1
Newtonian Celestial Mechanics

1.1
Prolegomena – Classical Mechanics in a Nutshell

1.1.1
Kepler's Laws

By trying numerous fits on a large volume of data collected earlier by Tycho Brahe and his assistants, Kepler realized in early 1605 that the orbit of Mars is not at all a circle, as he had expected, but an ellipse with the Sun occupying one of its foci. This accomplishment of "Kepler the astronomer" was an affliction to "Kepler the theologist", as it jeopardized his cherished theory of "celestial polyhedra" inscribed and circumscribed by spherical orbs, a theory according to which the planets were supposed to describe circles. For theological reasons, Kepler never relinquished the polyhedral-spherist cosmogony. Years later, he re-worked that model in an attempt to reconcile it with elliptic trajectories.

Although the emergence of ellipses challenged Kepler's belief in the impeccable harmony of the celestial spheres, he put the scientific truth first, and included the new result in his book *Astronomia Nova*. Begun shortly after 1600, the treatise saw press only in 1609 because of four-year-long legal disputes over the use of the late Tycho Brahe's observations, the property of Tycho's heirs. The most cited paragraphs of that comprehensive treatise are Kepler's first and second laws of planetary movement. In the modern formulation, the laws read:

- **The planets move in ellipses with the Sun at one focus.**
- **A vector directed from the Sun to a planet sweeps out equal areas in equal times.**

These celebrated conjectures should not overshadow another revolutionary statement pioneered in *Astronomia Nova* – the hypothesis that the Sun is not stationary in space but describes a trajectory across the stars. Pioneering this idea, Kepler-astronomer again came into a conflict with Kepler-theologian. The heliocentric views of Kepler rested on a religious basis. Kepler was convinced that the universe was an image of God, with the Sun corresponding to the Father, the "stellar sphere" to the Son, and the enclosed space to the Holy Spirit. Kepler's hypothesis that the Sun

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Sergei Kopeikin, Michael Efroimsky, George Kaplan.

could travel relative to the stars indicates how his scientific insight was overpowering his theological constructions.
Another famous book by Kepler, *Harmonices Mundi*, saw the light a decade later, in 1619. In the final volume of that treatise, Kepler publicized his finding that the ratio of the cubed semi-major axis to the squared orbital period is the same for all planets. In modern terms, the third law of Kepler is usually formulated as:

- **The cube of a planet's semi-major axis is proportional to the square of its orbital period: $a^3 \sim T^2$.**

This way, for a planet with period T_1 and semi-major axis a_1, and a planet with period T_2 and semi-major axis a_2, the following relation takes place: $(a_1/a_2)^3 = (T_1/T_2)^2$.

1.1.2
Fundamental Laws of Motion – from Descartes, Newton, and Leibniz to Poincaré and Einstein

The next milestone contribution to the science of mechanics was offered a quarter-century later by René Descartes.

Basing his reasoning on the scholastic argument that nothing moves by virtue of its own nature towards its opposite or towards its own destruction, Descartes (1644) in his *Principles of Philosophy* came up with three laws of bodily motion. The first of those stated "that each thing, as far as is in its power, always remains in the same state; and that consequently, when it is once moved, it always continues to move." The second law held that "all movement is, of itself, along straight lines." The third law was an attempt to describe colliding bodies and to introduce a conserving quantity.

The first two laws of Descartes, together, constitute what is currently termed *the law of inertia* or the first law of Newton. Indeed, the wording of the law of inertia, suggested by Newton (1687) in his *Principia* was an exact equivalent to the first and second laws by Descartes.

First Law: *Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.*

There are two parts to this statement – one which predicts the behavior of stationary objects and the other which predicts the behavior of moving objects. Mathematical formulation of the first law of Newton demands introduction of new concepts, the absolute time t and the absolute space endowed with a special class of coordinates, $\boldsymbol{x} = (x^i) = (x^1, x^2, x^3)$, introduced in space – the so-called, inertial reference frames. Motion of the body in space is described by vector function $\boldsymbol{x}(t)$. The first law of Newton simply states that velocity of the body $\boldsymbol{v} = d\boldsymbol{x}/dt$ is nil or remains constant in the inertial reference frame if the body is not subject to the action of a net force. In both cases, acceleration, $\boldsymbol{a} = d\boldsymbol{v}/dt$, of a freely moving or static body vanishes in the inertial frame,

$$\boldsymbol{a} = 0 \,. \tag{1.1}$$

Equation 1.1 makes it evident that there is a multitude of the inertial frames moving with respect to each other with constant velocities.

This *law of inertia,* however, marked the point beyond which Newton's thought sharply diverged from Descartes' heritage. In formulating the rules of dynamics, Newton succeeded where Descartes had failed. The third law of Descartes, while marking one of the first attempts to locate an invariant or unchanging feature of bodily interactions, was just short of what is now called the momentum conservation. Newton, on his part, proposed two other laws, the law of impulse and the law of reciprocal actions, which must be put together in order to ensure the conservation of body's momentum

$$\boldsymbol{p} = m\boldsymbol{v} \,, \tag{1.2}$$

where m is mass of the body, and $\boldsymbol{v}$ is its velocity. The law of impulse and the law of reciprocal actions are known as the second and third laws of Newton respectively. They are formulated as follows.

Second Law: *The time rate of change of body's linear momentum* $\boldsymbol{p}$ *is equal to the net force* $\mathcal{F}$ *exerted on the body,*

$$\frac{d\boldsymbol{p}}{dt} = \mathcal{F} \,. \tag{1.3}$$

The second law is valid in any frame of reference that is written in an invariant form that is valid in arbitrary frames of reference both inertial and noninertial. Therefore, the force $\mathcal{F}$ splits algebraically in two parts – the force of inertia, $\boldsymbol{F}_{\text{in}}$, and the external force, $\boldsymbol{F}$. Therefore,

$$\mathcal{F} = \boldsymbol{F} + \boldsymbol{F}_{\text{in}} \,. \tag{1.4}$$

The force of inertia $\boldsymbol{F}_{\text{in}}$ exists only in noninertial (accelerated and/or rotating) frames and has a pure kinematic origin, while the external force $\boldsymbol{F}$ describes real physical interaction between the body under consideration with other bodies. The external force $\boldsymbol{F}$ determines the dynamical part of the body's motion that is not related to the choice of the reference frame. If mass of the body is conserved, $dm/dt = 0$, the second law of Newton can be written in a more familiar form,

$$m\boldsymbol{a} = \mathcal{F} \,, \tag{1.5}$$

which establishes a more simplistic relationship than that (1.3) between the acceleration of the body, its constant mass, and the force applied.

By setting $\mathcal{F} = 0$ in Newton's law of impulse (1.5) written for a body of a constant mass, one would arrive to the conclusion, $\boldsymbol{a} = 0$, and would get an impression that the law of inertia (1.1) is a special case of the law of impulse: a vanishing net force yields a zero acceleration. It looks like the law of inertia is redundant, and can be derived instead of being postulated. Newton surely observed the possibility of such a conclusion, but nevertheless chose to add the law of inertia as a separate statement. Moreover, he placed this law first. The only reason why he could have done so was his intention to single out a special class of forces – the kinematic forces of inertia – from the rest of genuine dynamical interactions, and to introduce a special class of inertial reference frames in which the forces of inertia $\boldsymbol{F}_{\text{in}}$ vanish so that the second law of Newton is reduced to pure dynamical form

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F} \,. \tag{1.6}$$

This logic makes it clear that the first Newton's law is not a tautology following the second Newton's law, but a crucial element of the entire theory introducing a special class of reference frames excluding the inertial forces. It may look simple now, but it took Newton's successors centuries to arrive at the modern formulation of the law of inertia:

> "There exist reference frames called inertial, such that a particle at rest or with constant velocity in one inertial frame will remain at rest or have constant velocity in all inertial frames, provided the net *external* force $\boldsymbol{F}$ acting on the particle is nil."

Crucial in this formulation is that it deliberately omits any mentioning of the absolute motion. This is because all inertial frames are effectively equivalent in the sense that the second law of Newton (1.6) is invariant (remains the same) irrespectively of the choice of the inertial frame.

Descartes–Newton's idea of inertia differs from the modern understanding of this phenomenon in that they both regarded uniform motion and rest as different bodily states. Of a special interest is the position of Descartes who was partially a *relationist* and partially an *absolutist*.[1] On the one hand, he argued that space and matter are inseparable aspects of one phenomenon, and that motion is always the motion of bodies relative to one another. On the other hand, despite holding motion to be relational, Descartes also held there to be a privileged sense of motion ("true motion") over and above the merely relative motions. In distinction from Descartes, Newton was a pure *absolutist* whose system of views consistently stemmed from his belief that space (and, likewise, time) has existence of its own, independently of the bodies residing in it. The concept of *absolute space and time* proposed in *Principia* laid a foundation for a version of the æther theory developed by Newton in his *Opticks*, a book in which he proposed a corpuscular theory of light. As the theory

1) Absolutism is a philosophical paradigm, according to which space and time are fundamental entities existing independently of matter. Relationism denies this paradigm so that space and time exist only as a supplementary mathematical tool to express relationships between the material bodies (and material fields). Relationism denies the existence of privileged coordinates, but may admit that some configurations of the bodies (fields) may have a privileged value for the observer.

had trouble explaining refraction, Newton claimed that an "æthereal medium" was responsible for this effect. He then went further to suggest it might be responsible for other physical effects such as heat transfer.

The law of conservation of the linear momentum of a closed system of mutually-interacting bodies required from Newton to postulate his

Third Law: Whenever a particle exerts a force, $\boldsymbol{F}_{12}$, on another particle, the latter one *simultaneously* exerts a force, $\boldsymbol{F}_{21}$ on the former, with the same magnitude and in the opposite direction,

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21} \ . \tag{1.7}$$

Be mindful that although the magnitude of the forces are equal, the accelerations of the bodies are not: the less massive body will have a greater acceleration due to Newton's second law (1.5). Let us apply the third law to a system of two interacting particles having instantaneous linear momenta, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, respectively. The second Newton's law for two particles written down in an inertial reference frame is

$$\frac{d\boldsymbol{p}_1}{dt} = \boldsymbol{F}_{12} \ , \tag{1.8a}$$

$$\frac{d\boldsymbol{p}_2}{dt} = \boldsymbol{F}_{21} \ . \tag{1.8b}$$

Adding (1.8a) and (1.8b) together, and applying the third law of Newton (1.7) yields

$$\frac{d}{dt}(\boldsymbol{p}_1 + \boldsymbol{p}_2) = 0 \ , \tag{1.9}$$

which is equivalent to the statement that the total linear momentum of the system, $\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2$, is constant in any inertial reference frame. One has to pay attention that this constancy of the overall linear momentum is preserved only in the inertial frames since in a noninertial frame, the overall momentum may be not conserved because the inertial forces, $\boldsymbol{F}_{\text{in}}$, may not obey the third law of Newton. The law (1.9) can be extended to a system of N interacting particles.

An important property of the force postulated in the third law of Newton is that physical interaction between bodies is instantaneous. It was in perfect agreement with the experimental situation at the time of Newton. Much later, after the development of electrodynamics, special and general theories of relativity, and other field theories, it became clear that there must be no instantaneous forces in nature. This does not undermine the validity of the Newtonian mechanics which remains fully self-consistent and works very well in the limit of low velocities and small accelerations. Post-Newtonian celestial mechanics in the solar system can be treated in most cases on the premise of the instantaneous gravitational interaction. Only dedicated experiments require that one include the finite speed of propagation of gravity to get theoretical predictions consistent with observations.[2)]

2) See Section 7.9 for particular details and explanations.

The idea of absolute motion was also challenged in seventeenth century by Leibniz, Huygens, and other *relationists*. Leibniz found the Newtonian notion of absolute space unacceptable because two universes whose bodies occupied different absolute positions but identical relative positions would be indistinguishable from one another. Despite the objections, the indisputable authority of Newton kept the theory of æther afloat for more than two hundred years. As Ferraro (2007) put it: "The controversy between relationists and absolutists quieted down in the following centuries due to the success of the Newtonian science. Actually it remained in a latent state because it would resurge at the end of the nineteenth century." The issue indeed returned to the center of attention in 1887 after the Michelson–Morley experiment challenged the possibility of æther's existence. The discussion continued and even spilled out into the twentieth century, with Poincaré and Einstein being on the opposite sides of the fence.

Although in his Saint Louis address of 1904, Poincaré came up with an early formulation of the relativity principle, he never granted this principle the fundamental status Einstein gave it in 1905 (Kobzarev, 1975). Defending the idea of æther, Poincaré believed that some dynamical effects conspire to prevent us from observing it by mechanical or electromagnetic means. On 11 April 1912, three months before his death, Poincaré gave to the French Society of Physics a talk entitled "The Relations between Matter and Æther." This duality of Poincaré's concept of motion brings up strong parallels with Descartes.

Einstein, on his part, strictly followed the line of Leibniz, rejecting absolute motion. Thus, he had no need to introduce æther in special relativity because in special relativity this entity, in its classical meaning, was redundant. Years later, after general relativity was developed, Einstein (1920) admitted in his Leiden's address that the general theory of relativity does not yet compel us to abandon æther. Einstein said that "according to the general theory of relativity, space is endowed with physical qualities (the metric potentials); in this sense, therefore, there exists an æther." At the same time, Einstein acknowledged that this general-relativistic meaning of the word "differs widely from that of the æther of the mechanical undulatory theory of light." At any rate, Einstein's interpretation deprives æther of its ability to define a reference frame.

Einstein's viewpoint was later corroborated by Dirac on the grounds of hole theory of vacuum. According to Dirac (1951), vacuum is a substance of complex structure and therefore may be regarded as a physical medium, a kind of æther. However this medium is Lorentz-invariant and thus defines no preferred inertial frame of reference. This viewpoint has now become conventional in quantum physics (Lee, 1981). The reason why the term *æther* is seldom applied to quantum vacuum in the modern literature is the necessity to avoid confusion with the old concept of æther, one associated with absolute motion.

1.1.3
Newton's Law of Gravity

Having formulated the three fundamental laws of motion, Newton went on in his *Principia* to explore a particular force, gravity. In this endeavor, Newton was getting inspiration from the works by Kepler and from correspondence with Hooke.[3)]

In 1666, Robert Hooke explained to the Royal Society his concept on what made the planets describe closed orbits about the Sun. According to Hooke, a force was needed, not to push a planet along from behind, but to pull it in towards a fixed gravitating center, so as to make the planet describe a closed curve instead of moving off along a straight tangent line into outer space. Circa 1680, in his correspondence with Newton, Hooke hypothesized that above ground level the gravity force changes inversely as the square of the distance from the Earth's center, and that below ground the force falls off as the center is approached. Hooke inquired as to what curve should be followed by a body subject to a central force obeying the inverse-square law.[4)] Soon, Newton proved that an orbit in the form of a conic section, with the center of attraction located in one of the foci, necessarily implies an inverse-square attraction[5)] – a result perfectly fitting Kepler's first law. We shall never know what made Newton procrastinate for almost five years with making his calculation public. For the first time, the discovery saw light in the *Principia* in 1687. In Newton's own words,

> "I deduced that the forces, which keep the planets in their orbs must [be] reciprocally as the squares of their distances from the centers about which they revolve: and thereby compared the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the Earth; and found them answer pretty nearly."

In the modern notations, the law will read (see Figure 1.1)

$$\boldsymbol{F}_{12} = -G\frac{m_1 m_2}{r_{12}^3}\boldsymbol{r}_{12}\ , \tag{1.10}$$

where $\boldsymbol{F}_{12}$ is the gravitational force wherewith body 2 acts on body 1, $G = (6.67428 \pm 0.00067) \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ denotes the Newtonian gravity constant, m_1 and m_2 are the masses of the two interacting bodies located at positions with

3) For historical analysis of the life and work of Newton and the contemporary scholars, see Cohen and Smith (2002).

4) Newton, in his letter to Halley on June 20, 1686, seeking to rebut Hooke's claim to have provided him originally with the idea of inverse-square gravity law, emphasized that the idea had been published before by Boulliau. In fact, Boulliau did not believe in a universal attraction force. At the same time, in his book he indeed argued that had such a force existed, it would likely obey the inverse-square law (Boulliau, 1645).

5) Be aware that Newton proved that orbits being conics entail the inverse-square law. He did *not* prove that the inverse-square radial force results in orbits being conics (Weinstock, 1982).

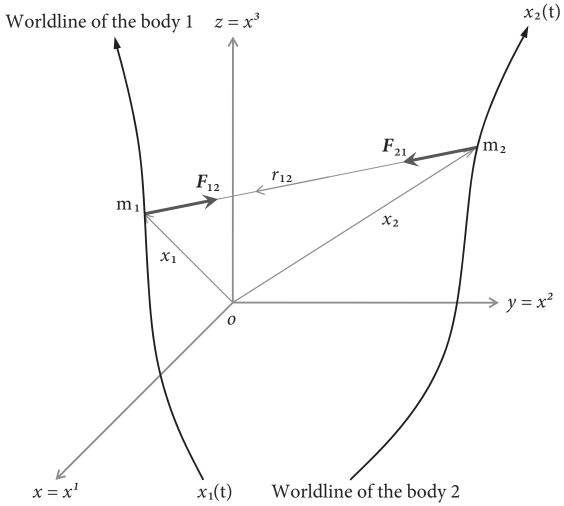

Figure 1.1 Newton's law of the universal gravitational attraction. The coordinates $x^i = (x, y, z)$ represent an arbitrary inertial reference frame with the origin O. This frame is assumed to be nonmoving and its axes are nonrotating. Time t is absolute and parameterizes the worldlines of the bodies.

spatial coordinates $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ respectively, the vector

$$\boldsymbol{r}_{12} = \boldsymbol{x}_1 - \boldsymbol{x}_2 \tag{1.11}$$

is aimed from the second body to the first one, the quantity r_{12} being this vector's Euclidean magnitude $r_{12} = |\boldsymbol{r}_{12}|$. According to the third Newton law of reciprocal action, body 1 acts on body 2 with a force

$$\boldsymbol{F}_{21} = -\boldsymbol{F}_{12} = -G\frac{m_1 m_2}{r_{21}^3}\boldsymbol{r}_{21} , \tag{1.12}$$

where $\boldsymbol{r}_{21} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = -\boldsymbol{r}_{12}$, and $r_{21} = |\boldsymbol{r}_{21}| = |\boldsymbol{r}_{12}| = r_{12}$.

Combining the gravity law with Newton's second law (the law of impulse), one gets

$$\frac{d}{dt}(m_1\dot{\boldsymbol{x}}_1) = -G\frac{m_1 m_2}{r_{12}^3}\boldsymbol{r}_{12} , \tag{1.13}$$

$$\frac{d}{dt}(m_2\dot{\boldsymbol{x}}_2) = -G\frac{m_2 m_1}{r_{21}^3}\boldsymbol{r}_{21} , \tag{1.14}$$

with overdot standing for an ordinary time derivative. If the masses of the bodies are constant,[6] (1.13)–(1.14) can be simplified to the form usually employed in celestial mechanics,

$$m_1\ddot{\boldsymbol{x}}_1 = -G\frac{m_1 m_2}{r_{12}^3}\boldsymbol{r}_{12} , \tag{1.15}$$

6) This is not always true. For example, mass of the Sun changes due to the emission of the solar wind and radiation. In many cases, however, the mass loss is slow and can be neglected.

$$m_2\ddot{\boldsymbol{x}}_2 = -G\frac{m_2 m_1}{r_{21}^3}\boldsymbol{r}_{21} \; . \tag{1.16}$$

Summing up (1.13) and (1.14), and integrating the result over the time, one arrives at the law of conservation of linear momentum of the gravitating two-body system,

$$m_1\dot{\boldsymbol{x}}_1 + m_2\dot{\boldsymbol{x}}_2 = \boldsymbol{P} \; , \tag{1.17}$$

where $\boldsymbol{P}$ is a constant vector of the linear momentum of the system. The *center of mass* of the two-body system is, by definition, a point given by the vector

$$\boldsymbol{X} = \frac{m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2}{m_1 + m_2} \; . \tag{1.18}$$

Integration of (1.17) with respect to time gives birth to a vector integral of motion of the center of mass,

$$M\boldsymbol{X} = \boldsymbol{D} + \boldsymbol{P}(t - t_0) \; . \tag{1.19}$$

Here,

$$M = m_1 + m_2 \tag{1.20}$$

is a constant total mass of the two-body system,

$$\boldsymbol{D} = M\boldsymbol{X}_0 \; , \tag{1.21}$$

and $\boldsymbol{X}_0$ is the constant position of the center of mass at the fiducial time, t_0, which is often called an *epoch* in dynamic astronomy. A constant vector

$$\boldsymbol{V} = \dot{\boldsymbol{X}} = \frac{\boldsymbol{P}}{M} \; , \tag{1.22}$$

is termed the *velocity of the center of mass*. Equation 1.19 tells us that the center of mass $\boldsymbol{X}$ of the two-body system moves uniformly along a straight line with the constant velocity $\boldsymbol{V}$.

Solving (1.11) and (1.18) elucidates that inertial coordinates of the bodies, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, can always be represented as a sum of two vectors, $\boldsymbol{X}$ and $\boldsymbol{r}_{12}$,

$$\boldsymbol{x}_1 = \boldsymbol{X} + \frac{m_2}{m_1 + m_2}\boldsymbol{r}_{12} \; , \qquad \boldsymbol{x}_2 = \boldsymbol{X} - \frac{m_1}{m_1 + m_2}\boldsymbol{r}_{12} \; , \tag{1.23}$$

where $\boldsymbol{r}_{12}$ is a vector of a relative position of body 2 with respect to body 1. Substituting these equations to the equations of motion (1.15) and (1.16) and accounting for the conservation of the integral of the center of mass of the two-body system, $\ddot{\boldsymbol{X}} = 0$, one obtains the equations of relative motion

$$\mu\ddot{\boldsymbol{r}}_{12} = -G\frac{M\mu}{r_{12}^3}\boldsymbol{r}_{12} \; , \tag{1.24}$$

where

$$\mu = \frac{m_1 m_2}{M} \,, \tag{1.25}$$

is called the *reduced mass*. Equations of relative motion for vector $\boldsymbol{r}_{21}$ is obtained by exchanging the body indices, $1 \leftrightarrow 2$, but it does not provide us with a new information. Equations of relative motion (1.24) are naturally termed the equations of motion of the *reduced two-body problem*. They could be also derived by subtracting (1.16) from (1.15). It may make an impression that the conservation of the integral of the center of mass is not important in derivation of the equations of the relative motion. However, this point of view is mistaken since if the integral of the center of mass were not exist, the equations of the relative motion would have extra terms associated with the force of inertia.

Inertial coordinates with the origin fixed at the center of mass of the gravitating system are named *barycentric*. In the barycentric frame of reference, the total momentum of the system is zero, $\boldsymbol{P} = 0$, while the position of the center of mass is constant and can be set to zero as well, $\boldsymbol{X}_0 = 0$. In this coordinate system, at any instant of time, one has $\boldsymbol{X} = \boldsymbol{V} = 0$, as follows from the conservation of momentum. Hence, in this frame, (1.23) are simplified as

$$\boldsymbol{x}_1 = \frac{m_2}{m_1 + m_2}\boldsymbol{r}_{12} \,, \quad \boldsymbol{x}_2 = -\frac{m_1}{m_1 + m_2}\boldsymbol{r}_{12} \,, \tag{1.26}$$

and the problem of motion is reduced to solving of only one differential equation of the relative motion (1.24) that can be re-written as

$$\ddot{\boldsymbol{r}}_{12} = -G\frac{M}{r_{12}^3}\boldsymbol{r}_{12} \,, \tag{1.27}$$

since the reduced mass μ is canceled out. In the barycentric reference frame, the problem of motion can be viewed as a problem of a single body with the reduced mass, μ, moving around a fixed center of gravity located at the barycenter of the two-body system and having a total mass, $M = m_1 + m_2$. In general, N-body problem, a similar procedure of introducing the relative coordinates can be employed to reduce the problem to an $(N-1)$-body setting.

1.2
The *N*-body Problem

Let us consider an isolated self-gravitating system consisting of a number of point-like massive bodies. In neglect of the gravitational pull from the Milky Way and the Hubble expansion of the universe, the solar system is a typical example but the consideration given in this chapter is applicable equally well to other isolated astronomical systems like a binary or multiple stellar system or a planetary system around other star. We shall enumerate the massive bodies with the capital letters $A, B, C, \ldots$ taking the values of $0, 1, 2, \ldots, N$, the index 0 being assigned to the primary body. Depending on a particular situation under consideration the primary can be either the Sun, or the Earth, or any other major planet.

1.2.1
Gravitational Potential

Let us begin at the discussion of the gravitational potential of a point-like mass m located at the origin of an inertial reference frame $\boldsymbol{x} = (x^i) = (x^1, x^2, x^3)$. Gravitational force of the mass m on a test particle of *unit* mass is given by the expression

$$f^i = -\frac{Gm}{r^3}x^i \,, \tag{1.28}$$

where $r = |\boldsymbol{x}| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ is the Euclidean distance from the mass to the field point $\boldsymbol{x} = (x^i)$. Elementary gravitational force, f^i, can be represented as a gradient of gravitational potential

$$\phi = \frac{Gm}{r} \,. \tag{1.29}$$

Therefore,

$$f^i = \frac{\partial \phi}{\partial x^i} = \left(\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \frac{\partial \phi}{\partial x^3}\right) \,. \tag{1.30}$$

If the mass m is displaced to the point with coordinates $\boldsymbol{x}' = (x'^i)$, the gradient expression (1.30) for gravitational force, f^i, remains the same but the value of the potential, ϕ, at the point x^i becomes

$$\phi = \frac{Gm}{|\boldsymbol{x} - \boldsymbol{x}'|} \,. \tag{1.31}$$

Let us now consider an extended massive body made up of a continuous distribution of matter having a compact support (enclosed in a finite volume) with a mass density $\rho(t, \boldsymbol{x})$. One assumes that the body's matter can move that explains the time dependence of the mass density which obeys the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i} = 0 \,, \tag{1.32}$$

where $v^i = v^i(t, \boldsymbol{x})$ is the velocity of an infinitesimally-small element of body's matter, and the repeated Roman indices mean the Einstein rule of summation from 1 to 3. Einstein's summation rule was invented to avoid the explicit (but in many cases unnecessary) appearance of the sign of summation, $\sum$, in tensor equations. It tacitly assumes that a pair of repeated (dummy) indices assume summation over corresponding values of the indices. In what follows, one uses the Einstein rule for summation of coordinate indices. For example, a scalar product of two vectors $\boldsymbol{a} = (a^i)$ and $\boldsymbol{b} = (b^i)$ will be written as $\boldsymbol{a}\cdot\boldsymbol{b} = a^i b^i \equiv \sum_{i=1}^{3} a^i b^i = a^1 b^1 + a^2 b^2 + a^3 b^3$.

