1

General Concepts in Physics of Excited and Ionized Gases

1.1

Ideal Plasma

1.1.1

Plasma as a State of Matter

The word “plasma” was introduced into science by the Czech physiologist J.E. Purkinje in the middle of the nineteenth century to denote the uniform blood fluid that is released from particles and corpuscles. This term was suggested for a uniform ionized gas of the positive column of a gas discharge by Langmuir [1–3] and now this term denotes any system with electrons and ions where charged particles determine the properties of this system. The most widespread form of plasma is an ionized gas which consists of atoms or molecules with an admixture of charged particles, electrons and ions. Such a plasma is the subject of this book.

To understand the conditions required for the existence of such a plasma under equilibrium conditions, we compare it with an identical chemical system. Let us consider, for example, atmospheric air consisting basically of nitrogen and oxygen molecules. At high temperatures, along with the nitrogen and oxygen, nitrogen oxides can be formed. The following chemical equilibrium is maintained in air:

$$N_2 + O_2 \leftrightarrow 2NO - 41.5\,\mathrm{kcal/mol} \tag{1.1}$$

Here and below, the sign ↔ means that the process can proceed either in the forward direction or in the reverse direction. According to the Le Chatelier principle [4, 5], an increase in the temperature of the air leads to an increase in the concentration of the NO molecules.

A similar situation exists in the case of formation of charged particles in a gas, but this process requires a higher temperature. For example, the ionization equilibrium for nitrogen molecules has the form

$$N_2 \leftrightarrow N_2^+ + e - 360\,\mathrm{kcal/mol} \tag{1.2}$$

Thus, the chemical and ionization equilibria are analogous, but ionization of atoms or molecules proceeds at higher temperatures than chemical transformations. To

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illustrate this, Table 1.1 contains examples of chemical and ionization equilibria. This table gives the temperatures at which 0.1% of molecules are dissociated in the case of a chemical (dissociation) equilibrium or 0.1% of atoms are ionized for an ionization equilibrium at a gas pressure of 1 atm. Thus, a weakly ionized gas, which we shall call a plasma, has an analogy with a chemically active gas. Therefore, although a plasma has characteristic properties, which we shall describe, it is not really a new form or state of matter as is often stated. An ideal plasma is a form of a gas, whereas a dense nonideal plasma (a plasma with strong coupling) is an analogue of a condensed atomic system.

In most cases a plasma is a weakly ionized gas with a small degree of ionization. Table 1.2 gives some examples of real plasmas and their parameters – the number densities of electrons N_e and of atoms N_a, the temperature (or the average energy) of electrons T_e, and the gas temperature T. In addition, some types of plasma systems are given in Figures 1.1 and 1.2. We note also that an equilibrium between charged particles and atoms or molecules may be violated if the plasma is located in an external field. In particular, electric energy is introduced in a gas discharge plasma and it is transferred to electrons in a first stage, and then as a result of collisions it is transferred from electrons to atoms or molecules. This form of injection of energy may lead to a higher electron energy compared with the thermal energy of atoms or molecules, and the plasma becomes a nonequilibrium plasma. Figure 1.3 gives some examples of equilibrium and nonequilibrium plasmas.

Table 1.1 Temperatures corresponding to dissociation of 0.1% of molecules or ionization of 0.1% of atoms at a pressure of 1 atm.

Chemical equilibrium	T, K	Ionization equilibrium	T, K
$2CO_2 \leftrightarrow 2CO + O_2$	1 550	$H \leftrightarrow H^+ + e$	7 500
$H_2 \leftrightarrow 2H$	1 900	$He \leftrightarrow He^+ + e$	12 000
$O_2 \leftrightarrow 2O$	2 050	$Cs \leftrightarrow Cs^+ + e$	2 500
$N_2 \leftrightarrow 2N$	4 500		
$2H_2O \leftrightarrow 2H_2 + O_2$	1 800		

Table 1.2 Parameters of some plasmas. N_e and N are the number densities of electrons and neutral atomic particles, respectively, and T_e and T are their temperatures.

