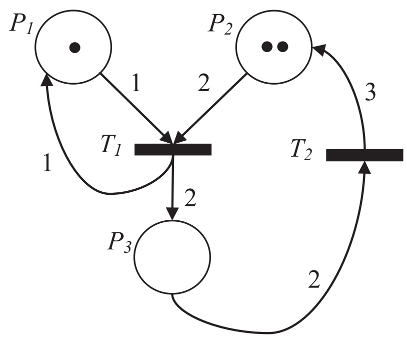

Figure 1.2. *A marked PN*

1.2.1. *Order relation on markings*

Let us consider two markings M_1 and M_2 of a PN.

We define the order relation between these markings as follows:

– $M_1 \geq M_2 \Longleftrightarrow M_1(P_i) \geq M_2(P_i), \forall Pi \in P$;

– $M_1 > M_2 \Longleftrightarrow M_1(P_i) \geq M_2(P_i), \forall P_i \in P$ and $\exists Pi | M_1(P_i) > M_2(P_i)$.

1.2.2. *Enabled transition*

The transition T_j is enabled for a given marking M if and only if:

$$M(P_i) \geq w^-(P_i, T_j), \forall Pi \in I(T_j)$$

In Figure 1.1, only the transition T_1 is enabled.

1.3. Dynamics of autonomous PNs

The previously defined notion of marking is the observation means of the evolution of the model. The position of the tokens will evolve according to a set of formal rules allowing the definition of some properties of the model. This will be recalled in the following, and more details may be found, for example, in [CAS 08, DAV 92, BES 01].

1.3.1. *Firing of a transition*

As PNs are models dedicated to discrete events systems, the firing of a transition may be considered as an event describing an elementary evolution of a system (see section 2.1 for the formal definition of labeled PN) characterized by the successive values of the marking before and after the firing. An enabled transition may be fired; from a given marking, each enabled transition could be fired but only one will be. The choice of the transition to be fired can be done arbitrarily. When a place has two output transitions their firings are in conflict. This notion of conflict (formally defined in Appendix A1.1) will be retrieved, for example, each time a failure occurs concurrently with a task activation or achievement. Some PN-dedicated software tools give the possibility of priority assignment to a transition concerned by a conflict, but this must be carefully handled to avoid the appearance of dead branches in the reachability graph. Two transitions $T_1, T_2 \in O(P_i)$ are not in conflict if they are not simultaneously enabled, which implies that these transitions have input places other than P_i.

The set of the enabled transitions must always be considered according to the current marking of the PN and not limited to a given place.

If M_b is the marking before the firing of T_j, the marking M_a after the firing is defined by:

– $\forall P_i \notin I(T_j) \cup O(T_j) \Longrightarrow M_a(P_i) = M_b(P_i)$

– $\forall P_i \in I(T_j) - (I(T_j) \cap O(T_j)) \Longrightarrow M_a(P_i) = M_b(P_i) - w^-(P_i, T_j)$

– $\forall P_i \in O(T_j) - (I(T_j) \cap O(T_j)) \Longrightarrow M_a(P_i) = M_b(P_i) + w^+(P_i, T_j)$

– $\forall P_i \in I(T_j) \cap O(T_j) \Longrightarrow M_a(P_i) = M_b(P_i) - w^-(P_i, T_j) + w^+(P_i, T_j)$

The firing of the transition T_j subtracts in place P_i as many tokens as indicated by $w^-(P_i, T_j)$ and adds in place P_k as many tokens as indicated by $w^+(P_k, T_j)$.

Figure 1.3 shows the PN of Figure 1.2 after the firing of transition T_2.

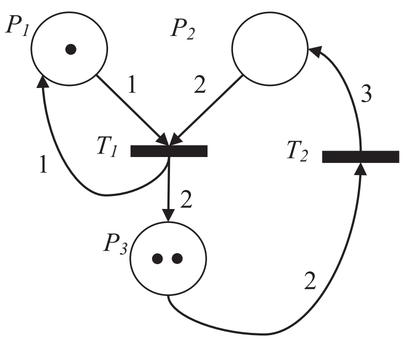

Figure 1.3. *PN of Figure 1.2 after firing of transition* T_2

1.3.2. *Transition matrix*

DEFINITION 1.3.– Let us define the backward matrix and forward matrix, as the following matrices with p lines and t columns:

$$W^- = \begin{bmatrix} w^-(P_1,T_1) & . & . & . & w^-(P_1,T_t) \\ . & . & . & . & . \\ . & . & w^-(P_i,T_j) & . & . \\ . & . & . & . & . \end{bmatrix} \quad [1.1]$$

$$W^+ = \begin{bmatrix} w^+(P_1,T_1) & . & . & . & w^+(P_1,T_t) \\ . & . & . & . & . \\ . & . & w^+(P_i,T_j) & . & . \\ . & . & . & . & . \\ w^+(P_p,T_1) & . & . & . & w^+(P_p,T_t) \end{bmatrix} \quad [1.2]$$

The transition matrix W is defined by:

$$W = W^+ - W^- \quad [1.3]$$

The transition matrix (p lines and t columns) is independent of the marking, each column simply shows the number of tokens to remove or add in a place when the corresponding transition fires. For Figure 1.1:

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 3 \\ 2 & -2 \end{bmatrix}$$

1.3.3. *Firing sequence*

DEFINITION 1.4.– A firing sequence is obtained when a set of transitions are successively fired, starting from an initial marking. It is represented by the concatenation of the successive names of the fired transitions.

