
CHAPTER 1

Introduction

Throughout the many years that I have worked with children and watched them learning mathematics and solving problems, I have never ceased to be amazed by the fascination that most children have for solving problems. When I observe them solving problems outside the classroom, no matter how difficult these might be, most children try very hard to make sense of the problem and to find a solution. The solution might not be right, but it is invariably reasonable from the child's viewpoint.

Once, a lively 8-year-old boy and I had just finished a problem-solving session, where he was tackling difficult multiplication problems in a teaching experiment, and we saw the headteacher as we walked to his classroom. 'Miss, I have been doing some really hard maths,' he said with a grin. It was the end of the day, and I came back to the room to collect my materials and leave. The headteacher was waiting for me. She was curious about what I had been doing in the experiment; the boy I had been working with was considered very weak in maths. He had, in fact, started the session showing signs of fear and stress. When I gave him the first problem, which should have been quite simple, he gave me an answer right away, a number, clearly without having had any time to think the problem through. But the teaching experiment I was trying out was not about giving numerical answers, but about showing through actions how to solve the problem. So I didn't have to say to him that it was not the right answer. I just had to pose new questions to him, which helped him see a path to the solution. When he completed the first exercise, he gave me the grin he later gave to the headteacher, relaxed, and said that this was actually fun. Anyone who has succeeded in solving a problem that seemed very hard to begin with can empathize with this boy: actually, succeeding in intellectual tasks is fun.

Working with deaf children solving mathematics problems has been no different. Once they become involved in problem-solving, they too experience the joy of intellectual success. So why write a book about teaching mathematics to *deaf* children? Is it not like teaching mathematics to hearing children? It is certainly easy to see why language learning is different for deaf and hearing children and perhaps even to extend the same

reasoning to understand why literacy learning has to be more difficult for deaf than for hearing children. But why mathematics?

This book analyses different aspects of mathematics learning: why we need to consider deaf children's strengths and difficulties above and beyond the expected difficulties that mathematics learning causes to all children. Much of the evidence discussed here is based on research with hearing children and analyses what is difficult for all children in mathematics learning. Concepts that are difficult for hearing children are also difficult for deaf children. But there are special considerations when teaching mathematics to deaf children. They relate both to the role of language in mathematics teaching and learning and to the preferences for cognitive processing that deaf children have, some of which differ from hearing children's preferences. It is hoped that at the end of this work you will find that you can see the children's viewpoint, although it should be recognized that much more research is needed for a thorough understanding of how best to teach mathematics to deaf children.

This introduction starts by analysing whether there is a specific problem for deaf children in learning mathematics and why this problem might exist. The research described in this first section is quantitative, in order to assess the size of the problem. The second half of the chapter presents a theoretical analysis of processes involved in children's mathematics learning, and selects specific issues that might make teaching mathematics to deaf children different from teaching hearing children. The subsequent chapters in this book analyse core mathematics concepts, their difficulties for all children, and the specific challenges that could be faced by deaf children. In each chapter, the analysis is followed by a description of successful attempts to find a better path for teaching those concepts to deaf children.

Is there a problem for deaf children in learning mathematics?

Educators in the UK have for a long time been monitoring the attainments of deaf children in mathematics. A report by the National Council of Teachers of the Deaf, published in 1957, compared the performance of 246 deaf pupils in four special schools in England with the standardized norms for hearing pupils. The pupils' progress in mathematics was assessed through the Schonell Arithmetic Test, which contained two assessments of arithmetic. The Schonell Arithmetic Test is standardized to produce an arithmetic age: for the average child the arithmetic age should be the same as the chronological age, though a small difference between the chronological and the arithmetic ages is often observed in either

direction. The average arithmetic age for the deaf pupils was two and a half years below their chronological age. This means that, for example, a 10-year-old would be performing in arithmetic similarly to a child aged between 7 and 8 years. This large difference between the chronological age and the arithmetic age does give cause for concern.

About a decade later, Wollman (1965) tested approximately one-third of the pupils from 13 schools for the deaf in the UK and a comparison group of 162 hearing pupils in secondary schools. Deaf pupils were once again found to perform significantly worse than the hearing cohorts. Wollman's study does not describe the arithmetic age of the pupils, as in the study by the National Council of Teachers of the Deaf, but makes a comparison between the means for the hearing and for the deaf pupils. In order to understand this comparison, it is necessary to think about how means work in describing a group.

A mean for a large group of pupils is normally at the middle of the distribution of scores: many pupils have a score that is the same as the mean, some will have higher and some will have lower scores. For most large groups and age-appropriate tests, it is possible to estimate how many pupils will score within a certain range above and below the mean. This estimation allows us to understand the significance of the difference between a child's score, for example, and the mean. The statistic that is used to help us understand the importance of a difference between a particular score and the mean is the *standard deviation*. For most groups, about two-thirds of the pupils will have scores in the range between one standard deviation below and one standard deviation above the mean. If a child's score is more than one standard deviation below the mean, this places the child roughly among the weakest 15% of the group. For all practical purposes, this means that this child needs help. In Wollman's study, the average for the deaf pupils in the mathematics test was about one standard deviation below the mean for the hearing pupils. This indicates that about 50% of the deaf pupils' performance was equivalent to that of the weakest 15% of the hearing pupils. These comparisons give us an idea of the size of the problem.