Type of plasma	N_e, cm^{-3}	N, cm^{-3}	T_e, K	T, K
Sun's photosphere	10^{13}	10^{17}	6000	6000
E layer of ionosphere	10^5	10^{13}	250	250
He-Ne laser	3×10^{11}	2×10^{16}	3×10^4	400
Ar laser	10^{13}	10^{14}	10^5	10^3

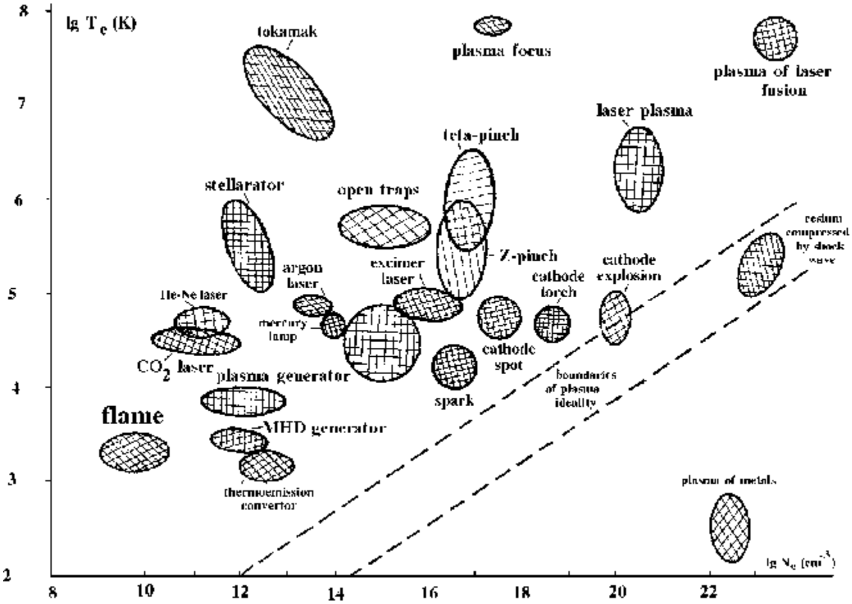

Figure 1.1 Characteristics of natural plasmas. MHD – magnetohydrodynamic.

It is seen that generation of an equilibrium plasma requires strong heating of a gas. One can create a conducting gas by heating the charged particles only. This

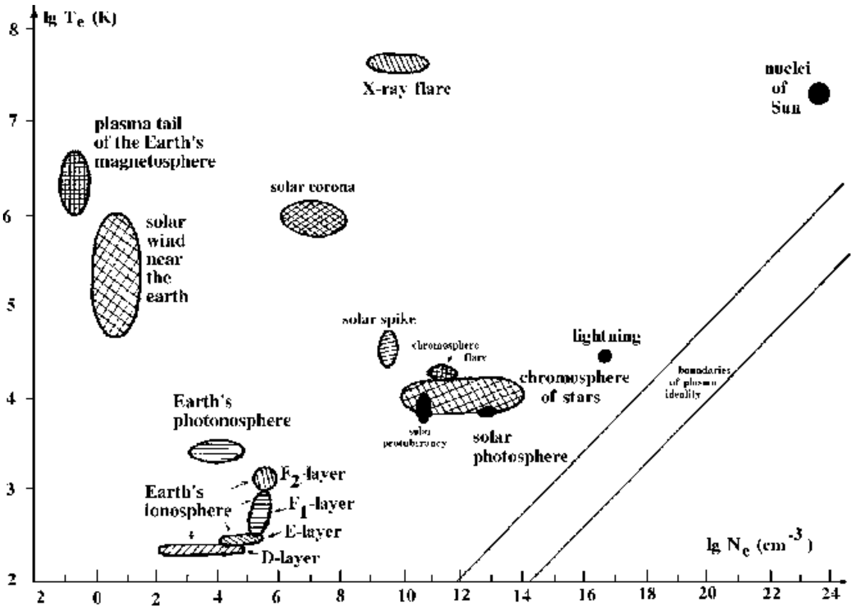

Figure 1.2 Characteristics of laboratory plasmas.