Integration of (1.32) over the finite volume $\mathcal{V}$ of the body tells us that the overall mass of the body,

$$M = \int\limits_{\mathcal{V}} \rho(t, \boldsymbol{x}) d^3x \,, \tag{1.33}$$

is constant. Indeed, taking the time derivative from both sides of (1.33) yields

$$\frac{dM}{dt} = \frac{d}{dt}\int_{\mathcal{V}} \rho(t, \boldsymbol{x}) d^3x = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d^3x + \oint_{\partial \mathcal{V}} \rho v^i dS_i \,, \tag{1.34}$$

where the surface integral is taken over the body's surface, $\partial\mathcal{V}$, and accounts for the presumable time-dependence of the boundary of integration due to the motion of matter. Applying now the Gauss theorem to the surface integral recasts (1.34) to

$$\frac{dM}{dt} = \int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i}\right) d^3x = 0 \,, \tag{1.35}$$

due to the equation of continuity. It proves that the overall mass of the extended body remains constant, that is, the mass of the body is the integral of motion of matter.

Gravitational potential U of the extended body is found as an integral taken over the body's volume comprised of the contributions of the "elementary" potentials (1.31) of the "point-like" elements of the body with mass m replaced with $m \to \rho d^3x$. It yields

$$U(t, \boldsymbol{x}) = G\int_{\mathcal{V}} \frac{\rho(t, \boldsymbol{x}')d^3x'}{|\boldsymbol{x} - \boldsymbol{x}'|} \,. \tag{1.36}$$

The equation for the gravitational force, F^i, exerted by the extended body on a probe unit mass at point x^i is defined as a gradient of the gravitational potential (1.36), that is,

$$F^i = \frac{\partial U}{\partial x^i} = \left(\frac{\partial U}{\partial x^1}, \frac{\partial U}{\partial x^2}, \frac{\partial U}{\partial x^3}\right) \,. \tag{1.37}$$

By taking second-order partial derivatives from the potential U, one can prove that the gravitational potential, U, obeys the second-order partial differential equation

$$\delta^{ij}\frac{\partial^2 U}{\partial x^i \partial x^j} = -4\pi G\rho \,, \tag{1.38}$$

where $\delta^{ij} = \mathrm{diag}(1, 1, 1)$ is the unit matrix, and one has used the Einstein summation rule to avoid the appearance of the double sum of summation, $\sum_{i=1}^{3}\sum_{j=1}^{3}$, in the right side of this equation. The differential operator acting on gravitational potential U in the left side of this equation is called the Laplace operator

$$\triangle \equiv \delta^{ij}\frac{\partial^2}{\partial x^i \partial x^j} \,, \tag{1.39}$$

and (1.38) is known as the Poisson equation, conventionally written as

$$\triangle U = -4\pi G\rho \,. \tag{1.40}$$

Gravitational potential (1.36) is the solution of the (inhomogeneous) Poisson equation that is valid both inside and outside of the body's volume. However, if one is interested in the gravitational potential and force only outside of the body, a homogeneous Laplace equation

$$\triangle U = 0 \,, \tag{1.41}$$

is sufficient.

1.2.2
Gravitational Multipoles

In many practical tasks of celestial mechanics and geodesy, one does not need the integral form of the Newtonian gravitational potential, but its multipolar decomposition describing gravitational field in terms of multi-index objects called multipoles. It can be obtained by expanding gravitational potential (1.36) outside of the body into an infinite Taylor series by making use of decomposition of the reciprocal distance $|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$ around the point $x'^{\,i} = 0$ with respect to the so-called harmonic polynomials. One has[7)]

$$\begin{aligned} \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} x'^{\,\langle i_1} x'^{\,i_2} \dots x'^{\,i_l\rangle} \frac{\partial^l}{\partial x^{i_1} \partial x^{i_2} \dots \partial x^{i_l}} \left(\frac{1}{r}\right) \\ &= \frac{1}{r} + \frac{x^i x'^{\,i}}{r^3} + \frac{1}{2}\left[\frac{3(x^i x'^{\,i})(x^j x'^{\,j})}{r^5} - \frac{r'^2}{r^3}\right] + \dots \,, \end{aligned} \tag{1.42}$$

where $r = |\boldsymbol{x}|$, $r' = |\boldsymbol{x}'|$, the multi-index notation of spatial indices has been used, each index $i_1, i_2, \dots, i_l$ runs from 1 to 3, and the angular brackets around the indices denote an algebraic operation making an object having such indices, symmetric and traceless (STF) tensor with respect to spatial rotations (see Appendix A). This expansion is more commonly written in terms of the Legendre polynomials $P_l(\cos\theta)$. Formula (1.42) can be easily converted to these polynomials after re-writing the scalar product of two vectors $x^i x'^{\,i}$ in trigonometric form, $x^i x'^{\,i} = r r' \cos\theta$, and substituting it to expression (1.42). It yields

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{r} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \left(\frac{r'}{r}\right)^l P_l(\cos\theta) \,. \tag{1.43}$$

Though this form of expansion of the reciprocal distance looks more simple, it requires further elaboration of $P_l(\cos\theta)$ in terms of the associated Legendre functions $P_l^m(\cos\theta)$ which is effectively equivalent to the expansion in terms of the harmonic polynomials. The harmonic polynomials have many mathematical advantages in theoretical studies (Hartmann *et al.*, 1994), and will be preferred almost everywhere in this book to describe the multipolar decompositions of gravitational potentials in classical and relativistic gravity theories.

7) Notice the usage of Einstein's summation rule for indices $i_1, i_2, \dots, i_l$ numerating spatial coordinates.

After substituting the Taylor expansion (1.42) in the definition (1.36) of the Newtonian gravitational potential, one obtains its multipolar expansion

$$U(t,\boldsymbol{x}) = G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}I^{\langle i_1 i_2\dots i_l\rangle}\partial_{i_1 i_2\dots i_l}\left(\frac{1}{r}\right) , \tag{1.44}$$

where one has used a shorthand notation for partial derivatives $\partial_i = \partial/\partial x^i$, $\partial_{i_1 i_2\dots i_l} = \partial_{i_1}\partial_{i_2}\dots\partial_{i_l}$, and $I^{\langle i_1 i_2\dots i_l\rangle}$ are the mass multipole moments of gravitational field of the body which are integrals from the density, $\rho(t,\boldsymbol{x})$, taken over the body's volume

$$I^{\langle i_1 i_2\dots i_l\rangle} = \int_{\mathcal{V}}\rho(t,\boldsymbol{x})x^{\langle i_1}x^{i_2}\dots x^{i_l\rangle}d^3x , \tag{1.45}$$

with $x^{\langle i_1}x^{i_2}\dots x^{i_l\rangle}$ representing the harmonic polynomial of the lth degree. The angular brackets around the indices of the polynomial denote a special kind of symmetry which is imposed on the harmonic polynomial by the condition that it must be a solution of a homogeneous Laplace equation, that is,

$$\triangle\left(x^{\langle i_1}x^{i_2}\dots x^{i_l\rangle}\right) = 0 . \tag{1.46}$$

It is this condition which demands for the lth order harmonic polynomial, $x^{\langle i_1}x^{i_2}\dots x^{i_l\rangle}$, to be a fully-symmetric and trace-free (STF) tensor with respect to rotations in three-dimensional Euclidean space (Blanchet and Damour, 1989; Pirani, 1965; Thorne, 1980). The word "trace-free" means that contraction of any pair of indices nullifies the STF tensor,

$$x^{\langle i_1}x^{i_1}x^{i_3}\dots x^{i_l\rangle} = \delta_{i_1 i_2}x^{\langle i_1}x^{i_2}\dots x^{i_l\rangle} \equiv 0 , \tag{1.47}$$

which is a mathematical property of any polynomial solution of the homogeneous Laplace equation (1.46). We provide more details on the structure of the harmonic polynomials in Appendix A.

Let us define a symmetric moment of inertia of the lth order,

$$I^{i_1 i_2\dots i_l} = \int_{\mathcal{V}}\rho(t,\boldsymbol{x})x^{i_1}x^{i_2}\dots x^{i_l}d^3x . \tag{1.48}$$

Then, the STF multipole moment (1.45) is expressed in terms of the moments of inertia as follows (Pirani, 1965)

$$I^{\langle i_1 i_2\dots i_l\rangle} = \sum_{k=0}^{[l/2]}a_k^l\,\delta^{(i_1 i_2}\delta^{i_3 i_4\dots}\delta^{i_{2k-1}i_{2k}}I^{i_{2k+1}\dots i_l)j_1 j_1\dots j_k j_k} , \tag{1.49}$$

where the round brackets around a group of indices denote full symmetrization with respect to permutation of the indices, $[l/2]$ denotes the integer part of $l/2$, the repeated indices denote Einstein's summation, and the numerical coefficient

$$a_k^l = (-1)^k\frac{l!}{(l-2k)!(2k)!!}\frac{(2l-2k-1)!!}{(2l-1)!!} . \tag{1.50}$$

The STF multipole moments $I^{\langle i_1 i_2 \dots i_l \rangle}$ are well-known in celestial mechanics, and other areas of theoretical physics. For example, the zero-order ($l = 0$) multipole moment I is simply a constant mass M of the body having been introduced in (1.33). Dipole moment ($l = 1$)

$$I^i = \int\limits_V \rho(t, \boldsymbol{x}) x^i d^3x \ , \tag{1.51}$$

defines position of the center of mass of the body with respect to the origin of the coordinates. The quadrupole moment ($l = 2$)

$$I^{\langle ij \rangle} = \int\limits_V \rho(t, \boldsymbol{x}) \left(x^i x^j - \frac{r^2}{3} \delta^{ij} \right) d^3x \ , \tag{1.52}$$

and the multipole moments of higher-order provide an integral characteristics of various asymmetries in the distribution of matter inside the body with respect to its equatorial and meridional planes.[8] If the origin of coordinates is placed to the center of mass of the body, the dipole moment vanishes, $I^i = 0$, and the multipolar expansion (1.44) can be written as follows

$$U(t, \boldsymbol{x}) = \frac{GM}{r} + G \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} I^{\langle i_1 i_2 \dots i_l \rangle} \partial_{i_1 i_2 \dots i_l} \left(\frac{1}{r} \right) \ , \tag{1.53}$$

where mass M is constant, but the multipole moments $I^{\langle i_1 i_2 \dots i_l \rangle}$ ($l \geq 2$) can depend on time. In many cases, contribution of higher-order multipoles to the overall gravitational field is fairly small and can be neglected, thus, leaving only the first term in the right side of (1.53). Extended body with spherically-symmetric distribution of mass has no multipole moments at all. Thus, its gravitational potential

$$U = \frac{GM}{r} \ , \tag{1.54}$$

is the same as that of the point-like mass M placed at the coordinate origin.

1.2.3 Equations of Motion

Let us derive the Newtonian equations of motion of extended bodies comprising the N-body system under consideration. We introduce a global inertial reference frame with time t and spatial coordinates $\boldsymbol{x} = (x^i) = (x^1, x^2, x^3)$ and assume that each body A occupies a finite volume $\mathcal{V}_A$ of space. The interior distribution of matter is characterized by mass density $\rho = \rho(t, \boldsymbol{x})$ and by the symmetric tensor of stresses $\pi^{ij}(t, \boldsymbol{x}) = \pi^{ji}(t, \boldsymbol{x})$, which is reduced in case of a perfect fluid to an

8) Notice that contraction of two indices of the quadrupole moment gives $I^{\langle ii \rangle} \equiv 0$, that is, its trace is indeed zero.

isotropic pressure $p = p(t, \boldsymbol{x})$ such that the trace of this tensor, $\pi^{ij} = p\delta^{ij}$. Macroscopic equations of motion of matter are[9)]

$$\frac{\partial(\rho v^i)}{\partial t} + \frac{\partial(\rho v^i v^j)}{\partial x^j} = -\frac{\partial \pi^{ij}}{\partial x^j} + \rho\frac{\partial U}{\partial x^i} , \tag{1.55}$$

where $v^i = dx^i/dt$ is velocity of matter and, $U = U(t, \boldsymbol{x})$, is gravitational potential that is a linear superposition of potentials of all bodies of the system

$$U(t, \boldsymbol{x}) = \sum_{B=0}^{N} U_B(t, \boldsymbol{x}) , \tag{1.56}$$

$$U_B(t, \boldsymbol{x}) = G \int\limits_{\mathcal{V}_B} \frac{\rho(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x' . \tag{1.57}$$

Making use of the equation of continuity (1.32), the equations of motion (1.55) can be recast to the form

$$\rho\frac{dv^i}{dt} = -\frac{\partial \pi^{ij}}{\partial x^j} + \rho\frac{\partial U}{\partial x^i} , \tag{1.58}$$

where the operator of the total derivative

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v^j\frac{\partial}{\partial x^j} , \tag{1.59}$$

describes differentiation along the worldline of the element of body's matter. We define the dipole moment of body A with respect to a point with coordinates $\boldsymbol{x}_A = (x_A^i)$ by the expression

$$I_A^i = \int\limits_{\mathcal{V}_A} \rho(t, \boldsymbol{x}) \left(x^i - x_A^i\right) d^3x . \tag{1.60}$$

The point x_A^i coincides with the center of mass of the body in the case when $I_A^i = 0$, and one imposes and keeps this condition for any instant of time. Hence, the time-dependent position $\boldsymbol{x}_A = \boldsymbol{x}_A(t)$ of the center of mass of body A is defined in the inertial coordinates by equation

$$m_A x_A^i = \int\limits_{\mathcal{V}_A} \rho(t, \boldsymbol{x}) x^i d^3x , \tag{1.61}$$

where

$$m_A = \int\limits_{\mathcal{V}_A} \rho(t, \boldsymbol{x}) d^3x , \tag{1.62}$$

is a constant mass of the body A. Equations of orbital motion of body A can be obtained after double differentiation of both sides of (1.61) with respect to time and

9) Observe the use of Einstein's summation rule in application to the repeated indices.

application of the macroscopic equation of motion of matter (1.58). For doing this calculation, an important formula giving a value of the time derivative of integral quantities will be required. More specifically, for any smooth function, $f = f(t, \boldsymbol{x})$, multiplied with density $\rho = \rho(t, \boldsymbol{x})$, the following differentiation rule is valid

$$\frac{d}{dt}\int\limits_{\mathcal{V}_A} \rho f\, d^3x = \int\limits_{\mathcal{V}_A} \rho \frac{df}{dt} d^3x \ , \tag{1.63}$$

where the total time derivative in the right side must be understood in the sense of (1.59). The proof of this rule is rather straightforward. The time derivative of the integral is

$$\frac{d}{dt}\int\limits_{\mathcal{V}_A} \rho f\, d^3x = \int\limits_{\mathcal{V}_A} \left(\frac{\partial \rho}{\partial t} f + \rho \frac{\partial f}{\partial t}\right) d^3x + \oint\limits_{\partial \mathcal{V}_A} \rho f v^i dS_i \ , \tag{1.64}$$

where the surface integral in the right side of this equation takes into account that the volume of the body changes as time passes on. By applying the equation of continuity (1.32) and the Gauss theorem, one can bring (1.64) to the following form,

$$\begin{aligned}\int\limits_{\mathcal{V}_A} \left(\frac{\partial \rho}{\partial t} f + \rho \frac{\partial f}{\partial t}\right) d^3x + \oint\limits_{\partial \mathcal{V}_A} \rho f v^i dS_i = \\ \int\limits_{\mathcal{V}_A} \left(-\frac{\partial(\rho v^i)}{\partial x^i} f + \rho \frac{\partial f}{\partial t} + \frac{\partial(\rho f v^i)}{\partial x^i}\right) d^3x \ ,\end{aligned} \tag{1.65}$$

which is immediately reduced to the right side of (1.63) after applying the Leibniz rule to the partial derivative

$$\frac{\partial(\rho f v^i)}{\partial x^i} = \frac{\partial(\rho v^i)}{\partial x^i} f + \rho v^i \frac{\partial f}{\partial x^i} \ . \tag{1.66}$$

Applying (1.63) two times to the center-of-mass definition (1.61), one obtains

$$m_A v_A^i = \int\limits_{\mathcal{V}_A} \rho(t, \boldsymbol{x}) v^i d^3x \ , \tag{1.67}$$

$$m_A a_A^i = \int\limits_{\mathcal{V}_A} \rho(t, \boldsymbol{x}) \frac{dv^i}{dt} d^3x \ , \tag{1.68}$$

where $v_A^i = dx_A^i/dt$ is velocity, and $a_A^i = dv_A^i/dt$ is acceleration of the body's center of mass, respectively. Now, one replaces the time derivative, dv^i/dt, in the right side of the integral in (1.68) with the macroscopic equations of motion (1.58), and split the gravitational potential, U, in two parts – internal and external,

$$U = U_A + \bar{U} \ , \tag{1.69}$$

where

$$\bar{U} = \sum_{\substack{B=0 \\ B \neq A}}^{N} U_B(t, \boldsymbol{x}) \,. \tag{1.70}$$

It yields

$$m_A a_A^i = -\int_{\mathcal{V}_A} \frac{\partial \pi^{ij}}{\partial x^j} d^3x + \int_{\mathcal{V}_A} \rho \frac{\partial U_A}{\partial x^i} d^3x + \int_{\mathcal{V}_A} \rho \frac{\partial \bar{U}}{\partial x^i} d^3x \,. \tag{1.71}$$

The first term in the right side of this equation vanishes,

$$\int_{\mathcal{V}_A} \frac{\partial \pi^{ij}}{\partial x^j} d^3x = \oint_{\partial \mathcal{V}_A} \pi^{ij} dS_j = 0 \,, \tag{1.72}$$

because stresses disappear on the surface of each gravitating body (Landau and Lifshitz, 1959). The integral of the derivative of the internal gravitational potential also vanishes,

$$\int_{\mathcal{V}_A} \rho \frac{\partial U_A}{\partial x^i} d^3x = -\int_{\mathcal{V}_A} \int_{\mathcal{V}_A} \rho(t, \boldsymbol{x}) \rho(t, \boldsymbol{x}') \frac{x^i - x'^i}{|\boldsymbol{x} - \boldsymbol{x}'|^3} d^3x \, d^3x' = 0 \,, \tag{1.73}$$

due to anti-symmetry of the integrand with respect to the exchange of coordinates, $\boldsymbol{x} \leftrightarrow \boldsymbol{x}'$. Thus, all internal forces exerted on the body cancel out exactly, and equations of motion of the center of mass of body A are reduced to

$$m_A a_A^i = \int_{\mathcal{V}_A} \rho \frac{\partial \bar{U}}{\partial x^i} d^3x \,. \tag{1.74}$$

External potential, $\bar{U}$, can be expanded in Taylor series around the point $\boldsymbol{x}_A$ with respect to the harmonic (STF) polynomials,

$$\bar{U} = \sum_{l=0}^{\infty} \frac{1}{l!} r_A^{\langle i_1} r_A^{i_2} \ldots r_A^{i_l \rangle} \partial_{i_1 i_2 \ldots i_l} \bar{U}(t, \boldsymbol{x}_A) \,, \tag{1.75}$$

where $r_A^i \equiv x^i - x_A^i$, the angular brackets around indices denote STF symmetrization defined in (1.49), the partial derivative $\partial_i = \partial/\partial x^i$, and the lth partial derivative $\partial_{i_1 i_2 \ldots i_l} \bar{U}(t, \boldsymbol{x}_A) \equiv \left[\partial_{i_1 i_2 \ldots i_l} \bar{U}(t, \boldsymbol{x})\right]_{x = x_A}$. The appearance of the harmonic polynomials in this expansion is justified because the external potential $\bar{U}$ satisfies the Laplace equation: $\triangle \bar{U} = 0$. Hence, the symmetric polynomial $r_A^{\langle i_1} r_A^{i_2} \ldots r_A^{i_l \rangle}$ must be apparently traceless. Substituting expansion (1.75) in (1.74) yields

$$m_A a_A^i = \sum_{l=0}^{\infty} \frac{1}{l!} I_A^{\langle i_1 i_2 \ldots i_l \rangle} \partial_{i i_1 \ldots i_l} \bar{U}(t, \boldsymbol{x}_A) \,, \tag{1.76}$$

which is the equation of motion of body A given in terms of its (time-dependent) STF multipole moments

$$I_A^{\langle i_1 i_2 \ldots i_l \rangle} = \int\limits_{\mathcal{V}_A} \rho(t, \boldsymbol{x}) r_A^{\langle i_1} r_A^{i_2} \ldots r_A^{i_l \rangle} d^3 x \,, \tag{1.77}$$

coupled with the partial derivatives of the external potential $\bar{U}$ taken at the center of mass of the body. Development of this theory is getting progressively complicated if one continues to keep all multipole moments of the bodies in equations of motion (Kopeikin, 1988). We shall show how to deal with these complications in Section 6.1. The present section is restricted with the case of spherically-symmetric bodies neglecting their tidal and rotational deformations. In such cases, the external potential $\bar{U}$ is simplified to a linear superposition of potentials of point-like masses

$$\bar{U}(t, \boldsymbol{x}) = \sum_{\substack{B=0 \\ B \neq A}}^{N} \frac{G m_B}{|\boldsymbol{x} - \boldsymbol{x}_B|} \,, \tag{1.78}$$

where $\boldsymbol{x}_B = \boldsymbol{x}_B(t)$ are time-dependent positions of the external bodies $B \neq A$ defined by an equation being similar to (1.60) where index A must be replaced with index B. Substituting the potential (1.78) in (1.76) and assuming that body A is also spherically-symmetric (so that only $l = 0$ monopole term, $I_A \equiv m_A$, remains), one arrives at the final form of dynamical equations of motion of $N + 1$ point-like masses m_A located at coordinate positions $\boldsymbol{x}_A$,

$$m_A \ddot{\boldsymbol{x}}_A = - \sum_{\substack{B=0 \\ B \neq A}}^{N} \frac{G m_A m_B}{r_{AB}^3} \boldsymbol{r}_{AB} \,, \tag{1.79}$$

with vector $\boldsymbol{r}_{AB} = \boldsymbol{x}_A - \boldsymbol{x}_B$ being directed from body B to A, $r_{AB} = |\boldsymbol{r}_{AB}|$.

1.2.4
The Integrals of Motion

The system of equations (1.79) admits ten integrals of motion: three integrals of the linear momentum $\boldsymbol{P}$; three integrals of the initial position of the center of mass, $\boldsymbol{X}_0$; three integrals of the angular momentum, $\boldsymbol{J}$, and one integral of the energy, E. The integrals of the linear momentum and the center of mass are obtained by summing up (1.79) over all the bodies of the system, followed by integration with respect to time. The net gravitational force in the sum is reduced to zero due to the third Newton's law, so one obtains the following two vectorial integrals of motion:

$$\sum_{A=0}^{N} m_A \dot{\boldsymbol{x}}_A = \boldsymbol{P} \,, \tag{1.80}$$

$$\sum_{A=0}^{N} m_A \boldsymbol{x}_A = \boldsymbol{P}(t - t_0) + \boldsymbol{D} \,, \tag{1.81}$$

with t_0 being the epoch, and the constant vector

$$\boldsymbol{D} = M\boldsymbol{X}_0 \,, \tag{1.82}$$

where the total mass of the system

$$M = \sum_{A=0}^{N} m_A \,, \tag{1.83}$$

is constant. To obtain one more integral of motion, take the dot-product of (1.79) with the velocity $\dot{\boldsymbol{x}}_A$, with the subsequent summation over all the bodies of the system,

$$\sum_{A=0}^{N} m_A \ddot{\boldsymbol{x}}_A \cdot \dot{\boldsymbol{x}}_A = -\sum_{A=0}^{N} \sum_{\substack{B=0 \\ B\neq A}}^{N} \frac{G m_A m_B}{r_{AB}^3} \boldsymbol{r}_{AB} \cdot \dot{\boldsymbol{x}}_A \,. \tag{1.84}$$

With aid of the equalities

$$\sum_{A=0}^{N} \sum_{\substack{B=0 \\ B\neq A}}^{N} \frac{G m_A m_B}{r_{AB}^3} \boldsymbol{r}_{AB} \cdot \dot{\boldsymbol{x}}_A = \frac{1}{2} \sum_{A=0}^{N} \sum_{\substack{B=0 \\ B\neq A}}^{N} \frac{G m_A m_B}{r_{AB}^3} \boldsymbol{r}_{AB} \cdot \dot{\boldsymbol{r}}_{AB} \,, \tag{1.85}$$

and

$$\frac{G m_A m_B}{r_{AB}^3} \boldsymbol{r}_{AB} \cdot \dot{\boldsymbol{r}}_{AB} = -\frac{d}{dt} \frac{G m_A m_B}{r_{AB}} \,, \tag{1.86}$$

(1.84) becomes

$$\frac{dE}{dt} = 0 \,, \tag{1.87}$$

with E standing for a scalar integral of motion – the energy:

$$E = \frac{1}{2} \sum_{A=0}^{N} m_A \dot{\boldsymbol{x}}_A^2 - \frac{1}{2} \sum_{A=0}^{N} \sum_{\substack{B=0 \\ B\neq A}}^{N} \frac{G m_A m_B}{r_{AB}} \,. \tag{1.88}$$

Clearly, the first term in the right side makes up the kinetic energy of the bodies, while the second one represents the gravitational potential energy. The former is always positive, while the latter is always negative.[10]

The last integral of motion – the angular-momentum vector $\boldsymbol{J}$ – is derived from (1.79) by taking the cross product of both sides of this equation with the position vector $\boldsymbol{x}_A$, summing up over all the equations, and subsequent integration over time. This entails

$$\boldsymbol{J} = \sum_{A=0}^{N} m_A \left(\boldsymbol{x}_A \times \dot{\boldsymbol{x}}_A \right) \,, \tag{1.89}$$

10) The gravitational potential energy being negative makes the relativistic masses of self-gravitating astronomical objects, like planets or stars, smaller than the algebraic sum of the rest masses of their constituent particles – baryons. See Section 6.1.3 for further details.

the sign $\times$ denoting the Euclidean cross product of two vectors. Constant vector $\boldsymbol{J}$ defines an invariant plane of the N-body problem called the *invariable plane of Laplace*.

