1
Introduction

1.1
Goals of the Book

A plasma is an assembly of charged particles, making its behavior inseparable from that of the electromagnetic field. When a plasma includes neutrals and when the collisions are numerous enough between charged and neutral particles, it causes the plasma to behave more like a neutral gas. This book will focus on the fully ionized plasmas, so emphasizing more the specific plasma properties.

Plasma evolution is governed by a loop: the charged particles move under the effect of the electromagnetic fields, and the particles, by their density and their velocities, create collective electromagnetic fields. This is true for any kind of plasma, collisional or not, fully ionized or not, and whatever the plasma and field parameters.

This "plasma loop" is sketched in Figure 1.1. One can observe on this sketch that two subloops can exist.

1. There is an electromagnetic loop, which can exist even in the absence of particles. In this case, the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are related to each other only by the vacuum Maxwell equations. The local source of the magnetic field is then just the displacement current $\varepsilon_0 \partial_t \boldsymbol{E}$ since there is no electric current due to particle motions. The signature of this electromagnetic loop is the existence of the electromagnetic waves in vacuum.
2. There is a collisional loop, which can exist in the absence of a collective field, and even with neutral particles (although the notion of collision is then different). The collisions between particles also allows information to propagate. The signature of this collisional loop is the existence of pressure (/sound) waves in the medium.

The general loop of Figure 1.1 can exist even with negligible collisions and with negligible displacement current. It is clear from this that in these conditions, any plasma evolution, for instance, any plasma wave, must always involve both kinds of evolution: particle and fields. In neutral gas like air, we are familiar with an almost complete separation between electromagnetic waves (light, radio, and so on), only involving $\boldsymbol{E}$ and $\boldsymbol{B}$, and sound waves, only involving the gas properties like mass

Collisionless Plasmas in Astrophysics, First Edition. Gérard Belmont, Roland Grappin, Fabrice Mottez, Filippo Pantellini, and Guy Pelletier.

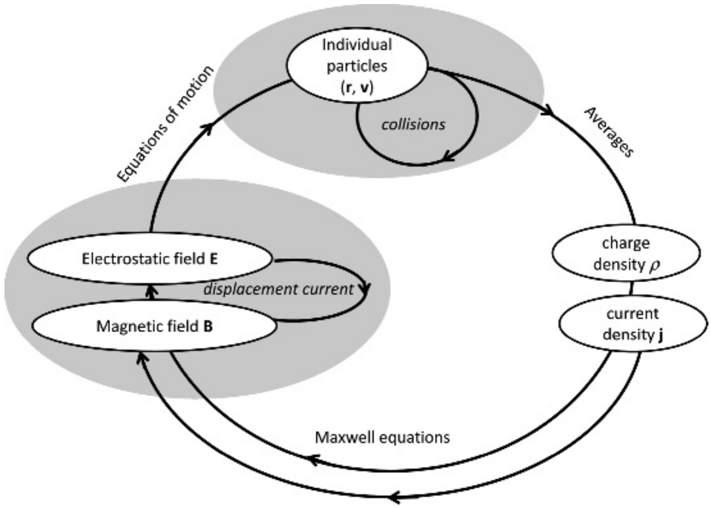

Figure 1.1 The plasma loop. The electromagnetic fields indicated in the sketch are the "collective" ones, that is, where the collision fields have been subtracted. The shaded areas concern, respectively, the electromagnetic subloop and the collisional subloop, which can exist in the absence of charged particles, but which are then not coupled to each other.

density ρ, fluid velocity $\boldsymbol{u}$, and pressure P. This separation is of course prohibited in a plasma.

The sketch of Figure 1.1 makes use of the notion of a "collective field". This notion will be defined in detail, but how can it be understood first from an intuitive point of view? In a small volume, the difference between the electron and ion densities makes a collective charge density which is a source for the electrostatic field, and the difference between their mean velocities makes a current which is a source for the magnetic field and the induced electric field. This loop is the intrinsic plasma loop. The displacement current, if not negligible, is never essential: it is just an additional complication to the fundamental phenomenon. Similarly, the presence of collisions is not essential to plasma phenomena, even if they bring specific properties to the plasma, which can allow for simplified modeling.

The collisions, when present, insure a continuous velocity redistribution between particles. It is the reason why they can allow simplified statistical descriptions: they make the thermodynamical functions such as entropy meaningful. In the absence of collisions, on the contrary, all these notions must be used with care. This book will particularly emphasize the collisionless limits of plasmas, in order to focus on the most intrinsic properties of the plasmas and understand what the descriptions are that remain valid without collisions and those which are specific to the collisional hypothesis.

The question of the collisionless limit is particularly crucial when considering the so-called fluid models. These models, such as MHD (magnetohydrodynamics), allow describing the plasma with a small number of macroscopic parameters, typically density, fluid velocity and pressure. Such a description is of course a huge

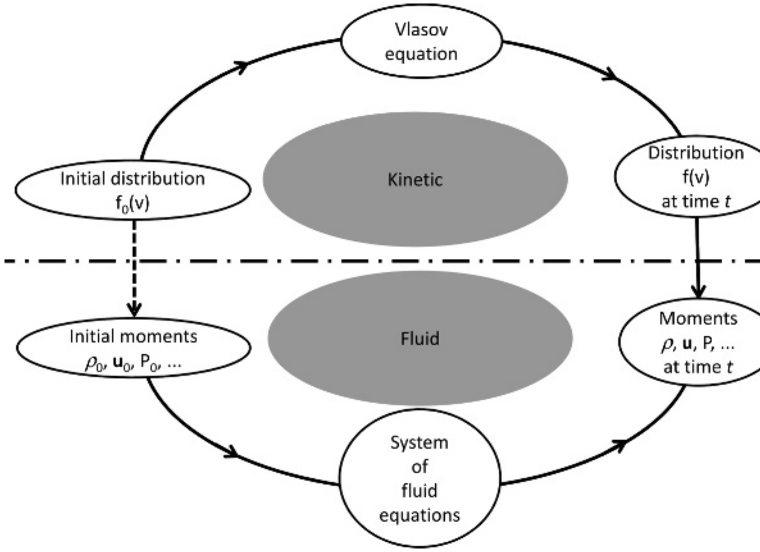

Figure 1.2 Principles of the fluid and kinetic methods in the case of an initial value problem. Note that, if the moments ρ, $\boldsymbol{u}$, P, can always be calculated from the distribution function $f(v)$, the opposite is not feasible without strong hypotheses.

reduction if compared to the description of all individual particles, but it is still an extremely big reduction with respect to the kinetic one, which describes the particle populations by their distribution function $f(\boldsymbol{v})$, that is, the density of probability of each velocity in small volumes. This reduction is, however, necessary, for computation time reasons, for any complex problem, in particular large scale and 3D. The validity of the fluid models is well established in the collisional case, but not in a general manner in the collisionless one. It is, therefore, important to understand what is universal in these models and what has to be questioned. We will show that all the weaknesses of these models lie in the so-called closure equation and emphasize the consequences of different choices for this equation.

Figure 1.2 shows the two main methods for modeling the behavior of a particle population. Both methods assume that one knows a valid kinetic equation, that is, a differential equation which describes the variations of the distribution function $f(\boldsymbol{v})$ with time t and space $\boldsymbol{r}$. In a collisionless plasma, this equation is the "Vlasov equation." In a collisional plasma, several equations such as the Boltzmann equation, can be used depending what approximate modeling has been adopted to describe the collisions.

Supposing, for instance, that we have to solve an initial value problem, the principles of the two methods are as follows:

1. Kinetic. To use this method, one is supposed to know the distribution function $f(t = 0)$ in the initial condition. The kinetic equation then allows one to determine $f(t)$ at any later time. Finally, as one is generally interested in the macroscopic parameters such as $\rho(t)$, $\boldsymbol{u}(t)$ and $P(t)$, the resulting distribution

function has to be integrated over velocities to determine them (they will be shown to be moments of $f(t)$).

2. Fluid. Starting for the initial macroscopic parameters such as $\rho(t = 0)$, $\boldsymbol{u}(t = 0)$ and $P(t = 0)$, one solves a differential system relating the variations of the moments to each other. The result is then directly the values of the moments at time t: $\rho(t)$, $\boldsymbol{u}(t)$ and $P(t)$.

For comparing the two methods, first one has to know the relationship between the "fluid moment" system and the original kinetic equation. We will show that all the equations of this system, except one, can be derived directly from the kinetic equation by integration. These moment equations are, therefore, as exact as the initial kinetic equation and do not introduce any further approximation with respect to it. Nevertheless, we will see that the systematic integration actually provides an infinity of moment equations (continuity equation, transport of momentum, pressure, and so on), but that each of them relate the moment of order n to the moment of order $n + 1$ (for instance, the pressure temporal variations to the heat flux spatial ones). For this reason, for solving a closed system with a finite number of equations, one is obliged to add a "closure equation", which is not obtained by integration. This is where all the approximation lies.

Another difference between the two methods has to be outlined, however: the fluid method supposes that the moments are known in the initial condition, while the kinetic one demands that the full distribution function is known. This makes a big difference. If only the initial moments are known, the later evolution is a priori not unique since a finite number of moments does not determine a unique distribution function. We will see that some evolutions are much more probable than others, but it is clear from this remark that, whatever the closure equation, the fluid method selects a particular class of distribution function perturbations. We will show in Chapter 5 that this point is crucial to understand why waves in a collisionless plasma are always damped with ordinary initial conditions (Landau damping [13]).

This book intends to be a basic textbook of plasma physics, and it, therefore, covers most classical topics of the domain, such as turbulence (weak/strong), magnetic reconnection, linear waves, instabilities, and nonlinear effects. In each domain, it starts from zero and tries to lead in a self sufficient manner to a view in accordance with the 2013 state of the art. Its main specificity is, however, to pay particular attention, in each domain, to the collisionless limit and the consequences of the different modelings, fluid or kinetic in this case. Many kinetic results in the collisionless limit may appear counterintuitive. For instance, it may appear surprising that the nondissipative Vlasov equation always leads to a damping of the waves; it is surprising as well to find a heat flux in the low solar corona, in a sense opposite to the temperature gradient. The main reason for all these surprises, is that our intuition, for many fundamental physical notions such as irreversibility, has been built in the more usual strongly collisional limit. This makes separating the universal concepts from those that are linked to this limit difficult. These basic notions, for this reason, are specially developed in the book, beginning with the nontrivial notion of collision and of mean free path in a plasma.

The book is designed for an audience of students and researchers. Those who discover the domain should find the essential basic notions. Those who already know them should find the necessary perspective to approach some profound questions concerning the collisionless limit. The book should also help understanding the necessary compromises to be made for modeling plasmas in different circumstances, the global fluid modeling being often necessary to complement the kinetic one, the latter being easily handleable only at small scales and for simple geometries such as 1D.

Most of the examples of the book for illustrating the theoretical concepts are taken in space physics (planetary magnetospheres) and in solar wind. Some others examples concern more remote astrophysical objects (see, for instance, Chapter 8). This choice of "natural plasmas" has been done for insuring homogeneity of the book and respecting the specialties of the authors. However, these examples must be understood only as illustrations. The concepts that are so illustrated are universal and of course not limited to them. Researchers working on laboratory plasmas, in particular, on magnetic confinement for nuclear fusion, are expected to find their interest as well in the presentation.

1.2 Plasmas in Astrophysics

1.2.1 Plasmas Are Ubiquitous

Most of the baryonic matter in the universe resides in the stars, whose hot interiors are made of plasma. Apart from the coolest ones, most star atmospheres are made of plasmas, as are their coronas. The outer parts of stellar coronas are made of tenuous plasmas generally in expansion, called stellar winds. Some of the gas clouds in galaxies can be ionized by neighboring stars. This is the case in the HII regions, forming vast clouds of hot and tenuous hydrogen rich plasmas. On a larger scale, in clusters of galaxies, the development of X-ray astronomy has revealed huge clouds of hot plasma filling the space between the galaxies.

If most of the planetary materials are made of neutral atoms and molecules, their nucleus is composed of a very dense nucleus of degenerate plasma partly supported by the Fermi pressure of free electrons. On the opposite side, the outskirts of the planetary atmospheres are an ionosphere, and possibly a magnetosphere, made of dilute plasmas in interaction with the wind of their star.

The physics of the fully ionized collisionless plasmas is the key element to understanding the corona and wind of stars (including the Sun), the magnetosphere of the planets, and a large variety of shock waves present in various astrophysical contexts.

1.2.2
The Magnetosphere of Stars

The lower layers of a star atmosphere are made of collisional plasma, and the ambient magnetic field has a complex structure involving many scales. It is often represented as a global simple magnetic field superimposed with a multiplicity of open or closed magnetic flux tubes. Some groups of magnetic flux tubes can be isolated, and constitute relatively coherent systems dominated by the plasma pressure forces and the magnetic field. Both the plasma and the magnetic fields evolve; they constitute a dynamical system.

It is not possible to measure directly the magnetic field in the star magnetosphere (it is only possible on the photosphere). Therefore, analytical and numerical models play an important role in their study. The models are generally based on the theory of dissipative plasmas, and the dissipation is attributed to collisions.

As the distance to the star increases, the density is reduced, while the temperature tends to increase (above the chromosphere) and then remains at a high level (typically 10^6 K). Therefore, farther from the star, the magnetosphere is less and less collisional. At the altitude where the magnetic flux tubes are open, the plasma flow velocity is high and supersonic; it is called a *stellar wind*. The *solar wind* is a collisionless plasma. With spacecrafts, in situ measurements of the magnetic field of waves, chemical composition and particle distribution functions have been performed down to sun distances of 0.3 au. As far as it has been measured, the solar wind was always supersonic and faster than MHD waves (see Section 5.1) and noncollisional.

From a theoretical point of view, the boundary conditions that define a star corona are a hot and collisional plasma with a generally complex magnetic field at its base, and a fast expanding plasma wind expanding into the interstellar medium on the other side. The star rotation must be taken into account for the consideration of the overall structure of the magnetosphere.

1.2.3
Shock Waves

As soon as a stellar wind meets another kind of medium, there is an interaction that is preceded by a shock, as long as the difference of velocities between the wind and the object exceeds the speed of sound and/or MHD waves. In collisionless plasmas, shock waves do not have visual signatures, but they can be radio emitters. Therefore, some of them can be studied remotely. In the solar wind, shock waves happen when a stream of fast wind reaches a slower one. They also develop upstream of planets or comets, provided that they are surrounded by an atmosphere. The shocks upstream of a solid obstacle are called *bow shocks*. The bow shock of the Earth is at a distance of about 10 earth radii, in a purely collisionless plasma. Bow shocks upstream of nonmagnetized planets such as Mars or Venus are, along the Sun–planet direction, very close to or inside the ionosphere that is a collisional plasma.

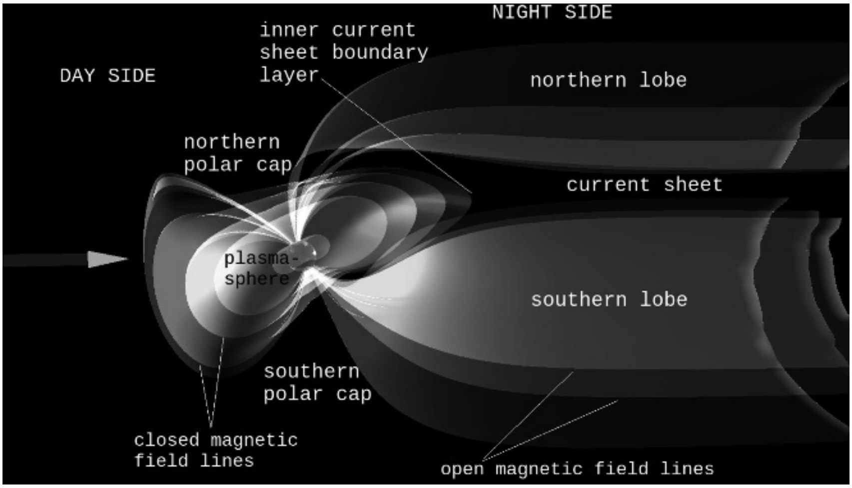

Figure 1.3 Shells of magnetic field lines (starting at the same magnetic latitude from the Earth's surface) showing various regions of the Earth's magnetosphere. The solar wind comes from the left-hand side. Image: courtesy of Bruno Katra, computed with the model [1].