But it is not only the quantitative aspect that gives cause for concern in the results of the study by Wollman. An analysis of where the deaf pupils succeeded and failed gives further reason for concern. It was found that a large proportion of pupils showed a skill in simple arithmetical processes; far more errors were attributable to lack of understanding than to mistakes in calculation. Thus the deaf pupils were learning how to carry out sums but not when to use them to solve problems. This is not an unusual difficulty for hearing pupils (see, for example, Brown, 1981; Bryant, 1985) but it was clearly a more serious obstacle for the deaf pupils.

Hine (1970) also found a considerable delay in mathematics age in a study of 104 deaf pupils. The delay for 10-year-olds was about 2 years but the delay for 15-year-olds was more serious: deaf 15-year-olds on average showed a competence comparable to that of hearing peers aged 10 to 11 years.

Unfortunately, the situation was much the same another decade later. Wood, Wood and Howarth (1983) carried out a survey, including 80 schools in England and Wales, on a group of 414 hearing-impaired pupils and a comparison group of 465 hearing pupils in secondary school. They used the Vernon and Miller Graded Arithmetic Test (Vernon and Miller, 1976), which they thought would have little contamination of reading difficulties. The hearing-impaired youngsters were about three and a half years behind the hearing pupils. Thus an average deaf school leaver would show a mathematical competence equivalent to that of a hearing pupil at age 12 to 13 years. This bleak result, which is even worse than that of the initial study by the National Council of the Teachers of the Deaf, does not mean that deaf pupils' mathematics achievement is getting worse over time. The tests used in the two studies are very different and the Vernon and Miller Test does require reading instructions in order to solve many of the items.

More recently, we (Nunes and Moreno, 1998a) compared the performance of 85 deaf pupils with the norms for the NFER-Nelson 7-11 series of mathematics assessments. The pupils were in Years 2 through to 5 (aged between 8 and 11 years) in special schools and mainstream schools with units for the hearing impaired in eight different schools in London. None of them had additional difficulties beyond the hearing loss. The mean for the deaf pupils was two standard deviations below the mean for the hearing pupils; this is equivalent to saying that the mean score for the deaf pupils was comparable to the result for the weakest 2% of hearing students. This very poor average performance was influenced by the fact that some of the deaf pupils had extremely low scores, which we included in the calculation of the mean. We then excluded from the calculation of the mean the 16 pupils whose score was so low that it was below the bottom 2% for their year group, and recalculated the mean. The result was an average score for the deaf pupils that was roughly one standard deviation below the mean for hearing pupils, and thus comparable to the weakest 15% of the hearing pupils.

Finally, the recent norming of the Stanford Achievement Test for deaf and hard-of-hearing students carried out in the USA (Traxler, 2000) confirms this bleak picture once again. In order to avoid giving students assessments that were too difficult and thus would prove frustrating, Traxler did not administer the grade-appropriate level assessment to all the participants but gave them assessments that were judged appropriate

on the basis of an initial screening. Well over half of the students aged 9 or older and as many as 90% of the 15-year-olds were given tests that were appropriate for hearing pupils at grades lower than their own. This by itself suggests an educational delay. The two mathematics tests administered, Problem Solving and Procedures, showed results that were similar in one way: the growth curves for the deaf and hard-of-hearing students are much flatter than those for hearing students. In Problem Solving, 80% of the students at age 14 showed a level of performance considered basic or below basic knowledge of mathematics problem-solving. In Procedures, 80% showed a level of performance that was considered below basic at the same age level. Thus very few deaf and hard-of-hearing students in the USA seem to be attaining a sound level of mathematics knowledge according to the Stanford Achievement Test.

The picture is therefore not an encouraging one. Deaf pupils' mathematics achievement has been documented as very weak for about five decades of research and there is no indication that educational changes that may have benefited hearing children's mathematics learning have succeeded in closing the gap between hearing and deaf pupils. This bleak picture cannot be taken lightly. Mathematics is a gate-keeper into many courses at university, where the minimum achievement is a grade C in the standardized examinations at age 16 (currently, GCSE). Numeracy tests are also given routinely in selection procedures for employment. Failure in mathematics is not an option: it is urgent that we reach a better understanding of why deaf pupils are not making better progress in mathematics, and how to change this situation.

The different shades of grey

Before proceeding any further, it is important to say that the picture of deaf children's mathematics achievement is not uniform, and that looking at means does not tell the whole story. In every study, there is variation between pupils, and in every study there are deaf pupils who perform at average and above-average levels for their age. In the study by Wood and colleagues (1983), 15% of the profoundly deaf pupils showed scores equal to the average for the hearing students or better.

Frostad (1996), in a study in Norway, observed a higher percentage of deaf students performing at average levels or better (37.5%) even though the average attainment was still significantly lower than that for hearing students. This figure was reported for all levels of deafness and it cannot be directly compared to the results described by Wood and colleagues, because different tests were used. Though better than the results observed by Wood and colleagues, these results are still an indication of difficulties for the deaf pupils, because in a standardized test 50% of the

sample who participated in the development of the test norms performed at the average level or better.

In our own work, we have also found variation. As indicated earlier, 16 of the 85 children – about 19% – had scores that were so low that they were not included in the table for conversion of raw scores into standardized scores. In order to scrutinize the distribution of the scores more carefully, we looked at the percentage of children in different bands of standardized scores. The test we used was standardized to produce a mean score of 100. Considering only the children who were severely or profoundly deaf, the results were: 25% were in the band below a standardized score of 50 (equivalent to about the weakest 2% of the hearing group); 8% were in the 51-to-70 band (which is at about the weakest 5% of the hearing group); 31% had standardized scores between 71 and 90 (corresponding to the weakest 40% for the hearing norms) and the remaining 36% had standardized scores of 90 or above (which were at average or above for the hearing group). Our results, obtained with English pupils much younger than those who participated in the study by Wood and colleagues, are more similar to the results observed by Frostad. But, as indicated earlier on, there is no possibility of direct comparison across the studies, owing to the use of different tests.