1.2.5
The Equations of Relative Motion with Perturbing Potential

It would be instructive to rewrite the equations of motion (1.79) in terms of the relative distances of the bodies from the primary body, the one denoted with the index $B = 0$. To this end, one introduces the relative-to-the-primary vectors

$$\boldsymbol{R}_A = \boldsymbol{x}_A - \boldsymbol{x}_0 \,, \quad \boldsymbol{R}_B = \boldsymbol{x}_B - \boldsymbol{x}_0 \tag{1.90}$$

where $\boldsymbol{x}_0$ denotes the position of the primary. Equation 1.79 written for the primary is[11]

$$\ddot{\boldsymbol{x}}_0 = \frac{G m_A}{R_A^3}\boldsymbol{R}_A + \sum_{\substack{B=1\\ B\neq A}}^{N} \frac{G m_B}{R_B^3}\boldsymbol{R}_B \,, \tag{1.91}$$

and for the other bodies

$$\ddot{\boldsymbol{x}}_A = -\frac{G m_0}{R_A^3}\boldsymbol{R}_A - \sum_{\substack{B=1\\ B\neq A}}^{N} \frac{G m_B}{r_{AB}^3}\boldsymbol{r}_{AB} \,, \tag{1.92}$$

where $\boldsymbol{r}_{AB} = \boldsymbol{x}_A - \boldsymbol{x}_B = \boldsymbol{R}_A - \boldsymbol{R}_B$ is a vector of relative distance directed from body B to A. The difference between (1.92) and (1.91) amounts to

$$\ddot{\boldsymbol{R}}_A = -\frac{G(m_0 + m_A)}{R_A^3}\boldsymbol{R}_A - \sum_{\substack{B=1\\ B\neq A}}^{N} G m_B \left(\frac{\boldsymbol{r}_{AB}}{r_{AB}^3} + \frac{\boldsymbol{R}_B}{R_B^3}\right) \,, \tag{1.93}$$

whose right side can be recast to a gradient form

$$\ddot{\boldsymbol{R}}_A = \frac{\partial U}{\partial \boldsymbol{R}_A} \,, \tag{1.94}$$

where

$$U = \frac{G(m_0 + m_A)}{R_A} + \mathfrak{W} \tag{1.95}$$

consists of an algebraic sum of a potential of a point-like mass, $m_0 + m_A$, and the *perturbing potential*

$$\mathfrak{W} = \sum_{\substack{B=1\\ B\neq A}}^{N} G m_B \left(\frac{1}{r_{AB}} - \frac{1}{R_B} - \frac{\boldsymbol{R}_A \cdot \boldsymbol{R}_B}{R_B^3}\right) \,. \tag{1.96}$$

11) Notice that the mass, m_0, of the primary cancels out.

The perturbing potential $\mathfrak{W}$ acting on a mass m_A is generated by all external masses m_B other than m_A or the primary with the mass m_0. It depends on the total gravitational potential of the external bodies with masses m_B taken at the position of body A, from which one subtracts a monopole term, $\sim R_B^{-1}$, and a dipolar component, $\sim (\boldsymbol{R}_A \cdot \boldsymbol{R}_B) R_B^{-3}$, which can be interpreted as a force of inertia $\boldsymbol{F}_{\rm in}$ emerging in the noninertial frame associated with the primary. It is interesting to notice that in the case of a two-body problem, the perturbing potential vanishes identically, $\mathfrak{W} \equiv 0$.

1.2.6
The Tidal Potential and Force

When the distance R_A happens to be much smaller than any of the distances R_B, the problem of relative motion of the body A around the primary becomes a two-body problem with the perturbation caused by the tidal forces from the external bodies. An example of such a motion is rendered by the Earth–Moon system that moves in the external gravitational field of the Sun and the major planets. In this case, the Earth assumes the role of the primary, the Moon plays the role of the secondary body A, while the external bodies $B \neq A$ are the Sun and the major planets of the solar system. An expansion of the perturbing potential $\mathfrak{W}$ in the Taylor series with respect to a small parameter R_A/R_B is obtained by expanding the function r_{AB}^{-1} about the point $R_A = 0$ in terms of the harmonic polynomials. This gives us

$$\begin{aligned}\frac{1}{r_{AB}} &= \sum_{l=0}^{\infty} \frac{1}{l!} R_A^{\langle i_1} R_A^{i_2} \dots R_A^{i_l \rangle} \left[\frac{\partial^l}{\partial R_A^{i_1} \partial R_A^{i_2} \dots \partial R_A^{i_l}} \left(\frac{1}{r_{AB}} \right) \right]_{R_A=0} \\ &= \frac{1}{R_B} + \frac{\boldsymbol{R}_A \cdot \boldsymbol{R}_B}{R_B^3} + \frac{1}{2} \left(\frac{3(\boldsymbol{R}_A \cdot \boldsymbol{R}_B)^2}{R_B^5} - \frac{R_A^2}{R_B^3} \right) + \dots ,\end{aligned} \tag{1.97}$$

where the angular brackets around spatial indices denote the STF (symmetric and traceless) tensor, and one has used vector notation $R_A^{i_l} = \boldsymbol{R}_A \quad (l = 1, 2, \dots)$, with each index i_l taking the values $(1, 2, 3)$ corresponding to the three Cartesian coordinates (x^1, x^2, x^3). Substituting this expansion into (1.96), one sees that both the monopole ($l = 0$) and dipole ($l = 1$) terms canceled out, so the tidal expansion of the disturbing potential acquires the following form

$$\mathfrak{W} = \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} R_A^{\langle i_1} R_A^{i_2} \dots R_A^{i_l \rangle} \frac{\partial^l \bar{U}}{\partial R_B^{i_1} \partial R_B^{i_2} \dots \partial R_B^{i_l}} , \tag{1.98}$$

where

$$\bar{U} = \sum_{\substack{B=1 \\ B \neq A}}^{N} \frac{G m_B}{R_B} \tag{1.99}$$

is the gravitational potential created by the external bodies at the position of the primary. The lowest-order term of the tidal potential $\mathfrak{W}$ is $l = 2$, which corresponds to the quadrupole moment in the expansion of the external gravitational potential $\bar{U}$ in the immediate neighborhood of the primary. It is worthwhile to point out that the partial derivatives in the expansion for the tidal potential possess the following property: contraction with respect to any couple of indices identically gives zero because the external potential $\bar{U}$ satisfies the homogeneous Laplace equation

$$\triangle \bar{U} \equiv \delta^{ij} \frac{\partial^2 \bar{U}}{\partial R_B^i \partial R_B^j} = 0 \,, \tag{1.100}$$

where the repeated indices assume summation from 1 to 3, and $\delta^{ij} = \mathrm{diag}(1, 1, 1)$ is the unit matrix (the Kronecker symbol).

The tidal force $\boldsymbol{F}_{\text{tide}} = \left(F^i_{\text{tide}}\right)$ exerted on the body A orbiting the primary is calculated as a partial derivative of the tidal potential

$$F^i_{\text{tide}} = \frac{\partial \mathfrak{W}}{\partial R_A^i} = \sum_{l=2}^{\infty} \frac{(-1)^l}{(l-1)!} R_A^{\langle i_1} R_A^{i_2} \ldots R_A^{i_{l-1}\rangle} \frac{\partial^l \bar{U}_A}{\partial R_B^{i_1} \ldots \partial R_B^{i_{l-1}} \partial R_B^i} \,, \tag{1.101}$$

so the equation of the relative motion of the body A around the primary is

$$\begin{aligned} \ddot{\boldsymbol{R}}_A &= -\frac{G(m_0 + m_A)}{R_A^3} \boldsymbol{R}_A + \boldsymbol{F}_{\text{tide}} \\ &= -\frac{G(m_0 + m_A)}{R_A^3} \boldsymbol{R}_A + \sum_{\substack{B=1 \\ B \neq A}}^{N} \frac{G m_B}{R_B^3} \left[\frac{3(\boldsymbol{R}_A \cdot \boldsymbol{R}_B) \boldsymbol{R}_B}{R_B^2} - \boldsymbol{R}_A \right] + \ldots \,, \end{aligned} \tag{1.102}$$

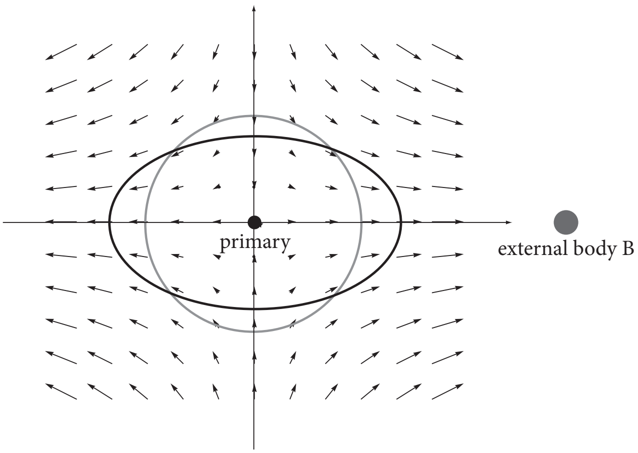

Figure 1.2 The vector field of the quadrupole tidal force is shown at different points in space around the primary kept fixed at the coordinate origin. The tidal force is caused by the external body B laid out on the x-axis far away from the primary. The circle depicts a circular orbit in the plane x–y that would be described by the body A about the primary in the absence of the perturbing tidal force. The quadrupole tidal force squeezes the circular orbit in the plane x–y along the direction toward the body B so that the orbit becomes an ellipse with the ratio of its axes equal to 2.

where one has only shown the leading (quadrupole) term in the tidal force perturbing the motion of the body A. If the orbit of the body A around its primary is circular and there is only one external body B lying on the x-axis, the quadrupole tidal force is maximal at the point of intersection of the orbit with x-axis, and is minimal at the points of intersection of the orbit with the y-axis. The ratio of the maximal-to-minimal values of the tidal force amounts to two. A plot of the quadrupole tidal force at different points in space around the primary is demonstrated in Figure 1.2.

1.3
The Reduced Two-Body Problem

In the simplest case of two bodies, only a primary of mass m_0 located at $\boldsymbol{x}_0$, and a secondary of mass m_1 located at $\boldsymbol{x}_1$ are present. The disturbance $\mathfrak{W}$ vanishes because the subscript B in (1.96) runs through one value solely, $B = 1$, and there are no other values to be taken by the index $B \neq A$. The motion becomes mathematically equivalent to the Newtonian one-body problem, that is, to movement about a *fixed* center of mass, $M = m_0 + m_1$, given by equation

$$\ddot{\boldsymbol{r}} = -\frac{GM}{r^3}\boldsymbol{r} \,, \tag{1.103}$$

where $\boldsymbol{r} \equiv \boldsymbol{r}_1$. Equation 1.103 has been derived independently in Section 1.1.3. A fortunate aspect of the two-body problem is that it is integrable in terms of elementary functions. The outcome is Newton's celebrated result: the generic solution is a conic with the gravitating center in one of its foci. This then grants one a possibility to thoroughly discuss multiple aspects of the orbital motion of the bodies making use of various parameterizations of the conics.

1.3.1
Integrals of Motion and Kepler's Second Law

Let the center of attraction be located in the origin of an inertial reference frame parameterized with axes $x^i = (x^1, x^2, x^3) = (x, y, z)$, as shown in Figure 1.3. The directions of the axes are defined via three unit vectors $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$ with the following components

$$\boldsymbol{e}_x = (1,0,0) \,, \quad \boldsymbol{e}_y = (0,1,0) \,, \quad \boldsymbol{e}_z = (0,0,1) \,. \tag{1.104}$$

The position of a moving body is given by the radius-vector $\boldsymbol{r}$. The projection of the velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$ of the body[12] onto the direction of the radius-vector $\boldsymbol{r}$ gives the rate $\dot{r}$ at which the radial distance evolves. In other words, the Euclidean dot product

12) Remember that a dot over any function of time denotes an ordinary derivative with respect to time, for example, $\dot{\boldsymbol{r}} = d\boldsymbol{r}/dt$, and so on.

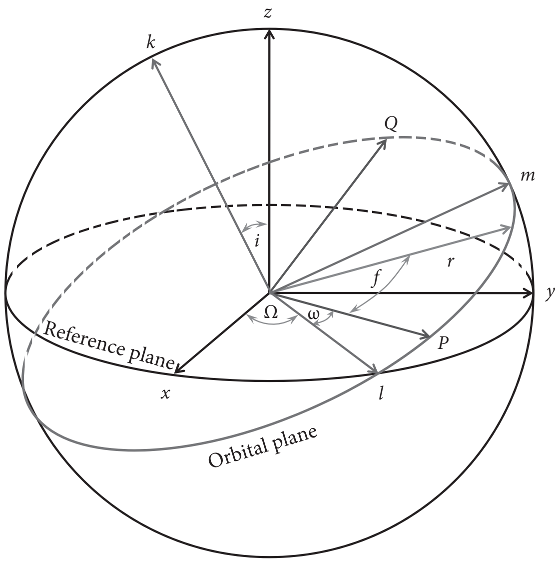

Figure 1.3 Inertial reference frame $(x^1, x^2, x^3) = (x, y, z)$ has its origin at the fixed center of gravity with mass M. The orbital plane is orthogonal to the unit vector $\boldsymbol{k}$ and intersects with the reference plane (x, y) along the apsidal line defined by the unit vector $\boldsymbol{l}$. Position of the moving body is characterized by the radius-vector $\boldsymbol{r}$.

$\dot{\boldsymbol{r}} \cdot \boldsymbol{r} = \dot{r} r$. Keeping this in mind, one sees that the dot-product of (1.103) by $\dot{\boldsymbol{r}}$ is

$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} + G M \frac{\dot{r}}{r^2} = 0 \, . \tag{1.105}$$

Integration of the latter results in a conservation law of the orbital energy,

$$\mathcal{E} = \frac{1}{2} \dot{\boldsymbol{r}}^2 - \frac{G M}{r} \, , \tag{1.106}$$

with the constant $\mathcal{E}$ being the energy per unit mass.

Taking the cross-product of both sides of (1.103) with vector $\boldsymbol{r}$, one trivially ends up with

$$\boldsymbol{r} \times \ddot{\boldsymbol{r}} = 0 \, , \tag{1.107}$$

integration whereof gives us another conservation law of the orbital angular momentum,

$$\boldsymbol{J} = \boldsymbol{r} \times \dot{\boldsymbol{r}} \, . \tag{1.108}$$

Here the constant vector, $\boldsymbol{J}$, is orthogonal to both $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ and is easily identifiable with the orbital angular momentum per unit mass. Conservation of this vector tells us that the plane defined by $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ stays unchanged. This circumstance gives birth to the term *orbital plane*. Notice that both magnitude and direction of vector $\boldsymbol{J}$ are conserved, so that each component J_x, J_y, and J_z of this vector is an independent integral of motion.

Orientation of the orbital plane is uniquely defined by the orientation of the angular momentum vector $\boldsymbol{J}$ that is parallel to the unit vector $\boldsymbol{k}$, that is $\boldsymbol{J} = J\boldsymbol{k}$ as shown in Figure 1.3. The orbital plane intersects with the reference plane (x, y) along the apsidal line defined by the unit vector $\boldsymbol{l}$ that is directed towards the ascending node of the orbit which assumes that the body in Figure 1.3 moves counter-clockwise if one watches the motion from the tip of vector $\boldsymbol{k}$. The unit vector $\boldsymbol{m} = \boldsymbol{k} \times \boldsymbol{l}$, and lies in the orbital plane. The triad of unit vectors $\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{k}$ is related to the triad of unit vectors $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$ defining orientation of three axes of the inertial frame, as follows

$$\boldsymbol{l} = \boldsymbol{e}_x \cos\Omega + \boldsymbol{e}_y \sin\Omega \ , \tag{1.109}$$

$$\boldsymbol{m} = -\boldsymbol{e}_x \cos i \sin\Omega + \boldsymbol{e}_y \cos i \cos\Omega + \boldsymbol{e}_z \sin i \ , \tag{1.110}$$

$$\boldsymbol{k} = \boldsymbol{e}_x \sin i \sin\Omega - \boldsymbol{e}_y \sin i \cos\Omega + \boldsymbol{e}_z \cos i \ . \tag{1.111}$$

Here, the angle Ω is the longitude of the ascending node of the orbit, and the angle i is the inclination of the orbit with respect to the reference plane.

Let us now introduce within the orbital plane the polar coordinates of the moving body which are the radial distance, r, and the argument of latitude, θ, that is the angle between vectors $\boldsymbol{r}$ and $\boldsymbol{l}$ reckoned counter-clockwise in the orbital plane from the direction $\boldsymbol{l}$. In terms of r and θ one has,

$$\boldsymbol{r} = r(\boldsymbol{l}\cos\theta + \boldsymbol{m}\sin\theta) \ , \tag{1.112}$$

$$\dot{\boldsymbol{r}} = \boldsymbol{l}\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right) + \boldsymbol{m}\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right) \ . \tag{1.113}$$

The angular momentum $\boldsymbol{J}$ being expressed in the polar coordinates becomes,

$$\boldsymbol{J} = \boldsymbol{k} r^2\dot{\theta} \ , \tag{1.114}$$

with its absolute value

$$J = |\boldsymbol{J}| = r^2\dot{\theta} = \text{const.} \tag{1.115}$$

In time δt, the radius-vector $\boldsymbol{r}$ sweeps out the angle $\delta\theta = \dot{\theta}\,\delta t$ and the area

$$\delta A = \frac{1}{2} r(r + \delta r)\sin(\delta\theta) = \frac{1}{2} r^2 \delta\theta \ . \tag{1.116}$$

After dividing each side of (1.116) by δt and taking the limit $\delta t \to 0$, one gets a differential equation for the area's temporal change,

$$\dot{A} = \frac{1}{2} r^2\dot{\theta} = \frac{J}{2} \ . \tag{1.117}$$

Integration of this equation ensures Kepler's second law of planetary motion: *equal areas are swept out in equal times*, as can be envisaged from the right side of (1.117), telling us that the time derivative of the area is equal to constant $J/2$. Thus, Kepler's second law has been shown to follow from Newton's theory. Our next step will be to demonstrate that Kepler's first and third laws do so as well.

1.3.2
The Equations of Motion and Kepler's First Law

Let us write the equation of motion (1.103) with respect to inertial Cartesian axes defined by the unit vectors $\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{k}$. In fact, due to the law of conservation of the angular momentum, only two components of this equation in the orbital plane will be present. By differentiating the law of conservation of the angular momentum (1.114), one obtains

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 . \tag{1.118}$$

Differentiating (1.113) with respect to time and making use of (1.118) allows us to express the acceleration of the body in the following form,

$$\ddot{\boldsymbol{r}} = \boldsymbol{r}\left(\frac{\ddot{r}}{r} - \dot{\theta}^2\right) . \tag{1.119}$$

Equation 1.115 also tells us that the time derivative of the angle θ is

$$\dot{\theta} = \frac{J}{r^2} . \tag{1.120}$$

Substituting this expression in (1.119) and the result of the substitution into the vectorial equation of motion (1.103) brings about a differential equation for the radial motion of the body

$$\ddot{r} + \frac{GM}{r^2} - \frac{J^2}{r^3} = 0 , \tag{1.121}$$

where J is the constant angular momentum of the orbital motion. The solution of this differential equation was not known at the time of Newton. For this reason, in his *Principia*, Newton only proved that orbits in the form of conics necessitate an inverse-square gravity law. To prove the inverse statement, that is, that the gravity law entails this type of orbits, Newton would have to cope with (1.121) which was not solved until 1710. On 13 December 1710, two Swiss mathematicians, Johann Bernoulli and Jakob Hermann[13], presented their solutions of (1.121) to a meeting of the Paris Academy of Sciences. Both speakers provided valid proofs (to which they had come upon independently) that Newton's gravity law yields orbits shaped as conics. For a historical account of those events, see Weinstock (1982).

13) The two were former disciples of Johann's older brother, Jakob Bernoulli; and Hermann was a distant relative of Euler.

In the middle of the eighteenth century, the solution of this problem was greatly simplified due to an elegant mathematical trick independently pioneered by d'Alembert and Clairaut. The first step of this method is to employ (1.120) as a means of switching from differentiation with respect to time t to differentiation with respect to the angle θ. The rationale beneath this replacement of variable is to find the shape of an orbital curve, that is, the dependence of r upon θ. Thus, one replaces the time derivatives of r with those with respect to θ

$$\dot{r} = \dot{\theta}\, r' \;, \tag{1.122}$$

$$\ddot{r} = \ddot{\theta}\, r' + \dot{\theta}^2 r'' \;, \tag{1.123}$$

where a dot signifies the time derivative, and the prime denotes the derivative with respect to θ. Making use of (1.118), (1.120), and (1.122), one can recast equation (1.123) to the following form:

$$\ddot{r} = \frac{J^2}{r^2}\left(\frac{r''}{r^2} - 2\frac{r'^2}{r^3}\right) \;. \tag{1.124}$$

The second crucial step is to replace r with a reciprocal radial variable $u = 1/r$, thus obtaining:

$$u' = -\frac{r'}{r^2} \;, \tag{1.125}$$

$$u'' = -\frac{r''}{r^2} + 2\frac{r'^2}{r^3} \;. \tag{1.126}$$

Comparing this equation with (1.124), and using the substitution $u = 1/r$, one obtains

$$\ddot{r} = -J^2 u^2 u'' \;. \tag{1.127}$$

This expression, along with $\dot{\theta} = J r^{-2} = J u^2$, helps us to transform (1.121) to the Binet equation

$$u'' + u = \frac{G M}{J^2} \;, \tag{1.128}$$

which is the equation of harmonic oscillator subject to a constant perturbation $G M/J^2$. Solution of this equation is a linear superposition of general solution of a homogeneous equation $u'' + u = 0$, and a particular solution of the inhomogeneous equation (1.128)

$$u = B\cos(\theta - \omega) + \frac{G M}{J^2} \;, \tag{1.129}$$

where B and ω are constants of integration depending on the initial conditions. This solution looks similar to a well-known analytic geometry expression for the reciprocal distance from a point on a conic to one of its foci,

$$\frac{p}{r} = 1 + e\cos f \;, \tag{1.130}$$

where f is the *true anomaly*, that is, the angular separation of the point from the periapse (subtended at the said focus), while $p = a(1-e^2)$ is a constant parameter being expressed in terms of the semi-major axis a and eccentricity e, and called *semilatus rectum*.[14]

To convert the resemblance to equivalence, one must choose the constants B and J in (1.129) in the following form

$$B = \frac{e}{a(1-e^2)} = \frac{e}{p} , \tag{1.131}$$

$$J = \sqrt{G M p} , \tag{1.132}$$

and equate the angular variables

$$f = \theta - \omega . \tag{1.133}$$

After these identifications, the orbital elements e and p turn out to be interconnected with the integrals of motion via formulae

$$p = \frac{J^2}{G M} , \tag{1.134}$$

$$e = \sqrt{1 + \frac{2\mathcal{E} J^2}{G^2 M^2}} , \tag{1.135}$$

so the constant B from (1.129) becomes

$$B = \frac{G M}{J^2} \sqrt{1 + \frac{2\mathcal{E} J^2}{G^2 M^2}} , \tag{1.136}$$

and the integral of the reduced total energy

$$\mathcal{E} = \frac{G M}{2p}(e^2 - 1) . \tag{1.137}$$

For different conics, the parameters of the orbit are defined as

$$\text{circle:} \quad p = a , \quad e = 0 , \tag{1.138a}$$

$$\text{ellipse:} \quad p = a(1-e^2) , \quad 0 < e < 1 , \tag{1.138b}$$

$$\text{parabola:} \quad p = 2q , \quad e = 1 , \tag{1.138c}$$

$$\text{hyperbola:} \quad p = a(e^2 - 1) , \quad e > 1 . \tag{1.138d}$$

14) To derive (1.130) for an ellipse, a circle, or a hyperbola, start with (1.147) written in a Cartesian coordinate system (ξ, η) whose origin is located in one of the foci, the axis ξ going through the foci, as shown in Figure 1.4. For a point on the conic, $\xi = r \cos f$ and $\eta = r \sin f$, plugging of which into (1.147) entails (1.130).

Equation 1.132 demonstrates that the angular momentum, J, of the orbit only depends on the focal parameter, p, and is always positive for any type of the orbit

$$J^2 = GMp > 0 . \tag{1.139}$$

On the other hand, (1.137) reveals that the reduced total energy $\mathcal{E}$ of the two-body system depends only on the semi-major axis, a, and has either positive or negative, or zero value for different types of conics

$$\text{ellipse:} \quad \mathcal{E} = -\frac{GM}{2a} < 0 , \tag{1.140a}$$

$$\text{parabola:} \quad \mathcal{E} = 0 , \tag{1.140b}$$

$$\text{hyperbola:} \quad \mathcal{E} = \frac{GM}{2a} > 0 . \tag{1.140c}$$

The case of a parabola is exceptional in that its eccentricity $e = 1$, and the semi-latus rectum is defined as $p = 2q$, where q is the minimal distance of the orbit to the gravitating center at body's closest approach. As parabolic (or near-parabolic) orbits are considered in extremely rare situations, they will be omitted below.[15]

The point of the closest approach of the orbit to the attracting center is called the *pericenter*, and the opposite point on the orbit is called the *apocenter*.[16] Let us define a unit vector $\boldsymbol{P}$ directed from the attracting center towards the pericenter and another unit vector $\boldsymbol{Q}$ lying in the orbital plane so that the triad $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{k}$ make a right-handed system of three vectors (see Figure 1.3). Vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ are related to vectors $\boldsymbol{l}$ and $\boldsymbol{m}$ by rotation at the angle ω

$$\boldsymbol{P} = \boldsymbol{l} \cos\omega + \boldsymbol{m} \sin\omega , \tag{1.141}$$

$$\boldsymbol{Q} = -\boldsymbol{l} \sin\omega + \boldsymbol{m} \cos\omega . \tag{1.142}$$

The angle ω is the same as in (1.133). It measures in the orbital plane the angular distance of the pericenter from the ascending node. In terms of the constant unit vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$, the radius-vector of the body is expressed as follows:

$$\boldsymbol{r} = r(\boldsymbol{P} \cos f + \boldsymbol{Q} \sin f) . \tag{1.143}$$

Derivation of velocity, $\boldsymbol{v} = \dot{\boldsymbol{r}}$, is achieved by direct differentiation of (1.143),

$$\dot{\boldsymbol{r}} = \dot{r}(\boldsymbol{P} \cos f + \boldsymbol{Q} \sin f) + r\dot{f}(-\boldsymbol{P} \sin f + \boldsymbol{Q} \cos f) . \tag{1.144}$$

15) One possibility of integrating such orbits is through switching to the Kustaanheimo–Stiefel variables (Stiefel, 1976). Treatment by more conventional means is offered, for example, by Osman and Ammar (2006).

16) In case of a planetary orbit around the Sun, these orbital points are called, respectively, the perihelion and aphelion. Corresponding points on the orbit of the Moon and on the orbit of artificial satellites of the Earth are called perigee and apogee.

