PART 1

Elements from Approximation Theory

1

Uniform Approximation

1.1. Canonical polynomials and uniform approximation

For fixed n, let $\mathcal{P}_n(I)$ denote the class of all polynomials with a degree equal to or less than n defined on I, a compact subset of $\mathbb{R}$, and let $P(x) := a_0 + ... + a_n x^n$. When set I is clearly identified by the context, we will write $\mathcal{P}_n$ instead. Some coordinates of the vector of coefficients $(a_0, ..., a_n)$ can take value 0.

For a continuous function f defined on I denote:

$$R := f - P$$

which is a continuous function on I, as is $|R|$.

Applying the Weierstrass theorem, it follows that $|R|$ attains its maximal value in I, for at least one point x. We denote:

$$\mu = \mu(a_0, a_1, ..., a_n) := \max_{x \in I} |R(x)| \geq 0$$

which yields to define

$$E^* := \inf_{(a_0, a_1, ..., a_n)} \mu(a_0, a_1, ..., a_n), \qquad [1.1]$$

the minimum value of the uniform error committed substituting f by a polynomial in $\mathcal{P}_n$. This lower bound exists.

DEFINITION 1.1.– A polynomial P^* in $\mathcal{P}_n$ with coefficients $(a_0^*, ..., a_n^*)$ such that

$$\|f - P^*\|_\infty := \sup_{x \in I} |f(x) - P^*(x)| = E^*,$$

where E^* is defined in [1.1] and is called a best approximating polynomial of f with degree n in the uniform sense.

This best approximation thus satisfies

$$P^* := \arg \inf_{P \in \mathcal{P}_n} \sup_{x \in I} |R(x)| . \qquad [1.2]$$

The polynomial P^* may be of a degree less than n.

Define

$$R^*(x) := f(x) - P^*(x) \qquad [1.3]$$

and

$$E^* = \max_{x \in I} |R^*(x)| . \qquad [1.4]$$

REMARK 1.1.– The Weierstrass theorem for continuous functions has been proved in many different ways, stemming from the original proofs by Weierstrass and Runge, followed by Bernstein. All proofs identify a specific sequence of polynomials; for example the family of Bernstein polynomials, the Lagrange interpolation polynomials or polynomials defined by convolutions. The difference between these various approaches lies in the rate of convergence which clearly differs from one sequence to another, yielding different approximation rates for the same degree. The quantity

$$E^* := E_n^*(f) := \inf_{P_n \in \mathcal{P}_n} \sup_{x \in [a,b]} |f(x) - P_n(x)|$$

which measures the adequacy of the approximation of the continuous function f is called the convergence rate.

1.2. Existence of the best approximation

This section answers the question of attainment, for the least uniform error, when approximating f by a polynomial in $\mathcal{P}_n$.

THEOREM 1.1.– Let f be a continuous function defined on I. For any integer n there exists P^* in $\mathcal{P}_n$ such that $\|f - P^*\|_\infty = \inf_{P \in \mathcal{P}_n} \|f - P\|_\infty$.

PROOF.– Since $\mathcal{P}_n$ is a linear space, the null polynomial $\mathbf{0}$ belongs to $\mathcal{P}_n$. Therefore if P^* exists

$$\|f - P^*\|_\infty \leq \|f - \mathbf{0}\|_\infty = \|f\|_\infty$$

holds.

Henceforth, P^* should belong to the set

$$\overline{S\left(f,\|f\|_{\infty}\right)} \cap \mathcal{P}_n := \left\{g \in \mathcal{C}^{(0)}\left(I\right) : \|g-f\|_{\infty} \leq \|f\|_{\infty}\right\} \cap \mathcal{P}_n.$$

This is a closed and bounded subset in $\mathcal{P}_n$. Since the dimension of $\mathcal{P}_n$ is finite, $\overline{S\left(f,\|f\|_{\infty}\right)} \cap \mathcal{P}_n$ is a compact set.

It is therefore enough to prove that the mapping

$$\Phi : \left(\mathcal{P}_n, \|.\|_{\infty}\right) \rightarrow \left(\mathbb{R}, |.|\right), P \mapsto \Phi\left(P\right) := \|f-P\|_{\infty}$$

is continuous.

From proposition A1.1 in Appendix 1, for Q in $\mathcal{P}_n$,

$$|\Phi(P)-\Phi(Q)| \leq \|P-Q\|_{\infty},$$

proving the claim.