In short, the three studies show that there is no uniform picture for the mathematics attainment of all deaf pupils. There is among deaf pupils, just as there is among hearing pupils, variation in success. Unfortunately the results show that the skies are not blue, but different shades of grey.

Does deafness cause difficulties in learning mathematics?

By now you must be wondering whether deaf children have difficulties in mathematics exactly *because* they are deaf. Some people argue that mathematics is a form of language and, because learning languages is difficult for deaf pupils, they find learning mathematics difficult. This argument would see a direct, and thus inevitable, connection between deafness and mathematics learning difficulty. Is this argument sound? What evidence is there to support this idea of a direct causal connection between deafness and difficulty in learning mathematics?

First, let us consider whether the argument is sound. The first part of the argument is that mathematics is a language *and* that it is thus difficult for deaf pupils because they find learning languages difficult. Both parts of this argument can be questioned.

Many people would argue that mathematics is not simply a language. Mathematics is a way of representing the world and communicating about number and space – and in this sense it could be considered as a language – but it involves more than that. Learning mathematics involves learning specific ways of thinking about number and space, and using these to process information. This discussion, though, is fruitless for our investigation of why deaf pupils find mathematics difficult. Whether it is a language or not, in a way it does not matter, because the second part of the argument is certainly flawed.

It is simply not true that deaf children have difficulty in learning languages. Deaf children of deaf parents, who sign to them from the time they are born, learn sign language as ably as hearing children learn oral language (Goldin-Meadow, 2003). Their vocabulary growth and mastery of the grammar of sign language are comparable to that of hearing children (Lillo-Martin, 1999; Newport and Meier, 1985; Petitto, 2000). Sign languages assume all the functions of spoken languages and are used not only to communicate about the here-and-now but also about worlds that are not present (Goldin-Meadow, 2003). Signs are not direct representations of objects or actions but are conventions, and just like spoken languages have to be learned.

The misconception that learning languages is difficult for deaf children comes from the fact that most deaf children – about 90% – are born to hearing parents, who do not know a signed language. Therefore there is a mismatch between the parents' knowledge and the children's language learning ability. The parents' input of a spoken language, which the child cannot receive, is as much beyond the reach of the deaf child as ultrasounds are to the human ear. The mismatch often continues for a long time, as many parents do not find out that their child is deaf until well into the child's second year of life, when the child does not show the expected production of spoken words. And when they do find out, they would themselves need time to learn sign language, if this is the type of education they choose for their child. Thankfully, this delay in diagnosis may soon be one less obstacle for deaf children because of the systematic introduction of diagnostic procedures early in the infant's life.

Thus it is now clear that deaf children do not have an inherent difficulty in learning languages but that the mismatch between the verbal input produced by hearing parents and the child's sensory abilities results in language delay. It should then be clear also that the idea that deafness causes difficulties in learning mathematics because of the very nature of mathematics and the very condition of deafness can be rejected.

However, is it possible that there is an indirect causal connection between deafness and difficulty in learning mathematics, perhaps because

spoken language is an essential part of how mathematics must be learned? What type of evidence would support the idea that there is a causal connection between deafness and mathematics learning? Different possibilities are considered in the sections that follow.

Correlational studies

One way in which psychologists investigate whether there is a causal connection between two human characteristics – in this case, deafness and mathematics difficulty – is to analyse whether it is possible to show a statistical relation between the two. If deafness causes mathematics difficulty, the level of hearing loss should be related to the level of difficulty. In most studies, finding that there is a mathematical relation between the two characteristics does not directly support the causal relationship between the two, but rather provides evidence that there *could* be a causal connection. The reason for a careful conclusion – that is, saying that there could be a causal connection – is that different possibilities exist to explain a mathematical relation between two characteristics. One is that the first characteristic causes the second: in this case, deafness causes mathematical difficulty. The second possibility is that the causal connection goes the other way around: in this case, mathematics difficulty causes deafness – a possible but very unlikely explanation of why there is a mathematical relation between deafness and mathematics difficulty. The third possibility is, nevertheless, plausible: that both characteristics are caused by a third factor, i.e. deafness and mathematics difficulty could both result from a third characteristic such as damage in brain functioning. In this case, the cause of deafness, and not the level of deafness, might show a connection with mathematics difficulty. Deafness is sometimes a genetic condition, which may or may not be associated with learning difficulties or other problems. Deafness might also be acquired, for example through meningitis, and the illness could cause other problems besides deafness.

Three of the studies referred to earlier on, which investigated the mathematics achievement of deaf pupils, also analysed the connection between the level of deafness and mathematics achievement. In order to investigate whether there is a statistical connection between two characteristics, correlations between them are calculated. Correlations vary between -1 , when there is a perfect correspondence between the two characteristics but an inverse one (as in this case, the greater the deafness, the smaller the mathematics competence), and $+1$; a correlation around 0 signifies that there is no statistical relationship between the two characteristics. Wollman (1965) found no correlation between the level of deafness and mathematics achievement. Wood and his colleagues (1983) found a weak negative correlation between the level of deafness and the

mathematics achievement, close to -0.3 , which indicates a feeble statistical relation between deafness and mathematical achievement. We (Nunes and Moreno, 1998a) did not find a statistical relation between deafness and mathematics achievement. Thus, three independent studies, which included a total of more than 500 deaf pupils, failed to provide evidence for a causal connection between level of hearing loss and mathematics competence. The hypothesis of a direct causal connection between deafness and mathematics competence thus becomes less plausible.