Making use of (1.130) describing the first Kepler's law along with equation $\dot{f} = \dot{\theta}$, the integral of the angular momentum taken in the form of (1.120) and (1.132) allows us to get the time derivative of the radial distance,

$$\dot{r} = \frac{e}{p} \sin f (r^2 \dot{f}) = \frac{Je}{p} \sin f = \sqrt{\frac{GM}{p}} e \sin f \,. \tag{1.145}$$

Substituting this expression in (1.144) and making use of the law of conservation of the angular momentum result in

$$\dot{\boldsymbol{r}} = \sqrt{\frac{GM}{p}} [-\boldsymbol{P} \sin f + \boldsymbol{Q}(\cos f + e)] \,. \tag{1.146}$$

1.3.3
The Mean and Eccentric Anomalies – Kepler's Third Law

To start with, introduce a coordinate system (ξ, η) with an origin fixed at the attraction center F, and with the unit vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ directed along the axes ξ and η, respectively. Draw an auxiliary circle of radius a equal to the semi-major axis, and centered at the midpoint O between the foci. Let the body be located on the conic at point B at time t, as shown in Figure 1.4. One draws a straight line parallel to the η-axis and passing through the orbiter. The line is orthogonal to the ξ-axis, and intersects the auxiliary circle at point C. The *eccentric anomaly* E is defined as

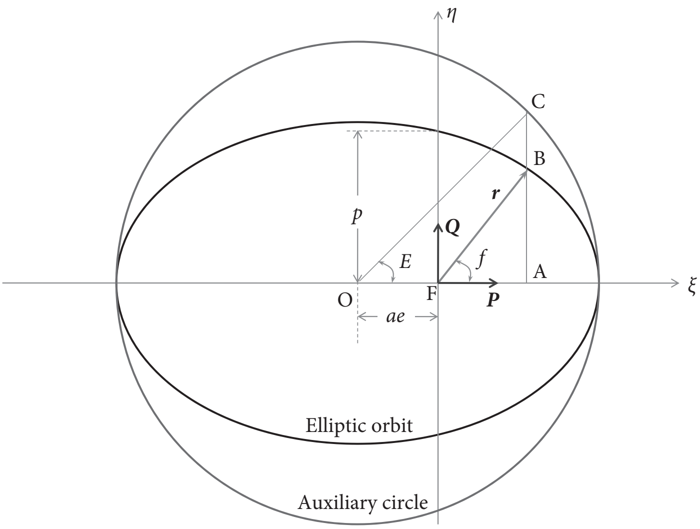

Figure 1.4 The orbiting body is at the point B. The eccentric anomaly is the angle E between directions OA and OC. The true anomaly f is the angle between the ξ-axis and the radius-vector $\boldsymbol{r}$. The distance between the center of the auxiliary circle and the conic is ae, where a is the semi-major axis and e is the eccentricity.

the angle subtended at the circle's center between the axis ξ and a straight line pointing at the point C on the circle.

As the distance between the origin and a focus is ea, one writes down the equation for the conic

$$\frac{(\xi + ae)^2}{a^2} + \frac{\eta^2}{a^2(1-e^2)} = 1 , \tag{1.147}$$

valid for a circle, an ellipse, or a hyperbola.

The abscissa of point B on the conic is equal to $\xi = r\cos f$. At the same time, it is equal to

$$\xi = a\cos E - ae = a(\cos E - e) \tag{1.148}$$

because the distance OF between the attraction center at point F and the center of the auxiliary circle at point O is equal to ae, in accordance with definitions of eccentricity e and the semi-major axis a. Equating the two expressions for the abscissa of the point B, and making use of (1.130) of the conic, one gets a formula interconnecting the two anomalies:

$$\cos E = \frac{e + \cos f}{1 + e\cos f} . \tag{1.149}$$

The ordinate of point B on the conic is $\eta = r\sin f$. Accounting for (1.147) of the conic and (1.148), one obtains

$$\eta = a\sqrt{1-e^2}\sin E . \tag{1.150}$$

Again, equating the two expressions for η gives us another interconnection,

$$\sin E = \frac{\sqrt{1-e^2}\sin f}{1 + e\cos f} , \tag{1.151}$$

which leads us to the expression for the distance from the focus as a function of the eccentric anomaly:

$$r = a(1 - e\cos E) . \tag{1.152}$$

In combination with (1.149), the latter renders:

$$r = \frac{a(1-e^2)}{1 + e\cos f} . \tag{1.153}$$

Equation 1.149 can be rewritten in the following equivalent forms:

$$1 - \cos f = (1+e)\frac{1-\cos E}{1 - e\cos E} , \tag{1.154a}$$

$$1 + \cos f = (1-e)\frac{1+\cos E}{1 - e\cos E} . \tag{1.154b}$$

With aid of the standard double-angle formulae, these relations can be reshaped correspondingly into

$$\sin^2\frac{f}{2} = \frac{1+e}{1-e\cos E}\sin^2\frac{E}{2}\,, \tag{1.155a}$$

$$\cos^2\frac{f}{2} = \frac{1-e}{1-e\cos E}\cos^2\frac{E}{2}\,. \tag{1.155b}$$

The ratio of these two formulae furnishes yet another elegant interconnection between the true and eccentric anomalies,

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\,. \tag{1.156}$$

Our next project is to link the anomalies with the time. To this end, one employs (1.120) and (1.133) to write

$$df = d\theta = \frac{J}{r^2}dt\,. \tag{1.157}$$

As the formulae (1.132) and (1.153) enable us to express J and r via the elements, one can rewrite (1.157) as

$$df = \sqrt{\frac{GM}{a^3}}\frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}dt\,. \tag{1.158}$$

Introducing a widely used quantity

$$n = \sqrt{\frac{GM}{a^3}}\,, \tag{1.159}$$

called *mean motion* or *mean angular frequency*, one can write down (1.158) as

$$n\,dt = \frac{(1-e^2)^{3/2}}{(1+e\cos f)^2}df\,. \tag{1.160}$$

In the special case of a bound orbit, that is, when the conic is a circle or an ellipse, the above formula shows an important property. As integration of its right side over a period, that is, from $f = 0$ through $f = 2\pi$, gives exactly 2π, one has

$$nT = 2\pi\,, \tag{1.161}$$

where T is the time of the orbital period. Obviously, the mean motion n is the angular velocity in the case of a circular orbit, and an *average* angular velocity (as seen from the focus) in the case of ellipse. Combining expression (1.161) with the definition (1.159), one arrives at Kepler's third law in a two-body problem,

$$T^2 = \frac{4\pi^2 a^3}{GM}\,, \tag{1.162}$$

which reads: *the square of the orbital period of a body is directly proportional to the cube of the semi-major axis of its orbit.*

Our next step is to interconnect the time with the eccentric anomaly. From the afore-proven equality (1.156), one deduces

$$d E = \frac{\sqrt{1-e^2}}{1+e\cos f} d f \ . \tag{1.163}$$

Moreover, (1.152) and (1.153) yield

$$1 - e\cos E = \frac{1-e^2}{1+e\cos f} \ . \tag{1.164}$$

Formulae (1.163), (1.164) and (1.160), being put together, entail

$$n d t = (1 - e\cos E) d E = d(E - e\sin E) \ , \tag{1.165}$$

where from a simple integration yields

$$E - e\sin E = n(t - t_0) + \mathcal{M}_0 \ , \tag{1.166}$$

where t_0 is the fiducial time, called the *epoch*, and $\mathcal{M}_0$ is the integration constant. The latter compels us to define, following Kepler, a convenient quantity

$$\mathcal{M} = \mathcal{M}_0 + n(t - t_0) \ , \tag{1.167}$$

called *mean anomaly*, with $\mathcal{M}_0$ now termed as the *mean anomaly at epoch*. According to (1.166), $\mathcal{M}$ obeys the *Kepler equation*

$$\mathcal{M} = E - e\sin E \ . \tag{1.168}$$

Therefore, it is clear that for elliptic and circular orbits, $\mathcal{M}$ changes by 2π over a period because the eccentric anomaly changes by 2π, and $\sin E$ is a periodic function with period 2π.

Finally, let us notice that (1.163) and (1.168) enable us to interconnect the mean and eccentric anomalies:

$$d\mathcal{M} = (1 - e\cos E) d E = \frac{(1-e^2)^{3/2}}{(1+e\cos f)^2} d f \ . \tag{1.169}$$

For bound orbits, this yields for one period of orbital revolution

$$\oint\limits_{\substack{\text{orbital}\\ \text{period}}} d\mathcal{M} = \int_0^{2\pi} \frac{(1-e^2)^{3/2}}{(1+e\cos f)^2} d f = 2\pi \ . \tag{1.170}$$

Thus, we once again see that $\mathcal{M}$ changes by 2π over a period.

Be mindful that one did *not* necessarily imply t_0 to be the instant of the first periapse passage τ. The time τ_N of the Nth periapse passage is defined from the condition that the eccentric anomaly

$$E = 2\pi(N-1) \ , \tag{1.171}$$

with N being an integer chosen so that $N = 1$ corresponds to $E = 0$ that is the first periapse passage. As evident from (1.168), condition (1.171) can also be rewritten as

$$\mathcal{M} = 2\pi(N-1) \ . \tag{1.172}$$

From here and (1.167), it is easy to demonstrate that the Nth periapse passage takes place at the time

$$\tau_N = t_0 + \frac{2\pi(N-1) - \mathcal{M}_0}{n} \ . \tag{1.173}$$

In celestial mechanics, the time of the first periapse passage, τ_1, is simply denoted as τ, which is

$$\tau = t_0 - \frac{\mathcal{M}_0}{n} \ . \tag{1.174}$$

Hence, the mean anomaly expressed in terms of the first periapse passage will look like

$$\mathcal{M} = n(t-\tau) \ . \tag{1.175}$$

Over one orbital revolution, the mean anomaly changes by 2π, while the time changes by period T. Thus, (1.175) naturally renders (1.161).

1.3.4 The Laplace–Runge–Lenz Vector

One has already learned that the reduced two-body problem obeying the Newton gravity law (1.103) permits four integrals of motion – the energy, $\mathcal{E}$, and the three components of the angular-momentum vector, $\boldsymbol{J}$. It is remarkable that the reduced two-body problem admits one more integral of motion. To demonstrate this fact, consider the so-called *Laplace–Runge–Lenz vector*

$$\boldsymbol{A}_{\mathrm{L}} = \dot{\boldsymbol{r}} \times \boldsymbol{J} - G M \frac{\boldsymbol{r}}{r} \ , \tag{1.176}$$

where $\boldsymbol{J}$ is the conserved angular-momentum vector. Despite its name, the Laplace–Runge–Lenz vector was discovered by neither of these three scholars. The honor of its discovery belongs to the aforementioned Jakob Hermann. It was also Hermann who demonstrated that $\boldsymbol{A}_{\mathrm{L}}$ is conserved in the two-body problem governed by Newton's gravity law of inverse squares (Hermann, 1710).

One way to explore this vector is to make use of (1.143), (1.146) defining the position $\boldsymbol{r}$ and velocity $\dot{\boldsymbol{r}}$ of the orbiting body in terms of the orthogonal unit vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ shown in Figures 1.3 and 1.4. Together, these equations yield the following expression for the angular-momentum vector defined in (1.108)

$$\boldsymbol{J} = \boldsymbol{k}\sqrt{G M p} \, , \tag{1.177}$$

where one has used (1.132) and employed the relationship $\boldsymbol{k} = \boldsymbol{P} \times \boldsymbol{Q}$. Insertion of (1.146) and (1.177) in the right side of (1.176) entails

$$\boldsymbol{A}_{\mathrm{L}} = G M e \boldsymbol{P} \, , \tag{1.178}$$

which tells us that the Laplace–Runge–Lenz vector is a constant vector directed towards the pericenter, the closest point of the orbit.

It would also be instructive to express the magnitude of the Laplace–Runge–Lenz vector through the other conserved quantities. Squaring both sides of (1.176) gives

$$A_{\mathrm{L}}^2 = G M^2 - \frac{2 G M}{r} \boldsymbol{r} \cdot (\dot{\boldsymbol{r}} \times \boldsymbol{J}) + J^2 \dot{r}^2 \, . \tag{1.179}$$

At the same time, permuting of the scalar triple product as

$$\boldsymbol{r} \cdot (\dot{\boldsymbol{r}} \times \boldsymbol{J}) = \boldsymbol{J} \cdot (\boldsymbol{r} \times \dot{\boldsymbol{r}}) = J^2 \tag{1.180}$$

enables us to rewrite the latter equation as

$$A_{\mathrm{L}}^2 = G M^2 + 2 J^2 \left(\frac{1}{2} \dot{r}^2 - \frac{G M}{r} \right) = G M^2 + 2 J^2 \mathcal{E} \, , \tag{1.181}$$

with $\mathcal{E}$ being the constant reduced energy per unit mass.

Conservation of both the absolute value and the direction of $\boldsymbol{A}_{\mathrm{L}}$ tells us that its three components, $(A_{\mathrm{L}x}, A_{\mathrm{L}y}, A_{\mathrm{L}z})$, are integrals of motion. However, only one of these three components can be regarded as an *independent* integral of motion. Indeed, as can be seen from (1.181), the magnitude of the Laplace–Runge–Lenz vector, $\boldsymbol{A}_{\mathrm{L}}^2 = A_{\mathrm{L}x}^2 + A_{\mathrm{L}y}^2 + A_{\mathrm{L}z}^2$, can be expressed through two known integrals – the energy, $\mathcal{E}$, and the magnitude of the angular-momentum vector, J. Another constraint follows from the fact that the vector $\boldsymbol{A}_{\mathrm{L}}$ belongs to the orbital plane and, thus, is always orthogonal to the vector of the angular momentum:

$$\boldsymbol{A}_{\mathrm{L}} \cdot \boldsymbol{J} = A_{\mathrm{L}x} J_x + A_{\mathrm{L}y} J_y + A_{\mathrm{L}z} J_z = 0 \, . \tag{1.182}$$

This explains why the conservation of the Laplace–Runge–Lenz vector increases the number of the *independent* integrals of motion in the reduced two-body problem not by three, but only by one – from four to five.[17] Together with the initial condition (expressed, for example, by fixing $\mathcal{M} = \mathcal{M}_0$ at the epoch t_0), this gives

17) Generally, a system that has k degrees of freedom and has, at the same time, more than k integrals of motion is called *super integrable*. However, a system with $2k - 1$ integrals is called *maximally super integrable*, the reduced two-body problem being the case with $k = 3$.

us six constants in total that should single out a particular trajectory of the body out of the entire multitude of parameterized conics. This observation agrees with the aforementioned fact that an arbitrary solution to the reduced two-body Kepler problem should depend upon six constants like the initial position and velocity of the body.

It can also be demonstrated that the existence of five independent integrals of motion makes it possible to integrate the equations of motion of the reduced two-body problem in quadratures. This fact turns out to be intimately related to the highly symmetrical nature of the reduced two-body problem. A central-force problem is trivially invariant under the spatial rotations making up the group SO(3), hence, the conservation of the angular momentum. However, the inverse-square force proportional to $1/r^2$, and the space harmonic oscillator with the force of elasticity proportional to r^2, possess a symmetry under a bigger group (Landau and Lifshitz, 1975). In the case of the inverse-square gravity law, this is: SO(4) for a negative energy $\mathcal{E} < 0$; or SO(1,3), for a positive energy $\mathcal{E} > 0$ (Dubrovin *et al.*, 1984, Section 34). Both SO(4) and SO(1,3) are rotational groups of symmetry in four-dimensional Euclidean and pseudo-Euclidean space respectively, and the Lie algebras of their generators have the dimension of six. This circumstance gives birth to six conserved quantities – the three components of $\boldsymbol{J}$ and the three components of $\boldsymbol{A}_{\mathrm{L}}$. Keep in mind though that these components and the energy are interconnected by the two constraints, (1.181) and (1.182). Further details on this interesting topic can be found in a series of excellent textbooks by Vozmischeva (2003) and Mathúna (2008).

1.3.5 Parameterizations of the Reduced Two-Body Problem

1.3.5.1 A Keplerian Orbit in the Euclidean Space

As shown in Section 1.2, the two-body problem is equivalent in the barycentric frame of reference to its reduced version (1.103) which, mathematically, looks as a motion of a particle of reduced mass μ about a fixed gravitating center of mass M. The generic solution to (1.103) is a Keplerian conic characterized by six constant parameters:

a – the semi-major axis,
e – the eccentricity,
$\mathcal{M}_0$ – the *mean anomaly at epoch*,
Ω – the angle of the ascending node,
i – the orbital inclination,
ω – the longitude of pericenter.

Three of these parameters, Ω, i, and ω, define the orientation of the orbit in space. Two parameters, a and e, fix the shape of the orbit. The remaining one, $\mathcal{M}_0$, determines the position of the body on the orbit at the initial epoch t_0. In an arbitrary inertial reference frame, there exist six additional constant parameters – the linear momentum (1.80) and the position of the center of mass (1.81). As a rule, a barycentric reference frame is chosen, so these integrals of motion get nullified and do not appear explicitly in any equation. Still, it is useful to keep in mind that these integrals actually exist.

Substituting (1.109)–(1.111), (1.141), (1.142) and (1.153) for expressions (1.143), (1.146) and applying simple trigonometric identities, the explicit form of the position and velocity of the body can be written down in the barycentric inertial coordinates as

$$\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z \ , \tag{1.183}$$

$$\dot{\boldsymbol{r}} = \dot{x}\boldsymbol{e}_x + \dot{y}\boldsymbol{e}_y + \dot{z}\boldsymbol{e}_z \ , \tag{1.184}$$

where

$$x = \frac{a(1-e^2)}{1+e\cos f}\left[\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos i\right] \ , \tag{1.185a}$$

$$y = \frac{a(1-e^2)}{1+e\cos f}\left[\sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos i\right] \ , \tag{1.185b}$$

$$z = \frac{a(1-e^2)}{1+e\cos f}\sin(\omega+f)\sin i \ , \tag{1.185c}$$

and

$$\dot{x} = -\frac{na}{\sqrt{1-e^2}}\big[\cos\Omega\sin(\omega+f) + \sin\Omega\cos(\omega+f)\cos i$$
$$+e(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i)\big] \ , \tag{1.186a}$$

$$\dot{y} = -\frac{na}{\sqrt{1-e^2}}\big[\sin\Omega\sin(\omega+f) - \cos\Omega\cos(\omega+f)\cos i$$
$$+e(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos i)\big] \ , \tag{1.186b}$$

$$\dot{z} = \frac{na}{\sqrt{1-e^2}}\left[\cos(\omega+f) + e\cos\omega\right]\sin i \ , \tag{1.186c}$$

with n being the mean motion (1.159).

Expressions 1.185–1.186 give us one possible form of the generic solution of equation of motion (1.103) – a form corresponding to parametrization of a conic by a set of six *Keplerian constants* $(a, e, \Omega, i, \omega, \mathcal{M}_0)$ and the variable true anomaly f. Since the true anomaly is a function of time, through the relation (1.160), then (1.185)–(1.186) define a dependence of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ upon the constants and the *time*, as required. This dependence, however, is *implicit*, and requires solution of the transcendental Kepler equation (1.168) along with a trigonometric equation (1.156).

The same solution can be parameterized via some other constants, for example, those of Delaunay: $\mathcal{M}_0, \omega, \Omega, \sqrt{GMa}, \sqrt{GMb}, \sqrt{GMb}\cos i$, where $b = a\sqrt{1-e^2}$ is the semi-minor axis of the conic. Another possibility is to consider the solution as a function of the initial conditions: then the constants

$(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$ are the six parameters defining a particular orbit. The latter option is natural when the integration is carried out numerically in the Cartesian coordinates, but is impractical for analytic treatments. Numerous other parameterizations have been introduced for various purposes. Whatever the set of the constants of integration chosen, their number should generally be six. A switch from the Keplerian constants to the Delaunay ones, or to any other parametrization, will still give the same geometric image of the curve in the coordinate (x, y, z) space. The velocity vector of the body being a tangent vector to that curve will not depend on the curve's parametrization either. However, the specific mathematical presentation of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ as functions of the new parameters will, of course, be different from those given by (1.185)–(1.186).

To avoid confusion, one would like to point out that the set of Delaunay constants differs from the set of Delaunay elements, as the latter set includes $\mathcal{M}$ instead of $\mathcal{M}_0$. In the same way, the set of Keplerian constants differs from the set of Keplerian elements: the former set contains $\mathcal{M}_0$, the latter $\mathcal{M}$. Employing the mean anomaly $\mathcal{M} = \mathcal{M}_0 + n(t - t_0)$ is convenient because this enables one to keep track of, via one variable, the explicit time dependence and the dependence upon the integration constant $\mathcal{M}_0$ – see the comprehensive treatise by Plummer (1918).

1.3.5.2 **A Keplerian Orbit in the Projective Space**

A generic solution to the reduced two-body problem is a section of cone, that is, a plane conic curve described by a quadratic polynomial. It would be instrumental to study a conic from the viewpoint of projective geometry, as this approach will allow us to treat any Keplerian orbit – circular, elliptical, parabolic, hyperbolic, and two-body collisional linear orbit – in a unified way. The approach adopted in this section has been proposed by Satō (1998).

A Keplerian orbit in the plane (ξ, η) is given by (1.143), (1.130)

$$\xi = \frac{q(1+e)\cos f}{1 + e\cos f}\,, \qquad \eta = \frac{q(1+e)\sin f}{1 + e\cos f}\,, \tag{1.187}$$

where $q \equiv r_{\min} = a(1-e)$ is the distance to the pericenter. Eliminating the true anomaly *f*, one obtains an ordinary quadratic form for the orbit

$$[(1-e)\xi + eq]^2 + \frac{1-e}{1+e}\eta^2 = q^2\,. \tag{1.188}$$

We now assume that coordinates ξ and η are dimensionless[18] and replace $\xi = \xi_1/\xi_0$, $\eta = \xi_2/\xi_0$ using homogeneous coordinates (ξ_0, ξ_1, ξ_2) that map (1.188) to the projective plane $\mathbb{P}^2$ (Casse, 2006)

$$-(1+e)q\xi_0^2 + 2e\xi_0\xi_1 + \frac{1-e^2}{(1+e)q}\xi_1^2 + \frac{1}{(1+e)q}\xi_2^2 = 0\,. \tag{1.189}$$

18) The dimensionless aspect of coordinates is an integral part of the definition of the projective plane $\mathbb{P}^2$ (Casse, 2006, Section 4).

Equation 1.189 can be written in a matrix form as

$$[\xi_0, \xi_1, \xi_2]\begin{bmatrix} -(1+e)q & e & 0 \\ e & (1+e)k & 0 \\ 0 & 0 & \dfrac{1}{(1+e)q} \end{bmatrix}\begin{bmatrix}\xi_0 \\ \xi_1 \\ \xi_2\end{bmatrix} = 0\,, \tag{1.190}$$

where $k \equiv 1/r_{\max} = (1-e)/[(1+e)q]$ is the reciprocal of the apocenter distance $r_{\max}$. We emphasize that there are only two free parameters (q, e) with k being expressed in terms of these two. Furthermore, parameters q and k are dimensionless in accordance with the definition of the homogeneous coordinates to which they are related through (1.190).

Equation 1.189 can be reduced to a canonical quadratic form by rotation in the projective space which diagonalizes the matrix in (1.190). The characteristic equation for eigenvalues λ of the matrix is

$$[(1+e)q\lambda - 1][\lambda^2 - (1+e)(k-q)\lambda - 1] = 0\,, \tag{1.191}$$

and it has three solutions

$$\lambda_0 = \frac{1}{2}\left[-(1+e)(q-k) - \sqrt{(1+e)^2(q+k)^2 + 4e^2}\right]\,, \tag{1.192}$$

$$\lambda_1 = \frac{1}{2}\left[-(1+e)(q-k) + \sqrt{(1+e)^2(q+k)^2 + 4e^2}\right]\,, \tag{1.193}$$

$$\lambda_2 = \frac{1}{(1+e)q}\,, \tag{1.194}$$

where one has used the identity $(1+e)^2 kq = 1 - e^2$ for transforming the root square terms in λ_0 and λ_1.

The next step is to replace parameters (q, e) with another set (ϕ, ψ) by making use of identifications

$$\cosh\phi\cos 2\psi = \frac{1}{2}(1+e)(q+k)\,, \tag{1.195a}$$

$$\cosh\phi\sin 2\psi = e\,, \tag{1.195b}$$

$$\sinh\phi = \frac{1}{2}(1+e)(q-k)\,, \tag{1.195c}$$

which makes the eigenvalues $\lambda_0 = -\exp\phi$, $\lambda_1 = \exp(-\phi)$. It is now rather straightforward to find the eigenvectors of the matrix in (1.190) that are

$$\boldsymbol{E}_0 = [\cos\psi, -\sin\psi, 0]\,, \quad \boldsymbol{E}_1 = [\sin\psi, \cos\psi, 0]\,, \quad \boldsymbol{E}_2 = [0, 0, 1]\,. \tag{1.196}$$

Canonical homogeneous coordinates of conic in the projective space $\mathbb{P}^2$ corresponding to these eigenvectors are

$$\{\zeta_0 : \zeta_1 : \zeta_2\} = \left\{\frac{1}{\sqrt{-\lambda_0}} : \frac{\cos\Theta}{\sqrt{\lambda_1}} : \frac{\sin\Theta}{\sqrt{\lambda_2}}\right\}\,, \tag{1.197}$$

where Θ is called the *projective anomaly* (Satō, 1998). Matrix of rotation from the canonical homogeneous coordinates $(\zeta_0, \zeta_1, \zeta_2)$ to the original ones is made of the components of the eigenvectors. The transformation has the following form of the rotation about the ξ_2-axis

$$\begin{aligned}\begin{bmatrix}\xi_0\\ \xi_1\\ \xi_2\end{bmatrix} &= \begin{bmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\zeta_0\\ \zeta_1\\ \zeta_2\end{bmatrix}\\ &= \begin{bmatrix}1+\exp\phi\tan\psi\cos\Theta\\ -\tan\psi+\exp\phi\cos\Theta\\ \sqrt{\exp 2\phi-\tan^2\psi}\sin\Theta\end{bmatrix}\exp\left(-\frac{\phi}{2}\right)\cos\psi\,. \qquad (1.198)\end{aligned}$$

Therefore, substituting

$$\alpha \equiv \exp\phi = \frac{1}{2}\left[(1+e)(q-k)+\sqrt{(1+e)^2(q+k)^2+4e^2}\right], \qquad (1.199)$$

$$\beta \equiv \tan\psi = \frac{2e}{(1+e)(q+k)+\sqrt{(1+e)^2(q+k)^2+4e^2}}, \qquad (1.200)$$

one obtains parametrization of the Keplerian conic in the projective space

$$\{\xi_0 : \xi_1 : \xi_2\} = \left\{1+\alpha\beta\cos\Theta : -\beta+\alpha\cos\Theta : \sqrt{\alpha^2-\beta^2}\sin\Theta\right\}, \qquad (1.201)$$

where α is the semi-major axis and β is a coordinate of the center of the conic in $\mathbb{P}^2$.