The function Φ, thus, is continuous; it has at least a minimal value P^* in the compact set $\overline{S\left(f,\|f\|_{\infty}\right)} \cap \mathcal{P}_n$. □

1.3. Characterization and uniqueness of the best approximation

The proof of the uniqueness theorem heavily relies on a theorem by Chebyshev which explores the number of changes of the sign of the error, which we state now. Consider P^* and E^* as defined in [1.2] and [1.4]. The set I is now an interval

THEOREM 1.2 (Borel–Chebyshev).– A polynomial P^* in $\mathcal{P}_n$, is the best uniform approximation of a function f in $\mathcal{C}^{(0)}\left([a,b]\right)$, if and only if the function $x \rightarrow R^*\left(x\right)$ equals E^* with alternating signs for at least $n+2$ values of x in $[a,b]$.

The proof of this important result is postponed to the end of the present section.

We introduce the following definitions:

DEFINITION 1.2.– The $n+2$ points defined in theorem 1.2 are called equioscilation points.

This class of points is a subset of the class of alternating points, defined by:

DEFINITION 1.3.– Given a function g, defined on some set X, a family of points $x_1, .., x_m$ in X is an alternating family of points for g if $|g(x_1)| = .. = |g(x_m)|$ and $sign\left(g(x_i)\right) = sign\left(g(x_{i+1})\right), 1 \leq i \leq m-1$.

Theorem 1.2 asserts that the approximation of a function f in $\left(\mathcal{C}^{(0)}\left([a,b]\right), \|.\|_{\infty}\right)$ requires handling the parameters $A = \{a_0, ..., a_n\}$ which are the coefficients of P for a generic P in V. Reducing the distance $|f(x) - P(x)|$ on a point x in $[a,b]$ may result in an increase of the distance on another point. The optimal approximation holds when the maximal distance $\sup_{x\in[a,b]} |f(x) - P(x)|$ holds equal on $n+2$ points. Furthermore, this maximal distance should be attained with alternating signs.

Theorem 1.2 can be extended when f belongs to a normed space $(\mathbf{V}, \|.\|)$, instead of $\left(\mathcal{C}^{(0)}\left([a,b]\right), \|.\|_{\infty}\right)$. A special case is when,

$$(\mathbf{V}, \|.\|) = \left(\mathcal{C}^{(1)}\left([a,b]\right), \|.\|_{[a,b]}^{1}\right)$$

where $\|f\|_{[a,b]}^{1} := \|f\|_{\infty} + \|f'\|_{\infty}$ (with $x \in [a,b]$).

We now introduce the following definition.

DEFINITION 1.4.– The critical points of the function f are defined as the set:

$$\mathbf{Cr}\left(f\right) := \left\{x \in [a,b] : |f\left(x\right)| = \|f\left(x\right)\|_{\infty}\right\},$$

the set of points where, f reaches its maximal absolute values in $[a,b]$.

We now state the important uniqueness result.

THEOREM 1.3.– Let f be a continuous function defined on $[a,b]$. The best uniform approximating polynomial is unique.

PROOF.– In order to prove uniqueness, assume that two such polynomials of best approximation, say P and Q exist. Then

$$S := \frac{P+Q}{2}$$

belongs to $\mathcal{P}_n$ and for any x in $[a,b]$

$$\begin{aligned} |f\left(x\right) - S(x)| &\leq \frac{1}{2}\left|f\left(x\right) - P\left(x\right)\right| + \frac{1}{2}\left|f\left(x\right) - Q\left(x\right)\right| \\ &\leq \inf_{P\in\mathcal{P}_n} \|f - P\|_{\infty} =: E^*. \end{aligned}$$

Furthermore, the polynomial S is the best uniform approximation of f.