Far from the Sun, the interface between the solar wind and the interstellar plasma is expected to include two shock waves. One has already been crossed by the Voyager 1 and 2 spacecrafts. This interface is called the *heliopause*.

The phenomenology of collisionless shocks is associated with particle acceleration, radio wave emissions and turbulence. They also exist on the borders of fast plasma flows associated with the remnants of supernovae. These shocks are potential sources of galactic cosmic rays (see Chapter 8).

1.2.4
Planetary Magnetospheres

From a theoretical point of view, planetary magnetospheres are the interface between a rotating spherical and magnetized body with a conducting surface (usually an ionosphere) and a stellar wind. The Earth and all the giant planets (Jupiter, Saturn, Uranus, Neptune) have a magnetic field; they are all surrounded by a magnetosphere. The largest magnetosphere in the solar system is that of Jupiter. It itself contains the smaller magnetosphere of its magnetized satellite Ganymede.

The most explored magnetosphere is, of course, that of the Earth, represented in Figure 1.3. As with other magnetospheres, it is first preceded by a *bow shock*, mentioned in Section 1.2.3. Behind the bow shock is a region of fast and turbulent plasma called the *magnetosheath*. In the magnetosheath, the majority of the magnetic field lines are convected in the same direction as the magnetic field (see Section 1.3.1.3 for the explanation of field lines motion), and they are not connected to the Earth. Then, a sharp transition is met: the *magnetopause*. Behind the magnetopause, all the magnetic field lines are connected to the Earth, at least on one end.

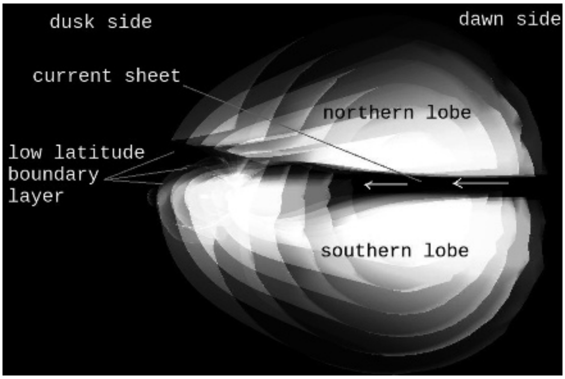

Figure 1.4 Shells of magnetic field lines (starting at the same magnetic latitude from the Earth's surface). A cut is made in the plane perpendicular to the solar wind direction. The white arrows represent the current density direction in the current sheet. On the dusk side, the low latitude boundary layer, which bounds the current sheet, is shown. There is a similar boundary on the dawn side. Image: courtesy of Bruno Katra, computed with the model [1].

The region enclosed by the magnetopause is properly called the magnetosphere. The magnetopause has a few singularities, where magnetic field lines connected in these regions to the Earth can easily (from a topological point of view) be connected to solar wind field lines. The most well known are the *polar caps*, but there are also the flanks of the magnetopause at low latitude, also called the *low latitude boundary layer* (Figure 1.4). The magnetosphere has an asymmetric profile. On the dayside, its extent is of the order of 10 earth radii. On the nightside, the magnetosphere is very elongated, forming the *magnetotail*. Two vast regions, where the magnetic field is almost aligned with the Earth–Sun direction, on the northern and southern sides have a very low density, and are called the *lobes*. The lobes are among the least dense regions of the solar system (about 0.1 particle/cm^3). Between the two lobes is a region of inversion of the direction of the magnetic field; it is the *neutral sheet* (see Figure 1.4). Because an electric current oriented in the east–west direction supports this magnetic field inversion, it is also called the *current sheet*. But this region is also much denser than the lobes, and it is called the *plasma sheet*. The various names of this region are a token of its importance in regards to the physics of the magnetosphere.

Closer to the Earth, and at low latitudes (below the polar cap) there is a denser region of plasma that corotates with the Earth, called the *plasmasphere*. The inner boundary of the plasma sheet is close to the nightside of the plasmasphere boundary, at a distance of about 6–10 earth radii. At higher latitudes, in a region where the magnetic field lines are still connected to the magnetotail, there is an occasional plasma acceleration that causes polar auroras on the ionosphere. This area (from the ionosphere up to a few earth radii of altitude along the field lines) is called the *auroral region*. At even higher altitudes, the plasma is connected to the solar wind via open field lines, in the *polar cap* and the *cusp* regions.

At the distance of the giant planets the solar wind is weaker (especially because of its density varying as d^{-2} where d is the Sun–planet distance). It, therefore, exerts a weaker pressure than on the Earth. Planetary rotation is another source of energy. For Jupiter, with a 10 h rotation and a strong magnetic field, the effect of the rotation dominates those of the solar wind. The plasma in corotation extends quite far from the planet, and the particles inertia in the rotating motion favors the settlement of an extended *ring current* region.

A ring current also exists around the Earth and is associated with the magnetospheric compression by the solar wind; therefore, its origin is of a different nature than fast rotating planets.

1.3 Upstream of Plasma Physics: Electromagnetic Fields and Waves

1.3.1 Electromagnetic Fields

The Maxwell equations describe the time and space variations of the electromagnetic field due to its sources: the charge density ϱ and the current density $\boldsymbol{j}$. In classical physics and in special relativity, the electromagnetic field can be split into two different fields, the electric field $\boldsymbol{E}$ and the magnetic (or induction) field $\boldsymbol{B}$. The four Maxwell equations relating their variations are respectively called the Maxwell–Gauss, Maxwell–Ampère, Maxwell–Faraday, and divergence-free equations, and they are:

$$\text{Gauss} \quad \nabla \cdot (\varepsilon_0 \boldsymbol{E}) = \varrho \tag{1.1}$$

$$\text{Ampère} \quad \nabla \times (\boldsymbol{B}/\mu_0) = \boldsymbol{j} + \partial_t(\varepsilon_0 \boldsymbol{E}) \tag{1.2}$$

$$\text{Faraday} \quad \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \tag{1.3}$$

$$\text{div-free} \quad \nabla \cdot \boldsymbol{B} = 0\,. \tag{1.4}$$

The constants ε_0 et μ_0 are called, respectively, the “dielectric permittivity” and “magnetic permeability” of the vacuum, and they appear in the Gauss and Ampère equations, which explicitly relate the fields to the ϱ and $\boldsymbol{j}$ sources. They are linked by the relation: $\varepsilon_0\mu_0 c^2 = 1$, which makes the constant c (speed of light) enter the system. In the Maxwell–Ampère equation, the term $\partial_t(\varepsilon_0 \boldsymbol{E})$ is called the displacement current. The pure “Ampère equation”, applicable in magnetostatic fields, does not include this term. Nevertheless, in short, we use here the name “Ampère equation” even in the nonstationary case.

In vacuum, the source terms ϱ and $\boldsymbol{j}$ are zero and the electromagnetic field is made of harmonic functions of space for each field, superposed with a linear superposition of electromagnetic waves propagating with the speed c. When sources are present (in particular in plasmas), the charge density changes the electric field, adding an "electrostatic" component; the electric current modifies the magnetic field and also the electric field via an "induced" component (whenever the magnetic field varies in time). The two kinds of sources are always related by the equation of charge conservation:

$$\nabla \cdot \boldsymbol{j} + \partial_t \varrho = 0 . \tag{1.5}$$

This equation can of course be derived from the equations of motion of the source charges, but also from the above Maxwell equations (divergence of Eq. (1.2) and temporal derivative of Eq. (1.4)), which outlines the necessary consistency between the electromagnetic fields and its sources.

From the Maxwell equations, an equation can be derived for the electromagnetic energy:

$$\partial_t E_{\mathrm{em}} + \nabla \cdot \boldsymbol{S} = -\boldsymbol{j} \cdot \boldsymbol{E} . \tag{1.6}$$

It relates the temporal variations of the electromagnetic energy

$$E_{\mathrm{em}} = \varepsilon_0 E^2/2 + B^2/2\mu_0 \tag{1.7}$$

to the divergence of the Poynting flux vector (energy arriving through the boundaries of a volume):

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}/\mu_0 \tag{1.8}$$

and to the term $-\boldsymbol{j} \cdot \boldsymbol{E}$, which represents the energy exchanges in volume between the electromagnetic field and the matter (for example the plasma).

1.3.1.1 The Scalar and Vector Potentials

The last two Maxwell equations (Faraday and divergence-free) are independent of the sources. They can usefully be integrated once, the former with respect to time, the latter with respect to space. This allows replacing the original fields $\boldsymbol{E}$ and $\boldsymbol{B}$ by two other functions: the scalar and vector potentials, Φ and $\boldsymbol{A}$, defined as:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{1.9}$$

$$\boldsymbol{E} = -\nabla \Phi - \partial_t \boldsymbol{A} . \tag{1.10}$$

With this formalism, the two last Maxwell equations are automatically satisfied and the first two (Gauss and Ampère) become:

$$\nabla^2 \Phi + \partial_t (\nabla \cdot \boldsymbol{A}) = -\varrho/\varepsilon_0 \tag{1.11}$$

$$\nabla^2 \boldsymbol{A} - \partial_t^2 \boldsymbol{A}/c^2 = -\mu_0 \boldsymbol{j} + \nabla \left(\nabla \cdot \boldsymbol{A} + \partial_t \Phi/c^2 \right) . \tag{1.12}$$

The potentials are prime integrals of the original fields; therefore, they are not unique. Each particular choice is characterized by a "gauge". The two most famous ones are called the Coulomb and Lorentz gauges. The Coulomb gauge is the simplest. It is defined by:

$$\nabla \cdot \boldsymbol{A} = 0 \,. \tag{1.13}$$

With this choice, Gauss and Ampère equations simplify into:

$$\nabla^2 \Phi = -\varrho/\varepsilon_0 \tag{1.14}$$

$$\nabla^2 \boldsymbol{A} - \partial_t^2 \boldsymbol{A}/c^2 = -\mu_0 \boldsymbol{j} + \nabla\left(\partial_t \Phi/c^2\right) \,. \tag{1.15}$$

In this gauge, the scalar potential is simply a solution of a Poisson equation (no propagation involved for Φ).

The Coulomb gauge is invariant in the nonrelativistic case (it keeps the same form in any inertial frame change), but not in relativity. The Lorentz gauge has better properties in this respect. It is defined by:

$$\nabla \cdot \boldsymbol{A} + \partial_t \Phi/c^2 = 0 \,. \tag{1.16}$$

It is slightly less simple, but it has the great advantage of dissociating Φ and $\boldsymbol{A}$ in their relations with the source terms. Indeed, Gauss and Ampère equations become in this case:

$$\nabla^2 \Phi - \partial_t^2 \Phi/c^2 = -\varrho/\varepsilon_0 \tag{1.17}$$

$$\nabla^2 \boldsymbol{A} - \partial_t^2 \boldsymbol{A}/c^2 = -\mu_0 \boldsymbol{j} \,. \tag{1.18}$$

In vacuum, the potentials defined in the Lorentz gauge just propagate at speed c. Moreover, this gauge is indeed invariant by any inertial frame change.

1.3.1.2 **Changes of Reference Frame**

The Maxwell equations are invariant in any change of reference frame, but the fields are not. In the nonrelativistic case, going from a frame R to a frame R' moving at a velocity $\boldsymbol{V}$ relative to R, the fields change as:

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} \tag{1.19}$$

$$\boldsymbol{B}' = \boldsymbol{B} \,. \tag{1.20}$$

In special relativity, one has to distinguish between the directions longitudinal and transverse relative to the velocity $\boldsymbol{V}$ of the frame change (respectively subscripts l and t):

$$\begin{aligned} E_\mathrm{l}' &= E_\mathrm{l} \\ \boldsymbol{E}_\mathrm{t}' &= \gamma\left(\boldsymbol{E}_\mathrm{t} + \boldsymbol{V} \times \boldsymbol{B}\right) \end{aligned} \tag{1.21}$$

$$\begin{aligned} B_\mathrm{l}' &= B_\mathrm{l} \\ \boldsymbol{B}_\mathrm{t}' &= \gamma\left(\boldsymbol{B}_\mathrm{t} - \boldsymbol{V} \times \boldsymbol{E}/c^2\right) \,. \end{aligned} \tag{1.22}$$

In these relations, the constant c (speed of light) appears explicitly, and also through the relativistic Lorentz factor $\gamma = 1/\sqrt{1 - V^2/c^2}$. It can be noted that the nonrelativistic case ($V \ll c$) corresponds to taking $\gamma = 1$ and neglecting $\boldsymbol{V} \times \boldsymbol{E}/c^2$ relatively to $\boldsymbol{B}_\mathrm{t}$.

Like the original fields, their first integrals, the potentials Φ and $\boldsymbol{A}$ are changed by a frame change. In the nonrelativistic case, they become:

$$\Phi' = \Phi - A_1 V \tag{1.23}$$

$$\boldsymbol{A}' = \boldsymbol{A} \,. \tag{1.24}$$

In special relativity:

$$\Phi' = \gamma (\Phi - A_1 V) \tag{1.25}$$

$$A_1' = \gamma \left(A_1 - \Phi\, V/c^2\right) \tag{1.26}$$

$$\boldsymbol{A}_\mathrm{t}' = \boldsymbol{A}_\mathrm{t} \,. \tag{1.27}$$

Finally, in the same referential change, the source terms become, in the nonrelativistic case:

$$\varrho' = \varrho - j_1 V/c^2 \tag{1.28}$$

$$\boldsymbol{j}' = \boldsymbol{j} - \varrho \boldsymbol{V} \tag{1.29}$$

and in special relativity:

$$\varrho' = \gamma \left(\varrho - j_1 V/c^2\right) \tag{1.30}$$

$$j_1' = \gamma (j_1 - \varrho V) \tag{1.31}$$

$$\boldsymbol{j}_\mathrm{t}' = \boldsymbol{j}_\mathrm{t} \,. \tag{1.32}$$

An important remark has to be made concerning Eqs. (1.28) and (1.30). As usual, the nonrelativistic case derives from the relativistic one by taking $\gamma = 1$. But it must be emphasized that the term $j_1 V/c^2$ exists in both relativistic and nonrelativistic cases: this term cannot be neglected with respect to ϱ in general. Neither the current nor the charge density remain invariant in an inertial referential change. The change in charge density at zero order in V/c is actually consistent with the change in electric field: its electrostatic part appears due to the appearance of the electric charge.

These changes of reference frame are exact when V is a time invariant uniform velocity. When $V = V(t)$ is variable, one has to associate, at every time t, a tangent change of reference frame associated with the instantaneous value of $V(t)$. In that case, the local equations can be still used. This is the case in Eqs. (1.19)–(1.22). (Because V is a parameter of the Lorentz transform, it is considered local.) When the derivatives (charge and current densities) or integrals (potentials) are considered, the corresponding derivatives and integrals of V can introduce terms that do not appear in the above formulas. This is illustrated in Section 1.3.1.4 where the case of a rotating plasma is considered.

1.3.1.3 Notion of "Magnetic Velocity"

The Maxwell equations involve the constant parameter c, which has the dimension of speed. But there is actually another speed which derives directly from the fields themselves:

$$\boldsymbol{v}_{\mathrm{m}} = \boldsymbol{E} \times \boldsymbol{B}/B^2 \,. \tag{1.33}$$

This velocity can be called the "magnetic velocity" since, locally, the electric field is zero in the frame moving at $\boldsymbol{v}_{\mathrm{m}}$; this means that the electromagnetic field is purely magnetic in this frame. This property gives to the velocity $\boldsymbol{v}_{\mathrm{m}}$ a major importance in plasma physics. In particular, in a quasi-homogeneous field, it is known that the particles rotate with a negligible drift velocity in this frame. This means that they follow, on average, the magnetic motion so defined. This is the origin of the so-called ideal Ohm's law used in MHD (see Chapter 3).