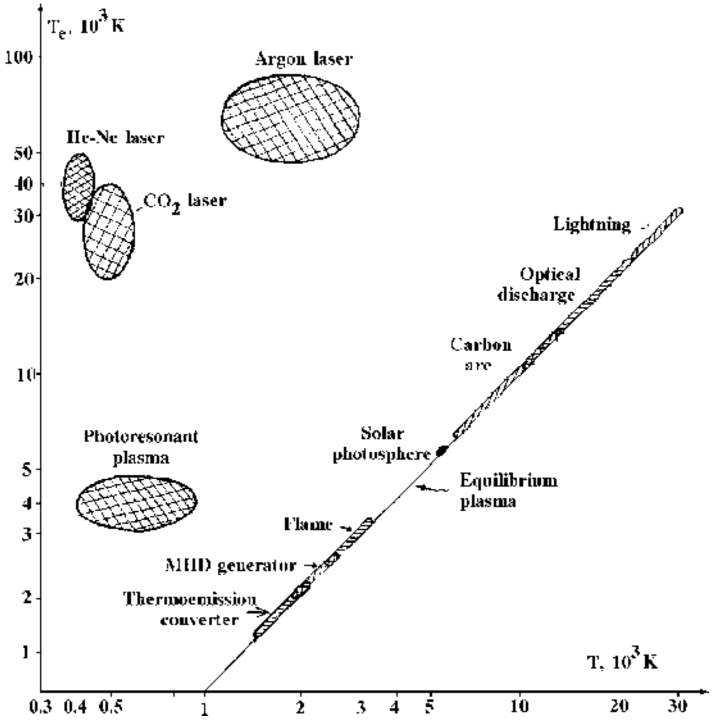

Figure 1.3 Electron and gas temperatures of laboratory plasmas. The straight line corresponds to an equilibrium plasma, whose electron and gas temperatures are equal.

takes place in gas discharges when an ionized gas is placed in an external electric field. Moving in this field, electrons acquire energy from the field and transfer it to the gas. As a result, the mean electron energy may exceed the thermal energy of neutral particles of the gas, and they can produce the ionization which is necessary for maintaining an electric current in the system. Thus, a gas discharge is an example of a plasma which is maintained by an external electric field. If the temperatures of electrons and neutral particles are identical, the plasma is called an equilibrium plasma; in the opposite case, we have a nonequilibrium plasma. Figure 1.3 gives some examples of equilibrium and nonequilibrium plasmas.

Thus, a plasma as a physical object has specific properties which characterize it. Because of the presence of charged particles, various types of interaction with external fields are possible and these lead to a special behavior of plasmas which is absent in ordinary gaseous systems. Furthermore, there are a variety of means for generation and application of plasmas, and these will be considered below.

1.1.2
The History of the Study of Electricity

Let us review briefly the history of plasma physics that is connected with the development of electrical techniques and electrical science [6]. One can date the start of the history of electric phenomena in gases to 1705, when the English scientist Francis Hauksbee made an electrostatic generator whose power allowed him to study luminous electric discharges in gases. In 1734, Charles Francois de Cisternay Dufay (France) discovered that air conducts electricity near hot bodies. In 1745, E.J. von Kleist (Germany) and P.V. Musschenbroek (Netherlands) constructed independently a type of electric capacitor named the Leyden jar. This made possible the study of electric breakdown in air. In 1752, the American scientist and statesman Benjamin Franklin created a theory of lightning on the basis of some experiments. He considered the lightning phenomenon as a flow of electricity through air that corresponds to contemporary understanding of this phenomenon as passage of an electric current through air.

The nineteenth century was a period in which electric processes and phenomena were studied intensely [7]. At that time the technique required to create a gas discharge was worked out and this paved the way for the subsequent development of the electrical sciences, including plasma physics. This included the creation of new gas discharge devices together with the development of sources of electricity and methods of electricity diagnostics. The sources of electricity were developed simultaneously in the creation and improvement of batteries, where electric energy resulted from chemical energy, and dynamos for transformation of mechanical energy into electric energy. In 1800, A. Volta (Italy) created a battery that was the prototype of the modern battery. It consisted of electrochemical cells joined in series, and each cell contained zinc and copper electrodes and an electrolyte between them, firstly H_2SO_4. Two types of ions, H^+ and SO_4^{2-}, are formed in this electrolyte, and protons capture electrons from the copper electrodes, forming bubbles of hydrogen (H_2), whereas negatively charged sulfate ions react with the zinc electrodes. This battery was improved many times and became a reliable source of electricity, and the general concept of this device was conserved in all subsequent devices of this type [8]. In the same period, the fuel element was also created (W.R. Grove, England, 1839). This generates electric energy as a result of the chemical reaction of H_2 and O_2.

On the path to the development of dynamos, J. Henry (USA) constructed an electric motor in 1831 that worked from a battery, that is, electric energy was transformed into mechanical energy, and M.H. Jacobi (Germany) showed that this dynamo could be used not only as a motor, but also as a generator of electricity, that is, it converts mechanical power into electric power. In 1870, Z.T. Gramme (France) constructed a magnetoelectric machine that was a prototype of contemporary generators of electricity in electric power plants. As a result of the creation and improvement of sources of electric energy, reliable equipment was made for various applications in manufacturing and the economy.