What if most types of deafness were not related to mathematics achievement, but there was a third factor, causing both deafness and mathematics difficulty? There is little research that can help answer this question. One of the difficulties in carrying out this type of investigation is that in a very large proportion of deaf children there is no identified cause of deafness: there is no family history of deafness, there is no birth trauma, no illness (such as meningitis, ear infection or prenatal German measles) nor genetic syndrome that explains why the child is deaf. In our study, this was true for just under 50% of the children. We found no association between cause of deafness and mathematics achievement, but our sample of children with a known cause of deafness was very small – 43 children – and the number of children with the same cause of deafness was thus even more reduced. With such small samples, it is not possible to find a connection between type of deafness and mathematics difficulty.

We can, however, follow this argument to its consequences to see what could be learned if a study showed that some types of deafness have a direct causal connection to mathematics difficulty. Let us suppose that it was in fact found that a particular cause of deafness – I will call it deafness type A – was definitely caused by some form of brain dysfunction that also caused difficulties in mathematics. This study would explain why children who have deafness type A have difficulty in learning mathematics, but not why the majority of deaf children show such difficulty. So, finding a connection between one type of deafness and mathematics difficulty does not help us understand the plight of the majority of the deaf pupils.

The only clue that could be found in the literature was a large review of the research on deafness and intellectual development. If deafness were to result in lower levels of intelligence, this would result, in turn, in lower mathematics achievement, because research has shown abundantly that there is a relationship between intellectual level and achievement in mathematics. Braden (1992, 1994) carried out a systematic, quantitative analysis of the results of different studies on the relationship between deafness and intelligence. He carefully considered the nature of the tests given – whether they involved verbal or non-verbal reasoning – as well as other important factors that would affect the validity of the study, such as the number of participants, the methods of delivery of the tests and data

analysis, and whether the cause of deafness had been known and analysed. Verbal reasoning skills require children to solve problems and display knowledge that involves language. For example, a verbal problem-solving task might involve analogical reasoning. In a verbal, semantic analogy task a child could be given a pair of words that have a particular meaning relation between them – for example, 'milk' and 'cheese'; the child is then given a third word, say 'leather', and asked to provide the fourth word. In order to give a correct answer, the child would have to consider that 'cheese is made from milk' and provide a word that shows the same relation to 'leather' – e.g. shoes or handbag. A non-verbal reasoning problem might be solving a puzzle, such as assembling a picture of a car from a number of pieces, or completing a picture where the discrete elements have a particular relationship to each other. Figure 1.1 provides an example.

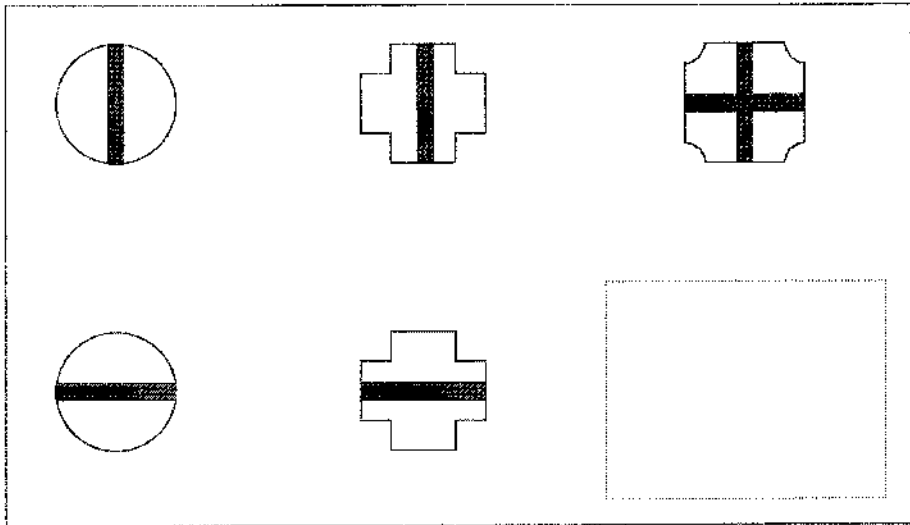


Figure 1.1 An example of visual analogy. What figure should be drawn into the rectangle?

Because there is a correlation between vocabulary knowledge and verbal reasoning tasks, it should come as no surprise if deaf people perform poorly on verbal reasoning tasks. But an analysis of verbal tasks is meaningless when comparisons between hearing and deaf people are made, because verbal tasks are clearly not 'fair tests'. There is typically some correlation between verbal and non-verbal reasoning because some of the cognitive operations that are involved in the tasks may be the same – Figure 1.1 shows that it is possible to use the cognitive operation of making analogies with both verbal and non-verbal materials. In spite of this

correlation and the difficulties that deaf people might have in solving verbal problems, there is no reason to expect that deaf people would be at a disadvantage in non-verbal reasoning tasks.