Semi-major axis a and eccentricity e of the Keplerian orbit in the Euclidean space are related with the parameters α and β of the projective space. The relationship is established after matching the Euclidean coordinates of pericenter and apocenter in (1.187) with similar points in the homogeneous coordinates

$$\xi = \frac{\xi_1}{\xi_0} = \frac{\alpha\cos\Theta-\beta}{1+\alpha\beta\cos\Theta} \qquad (1.202)$$

$$\eta = \frac{\xi_2}{\xi_0} = \frac{\sqrt{\alpha^2-\beta^2}\sin\Theta}{1+\alpha\beta\cos\Theta} \qquad (1.203)$$

$$r = \sqrt{\xi^2+\eta^2} = \frac{\alpha-\beta\cos\Theta}{1+\alpha\beta\cos\Theta}\,. \qquad (1.204)$$

It yields for the semi-major axis, a, and eccentricity, e, the following relationships

$$a = \frac{\alpha(1+\beta^2)}{1-\alpha^2\beta^2}, \quad e = \frac{\beta(1+\alpha^2)}{\alpha(1+\beta^2)}\,. \qquad (1.205)$$

Moreover, parameters

$$q = \frac{\alpha-\beta}{1+\alpha\beta}, \quad k = \frac{1-\alpha\beta}{\alpha+\beta}, \qquad (1.206)$$

and

$$1 - e = \frac{(\alpha - \beta)(1 - \alpha\beta)}{\alpha(1 + \beta^2)} \, , \quad 1 + e = \frac{(\alpha + \beta)(1 + \alpha\beta)}{\alpha(1 + \beta^2)} \, . \tag{1.207}$$

The orbit is circular for $\beta = 0$, elliptical for $\alpha\beta < 1$, parabolic for $\alpha\beta = 1$, and hyperbolic for $\alpha\beta > 1$. The orbit is degenerated to a straight line (collisional trajectory) if $\alpha = \beta$.

Kepler's equation for the projective anomaly Θ is obtained by using the law of conservation of angular momentum,

$$\xi\dot{\eta} - \dot{\xi}\eta = \sqrt{GMq(1 + e)} \, . \tag{1.208}$$

Substituting the formulas for the corresponding quantities into this law, one obtains

$$\frac{\alpha - \beta\cos\Theta}{(1 + \alpha\beta\cos\Theta)^2}\frac{d\Theta}{dt} = \sqrt{\frac{GM}{\alpha(1 + \beta^2)}} \, . \tag{1.209}$$

In the case of an elliptic orbit ($\alpha\beta < 1$), the integral with respect to Θ is reduced by the substitution

$$\tan\frac{\Theta}{2} = \sqrt{\frac{1 + \alpha\beta}{1 - \alpha\beta}} \tan\frac{E}{2} \, , \tag{1.210}$$

where E is the eccentric anomaly, into an ordinary form of Kepler's equation

$$E - e\sin E = n(t - t_0) + \mathcal{M}_0 \, , \tag{1.211}$$

with

$$n = \sqrt{\frac{GM}{\alpha^3}\left(\frac{1 - \alpha^2\beta^2}{1 + \beta^2}\right)^3} = \sqrt{\frac{GM}{a^3}} \, , \tag{1.212}$$

being the mean orbital motion defined earlier in (1.159). If $\alpha = \beta$ (a parabolic orbit), the integral is reduced by the substitution

$$s = \tan\frac{\Theta}{2} \, , \tag{1.213}$$

into a cubic equation

$$s^3 - 3\frac{1 - \alpha^2}{1 + \alpha^2}s = 6\sqrt{\frac{GM}{\alpha(1 + \alpha^2)^3}}(t - t_0) + \mathcal{M}_0 \, . \tag{1.214}$$

A hyperbolic case with $\alpha\beta > 1$ is treated similarly to the elliptic case after replacements: $1 - \alpha\beta \to \alpha\beta - 1$ and $\tan(\Theta/2) \to \tanh(\Theta/2)$.

1.3.6
The Freedom of Choice of the Anomaly

One has already encountered several options for defining the instantaneous position of a body on its orbit as a function of time. One option was to keep $\mathcal{M}_0$ among the integration constants, and to use the time t as a variable defining the position of the body at each moment. This method is seldom employed in practice because it is impossible to analytically express the components of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ as explicit functions of t. A better option would be to use the true anomaly f instead of the time. This method is more practical as it is directly implemented by the explicit expressions (1.185), (1.186). Similarly, one can keep $\mathcal{M}_0$ among the constants and use the eccentric anomaly E instead of the time. It is also possible to unite the integration constant $\mathcal{M}_0$ and the epoch t_0 into one constant parameter τ defined in (1.174) as the instant of the body's first passage through the pericenter. This is possible because in the unperturbed two-body problem, these two quantities show up in the linear combination, $\mathcal{M} = \mathcal{M}_0 + n(t - t_0)$, when the transition from the eccentric anomaly to the time is performed –see (1.166). Sometimes, it is also convenient to employ the angular anomaly subtended by the empty focus. A transition to this, the so-called *anti-focal anomaly*, was offered by Callandreau (1902a, 1902b), and has proven to be very useful for numerical integration of weakly-perturbed elliptic trajectories with low eccentricities (Fukushima, 2004).

Thus, one sees that the freedom of parametrization is not exhausted by one's preferences in choosing the constants of integration. Another freedom lies in one's choice of the "fast" variable – the anomaly. One can exploit this freedom in analytical calculations by parameterizing the relationship between the time and the anomaly and keeping it arbitrary through the calculation. The arbitrariness is eliminated by breaking the freedom at the end to simplify the resulting expressions or to serve another particular goal. The generalized anomaly W is *defined* as a solution to a simple differential equation

$$\frac{dE}{\sin E} = \frac{dW}{\sin W} , \tag{1.215}$$

that establishes the following trigonometric mapping between the eccentric anomaly E and the generalized anomaly W,

$$\tan\frac{W}{2} = \kappa \tan\frac{E}{2} , \tag{1.216}$$

with κ emerging as an integration constant and, thus, playing the role of a free constant parameter. It parameterizes a particular choice of the general anomaly W among a whole family of such anomalies. Equation 1.216 shows that the new anomaly W generalizes the relationship (1.156) between the true and eccentric anomalies, and relationship (1.210) does so between the projective and eccentric anomalies.

Making use of (1.216) in order to express the eccentric anomaly E in terms of the generalized anomaly W and the parameter κ, and substituting the so-obtained expression to (1.148), (1.150), and (1.152), one can derive the following formulae for the perifocal coordinates defined in Figure 1.4,

$$\xi = \frac{\kappa^2 - (1+e)/(1-e) + [\kappa^2 + (1+e)/(1-e)]\cos W}{\kappa^2 + 1 + (\kappa^2 - 1)\cos W} q \,, \tag{1.217}$$

$$\eta = \frac{2\kappa\sqrt{(1+e)/(1-e)}\sin W}{\kappa^2 + 1 + (\kappa^2 - 1)\cos W} q \,, \tag{1.218}$$

and for the distance from the gravitating center to the orbiter,

$$r = \frac{\kappa^2 + (1+e)/(1-e) + [\kappa^2 - (1+e)/(1-e)]\cos W}{\kappa^2 + 1 + (\kappa^2 - 1)\cos W} q \,, \tag{1.219}$$

where $q = a(1-e)$ is the distance from the pericenter to the focus wherein the attracting mass is located.

The generalized anomaly W was originally introduced into celestial mechanics by Subbotin (1936a, 1936b) who used a different parameter σ instead of κ.[19] Introducing an auxiliary quantity

$$\chi = \sqrt{1 - e^2 + \sigma^2} \,, \tag{1.220}$$

Subbotin (1936a, 1936b) was able to derive the following expressions for the perifocal coordinates:

$$\xi = a\left(\frac{\chi\cos W + \sigma}{\chi + \sigma\cos W} - e\right) \,, \tag{1.221}$$

$$\eta = a\frac{(1-e^2)\sin W}{\chi + \sigma\cos W} \,. \tag{1.222}$$

To establish a relationship between the parameter κ standing in (1.216)–(1.219) and the parameter σ introduced by Subbotin, equate the expression (1.217) with (1.221), and that (1.218) with (1.222) for the perifocal coordinates. This will render

$$\chi = \frac{\kappa^2 + 1}{2\kappa}\sqrt{1 - e^2} \,, \qquad \sigma = \frac{\kappa^2 - 1}{2\kappa}\sqrt{1 - e^2} \,. \tag{1.223}$$

From here, one sees that the parameter κ admits two values corresponding to one value of Subbotin's parameter σ,

$$\kappa_+ = \frac{\sigma + \chi}{\sqrt{1 - e^2}} \,, \tag{1.224}$$

$$\kappa_- = \frac{\sigma - \chi}{\sqrt{1 - e^2}} \,. \tag{1.225}$$

The value κ_+ corresponds to the generalized *focal* anomaly, W_+, and the value κ_- corresponds to the generalized *anti-focal anomaly*, W_-, that is,

$$\tan\frac{W_+}{2} = \kappa_+ \tan\frac{E}{2} \,, \qquad \tan\frac{W_-}{2} = \kappa_- \tan\frac{E}{2} \,. \tag{1.226}$$

19) In fact, Subbotin denoted his parameter α. Here, the notation σ is used to avoid confusion with the projective parameter α from the previous section.

One can see that

$$\tan \frac{W_-}{2} \tan \frac{W_+}{2} = -\tan^2 \frac{E}{2} \; . \tag{1.227}$$

Now, one can see that (1.221) and (1.222) are actually valid for the focal anomaly $W = W_+$. For the anti-focal anomaly W_-, the corresponding equations should read

$$\xi = a \left(\frac{\chi \sigma \cos W_-}{\chi - \sigma \cos W_-} - e \right) \; , \tag{1.228}$$

$$\eta = a \frac{(1 - e^2) \sin W_-}{\chi - \sigma \cos W_-} \; . \tag{1.229}$$

For more details on Subbotin's anomalies, see the paper by Sokolov (2009) who also corrected some misprints in Subbotin's works.

The advantage of parametrization (1.217)–(1.219) stems from the fact that it enables one to establish a correspondence between the generalized anomaly of Subbotin (1936a, 1936b) and the projective anomaly Θ of Satō (1998). Indeed, (1.217)–(1.219) tell us that the (focal) generalized anomaly W coincides with the eccentric anomaly E, for $\kappa = 1$ ($\sigma = 0$); with the true anomaly f, for $\kappa = \sqrt{(1+e)/(1-e)}$ ($\sigma = e$); and with the projective anomaly Θ, for $\kappa = \sqrt{(1+\alpha\beta)/(1-\alpha\beta)}$ corresponding to Subbotin's parameter $\sigma = \beta\sqrt{\alpha^2 - \beta^2}/(1+\beta^2)$.

The freedom of the anomaly choice in the Newtonian celestial mechanics remains greatly under-exploited, the work by Fukushima (2004) being a rare exception. We believe, though, that the future use of this freedom will yield fruits. We also think that the use of this freedom in the relativistic two-body problem may be equally productive, including this problem's practical application in data processing of binary-pulsar-timings. We shall return to this topic in Section 6.4.

1.4 A Perturbed Two-Body Problem

1.4.1 Prefatory Notes

Celestial mechanics of the two-body problem is insufficient in many practical applications where one has to take into account gravitational perturbations exerted on the Keplerian motion by external agents. Hence, one needs a mathematical extension of the unperturbed two-body formalism to perturbed settings. One such setting is that of a binary system embedded in the gravitational field of N external bodies which affect the orbital motion of the binary through tidal forces. Other types of perturbations include triaxiality of the interacting bodies as well as atmospheric drag, magnetic fields, tides, relativistic corrections, or forces of inertia if a noninertial reference frame is used for calculations. Since the disturbing forces are normally small compared to the gravitational interaction between the two bodies,

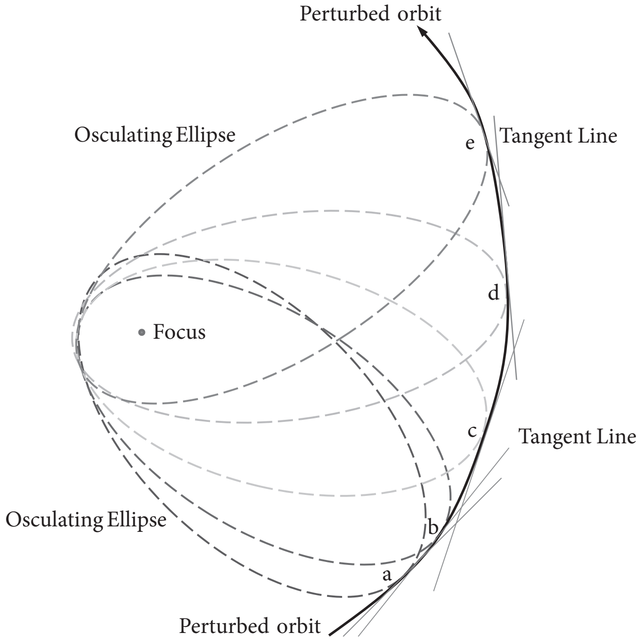

Figure 1.5 This picture illustrates the method of variation of parameters in application to the reduced two-body problem. A perturbed orbit can be presented as an envelope of a family of instantaneous conics sharing the common focus F. Each instantaneous conic is osculating – it touches the physical orbit, sharing with it the tangent line at the point of contact. The Keplerian parameters of the instantaneous conic evolve in time as the body moves through the positions a, b, c, d, e. Be mindful that the method of variation of parameters, in application to this problem, implies variation of the elements of the instantaneous conic, but not of the position of the focus. Therefore, the instantaneous conics are always confocal.

one may presume that each body moves along a conic that is *osculating* (tangent) at each instant of time to the actual physical trajectory and slowly evolving (see Figure 1.5). This approach was offered circa 1687 by Newton in his unpublished *Portsmouth Papers*. Very succinctly, in purely geometric terms, Newton also mentioned it in Corollaries 3 and 4 of Proposition 17 in the first book of his *Principia*. Implementation of this idea in the language of calculus was initiated by Euler (1748, 1753) and got its final shape in the works of Lagrange (1778, 1783, 1788a, 1788b, 1808a, 1808b, 1809).

Before explaining their developments, let us point out that the smallness of perturbations is, by itself, a rather shaky foundation for the varying-conic method. Indeed, one is immediately faced by the following questions:

1. To what degree of rigor can a perturbed orbit be modeled with a family of instantaneously osculating conics having the primary body in one of their foci?
2. Does this modeling admit an exact mathematical formulation?

3. Is this representation of the perturbed orbit by a family of the osculating conics unique?

These questions will not seem trivial if one recalls that the concept of evolving instantaneous conics had been introduced into practice (and that major developments of the disturbing-function theory had been accomplished) long before Frenet and Serret developed the theory of curves with their concept of the moving Frenet–Serret frame being closely associated with the *curvature* and *torsion* of the curve (Dubrovin *et al.*, 1984). This order of historical events explains the reason why the terms curvature and torsion of the orbit are rarely used in the dynamic astronomy books. Fortunately, Lagrange fortified his developments with the tools of calculus, which were powerful enough to completely surpass the theory of curves. Moreover, these tools in no way relied on the smallness of the disturbing forces. Hence, Lagrange's treatment of the problem already contains an affirmative answer to the first two questions. The answer to the third question, surprisingly, turns out to be negative. Below, this point is explained in more detail and one demonstrates that celestial mechanics permits internal freedom in description of perturbed orbits.

1.4.2
Variation of Constants – Osculating Conics

We shall start in the spirit of Lagrange (1808a, 1808b, 1809), but shall soon deviate on two points. First, in distinction from Lagrange, one will not assume that the disturbing force is conservative and depends upon the positions solely, but shall permit it to depend also upon velocities. Second, one's intention is to eventually relax the Lagrange constraint, that is, the assumption that the instantaneous conics should be tangent to the resulting perturbed curve. This will bring up orbital variables which will not be osculating, though mathematically useful.

In the modern, vectorial notations, Lagrange's line of reasoning looks as follows. A generic solution to the reduced two-body problem described by equation

$$\ddot{\boldsymbol{r}} + \frac{GM}{r^3}\boldsymbol{r} = 0\,, \tag{1.230}$$

is a Keplerian conic that is defined by the set of six orbital elements $\{C_i\} = C_1, \ldots, C_6$ implementing the chosen orbital parametrization. In some fixed inertial Cartesian coordinate system, this conic reads

$$\boldsymbol{r} = \boldsymbol{r}(C_1, \ldots, C_6, t)\,, \quad \dot{\boldsymbol{r}} = \boldsymbol{v}(C_1, \ldots, C_6, t)\,. \tag{1.231}$$

The expressions for orbital radius-vector, $\boldsymbol{r}$, and velocity, $\boldsymbol{v}$, were written down in the previous section. By definition, function $\boldsymbol{v}$ is the partial derivative of $\boldsymbol{r}$ with respect to time,

$$\boldsymbol{v} \equiv \left(\frac{\partial \boldsymbol{r}}{\partial t}\right)_{C_i=\text{const.}}\,. \tag{1.232}$$

Of course, since the orbital elements are constants of motion in the unperturbed two-body problem, the partial and ordinary time derivatives of vector $\boldsymbol{r}$ coincide.

The functions entering expression (1.231) can be used as an *ansatz* for solving the perturbed two-body problem

$$\ddot{\boldsymbol{r}} + \frac{GM}{r^3}\boldsymbol{r} = \boldsymbol{F} \ , \tag{1.233}$$

with vector $\boldsymbol{F}$ being a known disturbing force of whatever nature (including inertial forces). To solve (1.233), one assumes that the perturbed orbit coincides at each instant of time with an *instantaneous* Keplerian conic. This way, by going smoothly from one instantaneous conic to another, one endows the orbital parameters C_i with a time-dependence of their own,

$$\boldsymbol{r} = \boldsymbol{r}[C_1(t), \ldots, C_6(t), t] \ , \tag{1.234}$$

keeping the functional form of $\boldsymbol{r}$ the same as in (1.231). As the parameters C_i are now time-dependent, the velocity of the body,

$$\dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{r}}{\partial t} + \sum_{i=1}^{6} \frac{\partial \boldsymbol{r}}{\partial C_i}\frac{dC_i}{dt} = \boldsymbol{v} + \sum_{i=1}^{6} \frac{\partial \boldsymbol{r}}{\partial C_i}\frac{dC_i}{dt} \ , \tag{1.235}$$

acquires an additional input besides $\boldsymbol{v}$, while the term $\boldsymbol{v}$ retains the same functional form as it has in the unperturbed setting (1.231).

Substitution of expression (1.235) into the perturbed equation of motion (1.233) gives birth to three independent scalar differential equations of the second order. These three equations contain one independent variable – time, and six time-dependent parameters, $C_i(t)$, whose evolution is to be determined. Evidently, this cannot be done in a unique way because the number of the parameters exceeds, by three, the number of equations. This means that though the *perturbed* orbit given by the locus of points in space and by the values of velocity at each of these points is unique, its parametrization in terms of the orbital elements admits a certain freedom. The fact that the system of differential equations for the parameters $C_i(t)$ is underdetermined was noticed by Lagrange in his treatment. To make it solvable, he decided to amend it with three supplementary conditions imposed on functions C_i and their *first* time derivatives. His choice was

$$\sum_{i=1}^{6} \frac{\partial \boldsymbol{r}}{\partial C_i}\frac{dC_i}{dt} = 0 \ , \tag{1.236}$$

a so-called *Lagrange constraint* that is often imposed in the theory of ordinary differential equations. Imposition of this supplementary constraint was motivated by both physical considerations and by Lagrange's desire to simplify calculations. Since, physically, the perturbed orbit $\boldsymbol{r}$ with a time-dependent set of orbital elements $\{C_i(t)\}$ can, at each fixed time t, be interpreted as an instantaneous conic, Lagrange decided to make the $C_i(t)$ osculating, that is, to keep the instantaneous conics tangential to the perturbed trajectory, as displayed in Figure 1.5. This means that the physical trajectory of a body defined by $C_1(t), \ldots, C_6(t)$ must, at each instant of time, coincide *locally* with the unperturbed orbit that the moving body would follow if perturbations were to cease instantaneously. This can be achieved only when the dependence of the velocities upon the elements, in the perturbed setting, is the same as that in the original unperturbed case, $\dot{\boldsymbol{r}} = \boldsymbol{v}$. This, in turn, can be true only if the second term on the right side of (1.235) vanishes, that is, if one sets the extra condition (1.236). This vector condition, the *Lagrange constraint*, consists of three scalar equations which, together with the three equations of motion (1.233), constitute a well-defined system of six equations for six variables $C_1(t), \ldots, C_6(t)$.

As it was recently pointed out in Efroimsky (2002a, 2002b), the choice of the supplementary condition in the form of (1.236) is not always optimal. Moreover, as explained by Efroimsky and Goldreich (2003, 2004), in some important situations, this choice is simply unavailable. We shall address this topic below in Section 1.5. For now, though, one sticks to the supplementary condition in the form of Lagrange's constraint (1.236).

1.4.3
The Lagrange and Poisson Brackets

As a prerequisite to the subsequent calculations, it would be of use to introduce the so-called Lagrange and Poisson brackets of the orbital elements. The *Lagrange bracket* of two elements, C_k and C_i, is denoted by $[C_k\, C_i]$, and the entire set of the Lagrange brackets form a 6×6 matrix. Each element of the matrix is defined as a certain linear combination of scalar products of partial derivatives of the components of vectors $\boldsymbol{r}(C_1, \ldots, C_6, t)$ and $\boldsymbol{v}(C_1, \ldots, C_6, t)$ with respect to the orbital elements C_i. Namely,

$$[C_k\, C_i] \equiv \frac{\partial \boldsymbol{r}}{\partial C_k} \cdot \frac{\partial \boldsymbol{v}}{\partial C_i} - \frac{\partial \boldsymbol{r}}{\partial C_i} \cdot \frac{\partial \boldsymbol{v}}{\partial C_k} \,, \tag{1.237}$$

where the dot between two vectors denotes the Euclidean dot product.

The *Poisson bracket* of two elements, C_k and C_i, is denoted by $\{C_k\, C_i\}$. It is defined as a scalar product between two vectors that are partial derivatives of the orbital elements C_i with respect to coordinates and velocity of the body[20]

$$\{C_k\, C_i\} \equiv \frac{\partial C_k}{\partial \boldsymbol{r}} \cdot \frac{\partial C_i}{\partial \boldsymbol{v}} - \frac{\partial C_k}{\partial \boldsymbol{v}} \cdot \frac{\partial C_i}{\partial \boldsymbol{r}} \,. \tag{1.238}$$

The 6×6 matrix of the Poisson brackets is the *negative inverse* to the matrix of the Lagrange brackets,

$$\sum_{i=1}^{6} [C_j\, C_i]\{C_i\, C_k\} = -\delta_{jk} \,, \tag{1.239}$$

20) In differential geometry, the partial derivatives with respect to the coordinates and velocity are associated with covectors. This detail is ignored here because one works in a Euclidean space where vectors and covectors are formally equivalent to each other.

where $\delta_{jk} = \text{diag}(1,1,1,1,1,1)$ $(j,k = 1,2,\ldots,6)$ is the Kronecker symbol (the unit matrix) in six-dimensional Euclidean space of the orbital parameters, C_i. This relation can be easily derived if one expresses the orbital parameters in terms of radius-vector and velocity, $C_i = C_i(\boldsymbol{r},\boldsymbol{v})$, and applies the chain rule of differentiation,

$$\frac{\partial C_i}{\partial \boldsymbol{r}} \cdot \frac{\partial \boldsymbol{r}}{\partial C_j} + \frac{\partial C_i}{\partial \boldsymbol{v}} \cdot \frac{\partial \boldsymbol{v}}{\partial C_j} = \frac{\partial C_i}{\partial C_j} = \delta_{ij} \,, \tag{1.240}$$

in calculation of the product of two brackets in (1.239). Both the Lagrange and Poisson brackets are antisymmetric,

$$[C_k C_j] = -[C_j C_k] \,, \quad \{C_k C_j\} = -\{C_j C_k\} \,. \tag{1.241}$$

The antisymmetry of the brackets evidently yields vanishing diagonal elements:

$$[C_k C_k] = 0 \,, \quad \{C_k C_k\} = 0 \,. \tag{1.242}$$

A remarkable property of both the Lagrangian and Poisson brackets, which greatly facilitates their evaluation, is that they do not depend on time explicitly (Brouwer and Clemence, 1961; Schaub and Junkins, 2003), that is, their partial time derivatives vanish:

$$\frac{\partial}{\partial t}[C_k C_i] = 0 \,, \quad \frac{\partial}{\partial t}\{C_k C_i\} = 0 \,. \tag{1.243}$$

To verify this, let us take the partial time derivative of the Lagrange brackets (1.237). One obtains

$$\frac{\partial}{\partial t}[C_k C_i] = \frac{\partial \boldsymbol{r}}{\partial C_k} \cdot \frac{\partial \boldsymbol{a}}{\partial C_i} - \frac{\partial \boldsymbol{r}}{\partial C_i} \cdot \frac{\partial \boldsymbol{a}}{\partial C_k} \,, \tag{1.244}$$

where $\boldsymbol{a} = \partial \boldsymbol{v}/\partial t$ is the orbital acceleration on an instantaneous conic. For Keplerian conics,

$$\boldsymbol{a} = -\frac{GM}{r^3}\boldsymbol{r} = \frac{\partial}{\partial \boldsymbol{r}}\left(\frac{GM}{r}\right) \,, \tag{1.245}$$

with $r = |\boldsymbol{r}|$. Hence,

$$\begin{aligned} \frac{\partial}{\partial t}[C_k C_i] &= \frac{\partial \boldsymbol{r}}{\partial C_k} \cdot \frac{\partial}{\partial \boldsymbol{r}} \frac{\partial}{\partial C_i}\left(\frac{GM}{r}\right) - \frac{\partial \boldsymbol{r}}{\partial C_i} \cdot \frac{\partial}{\partial \boldsymbol{r}} \frac{\partial}{\partial C_k}\left(\frac{GM}{r}\right) \\ &= \frac{\partial^2}{\partial C_k \partial C_i}\left(\frac{GM}{r}\right) - \frac{\partial^2}{\partial C_i \partial C_k}\left(\frac{GM}{r}\right) = 0 \end{aligned} \tag{1.246}$$

since the second partial derivatives commute. It proves that the Lagrange brackets bear no explicit dependence on the time variable. The proof that the Poisson brackets do not depend explicitly on time follows immediately after taking partial time derivative from both sides of (1.239).

Table 1.1 The Lagrange brackets for the Keplerian osculating elements $(C_1, C_2, C_3, C_4, C_5, C_6) = (a, e, i, \Omega, \omega, \mathcal{M}_0)$. The notation $b = a\sqrt{1-e^2}$ stands for the semi-minor axis.