From the purely electromagnetic point of view, the magnetic velocity is also important for allowing, in certain circumstances, to define a "magnetic field line motion". The magnetic field is indeed often represented by its field lines. They are by definition tangent to $\boldsymbol{B}$. In rectangular coordinates they are the solutions of

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} \,. \tag{1.34}$$

They can always be defined at any time and any point (except at the null points, that is points where $B = 0$, if any). The concept of "magnetic field line motion" is meaningful in all cases when the lines are equipotential, that is when the component $E_{\parallel}$ parallel to the magnetic field is zero all along of them. In these conditions (and even in conditions slightly more general, see next section), one can prove that the field lines "move at velocity $\boldsymbol{v}_{\mathrm{m}}$". This means that if all points of a given line are moved at velocity $\boldsymbol{v}_{\mathrm{m}}$, they still are all on the same field line at any time later. In the limits of validity of the condition $E_{\parallel} = 0$, it is, therefore, a usual – and quite useful – concept to consider that the line at different times is the same, which just moves. In this way, one gives an identity to the field lines, which can be viewed as kinds of "rubbers" moving and deforming. This concept is particularly important when studying low frequency fluctuations in a plasma ("MHD range"). This property of "freezing" of the field lines in the velocity field $\boldsymbol{v}_{\mathrm{m}}$ is purely electromagnetic since no plasma parameter is involved, neither in the velocity definition nor in the demonstration (Maxwell equations are sufficient). Nevertheless, the condition of validity $E_{\parallel} = 0$ is actually imposed – or not – by the plasma. We will see in Chapter 3 that this condition is actually verified in a plasma at sufficiently large scales.

It can also be noted that the velocity $\boldsymbol{v}_{\mathrm{m}}$ is collinear and close (within a factor of 2) to the velocity of propagation $\boldsymbol{V}_{\mathrm{em}}$ of the electromagnetic energy E_{em}. The relation between $\boldsymbol{v}_{\mathrm{m}}$ and $\boldsymbol{V}_{\mathrm{em}}$ is evident if the Poynting flux, which is defined as $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}/\mu_0$ is written as $\boldsymbol{S} = \boldsymbol{V}_{\mathrm{em}} E_{\mathrm{em}}$, with $E_{\mathrm{em}} = \varepsilon_0 E^2/2 + B^2/2\mu_0$.

Proof that the field lines move at velocity $\boldsymbol{v}_{\mathrm{m}}$ when $E_{\parallel} = 0$. Let $\delta \boldsymbol{l}$ be a vector that connects two points P_1 and P_2 located on the same field line and separated by an infinitesimal distance. Because $\delta \boldsymbol{l}$ is parallel to the magnetic field, $\boldsymbol{C} = \delta \boldsymbol{l} \times \boldsymbol{B} = 0$. If this vector can be shown to be invariant in motion at velocity $\boldsymbol{v}_{\mathrm{m}}$, this will prove that $\delta \boldsymbol{l}$ is always parallel to $\boldsymbol{B}$, and, therefore, that P_1 and P_2 will remain on the same field line. Noting $D_t = \partial_t + \boldsymbol{v}_{\mathrm{m}} \cdot \nabla$, one has:

$$D_t(\boldsymbol{C}) = D_t(\delta \boldsymbol{l} \times \boldsymbol{B}) = D_t(\delta \boldsymbol{l}) \times \boldsymbol{B} + \delta \boldsymbol{l} \times D_t \boldsymbol{B} \, .$$

The first term can be expressed thanks to the definition of D_t:

$$D_t(\delta \boldsymbol{l}) = \boldsymbol{v}_{\mathrm{m}}(\boldsymbol{l} + \delta \boldsymbol{l}) - \boldsymbol{v}_{\mathrm{m}}(\boldsymbol{l}) = \delta \boldsymbol{l} \cdot \nabla \boldsymbol{v}_{\mathrm{m}} \, .$$

The second term can be expressed thanks to Faraday's equation:

$$D_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} + \boldsymbol{v}_{\mathrm{m}} \cdot \nabla \boldsymbol{B} \, .$$

Using the definition of $\boldsymbol{v}_{\mathrm{m}}$, one can express the electric field $\boldsymbol{E}$ as:

$$\boldsymbol{E} = \boldsymbol{E}_{\|} - \boldsymbol{v}_{\mathrm{m}} \times \boldsymbol{B} \, .$$

Replacing $\boldsymbol{E}$ by this expression and developing the curl of the cross product as explained in Appendix A.1, a little algebra provides:

$$\delta \boldsymbol{l} \times D_t \boldsymbol{B} = -\delta \boldsymbol{l} \times (\nabla \times \boldsymbol{E}_{\|}) - B \delta l \boldsymbol{b} \cdot \nabla(\boldsymbol{v}_{\mathrm{m}}) \times \boldsymbol{b} \tag{1.35}$$

$$D_t(\delta \boldsymbol{l}) \times \boldsymbol{B} = B \delta l \boldsymbol{b} \cdot \nabla(\boldsymbol{v}_{\mathrm{m}}) \times \boldsymbol{b} \, . \tag{1.36}$$

The vector $\boldsymbol{b}$ is the unit vector of the field line. The sum of the two equations finally provides the variation of C we were looking for:

$$D_t(\boldsymbol{C}) = -\delta \boldsymbol{l} \times (\nabla \times \boldsymbol{E}_{\|}) \, . \tag{1.37}$$

We can, therefore, conclude that $E_{\|} = 0$ is a sufficient condition to get the freezing-in property of the field lines in the $\boldsymbol{v}_{\mathrm{m}}$ velocity field: if P_1 and P_2 move with the magnetic field velocity $\boldsymbol{v}_{\mathrm{m}}$, they remain on the same field line. The condition $\delta \boldsymbol{l} \times (\nabla \times \boldsymbol{E}_{\|}) = 0$, more general and slightly less restrictive, is rarely used because, if the condition $E_{\|} = 0$ is often satisfied at large scale in a plasma because of the electron motion, there is no such physical justification for the more general condition.

1.3.1.4 **Space Plasmas in Corotation with Their Planet/Star**

The magnetized bodies in rotation, such as planets or stars, are ubiquitous in the universe. Their magnetic field generally comes from an internal "dynamo" source, but it can also be remnant fields in some occasions. These bodies are generally embedded in plasmas of external origin and one is justified to ask whether these plasmas will remain insensitive to the body rotation or if they will be drawn into this rotation. As shown in Chapter 3, the plasma always follows, at large scale, the "magnetic motion" $\boldsymbol{v}_{\mathrm{m}}$ of the field lines. Near the body surface, the magnetic field is generally rigidly anchored to it; consequently, the plasma can be considered in corotation with the magnetized body. Let us first see the consequences of the corotation of a plasma with a magnetized body.

The notion of corotation. Let $\boldsymbol{\Omega}$ be the body rotation velocity. The corotating plasma has a velocity $\boldsymbol{V} = \boldsymbol{\Omega} \times \boldsymbol{r}$. Because the magnetic field lines (close to the body) follow the same motion, $\boldsymbol{V}$ is also the motion of the magnetic field lines defined in Eq. (1.33). This sets the existence of the so-called corotation electric field,

$$\boldsymbol{E} = -\boldsymbol{V} \times \boldsymbol{B} = -(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{r}\boldsymbol{\Omega} - \boldsymbol{B} \cdot \boldsymbol{\Omega}\boldsymbol{r} \,. \tag{1.38}$$

This is the electric field that an observer would see in the inertial frame of reference where the body velocity is null.

It is important to mention at this point that if corotation is generally assumed at very close distance to the body, it is not granted at a larger distance. Even when the plasma moves with the magnetic field line velocity $\boldsymbol{v}_{\mathrm{m}}$, this velocity can be different from the corotation velocity $\Omega \times \boldsymbol{r}$, provided that the plasma has an appropriate retroaction on the shape of the magnetic field lines. It is shown in Section 1.3.3.3 that even in vacuum, the shape of the magnetic field lines depend on the rotation rate Ω of the body; therefore, it is easy to understand that this happens too with a plasma.

Terrestrial magnetosphere and ionosphere. Most of the Earth's ionosphere is in corotation. This means that the plasma in the ionosphere is exposed to the same alternation of nights and days as the Earth's surface. When the ionosphere is exposed to sunlight, the UV increase the ionization rate, while it is zero at night. In the range of altitudes above 400 km, the recombination rate of the ions is low in comparison to the duration of the night, and the plasma density of this ionospheric layer remains roughly constant. But in the range of 60–350 km, the recombination rate is higher, and the ionospheric plasma content at these altitudes varies periodically with the same period as the Earth's rotation with a minimum in the morning hours.

Above the ionosphere, in the range of latitudes $\sim \pm 60°$, the plasma is trapped along closed magnetic field lines. The plasma filling this region has escaped from the ionosphere. It is cold ($T \sim 1$ eV) in comparison to the $T \sim 10^2 - 10^4$ eV plasma found in the solar wind and other regions of the magnetosphere. It is in corotation with Earth. As with the ionosphere (also in corotation), it is asymmetric relative to local time. Its extension is typically 7 R_{E} on the evening side after having been refilled with ionospheric plasma, and 4 R_{E} on the morning side (after spending a night above a less dense and colder ionosphere). The plasma there is denser than anywhere else in the magnetosphere. This zone of corotating plasma is the *plasmasphere*. On the nightside of the Earth, the plasmasphere ends where the magnetotail begins. Compared to the magnetotail, the plasmasphere is a rather quiet region. The auroras observed in the midnight sector are magnetically connected to the magnetotail, therefore, at magnetic latitudes above those of the plasmasphere.

Other magnetospheres. Other magnetospheres contain a plasma in corotation. Actually, most of the magnetosphere of Jupiter is in corotation. More precisely, it is subcorotating. This means that the main component of the plasma velocity is $\boldsymbol{\Omega}'(r) \times \boldsymbol{r}$, with $\Omega'(r)$ close to but smaller than the angular velocity Ω of the planet. The plasma corotating with Jupiter contains the orbit of the closest Galilean satellite, Io, situated at a distance of six Jovian radii from Jupiter's surface. The observations tend to show that Jupiter's main auroral oval corresponds to the interaction of the (sub) corotating region with the noncorotating plasma.

From Eq. (1.1) and the Maxwell–Gauss equation, a charge density can be associated with the corotation electric field,

$$\varrho = -2\varepsilon_0 \boldsymbol{B} \cdot \boldsymbol{\Omega} - \varepsilon_0 (\boldsymbol{r} \times \boldsymbol{\Omega}) \cdot \nabla \times \boldsymbol{B} \,. \tag{1.39}$$

(We notice that the first term would not appear with a direct application of Eq. (1.28). This is because the charge density equation is not local, and the space derivative of the velocity $\boldsymbol{V}$, supposed null in Eq. (1.28), has been taken into account.) Considering the Maxwell–Ampère equation, and the fact that the partial time derivative of $\boldsymbol{E}$ is orthogonal to $\boldsymbol{r} \times \boldsymbol{\Omega}$, for a magnetic field that is not associated with an electric current (for instance a dipole field, see Section 1.3.3.1), we find the Goldreich–Julian density

$$\varrho = -2\varepsilon_0 \boldsymbol{B} \cdot \boldsymbol{\Omega} \,. \tag{1.40}$$

For Jupiter, this corresponds to a particle density $\varrho/e \sim 10^{-28}\,\mathrm{cm}^{-3}$ that is totally negligible. That is not the case with pulsars: they have a fast rotation rate (with a period of 1 s or less) and a strong magnetic field (typically 10^8 T) and the Goldreich–Julian charge density can correspond to an excess of 10^{16} electrons or positrons/m^3. The pulsars illustrate the fact that a plasma in corotation around a highly magnetized fast rotating body is nonneutral.

There is an absolute limit to the size of a corotation region, called the light cylinder radius $R_{\mathrm{LC}} = c/\Omega$. This is the distance at which the corotation velocity would be the speed of light. For all the objects in the solar system, R_{LC} is much larger than their magnetosphere. In the case of pulsars, the light cylinder is well inside the magnetosphere. In most models, it defines broadly the frontier between the inner magnetosphere, with a mixture of corotation and poloidal motion, and the wind, where the plasma motion is mostly radial.

1.3.2
Transverse and Longitudinal Electromagnetic Field

Usefully, the electric and magnetic fields can be considered, as any vector fields, as the sum of an irrotational (or longitudinal) component, and of a solenoidal (or transverse) component:

$$\boldsymbol{E} = \boldsymbol{E}_{\mathrm{l}} + \boldsymbol{E}_{\mathrm{t}} \quad \text{and} \quad \boldsymbol{B} = \boldsymbol{B}_{\mathrm{l}} + \boldsymbol{B}_{\mathrm{t}} \,. \tag{1.41}$$

These components are defined by:

$$\nabla \cdot \boldsymbol{E}_{\mathrm{t}} = 0, \nabla \cdot \boldsymbol{B}_{\mathrm{t}} = 0 \tag{1.42}$$

$$\nabla \times \boldsymbol{E}_{\mathrm{l}} = 0, \nabla \times \boldsymbol{B}_{\mathrm{l}} = 0\ . \tag{1.43}$$

The names "solenoidal" and "irrotational" are the most general ones. They are here often replaced by "transverse" and "longitudinal" by reference to the simple case of plane variations. These names are then defined with respect to the gradient (for example, the wave vector for a plane wave). It must not be confused with their use in the above section where "transverse" and "longitudinal" are defined with respect to the relative velocity between two different reference frames.

The Faraday equation involves only the transverse fields; the divergence-free and Gauss equations involve only the longitudinal ones. The Ampère equation can be applied separately to the two components. The Darwin approximation (see Section 1.3.10) is based on this decomposition of the electric field:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}_{\mathrm{t}} + \frac{1}{c^2}\partial_t \boldsymbol{E}_{\mathrm{t}} \tag{1.44}$$

$$\mu_0 \boldsymbol{J}_{\mathrm{l}} + \frac{1}{c^2}\partial_t \boldsymbol{E}_{\mathrm{l}} = 0\ . \tag{1.45}$$

Many wave properties can be analyzed in terms of this longitudinal and transverse decomposition.

1.3.3
Electromagnetic Fields in Vacuum

Plasma physics is a coupled system involving electromagnetic fields and charged particles. To better understand how the charged particles rule the electromagnetic field, it is worth recalling what are the properties of this field in the absence of particles, that is, in vacuum. This corresponds to the absence of second members of Maxwell equations: $\varrho = 0$ and $\boldsymbol{J} = 0$.

1.3.3.1 **Static Fields**

Because of the absence of charge and current density, one has: $\nabla \cdot \boldsymbol{E} = 0$ and $\nabla \times \boldsymbol{B} = 0$. For static fields, one has also: $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{E} = 0$, which show that electric and magnetic fields have the same properties. Then, both fields can be treated as gradients of scalar fields $\boldsymbol{E} = -\nabla \Phi$ and $\boldsymbol{B} = \nabla \Psi$.

For astrophysical spherical bodies (planets, stars), it is often useful to approximate at large distance the magnetic field as a dipole field. In spherical coordinates, $\boldsymbol{B}$ derives from $\Psi = M\cos(\theta)/r^2$. Then $B_r = -2M\cos(\theta)r^{-3}$, $B_\theta = -M\sin(\theta)r^{-3}$, $B_\phi = 0$, and $B = Mr^{-3}[1 + 3\sin^2(\theta)]^{1/2}$. A magnetic field line obeys the equation $r = R_{\mathrm{B}}\sin^2(\theta)/\sin^2(\theta_0)$ where R_{B} is the body radius and θ_0 determines a particular field line. The radius of curvature of the magnetic field lines is:

$$R_c = \frac{r_0}{3}\frac{\cos\lambda(1+3\sin^2\lambda)^{3/2}}{1+\sin^2\lambda} = \frac{a}{3\cos^2\Lambda}\frac{\cos\lambda(1+3\sin^2\lambda)^{3/2}}{1+\sin^2\lambda}\,. \tag{1.46}$$

At the equator ($\lambda = 0$), the curvature radius is a third of the distance to the dipole center.

1.3.3.2 **Waves**

Time dependent electromagnetic fields are the solutions of vectorial d'Alembert equations, directly deriving from Ampère and by the Faraday's laws.

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\partial^2_{tt}\boldsymbol{E}, \quad \text{and} \quad \nabla^2 \boldsymbol{B} = \frac{1}{c^2}\partial^2_{tt}\boldsymbol{B}\,. \tag{1.47}$$

Looking for sinusoidal plane waves, these equations lead to $\omega^2/(k^2c^2) = 1$, showing that these perturbations propagate at the speed of light in any direction. Going back to the original Ampère and Faraday's laws, it is easy checking that their amplitudes also have the following properties: $\boldsymbol{E}_0 = c\boldsymbol{B}_0$, $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$, $\boldsymbol{k}\cdot\boldsymbol{B}_0 = 0$ and $\boldsymbol{E}_0\cdot\boldsymbol{B}_0 = 0$. As all plane waves can be described as the sum of sinusoidal functions (linear equations), all plane waves have these same properties. For each particular sinusoidal solution, the amplitudes and the phases of $\boldsymbol{E}$ or $\boldsymbol{B}$ determine the polarization.