Consider a point z in $[a, b]$ where, by continuity

$$\left| f(z) - \frac{P(z) + Q(z)}{2} \right| = E^*.$$

It holds

$$E^* \leq \frac{1}{2} |f(z) - P(z)| + \frac{1}{2} |f(z) - Q(z)| \leq E^*$$

hence

$$\frac{1}{2} |f(z) - P(z)| + \frac{1}{2} |f(z) - Q(z)| = E^*.$$

Denote $\alpha := \frac{1}{2}(f(z) - P(z))$ and $\beta := \frac{1}{2}(f(z) - Q(z))$. It then holds

$$|\alpha + \beta| = |\alpha| + |\beta| .$$

Now $\frac{1}{2}|f(z) - P(z)| \leq E^*/2$ and so is $\frac{1}{2}|f(z) - Q(z)|$. Therefore, both equal $E^*/2$, i.e. $\alpha = \beta$. Therefore, $f(z) - P(z) = f(z) - Q(z)$ hence $P(z) = Q(z)$, which proves that z is the root of $P - Q$, a polynomial with degree less or equal n. Recall that S is a best uniform approximation of f. From Chebyshev–Borel's theorem, S admits $n+2$ maxima; therefore, we may find $n+2$ distinct points in $[a, b]$ similar to z. Since P and Q have degree less or equal n, they coincide. This closes the proof. □

1.3.1. *Proof of the Borel–Chebyshev theorem*

The proof obtained is through contradiction, assuming that the function $x \to R^*(x)$ takes h times the value E^* with alternating signs, with $h \leq n + 1$. Under this assumption, we will construct a polynomial $Q^*(x)$ for which

$$\max_{x \in [a,b]} |f(x) - Q^*(x)| < E^*,$$

is a contradiction.

Let

$$x_1, ..., x_h,$$

denotes the h points where $|R^*|$ equals E^*.

Define

$$\varepsilon_i := sign\left(f\left(x_i\right) - P^*\left(x_i\right)\right) \in \{-1, 1\}.$$

For any i between 1 and h, it holds

$$f\left(x_i\right) - P^*\left(x_i\right) = \varepsilon_i E^*.$$

Consider the system of h linear equations with $n+1$ variables $a_0, ..., a_n$ defined through,

$$\begin{cases} a_0 + a_1 x_1 + ... + a_{n-1} x_1^{n-1} + a_n x_1^n = \varepsilon_1 E^* \\ a_0 + a_1 x_2 + ... + a_{n-1} x_2^{n-1} + a_n x_2^n = \varepsilon_2 E^* \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_0 + a_1 x_h + ... + a_{n-1} x_h^{n-1} + a_n x_h^n = \varepsilon_h E^* \end{cases}.$$

Since, $h \leq n+1$, this system has at least one solution; if $h < n+1$, it has an infinite set of solutions.

Denoted by

$$a_0 = p_0, ..., \ a_n = p_n,$$

such a solution.

Since E^* is not zero, all the p_is cannot simultaneously take the value 0.

Using $(p_0, ..., p_n)$ define the following polynomial

$$\widetilde{P}(x) := \sum_{j=0}^{n} p_j x^j.$$

This polynomial shares a number of properties with $|R^*|$. First $\left|\widetilde{P}(x_i)\right| = |R^*(x_i)| = E^*$, for all $i = 1, 2, .., h+1$.

We now prove that a specific linear combination of $\widetilde{P}$ and P^* provides better uniform approximation of f with respect to P^*, which concludes the proof.

For any positive β, defined on $[a, b]$ the following continuous function

$$x \mapsto f\left(x\right) - P^*\left(x\right) - \beta\, \widetilde{P}(x).$$

The proof produces a value of β such that

$$\sup_{x\in[a,b]} \left| f(x) - P^*(x) - \beta\, \widetilde{P}(x) \right| < E^*. \qquad [1.5]$$

From a continuity argument, there exists some ϵ such that $0 < \epsilon < E^*$ with

$$|x' - x''| \leq \delta \Rightarrow \begin{cases} |R^*(x') - R^*(x'')| < \epsilon \\ \left| \widetilde{P}(x') - \widetilde{P}(x'') \right| < \epsilon \end{cases} \qquad [1.6]$$

when x' and x" both belong to $[a, b]$.

The proof aims to bound the supremum in [1.5], splitting $[a, b]$ as follows: we will consider neighborhoods $\mathcal{I}_\delta(x_i)$ of the points x_i where

$$\sup_{x\in[a,b]} |f(x) - P^*(x)|$$

attains its maximal value, and the complementary set of those in $[a, b]$.

STEP 1.– We bound $f(x) - P^*(x) = R^*(x)$ on neighborhoods of points x_i's, where the maximal error is attained with positive sign.