$[ea] = 0,$	$[\mathcal{M}_0 \Omega] = 0,$	$[\mathcal{M}_0 \omega] = 0,$
$[ia] = 0,$	$[ie] = 0,$	$[\omega \Omega] = 0,$
$[\Omega a] = \frac{nb}{2} \cos i,$	$[\Omega e] = -\frac{na^3 e}{b} \cos i,$	$[\Omega i] = -nab \sin i,$
$[\omega a] = \frac{nb}{2},$	$[\omega e] = -\frac{na^3 e}{b},$	$[\omega i] = 0,$
$[\mathcal{M}_0 a] = \frac{na}{2},$	$[\mathcal{M}_0 e] = 0,$	$[\mathcal{M}_0 i] = 0.$

Table 1.2 The Poisson brackets for the Keplerian osculating elements $(C_1, C_2, C_3, C_4, C_5, C_6) = (a, e, i, \Omega, \omega, \mathcal{M}_0)$. The notation $b = a\sqrt{1-e^2}$ stands for the semi-minor axis.

$\{ea\} = 0,$	$\{\mathcal{M}_0 \Omega\} = 0,$	$\{\mathcal{M}_0 \omega\} = 0,$
$\{ia\} = 0,$	$\{ie\} = 0,$	$\{\omega \Omega\} = 0,$
$\{\Omega a\} = 0,$	$\{\Omega e\} = 0,$	$\{\Omega i\} = -\frac{1}{nab \sin i},$
$\{\omega a\} = 0,$	$\{\omega e\} = -\frac{b}{na^3 e},$	$\{\omega i\} = \frac{\cos i}{nab \sin i},$
$\{\mathcal{M}_0 a\} = \frac{2}{na},$	$\{\mathcal{M}_0 e\} = \frac{b^2}{na^4 e},$	$\{\mathcal{M}_0 i\} = 0.$

Because of this remarkable property, it does not matter at which point of the instantaneous orbit one evaluates the brackets. Thus, one can chose the most convenient point of the orbit in order to reduce the amount of algebra involved. After the Lagrange brackets are found, the elements of the Poisson brackets (1.238) can be obtained by matrix inversion from (1.239). The results are presented in Table 1.1 and Table 1.2.

1.4.4
Equations of Perturbed Motion for Osculating Elements

One is now prepared to derive the equations describing evolution of the osculating elements C_i with the Lagrange constrain (1.236) imposed. As the second term on the right side of (1.235) vanishes, one can write the acceleration as

$$\frac{d^2 \boldsymbol{r}}{dt^2} = \frac{\partial \boldsymbol{v}}{\partial t} + \sum_{i=1}^{6} \frac{\partial \boldsymbol{v}}{\partial C_i} \frac{dC_i}{dt} = \frac{\partial^2 \boldsymbol{r}}{\partial t^2} + \sum_{i=1}^{6} \frac{\partial \boldsymbol{v}}{\partial C_i} \frac{dC_i}{dt} , \tag{1.247}$$

insertion whereof in the perturbed equation of motion (1.233) entails:

$$\frac{\partial^2 \boldsymbol{r}}{\partial t^2} + \frac{GM}{r^3} \boldsymbol{r} + \sum_{i=1}^{6} \frac{\partial \boldsymbol{v}}{\partial C_i} \frac{dC_i}{dt} = \boldsymbol{F} , \tag{1.248}$$

where $r \equiv |\boldsymbol{r}|$. The function $\boldsymbol{r}$ is, by definition, a Keplerian solution to the unperturbed two-body problem, with constant orbital elements. So this function obeys the unperturbed equation (1.230), which means that the sum of the first two terms on the left side of (1.248) must be equated to zero. This simplifies equation (1.248) to

$$\sum_{i=1}^{6} \frac{\partial \boldsymbol{v}}{\partial C_i} \frac{dC_i}{dt} = \boldsymbol{F} . \tag{1.249}$$

This is the equation of disturbed motion, written in terms of the osculating orbital elements $C_i = C_i(t)$. Together with Lagrange's constraint (1.236), they constitute a well-defined system of six equations that can be solved with respect to C_i for all $i = 1, 2, \ldots, 6$. However, the mathematical form of (1.249) is not optimal for finding its solution because the time derivatives dC_i/dt in the left side of this equation are algebraically coupled with vectors $\partial \boldsymbol{v}/\partial C_i$. Therefore, the next step is to decouple derivatives dC_i/dt from $\partial \boldsymbol{v}/\partial C_i$. It can be achieved with the formalism of the Lagrange brackets.

Let us make a Euclidean dot-product of both sides of (1.249) with vector, $\partial \boldsymbol{r}/\partial C_k$. It yields

$$\sum_{i=1}^{6} \left(\frac{\partial \boldsymbol{r}}{\partial C_k} \cdot \frac{\partial \boldsymbol{v}}{\partial C_i} \right) \frac{dC_i}{dt} = \frac{\partial \boldsymbol{r}}{\partial C_k} \cdot \boldsymbol{F} , \tag{1.250}$$

where $k = 1, 2, \ldots, 6$, while the dot between the two vectors denotes their Euclidean dot product. Making a dot-product of the Lagrange constraint (1.236) with vector $\partial \boldsymbol{v}/\partial C_k$ brings about

$$\sum_{i=1}^{6} \left(\frac{\partial \boldsymbol{v}}{\partial C_k} \cdot \frac{\partial \boldsymbol{r}}{\partial C_i} \right) \frac{dC_i}{dt} = 0 . \tag{1.251}$$

Subtraction of (1.251) from (1.250) results in

$$\sum_{i=1}^{6} [C_i C_k] \frac{dC_k}{dt} = \frac{\partial \boldsymbol{r}}{\partial C_i} \cdot \boldsymbol{F} . \tag{1.252}$$

With aid of (1.239), the derivative dC_i/dt can be decoupled so that one obtains

$$\frac{dC_i}{dt} = -\sum_{j=1}^{6} \{C_i C_j\} \frac{\partial \boldsymbol{r}}{\partial C_j} \cdot \boldsymbol{F} , \tag{1.253}$$

which is a system of six ordinary differential equations of the first order for the osculating elements of the perturbed orbit. The equations are valid in arbitrary Cartesian coordinates and for arbitrary parametrization of the Keplerian conic. For this reason, (1.253) have a wide range of applications in celestial mechanics. The method of variation of parameters is also used in the theory of ordinary differential equations for finding their general solutions.

1.4.5
Equations for Osculating Elements in the Euler–Gauss Form

Though (1.253) are invariant with respect to the change of coordinates and/or the orbital parametrization of the unperturbed orbit of a two-body system, choosing an appropriate parametrization can simplify their right side, thus, facilitating a solution. One of the most convenient parameterizations is given in terms of the Keplerian elements $(a, e, i, \Omega, \omega, \mathcal{M}_0)$ by formulae (1.185)–(1.186). From these formulae, one can calculate the matrix of the Lagrange brackets in a fairly straightforward way. The calculation can be greatly simplified if one recalls that the Lagrange brackets do not explicitly depend on time and, therefore, also upon the true anomaly, f, that may be set to nil, $f = 0$. The outcome of the calculation is displayed in Table 1.1 where one has shown only 15 elements of the matrix since the anti-symmetry of the Lagrange brackets immediately gives the other 15 off-diagonal elements. The remaining six diagonal elements are, of course, identically zero. As many elements of the matrix of the Lagrange brackets are zero, it is relatively easy to invert the matrix in order to calculate the Poisson brackets. The result is presented in Table 1.2. Calculating the partial derivatives $\partial \boldsymbol{r}/\partial a$, $\partial \boldsymbol{r}/\partial e$, and so on, from (1.185) and (1.186), forming their scalar dot-products with the perturbing force $\boldsymbol{F}$, and making transvection of these products with the Poisson brackets from Table 1.2 yield a system of ordinary differential equations for the Keplerian elements[21] in the *Euler–Gauss form*:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left(e F_{\mathrm{R}} \sin f + F_{\mathrm{T}} \frac{p}{r}\right) , \tag{1.254a}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[F_{\mathrm{R}} \sin f + F_{\mathrm{T}}(\cos f + \cos E)\right] , \tag{1.254b}$$

$$\frac{di}{dt} = \frac{r\cos(f+\omega)}{na^2\sqrt{1-e^2}} F_{\mathrm{N}} , \tag{1.254c}$$

$$\frac{d\Omega}{dt} = \frac{r\sin(f+\omega)}{na^2\sqrt{1-e^2}\sin i} F_{\mathrm{N}} , \tag{1.254d}$$

$$\frac{d\omega}{dt} = -\cos i \frac{d\Omega}{dt} + \frac{\sqrt{1-e^2}}{nae}\left[-F_{\mathrm{R}} \cos f + F_{\mathrm{T}}\left(1+\frac{r}{p}\right)\sin f\right] , \tag{1.254e}$$

$$\frac{d\mathcal{M}_0}{dt} = -\sqrt{1-e^2}\left(\frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt}\right) - \frac{2r}{na^2} F_{\mathrm{R}} , \tag{1.254f}$$

where the radial distance r has to be expressed in terms of the orbital elements of the two-body problem,

$$r = \frac{a(1-e^2)}{1+e\cos f} , \tag{1.255}$$

21) These equations are called the *planetary equations* in celestial mechanics.

while F_R, F_T, and F_N are the radial, transversal, and normal to the orbit components of the perturbing force $\boldsymbol{F}$,

$$F_\mathrm{R} = \boldsymbol{n} \cdot \boldsymbol{F} \,, \quad F_\mathrm{T} = (\boldsymbol{k} \times \boldsymbol{n}) \cdot \boldsymbol{F} \,, \quad F_\mathrm{N} = \boldsymbol{k} \cdot \boldsymbol{F} \,, \tag{1.256}$$

while the unit vector $\boldsymbol{n} = \boldsymbol{r}/r$ and the unit vector $\boldsymbol{k}$ is set to be orthogonal to the instantaneous orbital plane. The mean anomaly of the perturbed motion obeys the following equation,

$$\mathcal{M} = \mathcal{M}_0 + \int_{t_0}^{t} n(t')dt' \,, \tag{1.257}$$

where $\mathcal{M}_0$ is the solution of (1.254f), and $n(t)$ is the mean orbital frequency given by (1.159) where the semi-major axis $a = a(t)$ is the solution of the corresponding (1.254a).

Equation 1.254a–1.254f can be also independently derived by differentiating the two vectorial integrals of motion, the angular-momentum vector $\boldsymbol{J}$ and the Laplace–Runge–Lenz vector $\boldsymbol{A}_\mathrm{L}$, which are not conserved in the perturbed motion. Differentiating their vectorial definitions, (1.108) and (1.176), and making use of the equations of perturbed motion (1.233), result in:

$$\frac{d\boldsymbol{J}}{dt} = \boldsymbol{r} \times \boldsymbol{F} \,, \tag{1.258}$$

$$\frac{d\boldsymbol{A}_\mathrm{L}}{dt} = 2(\dot{\boldsymbol{r}} \cdot \boldsymbol{F})\boldsymbol{r} - (\boldsymbol{r} \cdot \boldsymbol{F})\dot{\boldsymbol{r}} - (\boldsymbol{r} \cdot \dot{\boldsymbol{r}})\boldsymbol{F} \,. \tag{1.259}$$

As the Lagrange's principle of the variation of orbital elements demands that all relationships of the unperturbed Keplerian orbit remain valid in the perturbed motion, one can also use the expressions (1.177) and (1.178) of these vector integrals for calculating the time derivatives. Differentiating (1.177) and (1.178) and keeping in mind that the unit vectors $\boldsymbol{k}$ and $\boldsymbol{P}$ are also functions of time via (1.111) and (1.141), leads to the following results,

$$\frac{d\boldsymbol{J}}{dt} = \frac{na}{2\sqrt{1-e^2}}\frac{dp}{dt}\boldsymbol{k} + na^2\sqrt{1-e^2}\left(\boldsymbol{l}\sin i\frac{d\Omega}{dt} - \boldsymbol{m}\frac{di}{dt}\right) \,, \tag{1.260}$$

$$\frac{d\boldsymbol{A}_\mathrm{L}}{dt} = GMe\left[\left(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt}\right)\boldsymbol{Q} + \left(\sin\omega\frac{di}{dt} - \cos\omega\sin i\frac{d\Omega}{dt}\right)\boldsymbol{k}\right] + GM\frac{de}{dt}\boldsymbol{P} \,, \tag{1.261}$$

where the unit vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ have been defined in (1.141), (1.142) and are also shown in Figure 1.3. After decomposing the perturbing force in the three components (1.256), and equating right sides of the corresponding equations, (1.258) and (1.260) as well as (1.259) and (1.261), it can then be demonstrated that the Euler–Gauss equations (1.254) are again obtained.

As a historical aside, one would mention that in his work on the lunar motion, Euler (1753) derived the planetary equations for the longitude of the node, Ω, the inclination, i, and the semilatus rectum, p, with the time derivatives of these three elements expressed through the three components of the disturbing force $\boldsymbol{F}$. Sixty years later, the method was amended by Gauss, who wrote down similar equations for the other three Keplerian elements, thus obtaining a full system of the planetary equations. Although many books refer to the system of equations (1.254) as the *Gauss* equations, it seems fair to pay the tribute evenly, calling them the *Euler–Gauss* equations.

1.4.6
The Planetary Equations in the Form of Lagrange

So far, one did not impose any limitation on the functional form of the perturbing force. Now, let us assume that the force is conservative and depends only on coordinates of perturbing bodies. This assumption was made by Lagrange (1778, 1783, 1788a, 1788b, 1808a, 1808b, 1809) in his treatment of planetary motions in the solar system. Being dependent solely on the positions of the moving bodies, this force can be expressed as a gradient of the *disturbing function*,

$$\boldsymbol{F} = \frac{\partial R}{\partial \boldsymbol{r}} = -\frac{\partial \mathfrak{W}}{\partial \boldsymbol{r}} , \tag{1.262}$$

with the *disturbing function*, R, being negative to the disturbing potential $\mathfrak{W}$ introduced above in (1.96). The chain rule for partial derivatives provides us with

$$\frac{\partial \boldsymbol{r}}{\partial C_i} \cdot \boldsymbol{F} = \frac{\partial \boldsymbol{r}}{\partial C_i} \cdot \frac{\partial R}{\partial \boldsymbol{r}} = \frac{\partial R}{\partial C_i} . \tag{1.263}$$

Hence, substituting the force (1.262) into (1.253) for osculating elements results in

$$\frac{d C_i}{dt} = -\sum_{k=1}^{6} \{C_i C_k\} \frac{\partial R}{\partial C_k} . \tag{1.264}$$

Finally, insertion of the Poisson brackets from Table 1.2 to (1.264) takes us to the celebrated system of planetary equations in the form of Lagrange:

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial \mathcal{M}_0} , \tag{1.265a}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial \mathcal{M}_0} - \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial \omega} , \tag{1.265b}$$

$$\frac{di}{dt} = \frac{\cos i}{na^2\sqrt{1-e^2}\sin i} \frac{\partial R}{\partial \omega} - \frac{1}{na^2\sqrt{1-e^2}\sin i} \frac{\partial R}{\partial \Omega} , \tag{1.265c}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i} \frac{\partial R}{\partial i} , \tag{1.265d}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} , \tag{1.265e}$$

$$\frac{d\mathcal{M}_0}{dt} = -\frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} . \tag{1.265f}$$

The advantage of the Lagrange planetary equations is that the right sides of (1.265) depend on a single function R, in contrast to the Euler–Gauss equations (1.254) whose right sides depend on three components of the disturbing force $\boldsymbol{F}$.

Disadvantages of employing the Lagrange (and the Euler–Gauss) equations are that the right ascension of the ascending node becomes indeterminate as the inclination tends to zero, and the argument of perigee becomes indeterminate as the eccentricity tends to zero. The difficulty is, of course, of a purely mathematical nature, and has nothing to do with the actual motion. It can be sidestepped by switching from the Keplerian elements to the, so-called, equinoctial orbital elements $(\mathfrak{p}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h}, \mathfrak{k}, \mathfrak{L})$, that are related to the Keplerian orbital parameters as follows (Broucke and Cefola, 1972; Walker *et al.*, 1985);

$$\mathfrak{p} = a(1-e^2) ,$$
$$\mathfrak{f} = e\cos(\Omega + \omega) ,$$
$$\mathfrak{g} = e\sin(\Omega + \omega)$$
$$\mathfrak{h} = \tan\frac{i}{2}\cos\Omega ,$$
$$\mathfrak{k} = \tan\frac{i}{2}\sin\Omega ,$$
$$\mathfrak{L} = \Omega + \omega + f .$$

The equinoctial orbital elements are useful for trajectory analysis and optimization of space flights. They are valid for circular, elliptic, and hyperbolic orbits. The Lagrange equations for equinoctial elements exhibit no singularity for zero eccentricity and orbital inclinations equal to 0 and 90°. However, two of the components are singular for an orbital inclination of 180°, though this can be handled by an appropriate re-definition.

1.4.7
The Planetary Equations in the Form of Delaunay

Another advantageous set of orbital elements is the Delaunay variables l, g, h, L, G, H. In terms of the Keplerian orbital elements, they are defined as

$l \equiv \mathcal{M}_0$	the mean anomaly at epoch,
$g \equiv \omega$	the argument of the pericenter,
$h \equiv \Omega$	the longitude of the ascending node,
$L \equiv \sqrt{GMa}$	
$G \equiv L\sqrt{1-e^2}$	the magnitude of the angular-momentum vector $\boldsymbol{J}$,
$H \equiv G\cos i$	the angular-momentum component normal to equatorial plane,

By using the chain rule of differentiation, one can easily rewrite the planetary equations (1.265) in terms of these variables:

$$\frac{dL}{dt} = \frac{\partial R}{\partial l} , \quad \frac{dl}{dt} = -\frac{\partial R}{\partial L} \tag{1.266a}$$

$$\frac{dG}{dt} = \frac{\partial R}{\partial g} , \quad \frac{dg}{dt} = -\frac{\partial R}{\partial G} , \tag{1.266b}$$

$$\frac{dH}{dt} = \frac{\partial R}{\partial h} , \quad \frac{dh}{dt} = -\frac{\partial R}{\partial H} , \tag{1.266c}$$

where one deliberately wrote the equations in pairs to emphasize their symplectic structure with the Hamiltonian being equal to the disturbing function R. In both sets of equations, (1.265) and (1.266), the element, $\mathcal{M}_0 = l$, can be substituted by the mean anomaly, $\mathcal{M}$, provided R is simultaneously substituted with $R' = R + GM/2a$. The advantage of the Delaunay equations is that they can be analyzed with the powerful mathematical technique of a symplectic geometry (Arnold, 1995). Just as in the Euler–Gauss and Lagrange planetary equations, the equations in the form of Delaunay become singular both in the limits of circular and/or equatorial orbits. Accordingly, a transition to the so-called canonical elements of Poincar'e can be an option (Brouwer and Clemence, 1961, p. 540).

1.4.8 **Marking a Minefield**

The most logical way of introducing the Delaunay variables would be to start out with the polar coordinates and their conjugate momenta, and to carry out the Hamilton–Jacobi procedure to find a canonical transformation to variables, which will remain mutually conjugated with respect to a vanishing Hamiltonian, that is, they will be canonical integrals of motion (Arnold, 1995; Landau and Lifshitz, 1975). These canonical variables are found through solution of the Hamilton–Jacobi equation, as demonstrated in numerous books – see, for example,[22] Plummer (1918, Sections 135–136), Kovalevsky (1967, Sections 25–32), or Vinti (1998, Chapters 6 and 9). Within this approach, the Lagrange-type planetary equations are derived from those of Delaunay.

Unfortunately, neither of these books address the following important question: will the Hamilton–Jacobi procedure always result in *osculating* canonical elements? As one will see in the subsequent sections, the answer to this question is affirmative if the disturbance depends solely on positions of bodies, and is negative for velocity-dependent perturbations. This happens because the Hamilton–Jacobi procedure implies that the resulting Delaunay elements are canonical, while the condition of canonicity becomes incompatible with the condition of osculation when the disturbance depends not only on the coordinates, but also upon velocities.

22) Plummer used notations β and β_2 for the negative Delaunay elements, $-l$ and $-g$, correspondingly.

As a result, the customary Delaunay and Lagrange planetary equations, when employed for such velocity-dependent perturbations, furnish orbital elements which are *not* osculating. In other words, the instantaneous conics parameterized by the so-obtained elements will not be tangent to the orbit. This means that these elements will render the correct position of the body, $\boldsymbol{r}$, but the partial derivative of $\boldsymbol{r}$ with respect to time will not provide its instantaneous velocity as in the case of the osculating elements. While the use of such nonosculating elements is sometimes beneficial mathematically, their physical interpretation is not always apparent. Interestingly, the Andoyer elements which are the analogues of the Delaunay elements employed in the canonical description of a rigid-body rotation, are subject to the same reservation.

1.5
Re-examining the Obvious

Don't ever take a fence down until you know the reason it was put up.

G. K. Chesterton

1.5.1
Why Did Lagrange Impose His Constraint? Can It Be Relaxed?

When deriving the planetary equations, Lagrange amended the equations of motion (1.249) with constraint (1.236) in order to make the overall system of the equations well-defined mathematically. In the case of the conservative perturbing force, which are represented as a gradient of the perturbing potential, it provided a maximal simplification of the resulting (1.264) for osculating elements. Besides, the physical interpretation of the elements, C_i, obeying constraint (1.236) was geometrically straightforward. Indeed, by assuming that at each instant of time the perturbed velocity $\dot{\boldsymbol{r}}$ is equal to the unperturbed Keplerian velocity $\boldsymbol{v}$, Lagrange set the instantaneous conics tangent to the perturbed orbital curve and, thus, made the appropriate orbital elements osculating – see Figure 1.5.

It would then be natural to inquire if the Lagrange constraint should *always* be imposed on the orbital elements. Specifically: are there situations in which this constraint is not the best choice or is incompatible with a particular form of the equations for orbital elements? Indeed, a careful examination by Efroimsky and Goldreich (2003, 2004) reveals that the attempt of modeling of an orbit by tangential (osculating) confocal conics may be in conflict with the canonical equations. More specifically, if the perturbing function $R = R(\boldsymbol{r}, \boldsymbol{v})$, depends not only on positions of the bodies but also on their velocities[23] the demand of osculation comes into a contradiction with one's desire to keep the Delaunay variables canonical. Analytical solution of the Delaunay equations will then furnish an answer that will

23) Examples of such forces are dissipative forces, the Coriolis forces in a precessing frame, or relativistic perturbations (see Section 6.4).

be mathematically consistent, but it will come out in terms of *nonosculating* orbital elements violating the Lagrange constraint (1.236). The Delaunay elements in this case will render a correct instantaneous position of the perturbed body (satellite, planet), but a wrong value of its instantaneous velocity. In celestial calculations, this "booby trap" is often encountered, though rarely noticed (Brumberg, 1972, 1991).

To explore the mathematical consequences of choosing a particular constraint imposed on the perturbed orbital parameters, one will deliberately permit a certain degree of nonosculation in the differential equations describing their evolution. This gives us some so-called *gauge freedom* in choosing the constraint so that by imposing a constraint different from that of the Lagrange, one can sometimes greatly simplify the resulting equations for the orbital, but no longer osculating, elements. The situation resembles that of one emerging in the Maxwell electrodynamics where a suitable choice of the gauge imposed on electromagnetic vector potential can considerably simplify calculations without changing the physical observables. Planetary equations, obeying a constraint more general than that of Lagrange, were derived in Efroimsky (2002a, 2002b). Before addressing that formalism, one would show an elementary example of the gauge freedom of differential equations due to Newman and Efroimsky (2003) in order to illustrate the idea underlying the method.

1.5.2
Example – the Gauge Freedom of a Harmonic Oscillator

A one-dimensional harmonic oscillator with coordinate, $x = x(t)$, and disturbed by a force $F(t)$ obeys the second-order differential equation,

$$\ddot{x} + \omega_0^2 x = F(t) \; , \tag{1.267}$$

where overdot denotes a time derivative and ω_0 is the oscillation frequency of unperturbed motion. We also impose some initial conditions, $x(0)$, and, $\dot{x}(0)$, at the initial instant of time $t_0 = 0$.

The method of variation of parameters suggests that the solution of (1.267) be sought for using a solution of a homogeneous equation with the integration constants replaced with yet unknown functions of time,

$$x = C_1(t) \sin \omega_0 t + C_2(t) \cos \omega_0 t \; , \tag{1.268}$$

where $C_1(t)$ and $C_2(t)$ are to be determined. Differentiation of $x(t)$ will lead us to

$$\dot{x} = \dot{C}_1(t) \sin \omega_0 t + \dot{C}_2(t) \cos \omega_0 t + \omega_0 [C_1(t) \cos \omega_0 t - C_2(t) \sin \omega_0 t] \; . \tag{1.269}$$

It is common, at this point, to set the sum, $\dot{C}_1(t) \sin \omega_0 t + \dot{C}_2(t) \cos \omega_0 t$, equal to zero in order to remove the indeterminacy which stems from having only one equation for the two variables, $C_1(t)$ and $C_2(t)$. This is equivalent to imposing the

Lagrange constraint which is convenient though not obligatory. Let us see what will happen if one does not impose this particular constraint by assuming that

$$\dot{C}_1(t)\sin\omega_0 t + \dot{C}_2(t)\cos\omega_0 t = \Phi(t)\,, \tag{1.270}$$

with $\Phi(t)$ being an arbitrary smooth function of time. We call (1.270) the *gauge condition* and $\Phi(t)$ the *gauge function* because picking up various $\Phi(t)$ leads to different solutions for $C_1(t)$ and $C_2(t)$ without changing the solution of the original (1.267) for function $x(t)$, as demonstrated below.