1.3.3.3 **Electromagnetic Wave of a Rotating Neutron Star, Pulsar Families**

Magnetized planets and stars are generally considered as conducting bodies in rotation. We have seen in Section 1.3.1.4 what happens when they are surrounded by a corotating plasma. We now examine the situation when there is no plasma in their environment, apart from a conducting plasma rotating with the body on its surface. In this simpler case, it is possible to compute the magnetic field far from the surface of the body.

The magnetic field near the surface is approximated as an inclined dipole making an angle I with the rotation axis. It turns with a frequency Ω that is equal, or closely related, to the spin frequency of the body. On the surface, the material is a conductor that behaves like a corotating plasma (it can be a thin ionosphere for planets, or a metal crust for pulsars). Therefore, there is a corotation electric field. Because the body is not surrounded by a plasma, the electromagnetic field outside is a solution of Eq. (1.47). The time derivative in Eq. (1.47) is caused by the rotation with a frequency Ω; it is:

$$-\left(\frac{\Omega}{c}\right)^2 \boldsymbol{B} = \nabla^2\boldsymbol{B}\,. \tag{1.48}$$

The characteristic length $R_L = c/\Omega$ is called the *light cylinder* radius. It is the distance, projected onto the equatorial plane, at which the corotation speed would equal the speed of light.

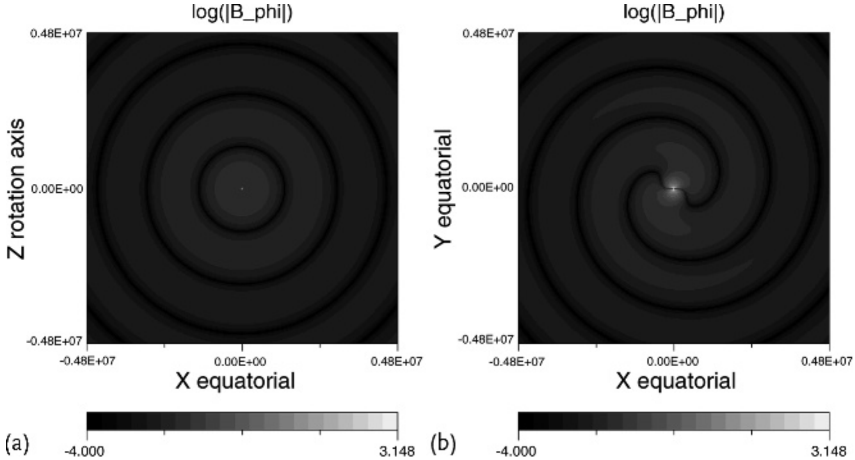

Figure 1.5 The azimuthal component of the magnetic field associated with a rotating conducting sphere with a dipole. The grey level scale represents $\log \|B_\phi\|$. (The minimal value -4 is set arbitrarily when B_ϕ becomes smaller.) The tilt angle between the magnetic and rotation axes is 50°, the star's radius is 10^4 m, and its rotation period is 10 ms. We can notice spiral shaped regions separated by a zone of low B_ϕ value. On each side of these low magnetic field regions, the sign of B_ϕ changes. The thickness of these spiral shaped regions is $2\pi R_{LC}$. This computation is based on the full solution, and not only the asymptotic solution. (a) $\log \|B_\phi\|$ on a meridional plane (relatively to the rotation axis). The vertical axis z is the rotation axis. (b) $\log \|B_\phi\|$ on the equatorial plane.

The complete calculation of the solution is rather tedious but can be completed fully analytically making use of spherical harmonics. At large distances $r \gg R_L$, the result can be put under the simplified form:

$$\begin{aligned}
B_r &= B_0 \frac{\Omega R^3}{c r^2} \sin I \sin\theta \sin\Phi \\
B_\theta &= B_0 \frac{\Omega^2 R^3}{2c^2 r} \sin I \cos\theta \cos\Phi \\
B_\phi &= -B_0 \frac{\Omega^2 R^3}{2c^2 r} \sin I \sin\Phi \\
E_r &= 0 \\
E_\theta &= -B_0 \frac{\mu_0 \Omega^2 R^3}{2cr} \sin I \sin\Phi \\
E_\phi &= -B_0 \frac{\mu_0 \Omega^2 R^3}{2cr} \sin I \cos\theta \cos\Phi \ .
\end{aligned} \tag{1.49}$$

This solution, displayed in Figure 1.5, is clearly not a plane wave. We have noted R the radius where the boundary solutions are defined (close to the star's surface). The electromagnetic field at these boundary conditions has been supposed to be a magnetic dipole B_s, inclined by an angle I over the rotation axis, associated with the corotation electric field $\boldsymbol{E} = -(\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{B}_s$. The continuity of the vertical magnetic field component, and of the horizontal electric field components are the constraints put on the electromagnetic field in vacuum and on the body surface: the vertical magnetic field is the same as for the dipole, the horizontal electric field is the same as for the corotation electric field. The phase Φ of the star rotation is defined as:

$$\Phi(r, \phi) = \phi + k(r - R)/R_{\mathrm{L}} - \Omega t . \tag{1.50}$$

From the above results, it can be checked that $\boldsymbol{E}$ and $\boldsymbol{B}$ vectors are perpendicular and that the wave radiates energy away at the rate

$$d_t E = \frac{2\pi\mu_0}{3c^3}\Omega^4 R^6 B_0^2 \sin^2 I . \tag{1.51}$$

It is an important result because it shows that the rate of energy radiated, which is an observable, depends very sharply on the rotation velocity. It can be used in the physics of magnetized fast rotating bodies such as white dwarfs (with $\Omega \sim 10^{-1}\,\mathrm{s}^{-1}$, for which the model was initially derived) and pulsars, with $\Omega \sim 10^1$–$10^4\,\mathrm{s}^{-1}$. It allows estimating the surface magnetic field B_0 of these objects. Thanks to pulsar timing over long lapses of time (with radio telescopes), it is possible to measure both Ω and $\dot{\Omega}$. For standard pulsars $\Omega \sim 6\,\mathrm{s}^{-1}$ and $\dot{\Omega} \sim -10^{-14}\,\Omega^2$. The energy loss $(1/2)M_{\mathrm{I}}\Omega\dot{\Omega}$ (where M_{I} is the momentum of inertia) associated with this slowing down is mostly due to the electromagnetic radiation defined in Eq. (1.51). Considering a radius $R \sim 10\,\mathrm{km}$, a star mass $M \sim 1$–$3\,M_{\mathrm{Sun}}$, one finds for standard pulsars a value $B_0 \sim 10^8\,\mathrm{T}$. This places the pulsars among the most magnetized bodies in the universe. A new class of pulsars with a typical rotation period of 10 s and a large time derivative of the angular velocity are found to reach $B_0 \sim 10^{11}\,\mathrm{T}$. They are called *magnetars*. Oppositely, pulsars with fast rotations rates ($\Omega \sim 10^3\,\mathrm{s}^{-1}$) have a lower $\dot{\Omega}$; they are found to be less magnetized, $B_0 \sim 10^5\,\mathrm{T}$. Several pulsar families can be so defined, depending upon the different correlations between rotation rates and surface magnetic field.

Of course, in reality, pulsars are not surrounded by vacuum. It was shown shortly after the discovery of the first pulsars that the electric force parallel to the magnetic field exceeds the gravitational forces by a factor 10^6. Therefore, pulsars are expected to host a magnetosphere filled with a plasma. Nevertheless, Eq. (1.51) is still used in first approximation to estimate the surface magnetic field. In some particular cases, the magnetic field has been estimated through other means, which confirmed the orders of magnitudes given here.

1.3.3.4 The Plasma as a Dielectric/Diamagnetic/Conducting Medium

Dielectric media The Maxwell–Gauss equation $\nabla \cdot (\varepsilon_0 \boldsymbol{E}) = \varrho$ describes how the spatial variations of the electric field are determined by the charge density in the medium. The electric field can actually be determined when this Maxwell equation is coupled with the equations of the medium which model how the charge density is determined by the electric field.

The charge density of the medium is in general a reaction to the electric field. In standard dielectrics such as silica, this reaction is just a “polarization” of the medium, which means that each atom, initially neutral, is transformed in a small electric dipole. This results in creating a density of electric dipole moment $\boldsymbol{P}$ called "dielectric polarization density". It can be shown that the corresponding charge density is the divergence of this vector field:

$$\varrho_{\mathrm{p}} = -\nabla \cdot \boldsymbol{P} . \tag{1.52}$$

If the polarization varies in time, it corresponds also to a "polarization current" $\boldsymbol{j}_{\mathrm{p}}$ given by:

$$\boldsymbol{j}_{\mathrm{p}} = \partial_t \boldsymbol{P} . \tag{1.53}$$

If the medium involves other types of charges, for instance free charges, which cannot be easily linked to the notion of electric dipole, this extra charge can be noted ϱ_{ex}:

$$\varrho = \varrho_{\mathrm{p}} + \varrho_{\mathrm{ex}} . \tag{1.54}$$

The further step is to assume that the polarization vector $\boldsymbol{P}$ depends on its cause, the electric field $\boldsymbol{E}$, through a simple relation:

$$\boldsymbol{P} = \boldsymbol{\chi}_{\mathrm{e}} \cdot \boldsymbol{E} . \tag{1.55}$$

It is worth noting that this relation is just algebraic and not differential, meaning that the medium polarization is supposed to depend on $\boldsymbol{E}$ linearly (if $\boldsymbol{\chi}_{\mathrm{e}}$ is constant) or nonlinearly otherwise, but not on the variations of $\boldsymbol{E}$.

When such a relation is satisfied, the medium is said to be "dielectric" and the coefficient $\boldsymbol{\chi}_{\mathrm{e}}$ is called the dielectric "susceptibility". In standard materials such as silica, the dielectric susceptibility can indeed be defined and measured; it is a known characteristic of the medium. It can, therefore, be used directly to model the field inside the material when it is put in a given environment, for instance, to study the role of a dielectric sheet inside a capacitor. When the polarization charge is independent of the electric field direction, the dielectric susceptibility is scalar: $\boldsymbol{\chi}_{\mathrm{e}} = \chi_{\mathrm{e}} \boldsymbol{I}$ ($\boldsymbol{I}$ being the identity tensor). Otherwise (anisotropic media), it is a full tensor.

Whenever such a dielectric susceptibility can be defined, the Maxwell–Gauss can be rewritten in the following form:

$$\nabla \cdot (\boldsymbol{\varepsilon} \cdot \boldsymbol{E}) = \varrho_{\mathrm{ex}} . \tag{1.56}$$

The tensor $\boldsymbol{\varepsilon} = \varepsilon_0 (\boldsymbol{I} + \boldsymbol{\chi}_{\mathrm{e}})$ is called the dielectric "permittivity", and the vector $\boldsymbol{D} = \boldsymbol{\varepsilon} \cdot \boldsymbol{E}$ is named "electric induction". In many instances, the extra charge ϱ_{ex} is zero.

Diamagnetic media Similar arguments can be used with respect to the magnetic field and its sources, that is concerning the Maxwell–Ampère equation:

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \partial_t (\varepsilon_0 \boldsymbol{E}) . \tag{1.57}$$

Under this form, one can notice that the notation $\boldsymbol{B}/\mu_0$, used in Eq. (1.2) and everywhere else in this book, has been replaced by the notation $\boldsymbol{H}$. Both notations are of course strictly equivalent as long as the equation is kept under this initial form, with the source term $\boldsymbol{j}$ on the RHS. Nevertheless, in the context of magnetic media, we will see that the notation $\boldsymbol{B}$ is classically reserved to another use, including a part of the current density, as the electric induction $\boldsymbol{D}$ includes a part of the charge density. In this context, $\boldsymbol{H}$ is simply called the "magnetic field", while $\boldsymbol{B}$ is called "magnetic induction".

In standard magnetic media such as ferrites, a part of the current density can be attributed to a "magnetization" of the medium, which can be put under the form:

$$\boldsymbol{j}_{\mathrm{m}} = \nabla \times \boldsymbol{M} \,. \tag{1.58}$$

The introduction of the magnetization density vector $\boldsymbol{M}$ is quite similar to the introduction of the polarization vector $\boldsymbol{P}$ for the charge density. It corresponds to a density of magnetic dipole moment, exactly as the first one corresponds to a density of electric dipole moment. It can also be viewed as an ensemble of small electric loops (sometimes called "Amperian currents"). In a plasma, it could be related to the circular motion of the particle around the magnetic field.

The magnetization current, which corresponds to spatial variations of the magnetic dipole density, has to be added to the polarization current presented above, which corresponds to time variation of the polarization vector. Other kinds of currents can be involved, in particular due to free charges when the medium is conducting. The corresponding extra current density can be noted $\boldsymbol{j}_{\mathrm{ex}}$. When both kinds of dipoles are present, the current density is:

$$\boldsymbol{j} = \boldsymbol{j}_{\mathrm{m}} + \boldsymbol{j}_{\mathrm{p}} + \boldsymbol{j}_{\mathrm{ex}} \,. \tag{1.59}$$

It should be noted that the part $\boldsymbol{j}_{\mathrm{m}}$ (magnetization current), contrary to the two other parts, does not correspond to any charge density since it has no divergence. The next step consists once again in supposing a simple relation between $\boldsymbol{M}$ and the magnetic field $\boldsymbol{H}$.

$$\boldsymbol{M} = \boldsymbol{\chi}_{\mathrm{m}} \cdot \boldsymbol{H} \,. \tag{1.60}$$

The tensor $\boldsymbol{\chi}_{\mathrm{m}}$ is called the magnetic susceptibility. In isotropic media, it is a scalar. Its sign can be positive or negative, which shows that the reaction of the medium can increase (paramagnetism) or decrease (diamagnetism) the magnetic field.

This allows rewriting the Maxwell–Ampère equation under the form:

$$\nabla \times \boldsymbol{H} = \boldsymbol{j}_{\mathrm{ex}} + \partial_t(\boldsymbol{D}) \,. \tag{1.61}$$

Or equivalently:

$$\nabla \times (\boldsymbol{\mu}^{-1} \cdot \boldsymbol{B}) = \boldsymbol{j}_{\mathrm{ex}} + \partial_t(\boldsymbol{\varepsilon} \cdot \boldsymbol{E}) \,. \tag{1.62}$$

This second form is equivalent at the condition of taking the new definition of the vector $\boldsymbol{B}$ (magnetic induction):

$$\boldsymbol{B}/\mu_0 = \boldsymbol{H} + \boldsymbol{M} = (\boldsymbol{I} + \boldsymbol{\chi}_{\mathrm{m}}) \cdot \boldsymbol{H} \,. \tag{1.63}$$

The relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ can, therefore, be written as $\boldsymbol{B} = \boldsymbol{\mu} \cdot \boldsymbol{H}$. The tensor $\boldsymbol{\mu} = \mu_0(\boldsymbol{I} + \boldsymbol{\chi}_{\mathrm{m}})$ is called the magnetic permeability. In standard materials such as ferrites, the magnetic permeability is a known characteristic of the medium. It allows, for instance, to determine the consequences of introducing a ferrite barrel inside a solenoid.

Conducting media In materials such as metals, the conduction current density can be related to the electric field via a relation of the form:

$$\boldsymbol{j} = \boldsymbol{\sigma} \cdot \boldsymbol{E} \,. \tag{1.64}$$

The tensor $\boldsymbol{\sigma}$ is called the conductivity tensor. It can be scalar or not depending whether the medium is isotropic or not. In an isotropic metal, the conductivity is just the inverse of the resistivity: $\boldsymbol{\sigma} = \boldsymbol{I}/\eta$.

It is to be noticed that, due to the charge continuity equation, the Maxwell–Gauss equation can be put also under the form of a relation between $\boldsymbol{j}$ and $\boldsymbol{E}$. There is therefore a relation between the dielectric susceptibility $\boldsymbol{\chi}_{\mathrm{e}}$ and the conductivity $\boldsymbol{\sigma}$. In the simplest case of no extra charge and a monochromatic wave ($\partial_t = -i\omega$), this relation takes the quite simple form: $\epsilon_0 \boldsymbol{\chi}_{\mathrm{e}} = -\boldsymbol{\sigma}/i\omega$.

1.3.3.5 Use of These Concepts in Plasmas

The equations relating ϱ, $\boldsymbol{j}$, $\boldsymbol{E}$, and $\boldsymbol{B}$ in a plasma are demonstrated in this book. They depend on the model used, which itself can depend on the time and space scale range. However, whatever the model, these relations generally do not have the above forms.