If for example starting from the initial marking M_0 the transitions T_1 then T_2 are fired to give the marking M_2, the sequence will be denoted as:

$$S = T_1T_2, \qquad M_0 \xrightarrow{S} M_2$$

REMARK 1.2.– The transition set T provided with the concatenation operation and a neutral element may be considered as a monoïd denoted by T^*. With such a notation, a firing sequence is one element of this monoïd: $S \in T^*$. This notation will sometimes be used later.

DEFINITION 1.5.– Let S be a firing sequence feasible from a marking M_i, the characteristic vector of the sequence denoted as N is a t size vector, whose j^{th} component represents how many times the transition T_j is fired in the sequence S:

$$N \in \mathbb{N}^t, \qquad N(T_j) = k \text{ if } T_j \text{ is fired } k \text{ times.}$$

1.3.4. *Reachable marking*

The M vector cannot take any value. From a given marking M_0, it is possible to list all the possible firing sequences. The obtained marking after each of these sequences is a reachable marking.

Let us note that $R(M_0)$ is the set of the reachable marking from the initial marking M_0:

$$R(M_0) = \{M_a \in \mathbb{N}^p : \exists S / M_0 \xrightarrow{S} M_a\}$$

1.3.5. *Fundamental equation*

In FSA, we defined the state changes by the mean of the transition function. In PNs, this function is defined as follows:

$$f : \mathbb{N}^p \times T \rightarrow \mathbb{N}^p$$

$f(M_k, T_j)$ is defined if and only if T_j is enabled, in this case, $f(M_k, T_j) = M_{k+1}$ with:

$M_{k+1}(P_i) = M_k(P_i) - w^-(P_i, T_j) + w^+(T_j P_i)$ for $P_i \in I(T_j) \cup O(T_j)$

As for the FSAs, we can extend f from the domain $\mathbb{N}^p \times T$ to the domain $\mathbb{N}^p \times T^*$ (T^* being the monoïd on the set T provided with the concatenation operation (see section 1.3.3)) and define for a given initial marking, the new obtained marking after a firing sequence of characteristic vector N.

We then obtain the fundamental matrix equation as:

$$M_k = M_i + W \cdot N \qquad [1.4]$$

For Figure 1.2, let us imagine from the initial marking, the firing sequence $T_2 T_1$. After the firing of T_2, the obtained marking is shown by Figure 1.3 and after the firing of T_1 it becomes as indicated by Figure 1.4.

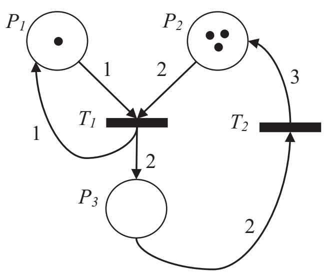

Figure 1.4. *PN state of Figure 1.3 after firing of transition* T_1

As the two components of the vector N are 1 and 1, each of the two transitions being fired one time, the obtained marking may be retrieved by the following calculus:

$$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -2 & 3 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1.3.6. *Properties of PN*

A set of definitions and properties are summarized here. For a complete description and formal demonstrations of properties, we can report to [DAV 89, CAS 08, DAV 92, BES 01, BRA 83]:

– *Boundedness:*

- a place of a PN is bounded for a given initial marking M_0 if for any accessible marking from M_0 the token number in this place remains finite. If $\forall M_n \in R(M_0)$, $M_n(P_i) \leq k$ with $k \in \mathbb{N}$, then P_i is k-bounded,

- a PN is bounded for a given initial marking M_0 if all the places are bounded for M_0.

If $\forall P_i \in P$, $\forall M_n \in R(M_0)$, $M_n(P_i) \leq k$ with $k \in \mathbb{N}$, then the PN is k-bounded.

These properties are dependent of the initial marking but sometimes a PN may be structurally bounded, that is to say bounded for any initial marking.

– *Liveness:*

- a transition T_j is alive for a given marking M_0 if $\forall M_n \in R(M_0)$, $\exists S : M_0 \xrightarrow{S} M_n / T_j \in S$ (there is always a firing of T_j),

- a PN is alive for a given marking M_0 if all its transitions are alive for M_0.

– *Blocking:*

- a blocking is a marking from which any transition is enabled. It corresponds to an absorbing state,

- a PN is blocking free for a given initial marking M_0 if no marking $M_n \in R(M_0)$ is a blocking.