The creation and development of sources of electricity determined the development of the gas discharge technique. The first type of discharge was an arc discharge at atmospheric pressure. The subsequent development of the pumping technique allowed the creation of burned gas discharges of various types. Crookes tubes [9] were an important development, where on the basis of a mercury vacuum pump the pressure of the remaining gas in a sealed-off tube may be decreased to 10^{-7} atm. The gas discharge technique allowed study of various processes involving electrons and ions in gases, including the cathode rays discovered by J. Pluecker (Germany) in 1851. Firstly, they were discovered from the sharp straightforward boundary of fluorescence resulting from applying a high voltage to electrodes of the tube with a low gas pressure. Later, it was proved that the gas was excited by a beam of charged particles that came from the cathode and therefore they were named cathode rays (*Kathodenstrahlen*) [10, 11]. In reality, cathode rays are electron beams and they were used subsequently as a specific instrument which became important in the new physics at the beginning of the twentieth century. For this technique the creation of an inductive coil by Rumkopf in 1851 was important as it provided a possibility to obtain high voltages in a simple way. All these developments allowed various forms of gas discharges to be obtained, including a glow discharge, and allowed them to be used for various technical applications. Along with gas discharges, this technique allowed beams of charged particles to be generated.

The analysis of some results obtained by using the new technique led to a new understanding of the nature of ionized gases and related physical objects. The understanding of electricity carriers followed from the electrolysis laws of Michael Faraday (England) in 1833–1834. The electrolysis laws give the ratio of the charge of the carrier to its mass. Along with the establishment of the fundamental laws of electrolysis, Faraday introduced new terms such as "ion" – the electricity carrier – and "anode" ("way up" from Greek) and "cathode" ("way down" from Greek) for the electrodes involved in electricity transfer. In continuation of these studies, Stoney (England) suggested the term "electron" for the quantum of elementary charge and estimated its value in 1874.

The development of electricity sources and gas discharges was a basis for the creation of the new science branches, such as spectroscopy, atomic physics, and nuclear physics, and also led to quantum mechanics, the atomic quantum theory, and nuclear physics at the beginning of the twentieth century. In turn, this new physics stimulated the subsequent gas discharge development [12, 13]. Finally, understanding the gas discharge phenomenon as a self-consistent one due to the ionization balance in a gas allowed Townsend to describe the passage of current through a gas [14–16]. According to I. Langmuir, an ionized gas excited in a gas discharge is characterized by different spatial regions, and the region adjacent to the walls was named by him as a "sheath", whereas a quasineutral ionized gas far from the walls was called a "plasma" [1–3] by analogy with blood plasma. These terms are used now [17, 18].

1.1.3
Methods of Plasma Generation

There are various methods to create a plasma as a result of action on matter, and we consider them briefly. A gas discharge plasma is the most widespread form of plasma and can have a variety of characteristics. It can be either stationary or pulsed, depending on the character of the external fields. An external electric field may cause electric breakdown of gas, which then generates different forms of plasma depending on the conditions of the process. In the first stage of breakdown, a uniform current of electrons and ions may arise. If the electric field is not uniform, an ionization wave can propagate in the form of an electron-avalanche streamer. In the next stage of the breakdown process, the electric current establishes a distribution of charged particles in space. This is one form of gas discharge.

After the external electric field is switched off, the plasma decays as a result of recombination processes of electrons with ions, and spatial diffusion of the plasma occurs. This plasma is called an afterglow plasma, and is used to study recombination and diffusion processes involving ions and excited atoms. A convenient way to generate plasma uses resonant radiation, that is, radiation whose wavelength corresponds to the energy of an atomic transition in the atoms constituting the excited gas. As a result of the excitation of the gas, a high density of excited atoms is attained, and collision of these atoms leads to formation of free electrons. Thus, the atomic excitation in the gas leads to its ionization and to plasma generation. This plasma is called a photoresonant plasma. The possibility of generating such a plasma has improved with the development of laser techniques. In contrast to a gas discharge plasma, a photoresonant plasma is formed as a result of excitations of atoms and therefore there are specific requirements for its formation. In particular, the temperature of the excited atoms can be somewhat in excess of the electron temperature. This plasma may be used for generation of multicharged ions, as a source of acoustic waves, and so on.