Braden concluded from his extensive review of studies that there is no evidence for a deficit in non-verbal reasoning skills among deaf people whose deafness is caused by identified genetic causes and who have no other impairment than deafness. When deafness is caused by illnesses that may have negative effects on the brain, there may be intellectual loss, though this is not a universal consequence.

This research is highly relevant to the question of a connection between deafness and mathematical ability. It has long been established that non-verbal reasoning is strongly correlated with mathematical ability. Braden's analysis thus leaves us with a puzzle. If deaf children have the intellectual ability to learn mathematics, what is getting in their way?

In order to consider possible answers to this question, it is necessary to think about how hearing children develop their mathematical competence and then ask to what extent the developmental processes involved could explain deaf children's difficulties. In the remaining section of this chapter some basic ideas about the development of mathematical competence are considered. The essence of the argument to be presented is that different mathematical concepts pose specific difficulties for all children learning mathematics. In each of the chapters that follows, a mathematical concept is considered and the specific difficulties that it poses for children are analysed. The pieces of the puzzle of what is getting in the way of deaf children's mathematics learning will be identified throughout the chapters. The concluding chapter will aim at putting the picture together.

General processes in the development of children's mathematical competence

Some preliminary thoughts

Let me start this section with an imagined situation. Suppose I ask you: What is eight times nine? How do you get the answer? Most hearing people who have attended school will have memorized this number fact. They know the verbal string 'eight times nine, seventy-two'. The fact that we know the answer to this problem as a memorized verbal string is seen as compelling evidence that oral language is an integral part of our knowledge of mathematics. This could lead us to think that deaf children have some difficulty in learning mathematics because they find it hard to memorize verbal strings.

Imagine a second situation. Suppose I gave you this story problem: 'Mary bought eight bars of chocolate. Each bar cost 9p. How much do the chocolate bars cost altogether?' What would you have to do to solve this story problem? Many people would answer that one would first have to identify the correct operation, multiplication, and then find the answer to what is eight times nine, in order to answer the question. This second example suggests that people have to identify a correct operation (is it addition, subtraction, multiplication or division?) and they apply it to the numbers to solve this problem.

Researchers and teachers have for a long time thought that solving story problems – and, by extension, solving mathematics problems in practical situations in everyday life – involves two steps. The first is to understand the description of the problem and identify the correct operation. The second is to carry out the operation. Both of these steps are seen as involving language. It is suggested that, in order to construct an interpretation of the situation, we need language. It is argued that this is so even if we were to go to a shop, for example, take the eight bars of chocolate from a box and see the price, 9p, written on them. In this practical situation, much as in the story problem, we would need to say to ourselves: 'Eight bars of chocolate at 9p each: that means eight times nine'. This description is intuitively appealing – reading it, we might think that this is the way *everyone* solves such problems. I will argue that there is some truth in this description for some of us, but that for most of us some aspects of this description are incorrect, even if they appear to be true.

There is little doubt that in order to solve a problem, even if it is a practical problem such as the one in the shop, we need to construct a *mental* representation of the situation. After all, we solve problems by thought operations, and our thinking works with mental representations. It also seems intuitively right that if we construct an incorrect mental representation of the problem, we are more likely to produce a wrong than a right answer. So we have to construct an adequate mental representation for the problem. But the idea that we then have to proceed to identify *the correct operation* is not supported by research. For many mathematical problems, including the story problem above, there is more than one logical way to find the answer. Coming up with eight times nine is only one way, though probably the preferred way for people who went to school and memorized the times table.

Much research with people who work in markets shows that this is not the preferred way for unschooled people. People solving practical problems in everyday life use different procedures, which depend on the number system that they use and on the procedures commonly adopted in their culture.

Gay and Cole (1967) were the pioneers of this line of investigation. They observed how the Kpelle, an indigenous group in Liberia, solved arithmetic problems. Their indigenous counting system, which they use to talk about and represent numbers, was not used to construct and memorize multiplication tables. Thus if they were asked to solve multiplication problems, their normal procedure for obtaining an answer was to use stones or fingers to represent the numbers and count. If asked how many chickens a man was selling, if he had three chickens in each of two baskets, the Kpelle would either count 3 two times on their fingers or make two groups of three pebbles and count them. Gay and Cole's work in Liberia brought home the message that multiplication tables are procedures, and that different procedures can be used to solve the same problem. One procedure is to create a representation of the numbers and count. Of course this is a tedious and error-prone procedure, particularly when the numbers are large. It is perhaps not appropriate for our society, where large numbers are quite common, but it is nevertheless correct.

Many years later, working with children in Brazilian street markets, we (Nunes, Schliemann and Carraher, 1993) had the opportunity to extend Gay and Cole's observations and produce new findings that helped us understand better the difference between a problem-solving procedure and the logic of the problem. In order to simplify the description of this work, the example given here is about a multiplication story, but it should be remembered that this is a general argument, which is not confined to multiplication problems.