Consider any interval $\mathcal{I}_\delta(x_i) := \left(x_i - \frac{\delta}{2}, x_i + \frac{\delta}{2}\right)$ with length δ and center x_i, where

$$R^*(x_i) = E^*$$

meaning that x_i is a point where P^* reaches its maximal (positive) value.

Let $x' = x_i$ and $x'' = x \in \mathcal{I}_\delta(x_i)$. In the interval $\mathcal{I}_\delta(x_i)$, since

$$|R^*(x') - R^*(x'')| < \epsilon,$$

we deduce that for all x in $\mathcal{I}_\delta(x_i)$

$$R^*(x_i) - R^*(x) < \epsilon \qquad [1.7]$$

where we used the fact that for any x in $\mathcal{I}_\delta(x_i)$, $E^* \geq |R^*(x)|$.

Hence, from [1.7]

$$E^* - \epsilon < R^*(x).$$

Since $R^*(x) < E^*$, it also holds that

$$E^* - \epsilon < R^*(x) < E^*. \qquad [1.8]$$

STEP 2.– We bound $\widetilde{P}$ on neighborhoods of points x_i's, where the maximal error is attained with a positive sign.

Consider the polynomial $\widetilde{P}(x)$ on $\mathcal{I}_\delta(x_i)$, which satisfies

$$\widetilde{P}(x_i) = E^*.$$

By continuity

$$\left|\widetilde{P}(x') - \widetilde{P}(x'')\right| < \epsilon$$

for ϵ defined in [1.6].

This yields,

$$-\epsilon < \widetilde{P}(x_i) - \widetilde{P}(x) < \epsilon,$$

i.e.

$$E^* - \epsilon < \widetilde{P}(x) < E^* + \epsilon. \qquad [1.9]$$

STEP 3.– We bound $f(x) - P^*(x) - \beta\,\widetilde{P}(x)$ on neighborhoods of points x_i's, where the maximal error is attained with a positive sign.

For any positive ω, [1.9] is equivalent to

$$\omega(E^* - \epsilon) < \omega\widetilde{P}(x) < \omega(E^* + \epsilon). \qquad [1.10]$$

From [1.8] and [1.10]

$$E^* - \epsilon - \omega(E^* + \epsilon) < R^*(x) - \omega\widetilde{P}(x) < E^* - \omega(E^* - \epsilon).$$

Choose any positive ω, such that

$$\omega < \frac{E^* - \epsilon}{E^* + \epsilon}. \qquad [1.11]$$

With this choice, for any x in $\mathcal{I}_\delta(x_i)$,

$$0 < R^*(x) - \omega \widetilde{P}(x) < E^* - \omega(E^* - \epsilon).$$

STEP 4.– We do the same as in the three previous steps taken on the neighborhoods of points x_i's, where the maximal error is attained with negative sign.

Until now, we have considered the points x_i's in $[a, b]$ where $R^*(x_i) = E^*$. We use the same construction as above, using the points x_j for which $R^*(x_j) = -E^*$, defining similarly intervals $\mathcal{I}_\delta(x_j)$ with length δ and centered at x_j.

A similar argument provides

$$-E^* + \omega(E^* - \epsilon) < R^*(x) - \omega \widetilde{P}(x) < 0.$$

Henceforth, in any interval of the form $\mathcal{I}_\delta(x_i)$, where $|R^*(x_i)| = E^*$, it holds

$$\left| R^*(x) - \omega \widetilde{P}(x) \right| < E^* - \omega(E^* - \epsilon). \qquad [1.12]$$

This inequality is valid on $\cup_{i=1}^h \mathcal{I}_\delta(x_i)$.

STEP 5.– We bound $f(x) - P^*(x) - \omega \widetilde{P}(x)$ on the complementary set of $\cup_{i=1}^h \mathcal{I}_\delta(x_i)$ in $[a, b]$.

We now explore the behavior of R^* outside the collection of the intervals $\mathcal{I}_\delta(x_i)$ where it assumes its extreme values.

Let

$$A := [a, b] \setminus \left(\cup_{i=1}^h \mathcal{I}_\delta(x_i) \right).$$

On A, the mapping $x \to |R^*(x)|$ attains its maximal value m such that $m < E^*$.

Since $A \subseteq [a, b]$ it holds $m \le E^*$. Furthermore $m \ne E^*$; indeed A and $\cup_{i=1}^h \mathcal{I}_\delta(x_i)$ are disjoint sets.