A laser plasma is created by laser irradiation of a surface and is described by some parameters such as the laser power and the duration of the process. In particular, if a short (nanosecond) laser pulse is focused onto a surface, material evaporates from the surface in the form of a plasma. If the number density of electrons exceeds the critical density (in the case of a neodymium laser, where the radiation wavelength is 1.06 μm, this value is 10^{21} cm^{-3}), the evolving plasma screens the radiation, and subsequent laser radiation heats this plasma. As a result, the temperature of the plasma reaches tens of electronvolts, and this plasma can be used as a source of X-ray radiation or as the source of an X-ray laser. Laser pulses can be compressed and shortened up to about 2×10^{-14} s. This makes possible the generation of a plasma in very short times, and it permits the study of fast plasma processes.

If the laser power is relatively low, the evaporating material is a weakly ionized vapor. Then, if the duration of the laser pulse is not too short (more than 10^{-6} s), there is a critical laser power (10^7–10^8 $\mathrm{W/cm^2}$) beyond which laser radiation is absorbed by the plasma electrons, and laser breakdown of the plasma takes place. For values of the laser power smaller than the critical value, laser irradiation of a surface is a method for generating beams of weakly ionized vapor. This vapor can be used for formation and deposition of atomic clusters.

A widely used method of plasma generation is based on passage of electron beams through a gas. Secondary electrons can be used further for certain processes. For example, in excimer lasers, secondary electrons are accelerated by an external electric field for generation of excited molecules with short lifetimes. The electron beam as a source of ionization is convenient for excimer and chemical lasers because the ionization process lasts such a short time.

A chemical method of plasma generation occurs in flames. The chemical energy of reagents is expended on formation of radicals or excited particles, and chemionization processes with participation of active particles generate charged particles. Transformation of the chemical energy into the energy of ionized particles is not efficient, so the degree of ionization in flames is low. Electrons in a hot gas or vapor can be generated by small particles. Such a process takes place in the products of combustion of solid fuels.

Introduction of small particles and clusters into a weakly ionized gas can change its electric properties because these particle can absorb charged particles, that is, electrons and positive ions or negative ions and positive ions recombine on these particles by attachment to them. This process occurs in an aerosol plasma, that is, an atmospheric plasma which contains aerosols. In contrast, in hot gases small particles or clusters can generate electrons.

A plasma can be created under the action of fluxes of ions or neutrons when they pass through a gas. Ionization near the Earth's surface results from the decay of radioactive elements which are found in the Earth's crust. Ionization processes and formation of an ionized gas in the upper atmosphere of the Earth are caused by UV radiation from the Sun. There are various methods of plasma generation that lead to the formation of different types of plasmas. Some methods of plasma generation and the types of plasmas resulting from them are given in Table 1.3.

Table 1.3 Methods of plasma generation.

Character of action	Matter	Type of plasma
Electric field	Gas	Stationary gas discharge plasma
Electromagnetic wave	Gas	Alternative gas discharge plasma
Resonant radiation	Atomic vapor	Photoresonant plasma
Excitation from chemical reactions	Chemically active mixture	Chemical (flame) plasma
Laser	Surface or particles	Laser plasma
Injection of electrons or ions	Surface or ionized gas	Beam plasma
Injection of nucleating vapor	Ionized gas	Cluster plasma
Injection of dust particles	Ionized gas	Dusty plasma
Ionization by hard radiation	Gas (air) with aerosols	Aerosol plasma

1.1.4
Charged Particles in a Gas

We shall consider primarily a plasma whose properties are similar to those of a gas. As in a gas, each particle of the plasma will follow a straight trajectory as a free particle most of the time. These free-particle intervals will occasionally be punctuated by strong interactions with surrounding particles that will cause a change in energy and the direction of motion. This situation occurs if the mean interaction potential of the particle with its neighbors is small compared with the mean kinetic energy of the particle. This is the customary description of the gaseous state of a system, and a plasma that also satisfies this description is called an ideal plasma.

Let us formulate now a quantitative criterion for a plasma to be ideal. The Coulomb interaction potential between two charged particles has the absolute value $|U(R)| = e^2/R$, where e is the charge of an electron or a singly charged ion and R is the distance between interacting particles. Thus, the interaction potential at the mean distance between particles $R_0 \sim N_\mathrm{e}^{-1/3}$ is $|U| \sim e^2 N_\mathrm{e}^{1/3}$, and because the mean thermal energy of the particles is of the order of T (T is the plasma temperature expressed in energy units), the condition of the gaseousness of a plasma due to interaction of charged particles is characterized by the smallness of the plasma parameter γ, which is

$$\gamma = \frac{N_\mathrm{e} e^6}{T^3} \ll 1 . \tag{1.3}$$

If the gaseousness condition (1.3) is fulfilled for a plasma, this plasma is ideal. In the following, we shall deal primarily with a plasma whose parameters satisfy (1.3).