We asked children who had different levels of experience in street markets to solve a series of arithmetic problems. In one example, we asked a youngster in third grade, JG,¹ who would have been taught the written computation procedure for multiplication, how much we would have to pay if he were selling a toy car for 50 *cruzados* (the Brazilian currency at the time) and we bought 15 toy cars. He did not write down the sum 15×50 , as he had learned to do in school. Instead, he counted in fifties as he pointed to his fingers in one hand: 'fifty, one hundred, one fifty, two hundred, two fifty'. He then paused, repeated 'two fifty', then said: 'five hundred.' Next he returned to the pointing to fingers and counting procedure: 'five fifty, six hundred, six fifty, seven hundred, seven fifty', and gave 'seven hundred fifty' as the answer. JG's procedure is an extension of the counting procedure observed among the Kpelle, but it is more efficient because it combines addition of equal groups with counting in

¹ In the Brazilian school system there is often a large variation in age levels within the same grade because children may start school later than the expected age of 7 or be retained for failure, most often in reading. Although the expected age of a child in the third year of school is 9, JG was 10 years old.

larger units. He counts in units of fifty, five times, then doubles that – obtaining the answer to 10×50 – then adds to this another lot of five fifties, and reaches the right answer. His procedure, which we termed *repeated groupings*, is different from recalling the answer from a multiplication table and also from calculating it in writing.

Counting, calculating with repeated groupings, and carrying out a multiplication sum are different procedures for solving problems, but none of these procedures is the essence of problem-solving. What makes the different procedures work is that they are all under the control of a logical understanding of the situation. Counting works for the Kpelle in the problem described earlier only if they construct two groups of three pebbles. If they were to construct one group of three pebbles when they hear that 'the man had three chickens in his baskets' and a second group of two pebbles when they hear that 'he had two baskets', and then count the pebbles, they would come up with the wrong answer. The logic is used in the choice of how to represent the numbers in the problem: making two groups of three pebbles is not a simple matching of pebbles to number words but represents a choice stemming from the understanding of how to represent the problem.

The same is true of the repeated grouping procedure: it works because it is under the control of a logical interpretation of the situation. JG established a system of correspondences. For each toy car, represented by a finger, he counted one fifty. When he reached five fifties, he doubled this and knew that two times five fifties is the same as ten fifties. Finally, he set correspondences between fingers and fifties until he reached 15 fifties. This analysis, and other examples presented later in the chapter about multiplicative reasoning, show that what is essential to the logic of multiplicative situations is the systematic correspondence illustrated in JG's procedure. Any procedure that preserves this correspondence will lead to the correct answer.

A theory that says that, in order to solve a story or a practical problem, it is necessary first to identify the correct operation and then to implement the calculation, leads to the prediction that children must make more mistakes when solving story and practical problems than when solving sums, because the former two (solving story problems and practical problems) would require solving the sum plus another additional step, which could also lead to error. The results of our research with Brazilian children (Nunes, Schliemann and Carraher, 1993) show without doubt that solving specific arithmetic operations is not the only path to solving school-type story problems or practical problems that come up in everyday life. We compared the children's ability to solve sums with their ability to solve school-type story problems and their ability to solve practical

problems. The questions were comparable across conditions. For example, one sum used was 35×10 . The equivalent story problem was: 'There are 35 children in each Year 1 class in a school. There are 10 Year 1 classes. How many children are in Year 1 in this school?' The equivalent practical problem was: The youngster interviewed was selling coconuts for 35 *cruzados* each; we said we wanted 10 coconuts and asked how much we would have to pay. The rate of correct response in these different conditions of testing was: 37% correct responses to arithmetic operations; 74% correct responses to school-type story problems; and 98% correct responses in the practical situation. These rates of correct responses are not consistent with the theory that children need to interpret the problem, and identify and implement the correct operation to succeed in problem-solving.

Thus, it is concluded that the idea that children have to represent the meaning of the situation and find *the right operation* is misguided. They have to understand the situation and find a procedure that is in line with the logic of the situation. This procedure in multiplicative situations may involve counting in ones, counting in larger units and repeated additions or multiplications.

Unfortunately, the idea that children must interpret a *verbal* description of situations and then identify *the correct operation* in order to solve arithmetic problems has resulted in misguided teaching approaches. One example is the method that suggests that children should be taught to identify 'key words' in story problems and, through these, identify the correct operation. Teachers who work with key words will produce lists of words that are connected to a particular operation; for example, the word 'altogether' is identified with addition and the words 'less' and 'fewer' are identified with subtraction. However, it is now quite well established that this method of teaching interferes with, rather than promotes, good reasoning in story problems. Think of the problem: 'Pete has three apples. Ann also has some apples. Pete and Ann have nine apples altogether. How many apples does Ann have?' If a child were to add upon seeing the word 'altogether', the answer would be wrong. Verschaffel and De Corte (1997) have shown that this problem does lead to a large number of wrong answers based on addition. If the problem is presented to the children with a different language, where the key word method works, the number of correct answers goes up. The alternative language that they describe is: 'There are nine apples. Three of them belong to Pete. The rest belong to Ann. How many apples does Ann have?' The fact that the children succeed when the problem is presented with a different language is not evidence in favour of the key word method: it actually shows that, if children are encouraged to base their representation of the

problem on isolated words, they cannot become good problem-solvers. Teaching children to connect particular words to particular sums cannot work, because there are many different ways to talk about the same situation.

To sum up, this section started out by considering a common belief about how people solve arithmetic problems. According to this commonly held view, when we encounter an arithmetic problem in everyday life or as a story problem in school, we first need to understand the problem and identify the correct arithmetic operation. Then we need to calculate the sum. This view of how we solve problems is partially right: it is true that we need to construct a logically sound representation of the problem. But it is also partially wrong: once a sound logical representation has been constructed, there are many different procedures to reach the correct answer – there is not just one correct arithmetic operation.