Let us show it by a simple example. The plasma models provide a set of differential equations. If one succeeds, for instance, in eliminating all variables but ϱ and $\boldsymbol{E}$ and their derivatives, one would actually find a relation between these variables. But this relation would be a *differential* equation, with derivatives of both variables at different orders, with respect to time and to space. This is not reducible to Eqs. (1.54) and (1.55).

After the previous example, this should prevent use of the notions of dielectric/diamagnetic/conducting medium for a plasma. No charge and no current can be included in full generality on the LHS of the Maxwell equations, and all of them must remain on the RHS as ϱ_{ex} and $\boldsymbol{j}_{\mathrm{ex}}$ source terms. Nevertheless, all these concepts can be used anyway, and often in a very efficient manner, in a particular domain: the calculation of the linear plasma waves. When calculating monochromatic waves, with one ω and one k, all the derivatives can be expressed as functions of these two parameters in an algebraic form and all the linear perturbations can always be expressed as functions of a single one. This allows calculating a relevant dielectric permittivity as well as a magnetic permeability or an electric conductivity as functions of ω and k.

A special mention must be made for the electric conductivity. In collisional and nonmagnetized plasmas, the plasma relation between $\boldsymbol{j}$ and $\boldsymbol{E}$ has actually the form (1.64). This is actually the only exception: as soon as a magnetic field is added, for instance, the simplest "Ohm's law", for a resistive medium, becomes:

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{j} \ . \tag{1.65}$$

This cannot be written under the form $\boldsymbol{j} = \boldsymbol{\sigma} \cdot \boldsymbol{E}$ without solving the whole system and expressing the variables $\boldsymbol{u}$ and $\boldsymbol{B}$ as functions of $\boldsymbol{E}$. And the form is then algebraic only for monochromatic waves in the linear limit. It can be noted also that, in this case, the magnetic field introduces an anisotropy to the system, which implies that the effective conductivity is not scalar.

1.3.4
Plane Waves in a Plasma

In collisionless media, as in any material or in a vacuum, waves can propagate, which means that any initial perturbation varying in space will evolve in time and space in a specific manner. We summarize here the most general methods allowing the calculation of the linear waves, their dispersion and their polarization. Some properties are independent of the medium and can be directly derived from the Maxwell equations: we will summarize only these ones in the present section. A more complete description of the waves demands to specify the plasma parameters and the scale range: some of the most important ones can be found in Chapter 5.

The method is based on three main steps:

1. Linearize the physical equation system around an equilibrium state. This does not change the Maxwell equations themselves (which give the electromagnetic fields variations as functions of the sources ϱ and $\boldsymbol{j}$) since they are linear. But it changes most of the plasma equations (which give the variations of ϱ and $\boldsymbol{j}$ as functions of the fields), because they are generally not linear.
2. Transform this differential system into an algebraic one (usually via Fourier transform)
3. Solve the algebraic system (eigenmodes and eigenvectors).

The first step relies on the fact that one deals with small amplitude fluctuations, which justifies a perturbative approach where only the first order is retained. The equilibrium state is most generally supposed to be a plain homogeneous and stationary state, where all fields and plasma parameters are constant.

The second step relies on the linearity of the above system: thanks to it, any solution is just a linear superposition of a basis of particular solutions. It is, therefore, sufficient to calculate these different particular solutions, each of which is characterized by a small number of free parameters. These parameters are typically the pulsation ω and the wave vector $\boldsymbol{k}$ when choosing monochromatic plane waves as particular solutions, that is, using Fourier transform (which is the most common method when the zero order state is homogeneous and stationary). The algebraic system obtained in this way is homogeneous (that is, without the RHS) since one studies the natural oscillations of the system, without forcing.

The third step then consists of solving the system as a function of ω and $\boldsymbol{k}$. Usually all the variables can be eliminated except one vector field, that we can call a "reference field". The equation relating the three components of that field with ω and $\boldsymbol{k}$ as parameters is a system of linear equations. Since the system is homogeneous, its determinant must be zero to get nontrivial solutions. This provides a relation between ω and $\boldsymbol{k}$, which is the dispersion equation. Each solution $(\omega, \boldsymbol{k})$ is called an eigenmode of the plasma and the corresponding eigenvector gives its *polarization*, that is, generally speaking, the perturbations of all variables (including the different components of the vectorial variables) as functions of only one.

In practice, the choice of the reference field requires attention. For high-frequency waves, when the displacement current is dominant on the particle current ($E/B \approx v_{\mathrm{phase}}/c \approx 1$), the more convenient reference field is the electric field. It allows doing the calculation with any densities and recovering the electromagnetic waves in vacuum as a limiting case of an evanescent plasma.

On the contrary, in the context of low frequency waves, that is, when the displacement current can be neglected in front of the particle one ($E/B \approx v_{\mathrm{phase}}/c \ll 1$), the choice of $\boldsymbol{E}$ as a reference field is no more convenient. Using a field that is negligible in some of the equations as a reference field carries the risk of an ill posed problem. It is the case in particular for MHD waves. The most usual choice of reference field is then the plasma velocity field $\boldsymbol{u}$. The choice of the magnetic field as a reference variable could seem also possible but it is not so pertinent because it does not allow to compute the electrostatic waves (which have no magnetic component).

It is worth noting that (i) even in the case of a stationary zero order state, the Fourier transform vs. time, at step 2, can sometimes be usefully replaced by a Laplace transform, which allows introducing the initial condition and investigating more clearly the questions of causality (see Chapter 5); (ii) With nonplane waves, step 2 cannot be systematically accomplished, and one remains with a set of both algebraic and differential equations.

1.3.5
Electromagnetic Components of Plane Plasma Waves

In a plasma, collisionless or not, all waves involve fluctuations in both the electromagnetic fields and in the plasma variables. The fields are expressed as functions of the plasma parameters by the Maxwell equations via the source terms ϱ and $\boldsymbol{j}$. Conversely, the plasma parameters depend on the fields via the particle trajectories: this provides a second part of the equation system, which we will call here the *plasma equations*. The plasma equations are generally complex and highly dependent on both the plasma parameters and the scale range of the phenomena under study. It is studied in the different chapters of the book. On the contrary, the Maxwell equations are quite general and some general wave properties can be deduced from them.

When applying the two first steps of the general program above to the Maxwell equations, one obtains:

$$\boldsymbol{k} \cdot \boldsymbol{E}_1 = -i\varrho_1/\varepsilon_0 \tag{1.66}$$

$$\boldsymbol{k} \times \boldsymbol{B}_1 + \omega \boldsymbol{E}_1/c^2 = -i\mu_0 \boldsymbol{j}_1 \tag{1.67}$$

$$\boldsymbol{k} \times \boldsymbol{E}_1 - \omega \boldsymbol{B}_1 = 0 \tag{1.68}$$

$$\boldsymbol{k} \cdot \boldsymbol{B}_1 = 0 \,. \tag{1.69}$$

The subscript 1 labels the first order fluctuations, and the notation $\boldsymbol{E}_1$ must be understood as a short one to represent the complex amplitude of the Fourier transform, which should be written $\widetilde{\boldsymbol{E}}_{1\omega,k}$ to be complete, and which is a function of ω and $\boldsymbol{k}$. All functions of the Fourier basis are supposed to vary with time and space as $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega)t}$. This means that, for deriving the above equations from the original differential ones, the derivatives with respect to time have been replaced by multiplications by $-i\omega$ and the nablas by $i\boldsymbol{k}$.

Some general properties can readily be drawn by distinguishing the longitudinal and transverse components of the vectors with respect to $\boldsymbol{k}$ (subscripts l and t). The above system can then be rewritten as:

$$B_{1\mathrm{l}} = 0 \tag{1.70}$$

$$k E_{1\mathrm{l}} = -i\varrho_1/\varepsilon_0 \tag{1.71}$$

$$\omega E_{1\mathrm{l}}/c^2 = -i\mu_0 j_{1\mathrm{l}} \tag{1.72}$$

$$\boldsymbol{k} \times \boldsymbol{B}_{1\mathrm{t}} - \omega \boldsymbol{E}_{1\mathrm{t}}/c^2 = -i\mu_0 \boldsymbol{j}_{1\mathrm{t}} \tag{1.73}$$

$$\boldsymbol{k} \times \boldsymbol{E}_{1\mathrm{t}} - \omega \boldsymbol{B}_{1\mathrm{t}} = 0 \,. \tag{1.74}$$

Equation (1.70) is independent of all the others. It is just the direct consequence of the divergence-free equation: waves have no magnetic field component of their perturbation along $\boldsymbol{k}$. Equations (1.71) and (1.72) relate the longitudinal component of the electric field $E_{1\mathrm{l}}$ to ϱ_1 and $j_{1\mathrm{l}}$. These two equations are equivalent if the charge continuity equation is taken into account. Equations (1.72) and (1.74) relate the transverse components of both fields to the transverse component of the electric current. The last two sets of two equations are independent of each other only if the plasma equations make $j_{1\mathrm{l}}$ independent of $\boldsymbol{E}_{1\mathrm{t}}$ and $\boldsymbol{j}_{1\mathrm{t}}$ independent of $E_{1\mathrm{l}}$. In terms of the conductivity tensor defined above ($\boldsymbol{j} = \boldsymbol{\sigma} \cdot \boldsymbol{E}$), this occurs when $\boldsymbol{\sigma}$ is a scalar. In these conditions, the electrostatic and electromagnetic waves are independent and can be superposed linearly: the electrostatic waves can be calculated by ignoring the magnetic field and the electromagnetic ones by ignoring the electrostatic component.

1.3.6
Some General Properties of Plane Wave Polarization and Dispersion

Keeping the current source term unknown, the magnetic field perturbation $\boldsymbol{B}_1$ can easily be eliminated between Eqs. (1.67) and (1.68). This elimination provides:

$$\left[\left(\omega^2 - k^2c^2\right)\mathbf{1} + \boldsymbol{k}\boldsymbol{k}c^2\right] \cdot \boldsymbol{E}_1 = -i\frac{\omega}{\varepsilon_0}\boldsymbol{j}_1 \,. \tag{1.75}$$

Taking $\boldsymbol{j}_1 = 0$, one readily recovers the vacuum solutions: two transverse waves with $\omega^2 - k^2c^2 = 0$ (the longitudinal solution $\omega = 0$ cannot exist in vacuum because of Eq. (1.71).

Beyond the vacuum case, very few results can be considered as really general. To obtain the dispersion relations in plasmas, the complete system, Maxwell plus plasma equations, has to be known. A way to get a step further without specifying the plasma equations is to use the fact that, whatever they are, they can always be noted in linear theory: $\boldsymbol{j}_1 = \boldsymbol{\sigma} \cdot \boldsymbol{E}_1$, using the $\boldsymbol{\sigma}$ "conductivity tensor" presented just above. This tensor will have of course to be calculated in each specific situation to obtain concretely the properties of the waves in a given situation.

Rewriting Eq. (1.75) with this notation, one obtains:

$$\boldsymbol{P}_E \cdot \boldsymbol{E}_1 = 0 \quad \text{with} \quad \boldsymbol{P}_E = \left[\left(\omega^2 - k^2c^2\right) \mathbf{1} + \boldsymbol{k}\boldsymbol{k}c^2 + i\frac{\omega}{\varepsilon_0}\boldsymbol{\sigma} \right] . \tag{1.76}$$

The dispersion relation, therefore, results, in general, from the nullity of the determinant $D(\omega, \boldsymbol{k})$ of a propagation matrix $\boldsymbol{P}_E$ which depends on the plasma equations through the tensor $\boldsymbol{\sigma}$. As already mentioned in Section 1.3.3.4, the conductivity is related to the dielectric susceptibility and permittivity and $\boldsymbol{\sigma}$ can be replaced by its expression in any of these functions.

Let us consider a magnetic field locally aligned in the z direction; the equation $D(\omega, \boldsymbol{k}) = 0$ can be expanded in the following way, as a function of the relative permittivity $\boldsymbol{\epsilon} = \boldsymbol{\varepsilon}/\varepsilon_0$.

$$\begin{vmatrix} \omega^2\epsilon_{xx} - k_{\|}^2c^2 & \omega^2\epsilon_{xy} & \omega^2\epsilon_{xz} + k_\perp k_{\|}c^2 \\ \omega^2\epsilon_{yx} & \omega^2\epsilon_{yy} - k^2c^2 & \omega^2\epsilon_{yz} \\ \omega^2\epsilon_{zx} + k_\perp k_{\|}c^2 & \omega^2\epsilon_{zy} & \omega^2\epsilon_{zz} - k_\perp^2 c^2 \end{vmatrix} = 0 \,, \tag{1.77}$$

where the wave vector $\boldsymbol{k}$ vector is included in the x, z plan, and $k_{\|}$ and $k_\perp$ are respectively its parallel and perpendicular projections relatively to the magnetic field direction: $\boldsymbol{k} = (k_\perp, 0, k_{\|})$. One can see on Eq. (1.77) how all the wave properties can be derived from the permittivity tensor. This tensor will be expressed and analyzed more explicitly in Chapter 5, when the role of the particle distribution is specified.

Even in simple cases, with simple plasma equations, a great diversity of waves can appear. They are generally characterized by their frequency ω and their wave vector $\boldsymbol{k}$. The equation relating these two quantities is called the *dispersion relation*. The ratio $v_\phi = \omega/k$ is the *phase velocity*. The ratio $N = kc/\omega = c/v_\phi$ is the *refraction index*. For certain values of the parameters, $N \gg 1$, that is $v_\phi/c \mapsto 0$. This is called a *resonance*. When $N \ll 1$ that is $v_\phi \mapsto \infty$, this is a *cut-off*. Examples will be given hereafter.

1.3.7 Electrostatic Waves

It is worth noting that plasma waves can involve, contrary to the vacuum waves, a longitudinal component E_{1l}. This component can even, for some wave modes such as the famous "Langmuir wave", be predominant. As soon as the transverse component of the electric field is negligible, Eq. (1.74) shows that its magnetic counterpart is also very small. It means that these modes are mainly caused by the space charge density ρ_1, while the electric current associated with the corresponding charge displacement can be neglected. This is why these waves are called "electrostatic waves", although their frequency is not zero. In this case, the linear dispersion equation is simply

$$\epsilon_{ll} = \epsilon_0 - \frac{\sigma_{ll}}{i\omega} = 0 \,. \tag{1.78}$$

These modes depend only on the longitudinal component of the dielectric tensor. The above dispersion relation can be derived directly from Eq. (1.76), or from the consideration of the dominant longitudinal component in Eq. (1.71).

1.3.8 Wave Packets and Group Velocity

With a single plane wave (of real frequency and wave vector), there is no propagation of any form of information, because the wave is present everywhere with the same intensity. In real life, waves do not fill the entire space. Instead of a monochromatic plane wave, let us rather consider packets of plane waves, belonging to a common branch of solutions of the dispersion equation $D(\omega, \boldsymbol{k}) = 0$. They are supposed to be linearly superposed, with their wave vectors spread in a finite interval. When this interval is small, one speaks of quasi-monochromatic waves. The amplitude of wave packets are represented, for any field A in the form

$$A(\boldsymbol{r}, t) = \int \tilde{A}(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} d\boldsymbol{k} \,. \tag{1.79}$$

The phase of each wave is defined as $\phi = \boldsymbol{k} \cdot \boldsymbol{r} - \omega t$. The wave packet has a maximum at a point $\boldsymbol{r}$ where all contributions are in phase, that is, where these phases reach a common value ϕ_0 which is independent of $\boldsymbol{k}$. For a quasi-monochromatic wave packet, where a Taylor expansion can be used, the phase is:

$$\phi = \phi_0 + d\boldsymbol{k} \cdot \nabla_k(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t) = \phi_0 + d\boldsymbol{k} \cdot (\boldsymbol{r} - \nabla_k(\omega)t) \,. \tag{1.80}$$

This shows the point $\boldsymbol{r}$ where the wave packet has its maximum amplitude given by $\boldsymbol{r} = \nabla_k(\omega)t$. This point moves at a velocity called the group velocity and defined by:

$$\boldsymbol{v}_g = \nabla_k \omega \,. \tag{1.81}$$

In a 1D problem, it can be simply written: $v_g = d\omega/dk$.