We define a weakly ionized gas as a gas with a small concentration of charged particles. Nevertheless, some properties of the weakly ionized gas are governed by the charged particles. For example, the degree of ionization in power-discharge molecular lasers is 10^{-7}–10^{-5}. In these lasers, the energy is first transferred from an external source of energy to electrons, and then it is transformed to the energy of laser radiation. As we shall see, a relatively small concentration of electrons determines the operation of this system.

Some properties of a weakly ionized gas are determined by the interaction between charged and neutral particles, whereas other properties are created by charged particles only. Although the concentration of charged particles in a plasma is small, the long-range Coulomb interaction between them may be more important than the short-range interaction between neutral particles. We consider below the plasma properties that are associated with the presence of charged particles. The short-range interaction of neutral particles is not important for these properties.

We now study the penetration of an external electric field into a plasma. Since this field leads to a redistribution of the charged particles of a plasma, it creates an internal electric field that opposes the external field. This has the effect of screening the plasma from an external field. To analyze this effect, we consider the Poisson equation for an electric field in a plasma, which has the form

$$\operatorname{div} \mathbf{E} = -\Delta\varphi = 4\pi e(N_\mathrm{i} - N_\mathrm{e}) \,. \tag{1.4}$$

Here $\mathbf{E} = -\nabla\varphi$ is the electric field strength, φ is the potential of the electric field, N_e and N_i are the number densities of electrons and ions, respectively, and ions are assumed to be singly charged. The effect of an electric field is to cause a redistribution of charged particles. According to the Boltzmann formula, the ion and electron number densities are given by

$$N_\mathrm{i} = N_0 \exp\left(-\frac{e\varphi}{T}\right) \,, \quad N_\mathrm{e} = N_0 \exp\left(\frac{e\varphi}{T}\right) \,, \tag{1.5}$$

where N_0 is the average number density of charged particles in a plasma and T is the plasma temperature. Substitution of (1.5) into the Poisson equation (1.4) gives

$$\Delta\varphi = 8\pi N_0 e \sinh\left(\frac{e\varphi}{T}\right) \,.$$

Assuming that $e\varphi \ll T$, we can transform this equation to the form

$$\Delta\varphi = \frac{\varphi}{r_\mathrm{D}^2} \,, \tag{1.6}$$

where

$$r_\mathrm{D} = \left(\frac{T}{8\pi N_\mathrm{e} e^2}\right)^{1/2} \tag{1.7}$$

is the so-called Debye–Hückel radius [19].

The solution of (1.6) describes an exponential decrease with distance from the plasma boundary. For example, if an external electric field penetrates a flat boundary of a uniform plasma, the solution of (1.6) has the form

$$\mathbf{E} = \mathbf{E}_0 \exp\left(-\frac{x}{r_\mathrm{D}}\right) \,, \tag{1.8}$$

where x is the distance from the plasma boundary in the normal direction. If the electron and ion temperatures are different, then (1.5) has the form

$$N_\mathrm{i} = N_0 \exp\left(-\frac{e\varphi}{T_\mathrm{i}}\right) \,, \quad N_\mathrm{e} = N_0 \exp\left(\frac{e\varphi}{T_\mathrm{e}}\right) \,,$$

and we will obtain the same results as above, except that the Debye–Hückel radius takes the more general form

$$r_\mathrm{D} = \left[4\pi N_0 e^2 \left(\frac{1}{T_\mathrm{e}} + \frac{1}{T_\mathrm{i}}\right)\right]^{-1/2} \,. \tag{1.9}$$

Now let us calculate the field from a test charge placed in a plasma. In this case the equation for the potential due to the charge has the form

$$\Delta\varphi \equiv \frac{1}{r}\frac{d^2}{dr^2}(r\varphi) = \frac{\varphi}{r_\mathrm{D}^2} \,,$$

where r is the distance from the charge considered. If this charge is located in a vacuum, the right-hand side of this equation is zero, and the solution has the form $\varphi = q/r$, where q is the charge. Requiring the solution of the above equation to be coincident with this one at $r \to 0$, we obtain for the potential of a test charged particle in a plasma

$$\varphi = \frac{q}{r} \exp\left(-\frac{r}{r_\mathrm{D}}\right) . \tag{1.10}$$

Hence, the interaction potential U of two charged particles with charges q_1 and q_2 is as follows:

$$U(r) = \frac{q_1 q_2}{r} \exp\left(-\frac{r}{r_\mathrm{D}}\right) . \tag{1.11}$$

Thus, the Debye–Hückel radius is a typical distance at which fields in a plasma are shielded by the charged particles. The field of a charged particle is eliminated on this scale by fields of surrounding particles.