Why should this perhaps subtle point matter? It matters a lot, because it is an important indicator of what we want to teach children so that they become good problem-solvers in mathematics. If it were correct that we need to teach them to identify *the right procedure*, our teaching would focus on procedures. However, if it is true that we need to teach children to construct sound logical representations of situations, our aim has to be to work with children's logic. These are two radically different pathways, even if in the second case we might also teach children about procedures.

In the next section a theory of how children develop their mathematical reasoning is presented. This theory guides the design of the activities for teaching deaf pupils that are discussed in the later chapters.

A theory of mathematical concepts for teaching mathematics

The connection between learning mathematics and logic will not come as a surprise to any readers of this book. The point about the central importance of logic for mathematical competence was made more than half a century ago by Jean Piaget (see, for example, Piaget, 1952), but it is probably made also by many unnamed teachers numerous times to their students in the classroom. It is easy to recognize that mathematical reasoning is a form of logical reasoning. This section suggests that this is true but that it is not the whole story. The framework used here is inspired by the work of the French psychologist, Gérard Vergnaud, who was one of Piaget's students. Vergnaud built on Piaget's conception of the use of logical reasoning in mathematics and brought other insights to the analysis of mathematical concepts. Vergnaud (1997) suggests that mathematical concepts – and thus teaching and learning mathematics – involve three dimensions to be mastered. Children need to understand the logic of the concept, the signs used in mathematics to talk and think about the con-

cept (which can be linguistic but can also be numeric, algebraic, graphic, schematic drawings etc.), and the situations in which the concept is used.

To expand on the first idea, let us consider why *logic* is important even for the simplest mathematical task, counting. In order to count correctly, children have to establish a correspondence between each object counted and the number words. This means counting all objects once and only once. They also have to understand that the last word they say when they count indicates the number of objects. The work of psychologists like Gelman and Gallistel (1978) suggests that at the age of 4 many children already understand these counting principles. They seem to be worse at doing the counting – that is, implementing the procedure themselves – than at checking whether someone else has done it correctly. In order to count correctly, they have to remember the counting words and simultaneously make sure that they are pointing at every object once and only once; under these circumstances, they may actually skip some or count some objects twice. However, when they watch someone else, they can check that the procedure is correct without the burden of execution (much like backseat drivers do!). Gelman and Meck (1983, 1986) and later Briars and Siegler (1984) asked children in the age range 3 to 5 years to say whether a puppet, manipulated by the experimenter, had counted the objects in a row ‘OK or not’. The puppet sometimes obeyed the counting principles and sometimes made mistakes. In both studies it was found that the children could judge correctly whether a puppet had counted OK or not, and that the children did better in this task than when they themselves did the counting. If the puppet skipped an object or counted the same object twice, they knew a mistake had been made, though they might have made the same type of mistake themselves, inadvertently, when counting. They also knew that the puppet had made a mistake if the puppet did not indicate the last number counted as the number of objects in the row. These interesting experiments with such young children show that even the simplest activity – counting to know how many objects – requires logic, and that even in such simple problems there is a difference between logic and carrying out a procedure.

But logic is not all, even for counting. When we count, we put a *sign* for the number in correspondence with an object. The sign can be a word – one, two, three etc. – or a gesture, in the case of a sign language. In either case, we need to create a representation for the object within a counting system. It is possible to reason about quantities without using signs to represent numbers. Much of Piaget’s work focused on this logic of quantities without the use of number signs. For example, in one of his tasks, Piaget (1952) asked children to place a marble into a jar at the same time as he himself placed a marble into another jar. The jars had different shapes: one was thinner than the other. After having placed many marbles

into the jars, the level reached by the marbles in the two jars differed: it went up higher in the thinner jar. Piaget then verified that the children remembered that they had placed a marble into their jar for every marble that he placed in his. Then he asked the children whether the number of marbles in the two jars was the same. Many children could answer this question correctly on the basis of logic, because they knew that there was a one-to-one correspondence between the jars. But they could not know how many marbles were in each of the jars because they had not counted. Thus, Piaget argued, there is no need to count in order to understand number.

Piaget is, of course, right: not all mathematical problems depend on our knowing the exact numbers in order to find the answer. But he overlooked another side of mathematical reasoning, one which involves arithmetic: in order to solve arithmetic problems, we do need to know the numbers. When we solve an arithmetic problem – say, a part-whole problem like the one about Pete's and Ann's apples presented in the previous section – we must understand its logic. In this case, the number of the whole – that is, how many apples Pete and Ann have altogether – is exactly the sum of the parts. But this is not sufficient to answer how many apples Ann has. It is also necessary to use signs for numbers – words or signed numbers – to be able to find the numerical answer. Thus, whenever we need to answer numerical problems, we need to know a number system, understand the conventions for number representation it involves, and how to manipulate it. Because mathematical signs are conventions, they have to be learned. So learning mathematical signs is an essential element in the development of children's mathematical competence. In the later chapters of this book the significance of different mathematical conventions will be discussed.

In the same way that it is possible to use different procedures to solve the same problem, there are also different mathematical conventions for representing mathematical information. Counting can be carried out in words in different languages, which are different conventions, and it can also be carried out in sign. There is now much evidence, which will be reviewed in the next chapter, that some counting systems are easier to learn than others. Think of the multiplication story problem presented earlier on; it was presented in words. The same information can be represented in visual form, with a small tag attached to each of eight chocolate bars. The information could also be presented in a table, where the number of chocolate bars would be placed in correspondence with the price of the chocolate. The research that will be reviewed later will show that the way in which information is presented to children does have an influence on how well they can solve problems. The critical point is that, in order for children to solve arithmetic problems and succeed in

mathematics, they must be able to represent numerical information and to manipulate these numerical representations. In order to succeed in school and later in the community and in their jobs, they must learn to use the systems of signs that are used to represent numerical information in school and in job settings. So mathematical concepts and reasoning involve not only logic; they also involve systems of signs used to solve problems and used in communication about numerical information.