On A it holds,

$$|R^*(x)| \le m. \qquad [1.13]$$

Let

$$M := \max_{x \in [a,b]} \left| \widetilde{P}(x) \right|.$$

We impose the additional condition on ω, defined in [1.10] through

$$0 < \omega < \frac{E^* - m}{2M}.$$

It then holds, for any x in A, that

$$\left|\omega \widetilde{P}\left(x\right)\right| < \frac{E^* - m}{2}. \qquad [1.14]$$

Apply the triangle inequality to $\left|f\left(x\right) - P^*\left(x\right) - \omega \widetilde{P}\left(x\right)\right|$, to obtain

$$\left|f\left(x\right) - P^*\left(x\right) - \omega \widetilde{P}\left(x\right)\right| \leq \left|f\left(x\right) - P^*\left(x\right)\right| + \omega \left|\widetilde{P}\left(x\right)\right|.$$

From [1.13] and [1.14] it follows that on A

$$\left|R^*\left(x\right) - \omega \widetilde{P}\left(x\right)\right| \leq \frac{m + E^*}{2}.$$

STEP 6.– We assemble the previous results to obtain that $P^*\left(x\right) - \omega \widetilde{P}\left(x\right)$ improves on P^*, a contradiction.

From that shown above and [1.12] we obtain for any x in $[a, b]$

$$\left|R^*\left(x\right) - \omega \widetilde{P}\left(x\right)\right| < \min\left\{\frac{m + E^*}{2}, E^* - \omega\left(E^* - \epsilon\right)\right\}.$$

Furthermore, since $m < E^*$ and by the very choice of ω

$$\min\left\{\frac{m + E^*}{2}, E^* - \omega\left(E^* - \epsilon\right)\right\} < E^*,$$

we obtain, for any x in $[a, b]$.

$$\left|R^*\left(x\right) - \omega \widetilde{P}\left(x\right)\right| = \left|f\left(x\right) - P^*\left(x\right) - \omega \widetilde{P}\left(x\right)\right| < E^*.$$

This means that

$$Q^*(x) := P^*\left(x\right) + \omega \widetilde{P}\left(x\right),$$

approximates f with an error smaller than that obtained through P^*, a contradiction.

This closes the proof.

As done previously, through theorem 1.2, the best polynomial approximation P^* of f is characterized by the fact that the function

$$x \to f(x) - P^*(x)$$

takes its common extreme value on at least $n+2$ points, with alternating sign, hence equioscillating points.

1.3.2. *Example*

EXAMPLE 1.1.– Define the Chebyshev polynomial of first kind defined on $[-1, 1]$ by,

$$T_n(x) := \cos(n \arccos x).$$

This polynomial has as equioscillation points the set

$$x_i := \cos(i\pi/n), 0 \leq i \leq n,$$

see Appendix 2. The polynomial $x \mapsto x^n - \frac{2}{2^n} T_n(x)$ has degree n and equioscillates $n+1$ times on the above x_i's. Hence, it is the best uniform approximation of the null function on $[-1, 1]$. Note that this best approximating polynomial is defined by fixing its value to 1. Indeed, when approximating the null function, all approximations are defined up to a multiplicative constant. This, obviously, is not the case for the approximation of any other function.

REMARK 1.2.– More generally, let $f : [-1; 1] \to \mathbb{R}, x \mapsto f(x) := \sum_{j=0}^{n+1} a_j x^j$, with $a_{n+1} \neq 0$. We determine the polynomial $\mathrm{P}_n^* \in \mathcal{P}_n$, which approximates f in $[-1; 1]$. This results in P_n^* such that $(f - P_n^*)/a_{n+1}$ is the best polynomial approximation of the null function with a degree less or equal $n+1$ with value 1. Since this polynomial is $2^{-n} T_{n+1}$, it holds $f - P_n^* = 2^{-n} T_{n+1}$. Hence, $P_n^* = \sum_{j=0}^{n+1} a_j x^j - 2^{-n} a_{n+1} T_{n+1}$.

EXAMPLE 1.2.– Let $f(x) = 9x^4 - x$. The best uniform approximation with a degree less or equal to 4 on $[-1; 1]$ is $P_4^*(x) = 9x^4 - x - \frac{9}{8}\left(8x^4 - 8x^2 + 1\right) = 8x^2 - x - 1$.