1.3.9 Propagation of Plane Waves in a Weakly Inhomogeneous Medium

The propagation of waves in an inhomogeneous medium is a complex topic that involves many possible phenomena: wavelength and frequency modifications, partial transmissions, absorption and reflection, loss of planarity, mode conversion (a wave pass from one branch of the dispersion equation solutions to another), and so on. Quite often, in an inhomogeneous medium, the hypothesis of planarity does not hold. Nevertheless, an asymptotic theory called the BGK theory after Bernstein, Green, and Kruskal, describes the propagation of planar waves when the gradient scales in the plasma are much larger than the wavelength.[1] The BGK theory provides a set of equations providing the wavelength, the frequency, the phase velocity, and the group velocity evolutions. It is also possible to do ray tracing.

In a stratified medium, where the propagation index N depends on z only, a ray with an initial vertical angle ϕ_0 in the x, z plane evolves along the path described as

$$x = \pm \int_0^z \frac{\sin \phi_0 d\zeta}{\sqrt{N^2(\zeta) - \sin^2 \phi_0}} . \tag{1.82}$$

1.3.9.1 An Example of Wave Propagation with a Cut-off Frequency

Anticipating Chapter 5, we can investigate the propagation of a wave with a propagation index given by $N^2(z) = 1 - \omega_p^2/\omega^2 = 1 - \alpha n$, where $n = n(z)$ is the plasma density, which is a function of the altitude. This relation is indeed the propagation of an electromagnetic wave in a plasma (ordinary radio wave). We consider the case of the lower ionosphere where the plasma density $n(z)$ increases with altitude. For simplification, we have a linear dependence. Then $N^2(z) = 1 - az$, where a^{-1} is the altitude at which the wave frequency equals the cut-off frequency of its branch of the dispersion relation. Equation (1.82) becomes $x = \pm(c\sqrt{b - z} - c\sqrt{b})$, where $b = (1 - \sin^2 \phi_0)/a$ and $c = \sin \phi_0/\sqrt{a}$. The rays have a parabolic shape, with a reflection below the altitude where $N(z)$ becomes null. More generally, in the vicinity of an altitude where $N^2(z)$ cancels, the expansion $N^2(z) \sim 1 - az$ is valid, and the above solution is quite general in the zone near the altitude of the cut-off. This fact is illustrated in Figure 1.6a that provides a numerical example of Eq. (1.82). We can conclude that a cut-off at a certain altitude corresponds to a reflection of the waves below the altitude where their frequency is the cut-off frequency.

1.3.9.2 Examples of Wave Propagation with a Resonance Frequency

We represent a resonance at the altitude z_0 by a function of the shape $N^2 = a(z - z_0)^{-l}$ where a is an arbitrary constant. This relation does not correspond to a particular plasma dispersion function; it is simply chosen to illustrate the effect of a resonance. For z close to z_0, the function to integrate is equivalent to $\sin \phi_0 (z_0 - \zeta)^{l/2} a^{-1/2}$, and its primitive is proportional to $(z_0 - \zeta)^{(l+2)/2}$. Therefore, in this vicinity, $z \sim z_0 - c x^{l/(l+2)}$. The ray approaches the altitude z_0 for a finite value of x, and has a vertical slope. Beyond this point, there is no propagation; therefore, the wave is absorbed at the resonance altitude. This is illustrated on the right-hand side of Figure 1.6 by a numerical solution of Eq. (1.82) for the case

1) This theory could be developed in the frame of the multiscale expansion introduced in Appendix A and used in Section 1.4.

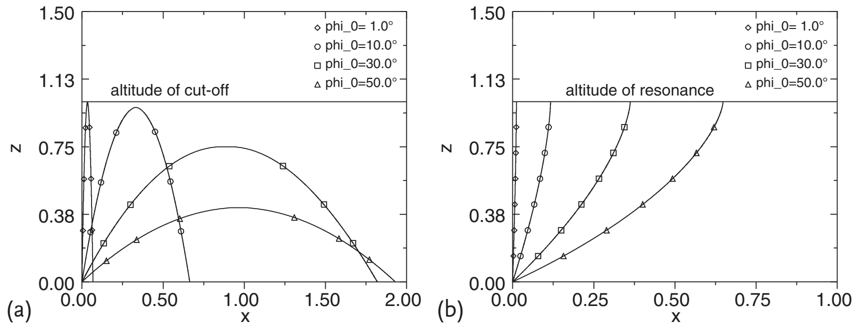

Figure 1.6 Ray tracing in a stratified medium for various initial angles. (a) The dispersion index $N(z) = \sqrt{1-z}$ has a cut-off at the altitude $z = 1$ marked by the horizontal line. We can see that the waves are reflected below this altitude. (b) The dispersion index $N(z) = 1/\sqrt{z-1}$ has a resonance at the altitude $z = 1$ marked by the horizontal line.

$l = 1$. We can see that the waves do not propagate beyond the altitude where the resonance is reached.

1.3.10
Useful Approximations of the Maxwell Equations in Plasma Physics

1.3.10.1 The Approximation of a Null Displacement Current Used in Classical MHD

On many occasions, we have defined a conductivity σ tensor. For dimensional analysis, one can reduce it to a scalar field σ and $J = \sigma E$. This is analogous to the introduction of an impedance Z defined by $ZI = U$ where I is the current circulating in a portion of plasma submitted to a potential drop U. Then, for dimensional analysis, $ZI = ZJS = U = EL$ where L is the length of the plasma portion and $S = L^2$ the area of surface perpendicular to it. Then, $J = E/(ZL)$, and we can write the Ampère equation

$$\nabla \times \boldsymbol{B} = \frac{Z_0}{Z} \frac{\boldsymbol{E}}{Lc} + \frac{1}{c^2} \partial_t \boldsymbol{E} \; , \tag{1.83}$$

where $Z_0 = c\mu_0 = 376.7\ldots\ \Omega$ is the impedance of free space. In terms of plane waves analysis, it is more simply written

$$\boldsymbol{k} \times \boldsymbol{B} = k \left(\frac{Z_0}{Z} + \frac{1}{c} \frac{\omega}{k} \right) \frac{\boldsymbol{E}}{c} \; . \tag{1.84}$$

For the comparison of orders of magnitude we make abstraction of the geometrical operators (see Section 3.3), and

$$B \equiv \left(\frac{Z_0}{Z} + \frac{v}{c} \right) \frac{E}{c} \; . \tag{1.85}$$

where $v/c = \omega/(kc)$ represents the typical velocity of the phenomenons under consideration (the phase velocity in the case of plane waves). Similarly, the Faraday equation is

$$\frac{E}{c} \equiv B\frac{v}{c} \tag{1.86}$$

and the Gauss equations is

$$k E \equiv \frac{\rho}{\epsilon_0} \, . \tag{1.87}$$

Concerning the Maxwell equations, the approximation made in MHD is the following: large spatial scales, low velocities relative to the speed of light, and a high plasma conductivity. The magnetic field amplitude is generally finite. In terms of the above equations, a small parameter ϵ is introduced and allows us to write

$$k = k_0 \epsilon \quad \text{and} \quad \frac{v}{c} = \beta\epsilon \quad \text{and} \quad \frac{Z_0}{Z} = \frac{\alpha}{\epsilon} \, , \tag{1.88}$$

where α, β, k_0 are finite, as well as B. Then from Eq. (1.85),

$$B \equiv \left(\frac{\alpha}{\epsilon} + \beta\epsilon\right) \frac{E}{c} \, . \tag{1.89}$$

As a result, the $\mu_0 \boldsymbol{J}$ term in the Ampère equation associated with $\alpha\epsilon$ dominates the displacement current, which is associated with $\beta\epsilon$. Therefore, in that context, the displacement current is neglected, and the Ampère equation reduces to

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \, . \tag{1.90}$$

Moreover, as the magnetic field is finite, $E/(cB) \equiv \delta\epsilon$ where δ has a finite value; therefore, E is of order one in ϵ. Of course, this does not imply that the electric field is null. But in the MHD equation, when one part of the electromagnetic field has to be eliminated from a set of equations, it is better to keep the magnetic field, because it is of a larger order of magnitude than the (normalized by c) electric field E/c (see Section 5.1).

Is the charge conservation Eq. (1.5) relevant to this MHD approximation? The charge conservation is derived from the divergence of the Ampère equation, combined with the Gauss equation. In the present case, the divergence of the Ampère equation is reduced to $\nabla \cdot \boldsymbol{J} = 0$. (This term is of order ϵ, while those associated with the displacement current is of order ϵ^3.) Therefore, the charge continuity equation is not relevant. Because there is no charge conservation, and because the use of the current density is unavoidable, it is not recommended to consider the charge density. Consistently, the Gauss equation, in the form of Eq. (1.87) shows that the charge density time derivative is of order ϵ^2, and should not be retained. Nevertheless, the Maxwell equations are completed, in MHD by a Ohm's law that generally provides a value for the electric field. Taking its divergence does not necessarily lead to a null charge density, but the charge density should be of order 2 or more.

1.3.10.2 The Approximation of a Null Displacement Current Used with a Multicomponent Plasma

The MHD approximation is made for large spatial scales. When one is also interested in smaller spatial scales, the above set of approximations can be replaced by the following:

$$\frac{Z_0}{Z} = \frac{\alpha}{\epsilon} \quad \text{and} \quad \frac{v}{c} = \beta\epsilon \ , \tag{1.91}$$

where α, β are finite, as well as B and k. Then Eq. (1.89) is still valid. Therefore, the displacement current is still negligible in the Ampère equation, and Eq. (1.90) still applies. Then, when B is finite, E/c is still of order ϵ. But the Gauss equation implies now that ρ is of the same order as E (when it was one order less in the MHD approximation). Then, one has to take ρ explicitly into account, as well as the Gauss equation, because they both involve fields of order ϵ. Because of Eq. (1.90), the charge continuity equation implies that $\rho = 0$. Therefore, in this approximation, Eq. (1.90) is explicitly completed by

$$\rho = 0 \ . \tag{1.92}$$

This is quasi-neutrality. This equation is especially important, because it allows us to set a relation between the densities of the different species. For instance, in the case of a plasma with a single ion species, the electron and the ion densities are related through the relation $n_{\mathrm{i}} = n_{\mathrm{e}}$. This is not the case in MHD where the plasma is considered as a single fluid.

The two approximations presented in this section are not compatible with electromagnetic waves in vacuum (because of the large conductivity assumption) and not with the electrostatic ones where ρ is finite and explicitly taken into account.

1.3.10.3 The Electrostatic Approximation

The electrostatic approximation supposes a large plasma impedance and small velocities. Then

$$\frac{Z_0}{Z} = \alpha\epsilon \quad \text{and} \quad \frac{v}{c} = \beta\epsilon \ , \tag{1.93}$$

where α, β, k_0 are finite, as well as E. The approximation on Z_0/Z consists of neglecting the current density. We can see that B is of order ϵ. To order zero, the Ampère equation is reduced to $\boldsymbol{\nabla} \times \boldsymbol{B} = 0$. As $\nabla \cdot \boldsymbol{B} = 0$, the solution is a uniform magnetic field, that cannot vary as a function of time.[2] The Maxwell equations reduce to

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{E} = \rho/\epsilon_0 \ ; \quad \boldsymbol{B} = \boldsymbol{B}_0 \ , \tag{1.94}$$

2) The Faraday equation shows that for a uniform magnetic field that depends on time, $\boldsymbol{\nabla} \times \boldsymbol{E}$ is uniform, and, therefore, the electric field diverges at infinity. Indeed, let us consider a sphere V of radius R. For finite values of $\partial_t \boldsymbol{B}$, the integral $\int_V \boldsymbol{\nabla} \times \boldsymbol{E}\, dV = \int_S \boldsymbol{n} \times \boldsymbol{E}\, dS$ varies as R^3, the sphere surface varies as R^2; consequently, E diverges as R, involving an infinite amount of electric energy. In order to avoid this divergence, $\partial_t \boldsymbol{B} = 0$ is necessary.

where $\boldsymbol{B}_0$ is constant and uniform. The electrostatic approximation can also be used locally. In that case, there exist solutions where the curl and the divergence of $\boldsymbol{B} = 0$ are null and $\boldsymbol{B}$ is not uniform. For instance, this is the case of a magnetic dipole, when the center of the dipole is out of the domain of validity of the approximation. In that case, for a bounded system,

$$\nabla\times\boldsymbol{E} = -\partial_t\boldsymbol{B} \quad\text{and}\quad \nabla\cdot\boldsymbol{E} = \rho/\epsilon_0\,; \quad \nabla\times\boldsymbol{B} = 0 \quad\text{and}\quad \nabla\cdot\boldsymbol{B} = 0\,. \tag{1.95}$$

The electrostatic approximation does not allow for the propagation of any electromagnetic wave, only purely electric waves can be considered. The continuity of the electric charge cannot be deduced any more from these reduced equations. Nevertheless, when used, this equation shows that $\nabla\cdot\boldsymbol{J}$ can be finite, but it has no consequence because the current density does not appear in the above equations.

The electrostatic approximation has various domains of application. It is widely used in the theory of auroral acceleration. The BGK theory is generally developed in this frame, as well as the theory of electrostatic double layers (see Chapter 6). It is also possible to establish a theory of electrostatic waves.

The propagation of electromagnetic waves into vacuum is not possible, and this is good news for those making numerical simulations who are interested only in low frequency waves: they won't have to bother with the computer time and memory requirements associated with a proper numerical computation of light waves.

1.3.10.4 **The Darwin Approximation**

The Darwin approximation, mainly used in numerical simulation, is designed to eliminate the light waves that do not interfere with the plasma, because they raise strong constraints on the time step: $c\Delta t < \Delta x$. But it is also designed to retain the physics of the waves that interact with the plasma as much as possible. In particular, it must retain the electrostatic waves and the continuity of the charge density.

The Darwin approximation proceeds in two steps. First, the electric field is cut in a longitudinal (or solenoidal) component $\boldsymbol{E}_\mathrm{l}$, and a transverse component $\boldsymbol{E}_\mathrm{t}$, defined by

$$\boldsymbol{E} = \boldsymbol{E}_\mathrm{t} + \boldsymbol{E}_\mathrm{l} \quad\text{and}\quad \nabla\cdot\boldsymbol{E}_\mathrm{t} = 0 \quad\text{and}\quad \nabla\times\boldsymbol{E}_\mathrm{l} = 0\,. \tag{1.96}$$

Then, one can notice that the Faraday equation involves only the transverse electric field, and the Gauss equation involves only the longitudinal field. The Ampère equation is applied separately to the two components:

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J}_\mathrm{t} + \frac{1}{c^2}\partial_t\boldsymbol{E}_\mathrm{t} \tag{1.97}$$

$$\mu_0\boldsymbol{J}_\mathrm{l} + \frac{1}{c^2}\partial_t\boldsymbol{E}_\mathrm{l} = 0\,. \tag{1.98}$$

The second stage in the Darwin approximation consists of neglecting the displacement current only in Eq. (1.97). Thus, it is no longer possible to have light waves.

The divergence of the second equation and the Gauss equation provides the equation of continuity of the charge density. In summary:

$$\boldsymbol{E} = \boldsymbol{E}_\mathrm{t} + \boldsymbol{E}_\mathrm{l}\,, \tag{1.99}$$

$$\nabla \times \boldsymbol{E}_\mathrm{l} = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{E}_\mathrm{l} = \rho/\epsilon_0 \tag{1.100}$$

$$\nabla \cdot \boldsymbol{E}_\mathrm{t} = 0 \quad \text{and} \quad \nabla \times \boldsymbol{E}_\mathrm{t} = -\partial_t \boldsymbol{B}\,, \tag{1.101}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}_\mathrm{t}\,. \tag{1.102}$$

In which order can one solve these equations? The longitudinal electric field can be deduced directly from the charge density. From the current density, one can know only the curl of the magnetic field. It is better to combine the Eqs. (1.101), (1.102), and the null divergence of the (transverse) magnetic field. There remains only a Laplacian equation,

$$\nabla^2 \boldsymbol{E}_\mathrm{t} = \mu_0 \partial_t \boldsymbol{J}_\mathrm{t}\,. \tag{1.103}$$

But usually, one cannot separate a priori the longitudinal and the transverse components. Therefore, an Ampère equation where this separation is not explicitly done is used:

$$\nabla^2 \boldsymbol{E}_\mathrm{t} = \mu_0 \partial_t \boldsymbol{J} - \frac{1}{c^2}\partial_t^2 \boldsymbol{E}_\mathrm{l}\,. \tag{1.104}$$

Then, as $\boldsymbol{E}_\mathrm{l}$ obtained as the solution of the Gauss equation is known, the only unknown is on the left-hand side of the equation, and it can be solved by inversion of the Laplacian.