The third dimension to be considered in the analysis of mathematical reasoning is the understanding of situations in which the reasoning is applicable. Piaget (1952) often used the example of transitive reasoning to explain the importance of situations for logic. If you know that stick A is longer than stick B, which in turn is longer than stick C, you can conclude that A is longer than C. This is an example of transitive reasoning. But not all relations are transitive, like the relation 'longer than'. The relation 'father of' is not transitive: if Paul is the father of John and John is the father of Mary, you must not conclude that Paul is the father of Mary. This is not a transitive relation.

This example seems so obvious that it may be necessary to consider others. Here, only one further example will be given, because every chapter that follows will consider the situations that relate to the logical ideas in the chapter. The relation 'equal to' is transitive but the relation 'different from' is not. If Jasmine has the same number of balloons as Julia and Julia has the same number of balloons as Deborah, we conclude with certainty that Jasmine and Deborah have the same number of balloons. But if all we know is that Jasmine does not have the same number of balloons as Julia and that Julia does not have the same number of balloons as Deborah, we do not know how Jasmine's and Deborah's balloon collections compare. They might have different numbers, but they might have the same number of balloons.

Piaget used these examples to argue that, even if logical principles were innate and programmed in the brain, as some have argued (e.g. Wynn, 1992; Geary, 1994), there is still an important role for experience in the development of mathematical competence. He argued that only experience with situations allowed children to understand the logic of the situations. This must be borne in mind later, when we discuss the situations in which children learn. Some researchers (e.g. Furth, 1971) have argued that deaf children suffer from a lack of experience in different situations. For example, it has been found that deaf children have a delay in the development of money concepts in comparison with same-aged hearing children (Austin, 1975). This delay could perhaps be due to the fact that hearing children can go out to shops and buy things on their own whereas the deaf children are less likely to do so because of the difficulties they might face when trying to communicate with the shop assistants.

What, then, if experience with money were shown to be important for children? This would place deaf children at a disadvantage, and other learning opportunities might have to be created to replace those missed in everyday life.

In summary, mathematical concepts involve *logic*, *systems of signs to represent the concept*, and *situations* where the concept is used, which help us understand the concept. All children might face difficulties with the logic, the systems of signs, and the understanding of the situations in which particular logical principles can be applied. The question examined in the subsequent chapters is whether the difficulties posed by these three dimensions of the concepts might be greater for deaf children than for hearing children. In order to answer this question, and identify the pieces of the puzzle, each of the dimensions will be considered.

Summary and conclusions

A review of results obtained by deaf pupils in mathematics achievement tests, considering data from a very large number of pupils over five decades and in different countries, suggests that deaf children do show weaker performance in mathematics than hearing children. This weaker performance is observed both in arithmetic and in problem-solving. However, this weak performance is not homogeneous: some deaf pupils (estimations vary between about 15% and 35%) perform as well as their hearing peers in mathematics.

The studies reviewed indicate that the reasons for this difficulty are not yet clear. It does not seem to be likely that deafness directly causes difficulties in mathematics learning. This direct causal link is not compatible with several observations. First, some profoundly deaf pupils perform well enough in mathematics tests. Second, there is no correlation or only a very weak correlation between level of hearing loss and mathematics achievement. Third, there is no evidence for a connection between causes of deafness and difficulties in mathematics. Fourth, deaf pupils' average performance in non-verbal intelligence tests is not below the norms for hearing pupils, and non-verbal intelligence tests are highly correlated with mathematics learning.

These results suggest that deaf children's difficulties are not inevitable but that they are at risk for developing mathematics difficulties. In order to understand the nature of this risk, it is necessary to consider how children solve problems and learn mathematics: what is difficult for all children and what could be especially difficult for deaf children.

A common view of how children solve mathematics problems is that they need first to interpret a problem verbally, then choose the correct

operation to solve the problem, then implement this operation – either through remembering verbally memorized number facts or through written computation procedures. This theory of arithmetic problem-solving is inconsistent with research results. It is clear that this theory does not make a distinction between the mathematical logic of problems and the procedures used to solve them. Research has shown that the same problem might be solved through the use of different procedures that all represent the logic of the situation.

An alternative description of mathematical competence was provided by the French psychologist, Gérard Vergnaud, who suggested that there are three aspects of mathematical competence that must be considered, both in research and in teaching children. The three aspects are the logic of the problem, the systems of signs used to think and talk about the problem, and the situations in which the logical principles can be validly used. All three aspects are mastered through experience and learning. So each of the chapters that follow in this book will be concerned with how children learn to use mathematical signs in different situations in order to solve problems.

A final word is required about the description of studies where the participants were deaf, before moving on to the next chapter. Many studies in the literature do not provide detailed information on the levels of hearing loss of the participants or the language used in the education, spoken or signed. So this information is often not presented. Sometimes it may also be omitted to avoid lengthy descriptions of samples. Descriptions of our own studies include this information but the information is not repeated when the same participants' results on different tasks are presented in different chapters. Readers wishing to obtain more information on participants of studies by other researchers are advised to consult the original work (full references are provided at the end of the book).