Substituting the gauge condition (1.270) in (1.269) and differentiating one more time results in

$$\begin{aligned}\ddot{x} = \dot{\Phi} &+ \omega_0[\dot{C}_1(t)\cos\omega_0 t - \dot{C}_2(t)\sin\omega_0 t] \\ &- \omega_0^2[C_1(t)\sin\omega_0 t + C_2(t)\cos\omega_0 t]\,.\end{aligned} \tag{1.271}$$

Replacing this result along with (1.268) in the original equation of motion (1.267) yields the dynamical equation rewritten in terms of the new variables, C_1 and C_2. Together with the *gauge condition* (1.270), it constitutes the following system of two differential equations,

$$\dot{\Phi} + \omega_0\left[\dot{C}_1(t)\cos\omega_0 t - \dot{C}_2(t)\sin\omega_0 t\right] = F(t)\,, \tag{1.272a}$$

$$\dot{C}_1(t)\sin\omega_0 t + \dot{C}_2(t)\cos\omega_0 t = \Phi(t)\,. \tag{1.272b}$$

This system can be algebraically solved with respect to time derivatives of functions C_1 and C_2:

$$\frac{dC_1}{dt} = \omega_0^{-1}\left[F\cos\omega_0 t - \frac{d}{dt}(\Phi\cos\omega_0 t)\right]\,, \tag{1.273a}$$

$$\frac{dC_2}{dt} = \omega_0^{-1}\left[-F\sin\omega_0 t + \frac{d}{dt}(\Phi\sin\omega_0 t)\right]\,, \tag{1.273b}$$

with the initial conditions

$$C_1(0) = \frac{\dot{x}(0) - \Phi(0)}{\omega_0}\,, \qquad C_2(0) = x(0)\,, \tag{1.274}$$

imposed on the variables C_1 and C_2 in terms of the known values of $x(0)$ and $\dot{x}(0)$ taken at the time $t_0 = 0$. Notice that the initial value, $\Phi(0)$, of the gauge function remains arbitrary. Clearly, (1.273) are a simple analogue to the Lagrange system of planetary equations, whence one expects that the concept of gauge freedom may be equally applicable to the planetary equations of celestial mechanics.

In the example under consideration, the unperturbed problem was deliberately chosen to be extremely simple – a harmonic oscillator. Therefore, one ended up with a very simple system of equations for variables, C_1 and C_2, with the gauge-dependent terms being the total time derivatives. While, in general, one would have arrived to a system of differential equations that can be integrated only numerically, in this simple case, the analytical integration is possible,

$$C_1(t) = C_1(0) + \omega_0^{-1} \left[\int_0^t F(t') \cos \omega_0 t' dt' - \Phi(t) \cos \omega_0 t \right] , \tag{1.275a}$$

$$C_2(t) = C_2(0) - \omega_0^{-1} \left[\int_0^t F(t') \sin \omega_0 t' dt' - \Phi(t) \sin \omega_0 t \right] . \tag{1.275b}$$

One sees that the solution for functions, $C_1(t)$ and $C_2(t)$, explicitly depends on the gauge function, Φ, which vanishes if the Lagrange constraint, $\Phi = 0$, is imposed. On the other hand, substitution of (1.275) in (1.268) leads to a complete cancellation of the Φ-dependent terms:

$$x = \omega_0^{-1} \int_0^t F(t') \sin \omega_0 (t - t') dt' + C_1(0) \sin \omega_0 t + C_2(0) \cos \omega_0 t , \tag{1.276}$$

where the first term is a particular solution of the inhomogeneous equation, and the last two terms are a general solution of a homogeneous equation (1.267). This simple exercise proves that the physical trajectory, $x = x(t)$, of the perturbed oscillator remains invariant irrespectively of the choice of the gauge function, $\Phi(t)$, though the mathematical description (1.275) of its motion in terms of the variables $C_1(t)$ and $C_2(t)$ *is* gauge-dependent and rather arbitrary up to the following transformation of the variables

$$C_1 \longrightarrow \tilde{C}_1 = C_1 + \Phi(t) \cos \omega_0 t , \quad C_2 \longrightarrow \tilde{C}_2 = C_2 - \Phi(t) \sin \omega_0 t . \tag{1.277}$$

This gauge-dependence of the variables C_1 and C_2, if not taken into account properly, may greatly influence the numerical error in finding solution for $x(t)$. Indeed, in settings more complicated than the perturbed harmonic pendulum, a choice of gauge may change the numerical error of integration by several orders of magnitude (Gurfil and Klein, 2006). Specifically, choosing the Lagrange constraint $\Phi(t) = 0$ is not necessarily optimal.

An equally important feature illustrated by this example may also concern timescales. Suppose the unperturbed oscillator's frequency, ω_0, is much higher than an upper cut-off frequency, γ_0, of the spectrum of the perturbing force $F(t) \sim A \cos \gamma_0 t$. Naively, one may expect that a "slow" disturbance would cause an appropriately slow modulation of $C_1(t)$ and $C_2(t)$ in the perturbed problem. The fact that this is not necessarily so can be easily seen after integration of (1.275), where the slow and fast frequencies mix under the integral. The perturbing force brings about the following "fast" components to the solution, that is,

$$C_1(t) \sim \frac{A \cos \gamma_0 t}{\omega_0^2 - \gamma_0^2} \sin \omega_0 t , \quad C_2(t) \sim -\frac{A \cos \gamma_0 t}{\omega_0^2 - \gamma_0^2} \cos \omega_0 t . \tag{1.278}$$

This tells us that, in principle, $C_1(t)$ and $C_2(t)$ can undergo fast changes even under a slowly-evolving, $\gamma_0 \ll \omega_0$, disturbance. To a numerist, this indicates that the integration step used in solving (1.275) should not be much larger than the one employed in the unperturbed setting. Again, in some situations, a clever choice of gauge function $\boldsymbol{\Phi}$ may relax this restriction. As demonstrated by Gurfil and Klein (2006), a special choice of gauge turns the integration problem, in the linear case, into a simple quadrature. This means that for a complicated systems with significant timescale differences, for which integration of the variational equations with the Lagrange constraint using a fixed time step is impossible, the variational form of the equations with the specially-adapted gauge, $\boldsymbol{\Phi}(t)$, can be integrated using a fixed time step.

1.5.3
Relaxing the Lagrange Constraint in Celestial Mechanics

1.5.3.1 The Gauge Freedom

Recall that a solution to the unperturbed equation of motion (1.230) of a restricted two-body problem is a conic whose functional form can be denoted with

$$\boldsymbol{r} = \boldsymbol{r}(C_1, \ldots, C_6, t) \ , \tag{1.279a}$$

$$\dot{\boldsymbol{r}} = \frac{\partial \boldsymbol{r}(C_1, \ldots, C_6, t)}{\partial t} \ , \tag{1.279b}$$

where, here and everywhere else, the overdot representing a total time derivative d/dt, C_i $(i = 1, 2, \ldots, 6)$ are the constants of integration that do not depend on time. In the presence of a perturbing force, $\boldsymbol{F}$, the two-body system obeys (1.233). Solving the perturbed equation (1.233) by the method of variation of parameters implies that the functional form of the solution for the perturbed radius-vector, $\boldsymbol{r}$, remains the same as in the unperturbed problem

$$\boldsymbol{r} = \boldsymbol{r}[C_1(t), \ldots, C_6(t), t] \ , \tag{1.280}$$

while the constants of integration become functions of time. The perturbed velocity of the body is given by the total time derivative

$$\dot{\boldsymbol{r}} = \boldsymbol{v} + \boldsymbol{\Phi} \ , \tag{1.281}$$

where the vector function,

$$\boldsymbol{v} = \frac{\partial \boldsymbol{r}[C_1(t), \ldots, C_6(t), t]}{\partial t} \ , \tag{1.282}$$

has the same functional form as the unperturbed two-body velocity (1.279b), and

$$\boldsymbol{\Phi} = \sum_{i=1}^{6} \frac{\partial \boldsymbol{r}}{\partial C_i} \dot{C}_i \ , \tag{1.283}$$

denotes a so-called *gauge function*, taking into account that the orbital elements of the perturbed motion are not kept constant any longer but become functions of time.

Standard procedure in application of the method of variation of variables is to use the Lagrange constraint, $\boldsymbol{\Phi} = 0$. Let us step away from the standard procedure and explore what will happen if one does not set the function $\boldsymbol{\Phi}$ to nil, but keeps it unconstrained. Then, one can proceed further to calculate the acceleration of the body,

$$\ddot{\boldsymbol{r}} = \frac{\partial \boldsymbol{v}}{\partial t} + \sum_{i=1}^{6} \frac{\partial \boldsymbol{v}}{\partial C_i} \dot{C}_i + \dot{\boldsymbol{\Phi}} \ . \tag{1.284}$$

After substituting this result in the perturbed equation of motion (1.233) and recalling that the method of variation of parameters implies $\partial \boldsymbol{v}/\partial t = -G M \boldsymbol{r}/r^3$, one will obtain three equations of motion for the variables $C_i(t)$,

$$\sum_{i=1}^{6} \frac{\partial \boldsymbol{v}}{\partial C_i} \frac{d C_i}{d t} + \frac{d \boldsymbol{\Phi}}{d t} = \boldsymbol{F} \ . \tag{1.285}$$

This equation should be compared with (1.249) that was derived on the basis of the Lagrange constraint. Equation 1.285 clearly demonstrates that the system of ordinary differential equations of the second order for the orbital elements, $C_i = C_i(t)$, admits rather large freedom of transformations associated with the *gauge function* $\boldsymbol{\Phi}$.

To find $C_i(t)$, one will have to solve a system comprised of the equations of motion (1.285) and the expression (1.283), which so far is merely a notation for the yet unspecified, vector function $\boldsymbol{\Phi}$. The identity (1.283) will become an additional differential equation for variables, $C_i(t)$, if one chooses a particular functional form for the gauge function $\boldsymbol{\Phi} = \boldsymbol{\Phi}[C_1(t), \ldots, C_6(t) t]$ as a function of time and the variables $C_i(t)$. The necessity to fix a functional form of $\boldsymbol{\Phi}$, that is, to impose three additional differential conditions upon $C_i(t)$, evidently follows from the fact that one has six variables $C_i(t)$ while the number of equations of motion (1.285) is only three. What functional form to attribute eventually to $\boldsymbol{\Phi}$ will depend on the specific type of the perturbation. This *gauge freedom* of the differential equations of the perturbed motion corresponds to a specific freedom of transformations in the space of six parameters C_i. A particular example of these transformations is delivered by the canonical transformations preserving the form-invariance of the Hamiltonian equations like the Delaunay equations (Arnold, 1995, Section 44). Our approach, however, goes beyond the canonical transformations and includes a more general class of transformations of the orbital elements which is discussed in Section 1.5.3.2. The gauge freedom of the solutions of differential equations of motion can be used in:

1. Computer simulations of orbits where the choice of gauge considerably influences the error propagation process. A good or bad choice of gauge function,

$\boldsymbol{\Phi}$, can optimize or destroy the numerical procedure. Specifically, the Lagrange gauge, $\boldsymbol{\Phi} = 0$, is not guaranteed to always be optimal.
2. Analytical treatment, in order to simplify the integration procedure, perhaps, reducing it to quadratures.

The functional dependence of $\boldsymbol{\Phi}$ can be chosen arbitrary insofar as its substitution in (1.283) entails no conflict with the equations of motion in the sense that if a specific function, $\boldsymbol{\Phi}$ is chosen in (1.283), exactly the same function should appear in the equations of motion (1.285), and vice versa. The caveat here is that taking a particular form of the equations of motion (1.285) also fixes the gauge function $\boldsymbol{\Phi}$ which may not be nil. If this fact is overlooked and the Lagrange constraint, $\boldsymbol{\Phi} = 0$, is used, it will lead to an erroneous solution for C_i. For example, taking the perturbed equations for the Delaunay canonical variables fixes the gauge. Under position-dependent disturbances, the gauge coincides with the Lagrange constraint $\boldsymbol{\Phi} = 0$, and the resulting Delaunay elements are osculating. However, in case of velocity-dependent perturbations, the Delaunay gauge turns out to be different from the Lagrange constraint. Therefore, the ensuing Delaunay elements must be treated as nonosculating. If one works with the gauge function $\boldsymbol{\Phi}$ properly, its particular choice will never influence the eventual solution for the physical variable $\boldsymbol{r}$, similar to that having been discussed in the previous Section 1.5.2 for the one-dimensional case of a harmonic oscillator.

1.5.3.2 The Gauge Transformations

As emphasized above, the split of the orbital velocity, $\dot{\boldsymbol{r}}$, given by (1.281) is not unique since there is no any limitation on the freedom of choice of the gauge function $\boldsymbol{\Phi}$. It means that the solution of the perturbed problem of motion given in terms of the orbital elements C_i admits a large freedom of the infinitesimal gauge transformations of the variables generated by various choices of $\boldsymbol{\Phi}$. The gauge transformation of the variables is given by equation

$$\tilde{C}_i = C_i(t) + \alpha_i(C_k, t)\,, \quad (i = 1, 2, \ldots, 6) \tag{1.286}$$

where α_i are smooth functions of the "old" variables $C_i = C_i(t)$ $(i = 1, 2, \ldots, 6)$, and the time t. The group of the transformations is defined by the condition that the coordinate position of the body has the same value under the change (1.286) of the variables,

$$\boldsymbol{r}\left[\tilde{C}_1(t), \ldots, \tilde{C}_6(t), t\right] = \boldsymbol{r}[C_1(t), \ldots, C_6(t), t]\,, \tag{1.287}$$

and the functional form of (1.283) remains the same, that is,

$$\boldsymbol{\Phi} = \sum_{j=1}^{6} \frac{\partial \boldsymbol{r}}{\partial C_j}\dot{C}_j\,, \quad \tilde{\boldsymbol{\Phi}} = \sum_{i=j}^{6} \frac{\partial \boldsymbol{r}}{\partial \tilde{C}_j}\dot{\tilde{C}}_j\,, \tag{1.288}$$

but the gauge functions $\boldsymbol{\Phi} = \boldsymbol{\Phi}(C_i, t)$ and $\tilde{\boldsymbol{\Phi}} = \tilde{\boldsymbol{\Phi}}(\tilde{C}_i, t)$ are different: $\boldsymbol{\Phi} \neq \tilde{\boldsymbol{\Phi}}$.

In order to derive a relationship between functions α_i and the gauge function $\boldsymbol{\Phi}$, let us expand the left side of (1.287) in the Taylor series with respect to α_i which is considered as a small parameter of the expansion. By canceling the radius-vector $\boldsymbol{r}(C_i, t)$ in both parts of the equation, one obtains an algebraic equation

$$\sum_{j=1}^{6} \frac{\partial \boldsymbol{r}}{\partial C_j} \alpha_j = 0 \,. \tag{1.289}$$

Taylor expansion of the second equation (1.288) and making use of (1.286) yields

$$\tilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi} + \sum_{j=1}^{6} \frac{\partial \boldsymbol{r}}{\partial C_j} \frac{\partial \alpha_j}{\partial t} \,. \tag{1.290}$$

Taking a partial time derivative from (1.289) provides us with a useful equality

$$\sum_{j=1}^{6} \frac{\partial \boldsymbol{r}}{\partial C_j} \frac{\partial \alpha_j}{\partial t} = -\sum_{j=1}^{6} \frac{\partial \boldsymbol{v}}{\partial C_j} \alpha_j \,. \tag{1.291}$$

Making use of this identity in (1.290) transforms it into

$$\sum_{j=1}^{6} \frac{\partial \boldsymbol{v}}{\partial C_j} \alpha_j = \boldsymbol{\Phi} - \tilde{\boldsymbol{\Phi}} \,. \tag{1.292}$$

The next step is to make a dot product of (1.289) with vector $\partial \boldsymbol{v}/\partial C_i$ and a dot product of (1.292) with vector $\partial \boldsymbol{r}/\partial C_i$. Then, subtract one equation from another. Accounting for definition of the Lagrange brackets (1.237), one arrives at an algebraic equation for the transformation functions α_i,

$$\sum_{j=1}^{6} [C_i C_j] \alpha_j = \frac{\partial \boldsymbol{r}}{\partial C_i} \cdot (\boldsymbol{\Phi} - \tilde{\boldsymbol{\Phi}}) \,, \tag{1.293}$$

which can be solved with the help of the matrix of the Poisson brackets (1.238). Indeed, after performing the matrix multiplication of (1.293) with the Poisson brackets and accounting for their property of orthogonality with the Lagrange brackets (1.239),

$$\alpha_i = \sum_{j=1}^{6} \{C_i C_j\} \frac{\partial \boldsymbol{r}}{\partial C_j} \cdot (\tilde{\boldsymbol{\Phi}} - \boldsymbol{\Phi}) \,. \tag{1.294}$$

This equation substituted in (1.286) allows us to calculate the correspondence between one set of the orbital elements, C_i, associated with the gauge function $\boldsymbol{\Phi}$, and another set of the elements, $\tilde{C}_i$, associated with the choice of another gauge function, $\tilde{\boldsymbol{\Phi}}$. For a fixed gauge function $\boldsymbol{\Phi}$, there is a residual gauge freedom of transformations of the orbital elements C_i given by the smooth functions α_i

which has no explicit dependence on time, that is, $\partial \alpha_i / \partial t = 0$. The residual gauge freedom is limited by the class of functions α_i that satisfy (1.289). The right side of (1.294) is functionally similar to the right side of (1.253) for the osculating elements. Hence, by expanding the difference $\tilde{\boldsymbol{\Phi}} - \boldsymbol{\Phi}$ in the radial, transversal and normal to-the-orbit components, one can write the right side of (1.294) in the form being similar with the right side of the Euler–Gauss equations (1.254).

1.5.4
The Gauge-Invariant Perturbation Equation in Terms of the Disturbing Force

Let us assume that one has picked up a particular function $\boldsymbol{\Phi} = \boldsymbol{\Phi}[C_1(t), \ldots, C_6(t), t]$. Then, the perturbed problem of motion is reduced to a system of two vector differential equations for six variables $C_i(t)$:

$$\sum_{i=j}^{6} \frac{\partial \boldsymbol{v}}{\partial C_j} \dot{C}_j = -\dot{\boldsymbol{\Phi}} + \boldsymbol{F} , \tag{1.295a}$$

$$\sum_{i=j}^{6} \frac{\partial \boldsymbol{r}}{\partial C_j} \dot{C}_j = \boldsymbol{\Phi} . \tag{1.295b}$$

Now, take the dot product of the first equation with $\partial \boldsymbol{r} / \partial C_i$, and the dot product of the second equation with $\partial \boldsymbol{v} / \partial C_i$. The difference between these two products will amount to

$$\sum_{j=1}^{6} [C_i C_j] \dot{C}_j = (\boldsymbol{F} - \dot{\boldsymbol{\Phi}}) \cdot \frac{\partial \boldsymbol{r}}{\partial C_i} - \boldsymbol{\Phi} \cdot \frac{\partial \boldsymbol{v}}{\partial C_i} , \tag{1.296}$$

where the left side contains the Lagrange brackets defined in (1.237). It is worth emphasizing that the Lagrange brackets are defined in a gauge-invariant, that is, $\boldsymbol{\Phi}$-independent fashion. Indeed, the dependence on $\boldsymbol{\Phi}$ could appear, if and only if, the brackets contained time derivatives from the variables $C_i(t)$. However, neither the function, $\boldsymbol{r}$, nor the function $\boldsymbol{v} = \partial \boldsymbol{r}(C_i, t) / \partial t$ include differentiation of parameters C_i with respect to time. The Poisson brackets defined in (1.238) are gauge-invariant for the same reason. Equation 1.296 implements the gauge-invariant generalization of the planetary equations (1.252) in the Euler–Gauss form.

Be mindful that $\boldsymbol{\Phi}$ is set to be a single-valued function $\boldsymbol{\Phi}(C_i, t)$ of the time t and parameters $C_i = C_i(t)$, but not of their time derivatives, $\dot{C}_i$. In principle, the gauge functions with dependence upon the parameters' time derivatives of all orders are also conceivable, especially in the post-Newtonian celestial mechanics of binary pulsars (Damour, 1983; Grishchuk and Kopeikin, 1986; Lorimer and Kramer, 2004) and coalescing binary stars (Pati and Will, 2000). Such gauge functions generate second and higher-order derivatives in the system of equations for parameters C_i (Damour and Schäfer, 1985; Grishchuk and Kopeikin, 1986) which solution is a highly nontrivial mathematical endeavor (Chicone *et al.*, 2001).

The full time derivative of the chosen $\boldsymbol{\Phi} = \boldsymbol{\Phi}(C_i, t)$ contains the time derivatives of the parameters C_i,

$$\dot{\boldsymbol{\Phi}} = \frac{\partial \boldsymbol{\Phi}}{\partial t} + \sum_{j=1}^{6} \frac{\partial \boldsymbol{\Phi}}{\partial C_j} \frac{d C_j}{d t} \; . \tag{1.297}$$

It will then be reasonable to move these derivatives to the left side of (1.296), thus, recasting the equation into

$$\sum_{j=1}^{6} \left([C_i C_j] + \frac{\partial \boldsymbol{r}}{\partial C_i} \cdot \frac{\partial \boldsymbol{\Phi}}{\partial C_j} \right) \frac{d C_j}{d t} = \frac{\partial \boldsymbol{r}}{\partial C_i} \cdot \boldsymbol{F} - \frac{\partial \boldsymbol{r}}{\partial C_i} \cdot \frac{\partial \boldsymbol{\Phi}}{\partial t} - \frac{\partial \boldsymbol{v}}{\partial C_i} \cdot \boldsymbol{\Phi} \; , \tag{1.298}$$

which is the general form of the gauge-invariant perturbation equation (Efroimsky and Goldreich, 2003, 2004). If the Lagrange gauge, $\boldsymbol{\Phi} = 0$, is imposed, (1.298) naturally coincides with (1.252) which is equivalent to the Lagrange equation (1.264) when the perturbing force has a potential $R = R(\boldsymbol{r})$ only depending on positions, $\boldsymbol{r}$, but not on the velocities, $\dot{\boldsymbol{r}}$, of the bodies.

Sometimes, other gauges become advantageous for analytical calculations. In those gauges, the orbital elements, $C_i(t)$, are nonosculating with the instantaneous conics are not tangent to the actual orbit. A useful example of nonosculating elements is the set of *contact orbital elements* which is discussed below. Other settings wherein employment of nonosculating variables considerably simplifies calculations are the Gyldén–Meshcherskii problem, that is, the orbital motion of a body of variable mass (Gurfil and Belyanin, 2008); the Lense–Thirring effect, that is, the relativistic motion of a satellite about a rotating mass (Ashby and Allison, 2007; Chashchina *et al.*, 2009; Ciufolini, 1986); evolution of relative orbits of spacecrafts under perturbations (Gurfil, 2007). An important example of such forces appears in the equations of motion of the post-Newtonian celestial mechanics, a topic to be discussed at length in Section 6.3 below, especially in conjunction with the different parameterizations of the relativistic two-body problem.

1.5.5
The Gauge-Invariant Perturbation Equation in Terms of the Disturbing Function

Let us assume that the perturbed dynamics of the reduced two-body problem can be described by the Lagrangian

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = \frac{\dot{\boldsymbol{r}}^2}{2} + \frac{G M}{r} + \Delta L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \; , \tag{1.299}$$

where the first two terms in the right side defines the unperturbed Lagrangian and the perturbation, $\Delta L = \Delta L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t)$, depends on both the position, $\boldsymbol{r}$, and velocity, $\dot{\boldsymbol{r}}$, of the body, and on the time t. The linear momentum of the body is defined by

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \frac{\partial \Delta L}{\partial \dot{\boldsymbol{r}}} \; . \tag{1.300}$$

If the perturbation ΔL is not singular, and one assumes that it is not, (1.300) can be solved, thus obtaining velocity $\dot{\boldsymbol{r}}$ as a function of the momentum $\boldsymbol{p}$ and position $\boldsymbol{r}$,

$$\dot{\boldsymbol{r}} = \boldsymbol{p} - \frac{\partial \Delta L}{\partial \dot{\boldsymbol{r}}} , \tag{1.301}$$

where the second term in the right side is a function of $\boldsymbol{p}$ and $\boldsymbol{r}$. One can derive the Hamiltonian function, H, by making use of the Legendre transformation of the Lagrangian supplemented by (1.301),

$$H(\boldsymbol{r}, \boldsymbol{p}, t) = \boldsymbol{p} \cdot \dot{\boldsymbol{r}} - L . \tag{1.302}$$

Straightforward calculation reveals that the Hamiltonian corresponding to the Lagrangian (1.299) is

$$H = \frac{\boldsymbol{p}^2}{2} - \frac{GM}{r} + \Delta H , \tag{1.303}$$

where the perturbation

$$\Delta H \equiv -\Delta L - \frac{1}{2}\left(\frac{\partial \Delta L}{\partial \dot{\boldsymbol{r}}}\right)^2 . \tag{1.304}$$

The disturbing function R, being often used in celestial mechanics, is defined as the Lagrangian's perturbation,

$$R(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \equiv \Delta L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) . \tag{1.305}$$

While the Hamiltonian's perturbation is denoted as

$$V(\boldsymbol{r}, \boldsymbol{p}, t) \equiv \Delta H(\boldsymbol{r}, \boldsymbol{p}, t) . \tag{1.306}$$

In virtue of (1.304), the interconnection of the Hamiltonian's perturbation with the disturbing function is written as

$$V = -R - \frac{1}{2}\left(\frac{\partial R}{\partial \dot{\boldsymbol{r}}}\right)^2 . \tag{1.307}$$

In many situations, the disturbance bears no dependence upon the velocity $\dot{\boldsymbol{r}}$. Therefore, the disturbing function in these cases coincides with the negative Hamiltonian's perturbation as the second term in (1.307) becomes nil. One assumes a more general case and intends to address disturbances with a velocity-dependence present. Hence, the necessity to use the full formula (1.307).

The Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{r}}} - \frac{\partial L}{\partial \boldsymbol{r}} = 0 , \tag{1.308}$$

written for the perturbed Lagrangian (1.299) are

$$\ddot{\boldsymbol{r}} + \frac{GM}{r^3}\boldsymbol{r} = \boldsymbol{F} , \tag{1.309}$$

with the perturbing force given by

$$\boldsymbol{F} = \frac{\partial R}{\partial \boldsymbol{r}} - \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\boldsymbol{r}}}\right) , \tag{1.310}$$

which should be substituted in the right side of (1.298). One notices that the perturbation R is an implicit function of the orbital elements,

$$R = R(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = R\left[\boldsymbol{r}(C_i, t), \dot{\boldsymbol{r}}(C_i, t), t\right] , \tag{1.311}$$

where $C_i = C_i(t)$. Hence, the partial derivative

$$\frac{\partial R}{\partial C_i} = \frac{\partial R}{\partial \boldsymbol{r}}\frac{\partial \boldsymbol{r}}{\partial C_i} + \frac{\partial R}{\partial \dot{\boldsymbol{r}}}\frac{\partial \boldsymbol{v}}{\partial C_i} + \frac{\partial R}{\partial \dot{\boldsymbol{r}}}\frac{\partial \boldsymbol{\Phi}}{\partial C_i} . \tag{1.312}$$

The time derivative entering (1.310) can be written as

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{\boldsymbol{r}}} = \frac{\partial}{\partial t}\left(\frac{\partial R}{\partial \dot{\boldsymbol{r}}}\right) + \sum_{j=1}^{6}\frac{\partial}{\partial C_j}\left(\frac{\partial R}{\partial \dot{\boldsymbol{r}}}\right)\frac{dC_j}{dt} . \tag{1.313}$$

Inserting the expression (1.310) for the force $\boldsymbol{F}$ in the generic equation (1.298), and making use of the expressions (1.312) and (1.313), one arrives at the general form of the gauge-invariant equation of orbital evolution for an arbitrarily-chosen set of orbital elements C_i (Efroimsky, 2005b; Efroimsky and Goldreich, 2003, 2004):

$$\sum_{j=1}^{6}\left\{[C_i C_j] + \frac{\partial \boldsymbol{r}}{\partial C_i}\cdot\frac{\partial \boldsymbol{\Psi}}{\partial C_j}\right\}\frac{dC_j}{dt} = -\frac{\partial V}{\partial C_i} - \left(\frac{\partial \boldsymbol{v}}{\partial C_i} + \frac{\partial R}{\partial \dot{\boldsymbol{r}}}\frac{\partial}{\partial C_i} + \frac{\partial \boldsymbol{r}}{\partial C_i}\frac{\partial}{\partial t}\right)\cdot\boldsymbol{\Psi} , \tag{1.314}$$

where one has used definition (1.307) of the Hamilton's perturbation V, and introduced a new notation

$$\boldsymbol{\Psi} \equiv \boldsymbol{\Phi} + \frac{\partial R}{\partial \dot{\boldsymbol{r}}} \tag{1.315}$$

for the gauge function shifted from its original value, $\boldsymbol{\Phi}$, by the partial derivative of the perturbing function, R, with respect to velocity of the body. The gauge function $\boldsymbol{\Psi}$ is arbitrary with the only limitation that comes from the decision to keep $\boldsymbol{\Phi}$ as a function of the time and the orbital elements but not of their time derivatives.