This approximation indeed allows the charge conservation and eliminates light waves. Many electrostatic and electromagnetic waves interacting with the plasma are possible, but not all of them. To be convinced of this limitation, we can derive the dispersion equation of linear waves in the Fourier space. The solution is

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) + \frac{\omega^2}{c^2}\frac{\boldsymbol{\epsilon}}{\epsilon_0} \cdot \boldsymbol{E} = 0\,. \tag{1.105}$$

with

$$\boldsymbol{\epsilon} = \epsilon_0 \left(\frac{\boldsymbol{k}\boldsymbol{k}}{k^2} + i \frac{\boldsymbol{\sigma}}{\omega \epsilon_0} \right)\,. \tag{1.106}$$

Without the Darwin approximation, we would have

$$\boldsymbol{\epsilon} = \epsilon_0 \left(\boldsymbol{I} + i \frac{\boldsymbol{\sigma}}{\omega \epsilon_0} \right)\,. \tag{1.107}$$

The two systems give the same solutions for electrostatic waves (as $\boldsymbol{k}$ and $\boldsymbol{E}$ are parallel). In vacuum ($\boldsymbol{\sigma} = \boldsymbol{0}$), we can see that the transverse waves (with $\boldsymbol{k} \perp \boldsymbol{E}$) have necessarily a null amplitude. This is consistent with the purpose of this approximation. But for long length waves (therefore, interacting with the plasma) with orthogonal $\boldsymbol{k}$ and $\boldsymbol{E}$, the solution can be greatly affected by this approximation.

1.4 Upstream of Plasma Physics: The Motion of Charged Particles

The motion of the particles in a fluid is determined by integration of the motion equation in the collective electromagnetic and gravitational fields over a time period that is inversely proportional to the collision frequency. The global motion of the fluid is derived from the particle motion. In a collisional medium, these equations have to be solved over the short time between two collisions, and each collision gives a set of random new initial conditions. When the collision rate is high, this can be treated by a statistical approach. When there is no collision, the particles follow a deterministic trajectory during the whole duration of the phenomenon under study. The initial distribution of positions and motion can be described statistically, but the determination of its evolution is deterministic and requires that we can compute trajectories over long periods of time.

In a collisionless plasma, the full dynamics of the particle is taken into account when the Lorentz force equations of motion are solved,

$$d_t \boldsymbol{r} = \boldsymbol{v} \quad \text{and} \quad d_t \boldsymbol{v} = \frac{q}{m}[\boldsymbol{E}(r) + \boldsymbol{v} \times \boldsymbol{B}(r)] \,. \tag{1.108}$$

It is possible to add to the electric field the addition of other potential forces, for instance, gravitational. But practically, the typical velocities in an astrophysical plasma around a planet or a star exceed by far the gravitational escape velocity. This is why the gravitation forces are often neglected.

1.4.1 The Motion of the Guiding Center

Nevertheless, a full description of the particles' trajectories is not necessary. Quite often, as for a particle into a wave, or in a slowly varying magnetic field, the particle has a rapid and periodic motion, added to a slow variation of the characteristics of the periodic motion, and a drift motion. We are interested by the general characteristics of the motion, without knowing the phase of its fast component. The slow component of the velocity is called the *guiding center* motion.

In many textbooks, the different components of the guiding center velocity are introduced separately as consequences of particular features of the electromagnetic field. Unfortunately, in this way, we are never sure whether we include all the terms, nor whether one was counted twice. This is why we present here a global theory of the guiding center.

A global, theory of the guiding center was published in 1963 in a reference textbook [2]. The so-called rigorous demonstration in that book requires previous knowledge of asymptotic expansion methods (this chapter is then preceded by a not-so-rigorous demonstration for physicists), and the book is no longer available. This is why it is worth presenting here an extensive derivation of the guiding center theory. We base our development on the multitimescales method. No previous knowledge of the theory of asymptotic developments is required. The general method is described in Appendix A.2 and it is applied in Appendix A.2.3 to the guiding center motion.

1.4.1.1 Principles of the Guiding Center Motion

The multiscale expansion method is based on a small parameter ϵ that relates the space and time derivatives of the electromagnetic field. If the electromagnetic field is uniform and constant, $\epsilon = 0$, and the velocity is the sum of a constant drift $\boldsymbol{E} \times \boldsymbol{B}/B^2$ and a rapid circular motion of frequency $\omega_\mathrm{c} = qB/m$ (the gyrofrequency) in the plane that is perpendicular to $\boldsymbol{B}$. Therefore, the high frequency of the system (noted ω_0 in Appendix A) is ω_c. Then, ϵ is considered as a small parameter provided that

$$\rho_\mathrm{L}|(\nabla B)_\perp| = \frac{v_\perp}{\omega_\mathrm{c}}|(\nabla B)_\perp| \ll B \,, \tag{1.109}$$

that is, formally equivalent to

$$m v_\perp |(\nabla B)_\perp| \ll q B^2 \,. \tag{1.110}$$

The characteristic fast time scale is $T_\mathrm{c} = 2\pi/\omega_\mathrm{c}$, during which the particle must see only a weak variation of the electromagnetic field

$$v_z \left(\frac{2\pi}{\omega_\mathrm{c}}\right) |(\nabla B)_\parallel| \ll B \quad \text{and} \quad \left(\frac{2\pi}{\omega_\mathrm{c}}\right) |\partial_t B| \ll B \,. \tag{1.111}$$

The parallel electric field $\boldsymbol{E}_\parallel$ is generally not the cause of a fast periodic motion. Therefore, it must induce a weak acceleration in comparison to those induced by the magnetic field

$$(q/m) E_\parallel \ll |\boldsymbol{v} \times \boldsymbol{B}| \sim \rho_\mathrm{L} \omega_\mathrm{c}^2 \,. \tag{1.112}$$

The velocity equation (Lorentz force) is

$$d_t \boldsymbol{v} = \boldsymbol{e}(\boldsymbol{r}, t_1, \ldots) + \boldsymbol{v} \times \boldsymbol{b}(\boldsymbol{r}, t_1, \ldots) \,, \tag{1.113}$$

where we note $\boldsymbol{e} = q\boldsymbol{E}/m$ and $\boldsymbol{b} = q\boldsymbol{B}/m$. We accept a contribution of order 0 of the magnetic field and the perpendicular electric field.

A development (detailed in the Appendix A) is made of the electromagnetic field

$$\boldsymbol{e}(\boldsymbol{r}, t_1, \ldots) = \boldsymbol{e}_\perp(\boldsymbol{R}, t_1, \ldots) + \epsilon(\boldsymbol{r}_0 \cdot \nabla)\boldsymbol{e}_\perp + \epsilon \boldsymbol{e}_{\parallel 1} + \ldots \tag{1.114}$$

$$\boldsymbol{b}(\boldsymbol{r}, t_1, \ldots) = \boldsymbol{b}(\boldsymbol{R}, t_1, \ldots) + (\boldsymbol{r}_0 \cdot \nabla)\boldsymbol{b} + \ldots \tag{1.115}$$

where the gradients are computed at the position $\boldsymbol{R}$ of the guiding center, and not of the particle.

The velocity equation is then written at orders ϵ^0 and ϵ^1 in the context of the multiscale expansion. This procedure is detailed in Appendix A.2.3. To lowest order, the slow component of the solution is the well known cross field drift velocity, also found in ideal MHD theory,

$$\boldsymbol{U}_0 = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} + U_{\|0}\boldsymbol{b}, \tag{1.116}$$

with $U_{\|0}$ still undetermined at this stage. The fast part at order zero describes a linear oscillator, whose solution is

$$\begin{aligned} u_{0x} &= +u_{\perp 0}\cos(\omega_c t + \psi) \\ u_{0y} &= -u_{\perp 0}\sin(\omega_c t + \psi) \\ u_{0z} &= 0\,. \end{aligned} \tag{1.117}$$

The slow part of Eq. (A25) is the basis of the derivation of the slow motion at first order of the guiding center velocity.

1.4.1.2 The Perpendicular Velocity of the Guiding Center

It is found that, in terms of ordinary time and space variables,

$$\begin{aligned} \boldsymbol{U}_\perp &= \boldsymbol{U}_0 + \epsilon \boldsymbol{U}_1 \\ &= \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} + \frac{m\boldsymbol{b}}{qB} \times \left[d_t \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} + U_\| d_t \boldsymbol{b} + \frac{\mu}{m}\nabla B \right]. \end{aligned} \tag{1.118}$$

The first term is the cross field drift $\boldsymbol{E} \times \boldsymbol{B}/B^2$, occurring as soon as the electric and the magnetic fields have components orthogonal to each other. All the contributions to the slow perpendicular drift are the product of $q\boldsymbol{b}/mB$ and a vector that is homogeneous to an acceleration, the effect of a force. We can develop the derivatives appearing in the force terms in order to prove a few effects

$$\begin{aligned} \boldsymbol{U}_\perp &= \frac{m\boldsymbol{b}}{qB} \times \left[\frac{q\boldsymbol{E}}{m} + d_t \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} + U_\| d_t \boldsymbol{b} + \frac{\mu}{m}\nabla B \right] \\ &= \frac{m\boldsymbol{b}}{qB} \times \left\{ \frac{q\boldsymbol{E}}{m} + d_t \boldsymbol{E}_\perp \times \frac{\boldsymbol{B}}{B^2} + \boldsymbol{E}_\perp \times d_t \left(\frac{\boldsymbol{B}}{B^2} \right) \right\} \\ &\quad + \frac{m\boldsymbol{b}}{qB} \times \left\{ U_{\|0} \left[\partial_t \boldsymbol{b} + U_{0\|} \boldsymbol{b} \cdot \nabla \boldsymbol{b} + \boldsymbol{U}_\perp 0 \cdot \nabla \boldsymbol{b} \right] + \frac{\mu}{m}\nabla B \right\}. \end{aligned} \tag{1.119}$$

The force $d_t \boldsymbol{E}_\perp \times \boldsymbol{B}/B^2$ is associated with the polarization drift velocity, that in virtue of the double cross product takes the simple form

$$\frac{m}{qB}\boldsymbol{b} \times \left(d_t \boldsymbol{E}_\perp \times \frac{\boldsymbol{B}}{B^2} \right) = \frac{m}{qB^2} d_t \boldsymbol{E}_\perp\,. \tag{1.120}$$

The polarization drifts, because its proportionality to the mass is mainly affected by ions and is often neglected for electrons, and as a result, it is a cause of electric current density in a plasma.

The development of $U_{\|0} d_t \boldsymbol{b}$ in Eq. (1.119) is associated to the velocities

$$\frac{m}{qB} U_{\|0} \boldsymbol{b} \times \partial_t \boldsymbol{b} + \frac{m}{qB} U_{\|0}^2 \boldsymbol{b} \times (\boldsymbol{b} \cdot \nabla \boldsymbol{b}) + \frac{m}{qB} U_{\|0} \boldsymbol{b} \times (\boldsymbol{U}_{0\perp} \cdot \nabla \boldsymbol{b})\,. \tag{1.121}$$

The second of these terms is called the curvature drift because it appears when the magnetic field lines have a finite curvature. It is generally the most important term of this group. All these terms are charge and mass dependent, they are potentially the cause of an electric current density.

The last term comes form the mirror force $-\mu \nabla B / m$ connected to the gradient of the modulus of the magnetic field. When the magnetic field increases, this force is repulsive, which is why it is called a magnetic mirror force, as if photons are reflected by a mirror. This term gives an indication about the tendency of plasmas to go preferentially in the regions of a weak magnetic field.

Apart from the curvature drift, the polarization drift, and the mirror force, we can see other terms. They appear in [2], but they are seldom mentioned in many other textbooks, whose approach of the guiding center motion, if pedagogic, is less scrupulous.

1.4.1.3 The Parallel Velocity of the Guiding Center

The equation of the parallel motion is

$$d_t \boldsymbol{U}_{\|0}(t) = \frac{q}{m}\boldsymbol{E}_{\|} - \frac{\mu}{m}\boldsymbol{\nabla}_{\|} B - \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} \cdot \frac{\partial \boldsymbol{b}}{\partial t}. \tag{1.122}$$

The first term is the parallel electric field. It is not specific to the guiding center theory. The term derived from the perpendicular magnetic gradient (the gradient of B in the direction perpendicular to $\boldsymbol{B}$) in the equation of the perpendicular velocity

$$\frac{\mu}{qB}\boldsymbol{b} \times \boldsymbol{\nabla} B \tag{1.123}$$

reappears in the equation of the parallel velocity Eq. (1.122).

The last term, connected to the partial time derivative of $\boldsymbol{b}$, is seen in [2] but is not visible in many other textbooks. The Eqs. (1.122) and (1.118) provide a fairly classical representation of the guiding center motion.

1.4.1.4 The Guiding Center Equation That Does Not Separate the Perpendicular and Parallel Components

From Eqs. (1.116) and (A61), it is possible to derive an equation of the global guiding center slow velocity $\boldsymbol{U} = \boldsymbol{U}_0 + \epsilon \boldsymbol{U}_1$, without separating the parallel and the perpendicular component, at order 1.

$$d_t \boldsymbol{U} = \frac{q}{m}[\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B}] - \frac{\mu}{m}\boldsymbol{\nabla} \boldsymbol{B}, \tag{1.124}$$

where the electromagnetic field $\boldsymbol{E}, \boldsymbol{B}$ is defined at the position $\boldsymbol{R}$ of the guiding center, and not at the instantaneous position $\boldsymbol{r}$ of the particle velocity. This equation determines the evolution of the slow component of the velocity. Equation (1.124) is seldom used in analytical calculations, but it can be very useful in numerical computations [3]. It is interesting when solved in a way that allows us to get rid of the fast varying part (coming from the $\boldsymbol{U} \times \boldsymbol{B}$ term). This can be done with an implicit numerical algorithm that combines a low-pass filtering with the computation of $\boldsymbol{U} \times \boldsymbol{B}$.

1.4.2 Adiabatic Invariants

1.4.2.1 First Adiabatic Invariant and Magnetic Trapping

In the slowly varying electromagnetic field of a magnetized medium, the shortest time scale concerning the particle motion is the gyrofrequency ω_c. An adiabatic invariant, defined in Appendix A.2, can be associated with the gyromotion of the particle. As only the motion defined by $\boldsymbol{p}_\perp$ is periodic, only this one must be retained in the definition of the first adiabatic invariant. Let $T = 2\pi/\omega_c = 2\pi m/qB$ be the period of the fast motion,

$$\mathcal{I} = \int_0^T p\,dq = \int_0^T m\boldsymbol{v}_\perp d\boldsymbol{r} = \int_0^T m\boldsymbol{v}_\perp d\boldsymbol{v}_\perp dt = \frac{2m\pi}{q}\frac{mv_\perp^2}{2B} . \tag{1.125}$$

Practically, the retained definition of the first invariant is the magnetic moment $\mu = mv_\perp^2/2B$ (see also Eq. (A60)).

If the particle evolves in a purely magnetic field, the total kinetic energy $v_\parallel^2 + v_\perp^2$ as well as μ is conserved. Then the pitch angle α, related to the above quantities through the equation $\sin\alpha = v_\parallel/ve_\perp$, depends directly on the amplitude of the local magnetic field:

$$\frac{\sin^2\alpha}{B} = \text{constant} = \frac{\sin^2\alpha_0}{B_0} . \tag{1.126}$$

Here, B_0 and α_0 are reference values. They often correspond to the region where the magnetic field value B_0 is minimal, thus defining the minimal value α_0 of the pitch angle. In a magnetosphere, the value B_0 is generally found at the magnetic equator, and α_0 is the equatorial pitch angle.

When a particle moves along a magnetic field line towards a region of increasing magnetic field, the pitch angle tends to increase too, until it reaches the value $\pi/2$. Then the particle can not go any further, otherwise $\sin\alpha$ would be larger than one. Equation (1.122) clearly states that at this stage, the particle bounces back into the region of lower magnetic field. Then, the first adiabatic invariant tells us that the tendency of the particles is to be trapped in the region of a low amplitude magnetic field, unless their initial pitch angle is small enough. This process is called the *mirror effect*. Particles bouncing back because of the increase of the magnetic field are said to be *mirrored*. The others are *untrapped*, or *passing* particles.