Liveness and blocking are properties dependant on the initial marking M_0.

1.3.7. *Other properties*

Some other properties are summarized in Appendix A.1.

1.3.8. *Invariants in a PN*

1.3.8.1. *Conservative component and marking invariant*

It should be noted that sometimes the weighted sum of the markings of a subset of places remains constant. This is an invariant of this subset which is called conservative component of the PN. As it is independent of the initial marking, this is a property of the unmarked PN (the value of this constant may only depend on the initial marking). In most cases, this is the characteristic of a physical property of the modeled system.

A P-semi-flow is a vector F of integers of dimension p (number of places of the PN) so that:

$$F^T \cdot W = 0 \tag{1.5}$$

According to the fundamental equation $M_k = M_i + W \cdot N$ (for any accessible marking from M_i by a firing sequence S characterized by the vector N): $F^T \cdot M_k = F^T \cdot M_i + F^T \cdot W \cdot N$. If $F^T \cdot W = 0$ we obtain:

$$F^T \cdot M_k = F^T \cdot M_i \tag{1.6}$$

which is the marking invariant.

The integers of the vector F may be considered as weights assigned to the places of the PN. The subset of places whose weights are null is the PN conservative component support of the P-semi-low. It will be noted P_F.

Any linear combination of a semi-flow is itself a semi-flow.

Let $P_F = \{P_1, P_2, ..., P_r\}$ be a conservative component of a PN and $F = [q_1, q_2, ..., q_r]^T$ the corresponding weighting vector. All the places of the conservative component are bounded and we get: $M(P_i) \leq F^T \cdot M_0/q_i$.

For example, it is easy to verify that in the PN of Figure 2.1 (see section 2.2) the subset of places $\{P_4, P_5\}$ is a conservative component, the sum of their marking is always equal to 1 (the initial marking of P_4).

1.3.8.2. *Repetitive component and firing invariant*

In the same way, a T-semi-flow is defined: $W \cdot F = 0$.

Here, the weighting vector of integers is a vector N (dimension = t, N_i being the firing number of T_i) associated with a transition sequence S. Let us note T_S the transition subset fired at least once in the sequence S.

T_S is a stationary repetitive component if and only if

$$W \cdot N = 0 \qquad [1.7]$$

T_S is an increasing repetitive component if and only if $W \cdot N > 0$.

If $W \cdot N = 0$, then N is a T-semi-flow but any semi-flow does not necessarily correspond to a repetitive component because it must correspond at least to a firing sequence.

If S is a repetitive sequence from the marking $M_1 \in R(M_0)$ and if S is also a firing sequence from $M_2 \in R(M_0)$, then S is also a repetitive sequence from M_2 (see fundamental equation).

For Figure 2.1 (see section 2.2), the transition set $\{T_2, T_3\}$ is a repetitive component because a firing of T_2 leads to a firing of T_3 and so on.

1.3.9. *Reachability graph*

The evolution of the marking due to transition firings may be represented by a graph called a reachability graph.

The reachability graph of a PN $RG(M_0)$ is a graph whose nodes are associated with the successive values of the marking vector from initial marking M_0 and whose arcs correspond to the firings of transitions. All the properties of a PN may be retrieved on the reachability graph.

Let us consider the PN of Figure 1.5 (left) and its initial marking with two tokens in place P_1 and one in place P_3. The corresponding reachability graph is on the right-hand side of the figure. The node $[2,0,1,0]^T$ corresponds to the initial marking that may evolve by firing of transitions T_1 or T_2 to reach respectively the markings $[1,1,1,0]^T$ and $[2,0,0,1]^T$, and so on.

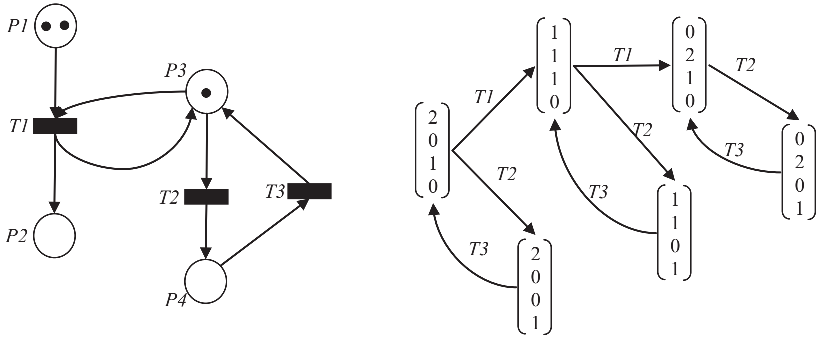

Figure 1.5. *A marked PN and its reachability graph*

In the current example, the reachability graph is finite but it is possible that it is not the case, meaning that the PN is not bounded (see section 1.3.6). It is then possible to define a finite covering graph by the identification of cycles in the reachability graph [DAV 89, DAV 92, DIA 01].