1.5.6
The Delaunay Equations without the Lagrange Constraint

As an example, let us consider the generic perturbation equation (1.314) for the Delaunay elements defined in Section 1.4.7. We permit the perturbation to depend both on the position and the velocity of the orbiting body, but do not impose the condition of osculation. Thus, the gauge function $\boldsymbol{\Psi}$ remains arbitrary. The system of the generic Delaunay equations read (Efroimsky and Goldreich, 2003):

$$\frac{dL}{dt} = -\frac{\partial V}{\partial l} - \frac{\partial R}{\partial \dot{\boldsymbol{r}}} \cdot \frac{\partial \boldsymbol{\Psi}}{\partial l} - \frac{\partial \boldsymbol{r}}{\partial l} \cdot \frac{d\boldsymbol{\Psi}}{dt} - \frac{\partial \boldsymbol{v}}{\partial l} \cdot \boldsymbol{\Psi} , \tag{1.316a}$$

$$\frac{dl}{dt} = \frac{\partial V}{\partial L} + \frac{\partial R}{\partial \dot{\boldsymbol{r}}} \cdot \frac{\partial \boldsymbol{\Psi}}{\partial L} + \frac{\partial \boldsymbol{r}}{\partial L} \cdot \frac{d\boldsymbol{\Psi}}{dt} + \frac{\partial \boldsymbol{v}}{\partial L} \cdot \boldsymbol{\Psi} , \tag{1.316b}$$

$$\frac{dG}{dt} = -\frac{\partial V}{\partial g} - \frac{\partial R}{\partial \dot{\boldsymbol{r}}} \cdot \frac{\partial \boldsymbol{\Psi}}{\partial g} - \frac{\partial \boldsymbol{r}}{\partial g} \cdot \frac{d\boldsymbol{\Psi}}{dt} - \frac{\partial \boldsymbol{v}}{\partial g} \cdot \boldsymbol{\Psi} , \tag{1.316c}$$

$$\frac{dg}{dt} = \frac{\partial V}{\partial G} + \frac{\partial R}{\partial \dot{\boldsymbol{r}}} \cdot \frac{\partial \boldsymbol{\Psi}}{\partial G} + \frac{\partial \boldsymbol{r}}{\partial G} \cdot \frac{d\boldsymbol{\Psi}}{dt} + \frac{\partial \boldsymbol{v}}{\partial G} \cdot \boldsymbol{\Psi} , \tag{1.316d}$$

$$\frac{dH}{dt} = -\frac{\partial V}{\partial h} - \frac{\partial R}{\partial \dot{\boldsymbol{r}}} \cdot \frac{\partial \boldsymbol{\Psi}}{\partial h} - \frac{\partial \boldsymbol{r}}{\partial h} \cdot \frac{d\boldsymbol{\Psi}}{dt} - \frac{\partial \boldsymbol{v}}{\partial h} \cdot \boldsymbol{\Psi} , \tag{1.316e}$$

$$\frac{dh}{dt} = \frac{\partial V}{\partial H} + \frac{\partial R}{\partial \dot{\boldsymbol{r}}} \cdot \frac{\partial \boldsymbol{\Psi}}{\partial H} + \frac{\partial \boldsymbol{r}}{\partial H} \cdot \frac{d\boldsymbol{\Psi}}{dt} + \frac{\partial \boldsymbol{v}}{\partial H} \cdot \boldsymbol{\Psi} . \tag{1.316f}$$

In case of a disturbance only depending on the position, one has $V = -R$. In this situation, it would be most convenient to fix the gauge in the Lagrange-constrain form: $\boldsymbol{\Phi} = 0$. This sets the Delaunay elements osculating, while (1.316) become the well-known canonical equations of Delaunay (1.266). When the disturbance also depends upon the velocity, (1.316) can still be reduced to the canonical form (1.266) by choosing another gauge, $\boldsymbol{\Psi} = 0$, though this can be done only at the cost of osculation loss. Indeed, while after imposing such a gauge the (1.316) will look similar to (1.266), but the instantaneous conics parameterized by the Delaunay elements will be nontangent to the perturbed orbit. This situation is depicted in Figure 1.6.

Our example reveals that a blithe use of the Delaunay elements in problems with velocity-dependent perturbations $R = R(\boldsymbol{r}, \dot{\boldsymbol{r}})$ may lead to erroneous geometric interpretation of the orbital motion as the loss of osculation may be not noticed. Another moral of the story is that often, the Delaunay elements are considered in the framework of the Hamilton–Jacobi theory of canonical transformations which treats these elements as canonical variables obeying the Hamilton equations. However, (1.316) are not necessarily Hamiltonian depending upon the gauge chosen. Hence, the canonicity of the Delaunay elements should not be taken for granted without checking upon which gauge conditions have been imposed. For example, the nonosculating orbital inclination can differ in the first order from the osculating inclination of the orbit that can be important for correct interpretation of the theory of Iapetus's inclination evolution developed by Ward (1981).

The main conclusion is: whenever one encounters a disturbance that depends not only upon positions, but also upon velocities or momenta, implementation of the canonical-perturbation method necessarily yields equations that render nonosculating canonical elements. It is possible to keep the elements osculating, but only at the cost of sacrificing canonicity. For example, under velocity-dependent orbital perturbations (like inertial forces, atmospheric drag, or relativistic

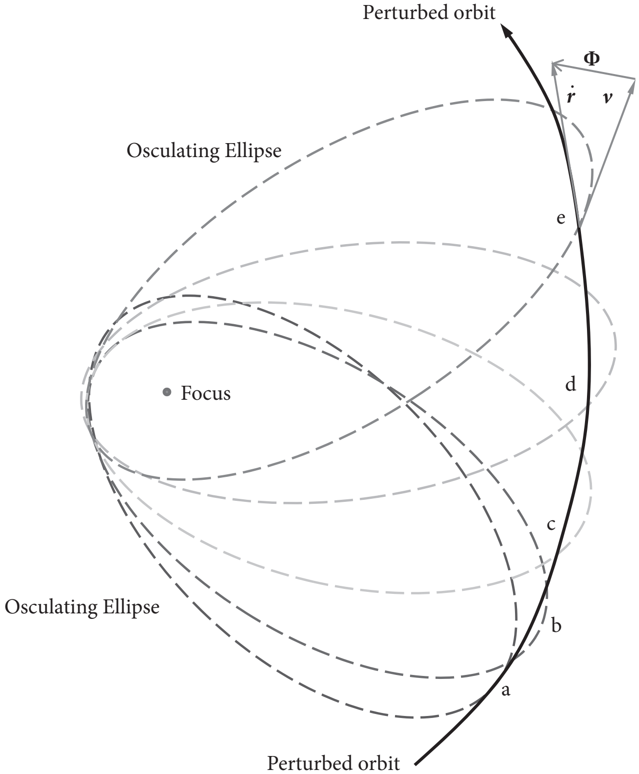

Figure 1.6 This picture illustrates the method of variation of parameters in the reduced two-body problem without imposing the condition of osculation. A perturbed orbit is a set of points, each of which is donated by a representative of a sequence of confocal instantaneous conics a, b, c, d, e, that are *not* supposed to be tangent, nor even coplanar to the orbit. As a result, the physical velocity $\dot{\boldsymbol{r}} = d\boldsymbol{r}/dt$ that is tangent to the perturbed orbit differs from the Keplerian velocity $\boldsymbol{v}$ that is tangent to the conic. The sequence of nonosculating conics is characterized by vector $\boldsymbol{\Phi}(\tilde{C}_1, \ldots, \tilde{C}_6, t) = \dot{\boldsymbol{r}}(\tilde{C}_1, \ldots, \tilde{C}_6, t) - \boldsymbol{v}(\tilde{C}_1, \ldots, \tilde{C}_6, t)$ expressed as a function of time and six (nonosculating) orbital elements.

correction), the equations for osculating Delaunay elements ($\boldsymbol{\Phi} = 0$ constraint is imposed) will no longer be Hamiltonian (Efroimsky, 2002a, 2002b).

For the first time, nonosculating orbital variables were encountered probably by Poincaré in his studies of the three-body problem, though he never explored these variables from the viewpoint of a non-Lagrange constraint choice. Having performed a transition from the barycentric to the heliocentric reference frame (Poincaré, 1896, 1897) noticed a subtle difference between the instantaneous conics parameterized by the canonical Delaunay variables defined in the two frames. A conic parameterized by the Delaunay elements in the barycentric frame deviated from the perturbed trajectory at the rate of t^2 of the time t. At the same time, a conic parameterized by the Delaunay elements in the heliocentric frame deviated from the perturbed trajectory at the linear rate t. As became evident later, the variables in the barycentric frame were osculating, while the heliocentric variables were not. Evolution of the heliocentric set of variables was governed by a Hamiltonian perturbation that turned out to be velocity-dependent, which is natural because the heliocentric frame is noninertial with the Hamiltonian perturbation depending on linear momenta of the bodies. In the language of symplectic geometry, Poincaré's finding, including the issue of choosing either osculating or nonosculating elements in the three-body problem, was briefly addressed in Abdullah and Albouy (2001).

1.5.7
Contact Orbital Elements

The generic equation (1.314) evidently reveal the convenience of the constraint

$$\boldsymbol{\Psi} = 0 \Longleftrightarrow \boldsymbol{\Phi} = -\frac{\partial R}{\partial \dot{\boldsymbol{r}}} \; . \tag{1.317}$$

It cancels many terms in (1.314), reducing them to

$$\sum_{j=1}^{6} [\tilde{C}_i \tilde{C}_j] \frac{d\tilde{C}_j}{dt} = -\frac{\partial V}{\partial \tilde{C}_i} \; , \tag{1.318}$$

where one has denoted $\tilde{C}_i$, the orbital elements corresponding to the constraint (1.317) in order to distinguish them from the osculating elements C_i. The so-defined orbital elements $\tilde{C}_i$ are called the *contact elements*. They are often used for analysis of orbits perturbed by velocity-dependent forces (Efroimsky, 2005a).

The term "contact elements" was offered in celestial mechanics by Brumberg *et al.* (1971). Later, Kinoshita (1993) employed these variables. At that time, though, it was not yet clear that such variables obey conditions (1.317). It can be proven that variables obeying the same conditions also show up when one tries to preserve the interrelation $|\boldsymbol{J}| \equiv |\boldsymbol{r} \times \boldsymbol{p}|$ in the frame precessing at a rate $\boldsymbol{\mu}$ where the momentum per unit mass, $\boldsymbol{p} = \dot{\boldsymbol{r}} + \boldsymbol{\mu} \times \boldsymbol{r}$ is not equal to $\dot{\boldsymbol{r}}$, making this situation similar to the case of the velocity-dependent perturbations of orbital motion (Goldreich, 1965). Calculations carried out in terms of these variables are often greatly simplified. At the same time, one should be aware that the instantaneous conics parameterized by these variables are not tangent to the actual trajectory. Brumberg *et al.* (1971), Kinoshita (1993), and Goldreich (1965) employed the contact variables to describe motion of a satellite orbiting a precessing oblate massive body. Although the instantaneous values of the contact variables differ from their osculating counterparts already in the first order, their averages differ only in the second order, *provided the motion is periodic*. However, in other situations, the absence of a periodic precession can invalidate geometric interpretation of the averaged values of these elements already in the first order.

Derivatives of the contact elements can be decoupled from the Lagrange brackets in (1.318) with the help of the matrix orthogonality condition (1.239). Thus, one obtains a system of ordinary differential equations for the contact elements,

$$\frac{d\tilde{C}_i}{dt} = \sum_{j=1}^{6} \{\tilde{C}_i \tilde{C}_j\} \frac{\partial V}{\partial \tilde{C}_j} , \tag{1.319}$$

which looks similar to the Lagrange planetary equation (1.264), except that now the perturbation, V, is given by more complicated expression (1.307) that involves the partial derivative of the disturbing function, R, with respect to velocity of the body, and the elements $\tilde{C}_i$ obey the constraint (1.317), instead of the Lagrange constraint, $\boldsymbol{\Phi} = 0$. More specifically, for the set of the contact elements $(\tilde{a}, \tilde{e}, \tilde{\imath}, \tilde{\Omega}, \tilde{\omega}, \tilde{\mathcal{M}}_0)$, one gets (Brumberg, 1972, 1991):

$$\frac{d\tilde{a}}{dt} = \frac{2}{\tilde{n}\tilde{a}} \frac{\partial V}{\partial \tilde{\mathcal{M}}_0} , \tag{1.320a}$$

$$\frac{d\tilde{e}}{dt} = \frac{1-\tilde{e}^2}{\tilde{n}\tilde{a}^2\tilde{e}} \frac{\partial V}{\partial \tilde{\mathcal{M}}_0} - \frac{\sqrt{1-\tilde{e}^2}}{\tilde{n}\tilde{a}^2\tilde{e}} \frac{\partial V}{\partial \tilde{\omega}} , \tag{1.320b}$$

$$\frac{d\tilde{\imath}}{dt} = \frac{\cos\tilde{\imath}}{\tilde{n}\tilde{a}^2\sqrt{1-\tilde{e}^2}\sin\tilde{\imath}} \frac{\partial V}{\partial \tilde{\omega}} - \frac{1}{\tilde{n}\tilde{a}^2\sqrt{1-\tilde{e}^2}\sin\tilde{\imath}} \frac{\partial V}{\partial \tilde{\Omega}} , \tag{1.320c}$$

$$\frac{d\tilde{\Omega}}{dt} = \frac{1}{\tilde{n}\tilde{a}^2\sqrt{1-\tilde{e}^2}\sin\tilde{\imath}} \frac{\partial V}{\partial \tilde{\imath}} , \tag{1.320d}$$

$$\frac{d\tilde{\omega}}{dt} = -\frac{\cos\tilde{\imath}}{\tilde{n}\tilde{a}^2\sqrt{1-\tilde{e}^2}\sin\tilde{\imath}} \frac{\partial V}{\partial \tilde{\imath}} + \frac{\sqrt{1-\tilde{e}^2}}{\tilde{n}\tilde{a}^2\tilde{e}} \frac{\partial V}{\partial \tilde{e}} , \tag{1.320e}$$

$$\frac{d\tilde{\mathcal{M}}_0}{dt} = -\frac{1-\tilde{e}^2}{\tilde{n}\tilde{a}^2\tilde{e}} \frac{\partial V}{\partial \tilde{e}} - \frac{2}{\tilde{n}\tilde{a}} \frac{\partial V}{\partial \tilde{a}} . \tag{1.320f}$$

As the right side of the resulting equation (1.319) only contains the Hamiltonian variation V, it may be logical to christen the constraint (1.317) the *Hamiltonian gauge*. Insertion of this gauge in the expression (1.281) for the perturbed velocity makes this velocity read,

$$\dot{\boldsymbol{r}} = \boldsymbol{v} - \frac{\partial R}{\partial \dot{\boldsymbol{r}}} . \tag{1.321}$$

Comparing this with (1.300) and (1.305), one sees that in the Hamiltonian gauge, the partial time derivative of the perturbed coordinates, $\boldsymbol{v} = \partial \boldsymbol{r}/\partial t$, is equal to the canonical momentum,

$$\boldsymbol{v}\left[\tilde{C}_1(t), \ldots, \tilde{C}_6(t), t\right] = \boldsymbol{p}\left[\tilde{C}_1(t), \ldots, \tilde{C}_6(t), t\right] . \tag{1.322}$$

Equation 1.322 allows us to interchange the velocities and the corresponding momenta in the expressions for the Lagrange and Poisson brackets whenever one is working in the Hamiltonian gauge. It also tells that the contact elements $\tilde{C}_i$ represent an osculating instantaneous orbit in the phase space $(\boldsymbol{r}, \boldsymbol{p})$ in contrast to the canonical osculating elements in the Lagrange gauge, which represent an osculating conic in the configuration space $(\boldsymbol{r}, \dot{\boldsymbol{r}})$. The relationship between the osculating elements C_i in the configuration space and the contact elements $\tilde{C}_i$ in the phase space can be found from (1.294). Indeed, substituting the Lagrange constraint, $\boldsymbol{\Phi} = 0$, and the Hamiltonian gauge, $\tilde{\boldsymbol{\Phi}} = -\partial R/\partial \dot{\boldsymbol{r}}$, one obtains

$$\tilde{C}_i - C_i = -\sum_{j=1}^{6} \{C_i C_j\} \frac{\partial \boldsymbol{r}}{\partial C_j} \cdot \frac{\partial R}{\partial \dot{\boldsymbol{r}}} . \tag{1.323}$$

The right side of this equation looks the same as that (1.253) with the "force" $\boldsymbol{F} = \tilde{\boldsymbol{\Phi}} = -\partial R/\partial \dot{\boldsymbol{r}}$. Therefore, the differences between the elements, $(\tilde{a}, \tilde{e}, \tilde{\imath}, \tilde{\Omega}, \tilde{\omega}, \tilde{\mathcal{M}}_0)$ and $(a, e, i, \Omega, \omega, \mathcal{M}_0)$ are given by the right side of (1.254).

It is possible to prove that the Hamiltonian gauge condition (1.317) is compulsory imposed by the canonical perturbation theory of the Hamiltonian equations for any dynamic system. Indeed, let us assume that the orbital elements $\tilde{C}_1, \ldots, \tilde{C}_6$ are associated with generalized coordinates $\boldsymbol{Q} = (Q_i) = (\tilde{C}_1, \tilde{C}_2, \tilde{C}_3)$ and generalized momentum $\boldsymbol{P} = (P_i) = (\tilde{C}_4, \tilde{C}_5, \tilde{C}_6)$, which obey the Hamiltonian equations

$$\frac{dQ_i}{dt} = \frac{\partial \tilde{H}}{\partial P_i} , \qquad \frac{dP_i}{dt} = -\frac{\partial \tilde{H}}{\partial Q_i} , \tag{1.324}$$

where the perturbed Hamiltonian

$$\tilde{H} = \tilde{H}(\boldsymbol{Q}, \boldsymbol{P}, t) = V(\boldsymbol{r}, \boldsymbol{p}, t) + \partial \chi/\partial t , \tag{1.325}$$

and $\chi = \chi(\boldsymbol{r}, \boldsymbol{p}, t)$ is a generating function defining the canonical transformation from $(\boldsymbol{r}, \boldsymbol{p})$ to $(\boldsymbol{Q}, \boldsymbol{P})$ (Arnold, 1995; Landau and Lifshitz, 1969) so that radius-vector $(r^i) = \boldsymbol{r}(\boldsymbol{Q}, \boldsymbol{P}, t)$ and momentum $(p^i) = \boldsymbol{p}(\boldsymbol{Q}, \boldsymbol{P}, t)$.

The total time derivative of $\boldsymbol{r}$ is expressed as follows

$$\dot{\boldsymbol{r}} = \frac{\partial \boldsymbol{r}}{\partial t} + \frac{\partial \boldsymbol{r}}{\partial Q_i}\frac{dQ_i}{dt} + \frac{\partial \boldsymbol{r}}{\partial P_i}\frac{dP_i}{dt} = \frac{\partial \boldsymbol{r}}{\partial t} + \{\boldsymbol{r}\tilde{H}\} , \tag{1.326}$$

where

$$\{\boldsymbol{r}\tilde{H}\} = \frac{\partial \boldsymbol{r}}{\partial Q_i}\frac{\partial \tilde{H}}{\partial P_i} - \frac{\partial \boldsymbol{r}}{\partial P_i}\frac{\partial \tilde{H}}{\partial Q_i} , \tag{1.327}$$

is the Poisson brackets of $\boldsymbol{r}$ and $\tilde{H}$ expressed in terms of the canonical variables (Q_i, P_i), and the Einstein summation rule is used for the repeated indices. However, due to the invariance of the Poisson brackets with respect to the canonical transformations (Landau and Lifshitz, 1969), one has

$$\{\boldsymbol{r}\tilde{H}\} = \{\boldsymbol{r}V\} = \frac{\partial \boldsymbol{r}}{\partial r^i}\frac{\partial V}{\partial p^i} - \frac{\partial \boldsymbol{r}}{\partial p^i}\frac{\partial V}{\partial r^i} = \frac{\partial V}{\partial \boldsymbol{p}} . \tag{1.328}$$

However, the partial derivative $\partial V/\partial \boldsymbol{p} = \partial R/\partial \dot{\boldsymbol{r}}$ (Landau and Lifshitz, 1969, Section 40). Hence, going back to (1.326) it reveals that any system of canonical variables leads to (1.321) which implies the Hamiltonian constraint (1.317) telling us that the variables $(\boldsymbol{Q}, \boldsymbol{P})$ must be interpreted as the contact elements $\tilde{C}_1, \ldots, \tilde{C}_6$.

This gauge-stiffness[24)] property of the system of the Hamiltonian equations should be taken into account in the theory of the gauge transformations of the parameter space.

1.5.8
Osculation and Nonosculation in Rotational Dynamics

Interestingly, the phenomenon of osculation versus nonosculation emerges not only in the theory of orbits, but also in rotational dynamics, provided the method of variation of parameters is employed (Efroimsky and Escapa, 2007). This should not be surprising because the mathematics underlying rotational mechanics is virtually identical to that underlying orbital mechanics. In orbital mechanics, a perturbed trajectory of a body consists of points, each of which is donated by a representative of a sequence of instantaneous Keplerian conics. If one now disembodies this idea of its particular implementation, one should agree that:

1. a trajectory may be assembled of points contributed by a family of algebraic curves of an essentially arbitrary type, not necessarily conics;
2. it is not obligatory to set the family of curves tangent to the perturbed trajectory. In fact, it is often beneficial to choose them to be nontangent.

In its generality, the approach can be applied, for example, to describe the time evolution of Euler's angles characterizing orientation of a rotating body with respect to inertial space. A disturbed rotation can be thought of as consisting of a series of small turns along different Eulerian cones, each of which is an orbit on the Euler angles' manifold corresponding to an unperturbed state of the angular momentum (spin) of the body. Just as in orbital mechanics, a transition from one instantaneous Keplerian conic to another is caused by a disturbing force. Therefore, a transition from one instantaneous Eulerian cone to another is governed by either an external torque, or the torque due to precession of the frame, or other perturbations like redistribution of matter within the rotating body. Thus, in rotational mechanics, the Eulerian cones play the same role as the Keplerian conics do in the orbital dynamics. Most importantly, a perturbed rotation may be parameterized by the elements of the Eulerian cones in an osculating or in a nonosculating manner that is picked up by imposing a constraint on the rotational elements that is similar to choosing the gauge function, $\boldsymbol{\Phi}$, in the orbital dynamics. In many cases, the osculating Eulerian cones are convenient, but the nonosculating parametrization may sometimes be more beneficial.

When the equations for the rotational elements are required to be canonical, the so-called Andoyer variables are typically chosen. However, like in the case of the Delaunay orbital elements, the Andoyer variables may share the important peculiarity: under certain circumstances, the standard Hamiltonian equations of rotational motion render the elements nonosculating. In the theory of orbits, the standard form

24) This term was suggested by Peter Goldreich.

of the Lagrange and Delaunay planetary equations yield contact elements when perturbations depend on velocities. To keep the elements osculating in the configuration space, the equations must be amended with extra terms that are partial derivatives of the disturbing function with respect to velocities. It complicates the planetary equations and makes, for example, the Delaunay equations noncanonical. In rotational dynamics, whenever a perturbation depends upon the angular velocity, the canonical approach makes the Andoyer variables entering the Hamiltonian equations to be nonosculating to the Eulerian cones. To make them osculating, extra terms should be added to the standard Hamiltonian equations, but then the equations will no longer be canonical (Efroimsky and Escapa, 2007).

1.6 Epilogue to the Chapter

As with any physical theory, Newtonian mechanics have a restricted realm of applicability. The first example of this realm's limitations, the problem of Mercury's apsidal precession, was encountered by astronomers back in the second part of the nineteenth century. By then, it had long been known that within the unperturbed Keplerian two-body problem, the Laplace–Runge–Lenz vector is preserved and is always pointing toward the pericenter. Hence, if one neglects the planets' mutual disturbances, the periapses of their orbits would remain idle. The disturbances however make the periapses move. The effect is especially pronounced in the case of Mercury, which has a small mass and therefore is most sensitive to the pull of the other planets. At the same time, since Mercury has an orbit of a high eccentricity and a small period, the advance of its pericenter is fairly easy observable, and it was accurately measured back in the nineteenth century. However, the rate of this advance turned out to differ from the predictions of the classical planetary theory by about 43 arcsec per century. To explain the discrepancy, astronomers had to wait until the theory of general relativity was created. Explanation of Mercury's anomalous apsidal precession then became one of the first triumphs of Einstein's theory.

During the twentieth century, astronomers came across many other examples of celestial motion, for whose accurate description in terms of the Newtonian mechanics turned out to be insufficient. An incomplete list includes the orbital motion of neutron stars and accretion-disk particles in binary systems; the motion of artificial satellites, the Moon, asteroids, and inner planets in the Solar System; and propagation of light through gravitational field. Recently, several so-far-unexplained anomalies in the orbital motion of spacecraft, planets, and the Moon have been registered (Anderson and Nieto, 2010). They may indicate that even more subtle relativistic effects in the orbital motion of the bodies should be taken into account. Still, the problem of Mercury's pericenter advance marks the starting point whence the science of relativistic celestial mechanics reckons its history.

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