Now we shall check the validity of the condition $e\varphi \ll T$, which allowed us to simplify the equation for the electric field strength. Because this condition must work at distances of the order of r_D, it has the form

$$\frac{e^2}{r_\mathrm{D} T} \sim \left(\frac{e^6 N_0}{T^3}\right)^{1/2} \ll 1 .$$

This condition according to criterion (1.3) characterizes an ideal plasma.

We can determine the number of charged particles that participate in shielding the field of a test particle. This value is of the order of magnitude of the number of charged particles located in a sphere of radius r_D. The number of charged particles is, to within a numerical factor,

$$r_\mathrm{D}^3 N_0 \sim \sqrt{\frac{T^3}{N_0 e^6}} \gg 1 .$$

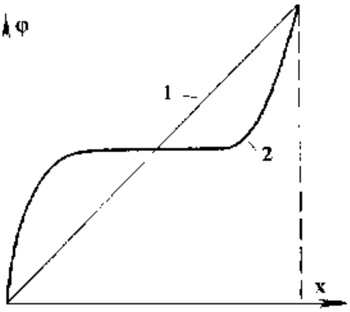

Figure 1.4 The distribution of the electric potential in a gap containing an ionized gas. 1 – the Debye–Hückel radius is large compared with the gap size; 2 – the Debye–Hückel radius is small compared with the gap size.

Hence, this value is large for an ideal plasma.

Thus, the Debye–Hückel radius is the fundamental parameter of an ideal quasineutral plasma. It is the distance over which an ensemble of charged plasma particles shields external electric fields or fields of individual plasma particles. Correspondingly, the character of the distribution of external fields in a plasma differs from that in a gas, as shown in Figure 1.4 for a gaseous or plasma gap.

1.1.5
Definition of a Plasma

Let us consider a gas-filled gap subjected to an external electric field. If the gas does not contain charged particles, the field is uniform in the gap. In the presence of charged particles in the gas, an external electric field is shielded near the edges of the gap (see Figure 1.4) at distances of the order of the Debye–Hückel radius. Thus, the character of the distribution of an electric field inside the gas is different in these two cases. Note that here we assume the plasma to be quasineutral up to its boundaries, an assumption that can be violated in real cases. On the basis of the above considerations, one can define a plasma as a weakly ionized gas whose Debye–Hückel radius is small compared with its size L, that is,

$$L \gg r_D \,. \tag{1.12}$$

For an electron temperature of $T_e \sim 1\,\mathrm{eV}$ and a gap size of the order of 10 cm this criterion gives an electron number density $N_e \gg 10^4\,\mathrm{cm}^{-3}$, which is a very small value (see Figures 1.1 and 1.2). For example, the electron number density of a glow discharge lies in the range 10^7–$10^{12}\,\mathrm{cm}^{-3}$. Table 1.4 contains parameters of some plasma types in terms of a typical number density N_e and temperature T_e of electrons, and also important parameters of the plasma types under consideration, parameters such as the Debye–Hückel radius of the plasma r_D and its typical size L. These examples are a representative sampling of real plasmas.

Table 1.4 Parameters of some plasmas.

Type of plasma	**N_e, cm^{-3}**	**T_e, K**	**r_D, cm**	**L, cm**
E layer of ionosphere	10^5	300	0.3	10^6
He-Ne laser	10^{11}	3×10^4	3×10^{-3}	1
Mercury lamp	10^{14}	5×10^4	3×10^{-5}	0.1
Sun's chromosphere	5×10^8–5×10^9	3×10^4–3×10^5	0.01–0.1	10^9
Lightning	10^{17}	3×10^4	3×10^{-6}	100