EXAMPLE 1.3 (see [DZY 08]).– Let $f(x) = \sin x, x \in [-4\pi; 4\pi]$. Determine the polynomial $P_n^* \in \mathcal{P}_6$, corresponding to f in $[-4\pi; 4\pi]$. The function $\sin x$ alternates its sign on the points $x_j = -\frac{7}{2}\pi + k\pi$ and $k = 0, 1, ..., 7$. Let $\mathbf{0}_6 : [-4\pi; 4\pi] \to \mathbb{R}, x \mapsto \mathbf{0}_6(x) := 0$; then on $x_j \in [-4\pi; 4\pi]$, $j = 0, 1, ..., 7$, it holds: $\sin x_j - \mathbf{0}_6 = (-1)^j \|\sin x_j\|_{\infty,[-4\pi;4\pi]} = (-1)^j$. Hence, form the Chebyshev–Borel theorem, the null polynomial $\mathbf{0}_6 \in \mathcal{P}_6([-4\pi; 4\pi])$ is the best uniform approximation of $\sin x$ in $\mathcal{P}_6([-4\pi; 4\pi])$.

EXAMPLE 1.4.– Determine in

$$\mathcal{P}_n(1) := \{P \in \mathcal{P}_n([a;b]), 0 < a < b : P(0) = 1\}$$

the best approximation of the null function. Intuitively, since $T_n\left(\frac{2x-a-b}{b-a}\right)$ is the best approximation of the null function on $[a;b]$, we may say that $T_n\left(\frac{2x-a-b}{b-a}\right)/T_n\left(\frac{-a-b}{b-a}\right)$ is the solution. Indeed, the constant $1/T_n\left(\frac{-a-b}{b-a}\right)$ makes $T_n\left(\frac{2x-a-b}{b-a}\right)/T_n\left(\frac{-a-b}{b-a}\right)$ evaluated at 0 equal to 1. We prove that $T_n\left(\frac{2x-a-b}{b-a}\right)/T_n\left(\frac{-a-b}{b-a}\right)$ is indeed the solution. By contradiction, if $Q \in \mathcal{P}_n$ and $Q(0) = 1$ is the solution then $T_n\left(\frac{2x-a-b}{b-a}\right)/T_n\left(\frac{-a-b}{b-a}\right) - Q_n(x)$ alternates values in at least $n+1$ points in $[a;b]$. This function thus has n roots in $[a;b]$ at least. On $x = 0$, $Q(0) = 1$. Hence $T_n\left(\frac{2x-a-b}{b-a}\right)/T_n\left(\frac{-a-b}{b-a}\right) - Q(0) = 0$. Hence $T_n\left(\frac{2x-a-b}{b-a}\right)/T_n\left(\frac{-a-b}{b-a}\right) - Q(x)$ has at least $n+1$ root. But this function has degree n, thus $T_n\left(\frac{2x-a-b}{b-a}\right)/T_n\left(\frac{-a-b}{b-a}\right) = Q(x)$ for all x.

REMARK 1.3.– In the example above, the minimal value of the functional $\|.\|_\infty$ is on the frontier of the unit sphere of $\mathcal{P}_n([a,b])$. Such points are called extreme points. See Chapter 5 for extensions.

EXAMPLE 1.5.– Let

$$f : [-1,1] \to \mathbb{R}, x \mapsto f(x) := x^4.$$

The best uniform approximation of f in $\mathcal{P}_3([-1,1])$ is given by,

$$P^*(x) := x^2 - \frac{1}{8}.$$

The derivative of the error function

$$R^*(x) := x^4 - x^2 + \frac{1}{8}$$

takes value 0 in $x_1 = 0, x_2 = \frac{\sqrt{2}}{2}$ and $x_2 = -\frac{\sqrt{2}}{2}$.

Furthermore

$$R^*(-1) = \frac{1}{8}, R^*(0) = \frac{1}{8}, R^*\left(\frac{\sqrt{2}}{2}\right) = -\frac{1}{8} \text{ and } R^*\left(-\frac{\sqrt{2}}{2}\right) = -\frac{1}{8}, R^*(1) = \frac{1}{8}.$$

Henceforth, there exist five points where the absolute value of the error is maximal. The error alternates sign.