The particles with $\sin\alpha_0 < (B_0/B_{\max})^{1/2}$ are not reflected. In the vicinity of a planet, $B_{\max}$ corresponds to the topside ionosphere. Below, the plasma becomes collisional, and the particles are captured through binary interactions in the ionosphere. The capture process, when the particle has enough energy can be a source of visible light, UV, and X-rays. The particles at the origin of the polar auroras, on the Earth and on other magnetized planets, are untrapped particles. They are said to be in the *loss cone* of the velocity distribution function.

For mirrored particles, whose position along the magnetic field line is defined by their curvilinear abscissa s, the bouncing period τ_b between the two mirror points m_1 and m_2 is

$$\tau_b = \int_{m_1}^{m_2} \frac{ds}{v_{\|}(s)} = \frac{2}{v} \int_{m_1}^{m_2} \frac{ds}{\sqrt{1 - B(s)/B_{max}}} . \tag{1.127}$$

The regions of closed field lines contain a trapped plasma. The fact that they contain particles of high energy (a few MeV) was one of the first discoveries about the magnetosphere, made in 1958 with the space probe Explorer I and was very advertised. These regions of trapped plasma are called the *Van Allen belts*, or *radiation belts*. The plasma of low energies (below a few keV) is also trapped and can remain for weeks. The trapped plasma region has a torus shape, centered, in the case of Earth, at a distance of about 5 R_E (where $R_E = 6400$ km is the Earth's radius). This region is inside the plasmasphere mentioned in Section 1.3.1.4. At close distance to the magnetized planets, and not too close to the poles, we have seen that the magnetic field lines are closed on both sides, and that $\boldsymbol{B}$ can be approximated as a dipole (see Section 1.3.3.1). For a dipole, the field lines of equation $r = R_B \cos^2(\lambda)/\cos^2(\lambda_0)$ reach the body surface (radius R_B) at the latitude $\lambda_p = \arccos L^{-1/2}$ where $L = r_0/R_B$ is the *McIllwain parameter*, r_0 being the equatorial distance of the magnetic field line. In the dipole field approximation as well as other models, L is used to characterize a field line. For a dipole field, the pitch angle can be expressed as a function of the magnetic latitude λ

$$\sin^2 \alpha(\lambda) = \frac{\cos^6 \lambda}{\sqrt{1 + 3\sin^2 \lambda}} \sin^2 \alpha_0 . \tag{1.128}$$

Knowing that at the mirror points of latitude λ_m, $\sin^2 \lambda_m = 1$ this equation gives the possibility of computing the latitude of the mirror points. The latitude of the mirror points is independent of L and, therefore, it is independent of the magnetic field line. The boundary l of the loss cone is defined by $\lambda_m > \lambda_e$, that can be expressed as $\sin^2 \alpha_{0l} = (4L^6 - 3L^5)(-1/2)$. The bounce period is

$$\tau_b = \frac{4r_0}{v} \int_0^{\lambda_m} \frac{\cos \lambda (1 + 3\sin^2 \lambda)^{1/2}}{\cos \alpha(\lambda)} d\lambda \sim \frac{4r_0}{v} . \tag{1.129}$$

The typical bounce period is ~ 1 s for 1 keV electrons and ~ 1 min for ions trapped around the Earth. The motion of these particles comprises the cross fields drift that coincide with the corotation velocity (provided that the electric field is caused only by corotation). The sum of their gradient and curvature drifts is

$$v_{GC} = \frac{mv^2}{2qBR_c}(1 + \cos^2 \alpha) \tag{1.130}$$

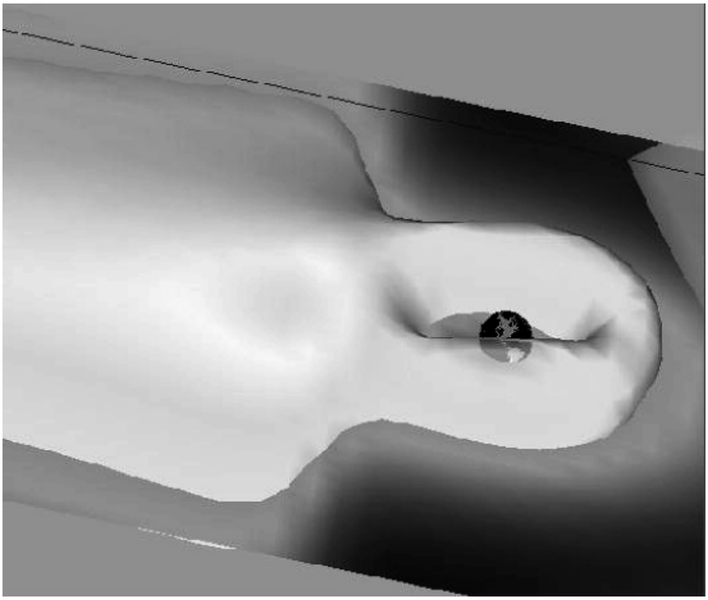

Figure 1.7 Isocontour of the current density magnitude $|j| \propto |\nabla \times \boldsymbol{B}|$ in the inner magnetosphere. The magnetic field is computed with the model [1]. On the left side of the figure is the current sheet layer, extended in the east–west direction. The Earth is surrounded by the ring current clearly visible on the figure. This current is associated with a particle drift in the azimuthal direction. (Courtesy: P. Robert, LPP/CNRS)

where the variables B, R_c, α depend on the latitude λ and R_c is the curvature radius of the magnetic field line $R_c = \|d^2\boldsymbol{r}/ds^2\| = (r_0/3)\cos\lambda(1+3\sin^2\lambda)(3/2)/(2-\cos^2\lambda)$. Then, averaged over a bounce period, this corresponds to an azimuthal drift

$$\langle\dot{\varphi}\rangle = \frac{4}{\tau_b}\int_0^{\lambda_b}\frac{d\lambda}{\cos\alpha}\frac{ds}{d\lambda} \sim (3/2)\frac{mv^2 r_0}{qB_b R_b^3} \tag{1.131}$$

where B_b is the surface magnetic field on the equator of the body, and R_b the body radius. Since average drift is charge dependent, it is at the origin of an azimuthal electric current that turns completely around the body in the regions of closed magnetic field lines (that is, at low altitude and low latitudes). This current is called the *ring current*, visible in Figure 1.7.

1.4.2.2 **Second Adiabatic Invariant**

When a particle is trapped between two mirror points, its guiding center bounces between the mirror points with the period τ_b. This is the highest frequency involved in the motion of the guiding center (not of the particle). Therefore, an adiabatic invariant can be associated with this periodic motion, as far as the variation of the magnetic field is slow relative to the period τ_b. Following the same procedure as for the first adiabatic invariant, we select the component of the impulsion that has the periodic motion. This time, this is the parallel component. The associated generalized coordinate is the curvilinear abscissa along the magnetic field line. Therefore, the second adiabatic invariant, noted J, is

$$J = \int_{m_1}^{m_2} p_{\|} ds \tag{1.132}$$

where m_1 and m_2 are the abscissas of the mirror points. As we have seen, the typical period for a 1 keV particle in the Earth's radiation belt is about 1 sec. Even during sudden events, such as the magnetic field reconfigurations associated with substorms, the typical time of evolution of $\boldsymbol{B}$ is on the order of minutes. Therefore, J can be considered an electron adiabatic invariant. For the ions, the bouncing period is on the order of a minute, and J is not necessary an adiabatic invariant for them. But it can still be an adiabatic invariant for ions of much higher energy (a few MeV for the high-energy component of the radiation belts).

1.4.2.3 Third Adiabatic Invariant and the Magnetic Storms

The azimuthal drift of the guiding centers given explicitly in Eq. (1.131) also induces a periodic motion, which can be associated with a third adiabatic invariant. It can be shown that this third invariant is the magnetic flux enclosed by the guiding center during its azimuthal motion,

$$\Phi = \int \boldsymbol{A} \cdot d\boldsymbol{l} \,. \tag{1.133}$$

In the inner magnetosphere of the Earth, for the particles in the range 1–100 keV, the period associated with the azimuthal motion is typically hours, while the variations of the magnetospheric magnetic field are generally faster. The flux Φ is an adiabatic invariant only for the high-energy particles (a few MeV) of the radiation belts.

1.4.3 The Motion of a Particle in a Wave

In most theories of wave collisionless plasma interaction, the motion of charged particles in the wave electromagnetic field must be evaluated. A complete theory of particles in a wave would involve the simultaneous appearance of various concepts. Here, in simplified contexts, we show a few of the most important effects connected with the motion of a particle in a wave.

We start with particles propagating much slower than the wave phase velocity, dealing with the effect of the modulation of the wave amplitude. This effect, as the guiding center theory, is developed in the mathematical context of the multi-timescale asymptotic expansion.

Then the notion of wave trapping is presented. It appears with quasi-monochromatic waves, even in unmagnetized plasmas, and is important for particles with a velocity of the same order of magnitude as the phase velocity.

1.4.3.1 The Nonresonant Particle in a Modulated Sinusoidal Electromagnetic Field

In this section, the concept of a guiding center is left out. The multiscale expansion method is used in a different context, more related to microphysics: the behavior of a particle into the electromagnetic field of a wave. Here, the wave is supposed to have a phase velocity larger than the particle velocity, but its envelope is not uniform. The wave frequency, considered the high frequency, is ω, and we consider that the magnetic field has no uniform component. The electric field is

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0(\boldsymbol{r}, t)\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) \tag{1.134}$$

$E_0(\boldsymbol{r}, t)$ varies slowly according to space and time, and the wave envelope propagates slowly. In terms of multiscale expansion, it is straightforwardly written

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0(\boldsymbol{r}_1, t_1, \ldots)\cos(\omega t_0 - \boldsymbol{k}_1\cdot\boldsymbol{r}_1)\ . \tag{1.135}$$

An example of such a waveform, taken in the solar wind, is shown in Figure 1.8. The Maxwell–Ampère equation can be solved, giving a magnetic field

$$\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{B}_{c,0}(\boldsymbol{r}_1, t_1, \ldots)\cos(\omega t_0 - \boldsymbol{k}_1\cdot\boldsymbol{r}_1) + \boldsymbol{B}_{s,0}(\boldsymbol{r}_1, t_1, \ldots)\sin(\omega t_0 - \boldsymbol{k}_1\cdot\boldsymbol{r}_1) \tag{1.136}$$

where

$$\boldsymbol{B}_{c,0} = \epsilon\frac{\boldsymbol{k}_1\times\boldsymbol{E}_0}{\omega} \quad \text{and} \quad \boldsymbol{B}_{s,0} = -\epsilon\frac{\nabla_1\times\boldsymbol{E}_0}{\omega}\ . \tag{1.137}$$

Then, the equation of motion involving the Lorentz force is written up to the first order. The zero order equation is solved explicitly,

$$\boldsymbol{u}_0 = \frac{q}{m\omega}E_0\sin\phi \quad \text{and} \quad \boldsymbol{r}_0 = -\frac{q}{m\omega^2}E_0\cos\phi\ . \tag{1.138}$$

We have introduced here the notation $\phi = \omega t_0 - \boldsymbol{k}_1\cdot\boldsymbol{r}_1$. The first order equation is averaged, and one finds

$$\partial_{t_1}U_0 = -\frac{q^2}{2m^2\omega}[\boldsymbol{E}_0\cdot\nabla_1\boldsymbol{E}_0 + (\nabla_1\times\boldsymbol{E}_0)\times\boldsymbol{E}_0] = -\frac{q^2}{2m^2\omega}\nabla_1 E_0^2\ . \tag{1.139}$$

Going back to the ordinary variables, the following equation gives the equation of slow motion,

$$d_t U = -\frac{q^2}{2m^2\omega}\nabla E_0^2\ . \tag{1.140}$$

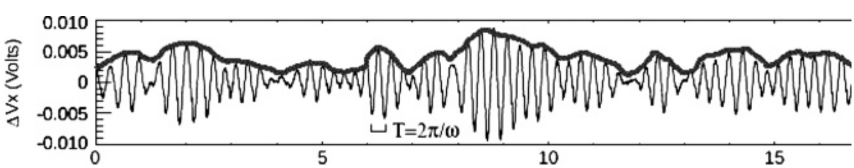

Figure 1.8 Electric potential (in Volts) of a typical wave form observed in the solar wind. From [4], see also Figure 6.31. This signal can be interpreted as a sinusoid of high frequency $T = 2\pi/\omega$ modulated by a slowly varying envelope $E_0(t)$ represented here by the grey line.

It says that the particle is submitted to a force acting on the long run that derives from the modulation of the wave envelope, that pushes the particles in the regions of lower wave energy. This is called the *ponderomotive force*.

1.4.3.2 **The Particle in a Sinusoidal Electrostatic Wave Field**

We consider the simple case of an electrostatic wave, and we compute the motion in the reference frame of the wave (the one that moves at the wave phase velocity). The equation of motion is

$$d_\tau^2 x(\tau) = \frac{q E_0}{m} \sin(k x(\tau)) \ . \tag{1.141}$$

where E_0 is the wave electric field amplitude, and k is the wave vector. For small wave amplitude E_0, the amplitude of the motion is small too. It is a harmonic oscillation, called bounce motion, of frequency

$$\omega_\mathrm{B} = \left(\frac{q k E_0}{m}\right)^{1/2} \ . \tag{1.142}$$

In the following development, we do not suppose small oscillation, but the harmonic bounce frequency ω_B is used to parametrize the system.

The equation of motion has a first integral

$$\frac{m}{2}\left(d_\tau x(\tau)\right)^2 + \frac{q E_0}{k} \cos[k x(\tau)] = W \ . \tag{1.143}$$

This is the particle energy. This integral is used to characterize the particle trajectories. Two cases must be considered separately, whether $e E_0 < k W$ or $e E_0 > k W$. In the first case, the particle can keep a nonnull velocity whatever its position. Therefore, the particle motion, if perturbed by the wave, keeps a constant direction. The particles with an energy W larger than $e E_0/k$ are, therefore, *untrapped* by the wave. The other particles, with $e E_0 > k W$, can move through a portion of space limited by $\|k x\| < \arccos(k W/e E_0)$, they are *trapped* by the wave. In the limit $W \sim 0$, the trapped particle velocities oscillate with the frequency ω_B. For larger energies, the bounce frequency is derived through the use of Jacobi elliptical functions.

The motion of untrapped particles is based on Eq. (1.143) and on the new variables $\xi = k x/2$, $w = k W/e E_0$. For untrapped particles, $w > 1$. In terms of ξ and w, the first invariant equation becomes

$$\left(d_\tau \xi(\tau)\right)^2 = \alpha^2 (1 - \beta \sin^2 \xi) \tag{1.144}$$

where $\alpha^2 = e E_0 k(1 + w)/2m = \omega_\mathrm{B}^2 (1 + w)/2$ and $\beta = 2/(1 + w) < 1$. It can be integrated,

$$\alpha t = \int_{k x(0)/2}^{k x(t)/2} \frac{d\xi}{\sqrt{1 - \beta \sin^2 \xi}} = F(k x(t)/2, \beta) - F(k x(0)/2, \beta) \ . \tag{1.145}$$

The integral is the elliptic integral of the first kind (usually met in physics textbooks when dealing with the oscillations of the pendulum). It reciprocal is the Jacobi amplitude am $= F^{-1}(t, \beta)$, and

$$k x(t) = 2\mathrm{am}(\alpha t + F(k x(0)/2, \beta)) \,. \tag{1.146}$$

For trapped particles, a new variable ζ is defined by $\sin\zeta = \beta \sin\xi$. It is valid for a limited domain of values of ξ, compatible with the fact that the particles are trapped and that the values of $\|\xi\|$ are bounded. Then, Eq. (1.144) becomes

$$(d_\tau \zeta(\tau))^2 = \alpha^2 \left(1 - \frac{1}{\beta} \sin^2 \zeta\right) \,. \tag{1.147}$$

Its solution is

$$\alpha t = F\left(\arcsin\left(\beta \sin k x(t)\right), \frac{1}{\beta}\right) - F\left(\arcsin\left(\beta \sin k x(0)\right), \frac{1}{\beta}\right) \,. \tag{1.148}$$

We can see that the trapped particles have a nonharmonic motion, and their frequency depends on their amplitude β. As we have seen, for a vanishing amplitude, the frequency is ω_{B}. For larger amplitudes, the frequency is lower. Consequently, the phases characterizing each individual motion mix, hindering coherent collective behavior of the trapped particles. This point, which goes beyond the motion of isolated particles, is discussed more extensively in Section 6.3.2.