1.1.6
Oscillations of Plasma Electrons

The Debye–Hückel radius is the parameter that characterizes an ideal quasineutral plasma. We can estimate a typical time for the response of a plasma to an external field. For this purpose we study the behavior of a uniform infinite plasma if all the plasma electrons are shifted at the initial time by a distance x_0 to the right starting from a plane $x = 0$. This creates an electric field whose strength corresponds to the Poisson equation (1.4):

$$\frac{dE}{dx} = 4\pi e(N_\mathrm{i} - N_\mathrm{e}) \ .$$

Assuming the electric field strength at $x < 0$ is zero, the Poisson equation gives an electric field strength for $x > x_0$ of $E = 4\pi e N_0 x_0$, where N_0 is the average number density of charged particles in the plasma. The movement of all the electrons under the influence of the electric field leads to a change in the position of the boundary. The equation of motion for each of the electrons can be written as

$$m_\mathrm{e} \frac{d^2(x + x_0)}{dt^2} = -eE \ ,$$

where m_e is the electron mass and x is the distance of an electron from the boundary. Because x is a random value, not dependent on the phenomenon being considered, one can assume this value to be independent of time. Thus, the equation of motion of an electron is

$$\frac{d^2 x_0}{dt^2} = -\omega_\mathrm{p}^2 x_0 \ ,$$

where [2, 20–23]

$$\omega_\mathrm{p} = \left(4\pi N_0 \frac{e^2}{m_\mathrm{e}}\right)^{1/2} \tag{1.13}$$

is called the plasma frequency, or Langmuir frequency.

The solution of the equation obtained predicts an oscillatory character for the electron motion. Accordingly, $1/\omega_\mathrm{p}$ is a typical time for a plasma to respond to an external signal. Note that the value $r_\mathrm{D}\omega_\mathrm{p} = \sqrt{2T/m_\mathrm{e}}$ is the thermal electron velocity. From this it follows that a typical time for a plasma to respond to an external signal is the time during which the electrons experience a displacement of the order of the Debye–Hückel radius. Thus, we have two fundamental parameters of an ideal quasineutral plasma: the Debye–Hückel radius r_D, which is a shielding distance for fields in a plasma, and the plasma frequency ω_p, so ω_p^{-1} is a typical time for the plasma to respond to external signals.

Thus, the Debye–Hückel radius and the plasma frequency are two fundamental parameters which reflect a long-range character of interaction of charged particles in an ionized gas. These parameters do not depend on a short-range interaction of neutral particles in an ionized gas, and therefore the character of reacting to external fields as well some plasma collective phenomena in ionized gases are determined by charged particles only and are independent of the presence of atoms or molecules in an ionized gas. As a result, collective phenomena in ionized gases have a universal character and may be identical for various ionized gases if the gas criterion holds true for neutral and charged particles.

1.1.7
Interaction of Charged Particles in an Ideal Plasma

We now calculate the average potential energy of a charged particle in an ideal plasma and the distribution function for the interaction potential of charged particles. From the interaction potential (1.11) for two charged particles, we have for its average value

$$\overline{U} = \int_0^\infty e\varphi \left[N_0 \exp\left(-\frac{e\varphi}{T}\right) - N_0 \exp\left(\frac{e\varphi}{T}\right)\right] d\mathbf{r} \,.$$

We assume the charges of electrons and ions in a plasma to be $\pm e$ and use (1.10) for the electric potential φ from an individual plasma particle of charge $+e$, that is,

$$\varphi = \frac{e}{r} \exp\left(-\frac{r}{r_\text{D}}\right) \,.$$

The above formula for the average interaction potential in an ideal plasma accounts for pairwise interactions of all the charged particles in a volume that have a Boltzmann distribution. In the case of an ideal plasma, the principal contribution to the integral occurs at small interactions $e\varphi \ll T$, which gives

$$\overline{U} = -\frac{2N_0}{T} \int_0^\infty (e\varphi)^2 4\pi r^2 dr = -\frac{4\pi N_0 r_\text{D}}{T} = -\frac{e^2}{2r_\text{D} T} \,, \tag{1.14}$$

where we used expression (1.7) for the Debye–Hückel radius. Thus, the average energy of a charged particle in an equilibrium plasma is

$$\overline{\varepsilon} = \frac{3T}{2} - \frac{e^2}{2r_\text{D}} \,. \tag{1.15}$$

To estimate fluctuations ΔU of the interaction potential for a charged particle in a plasma, we assume this value to be determined by positions of other charged particles in a sphere of the Debye–Hückel radius centered on the test particle. The mean number of charged particles in this region is $n \sim N_0 r_\text{D}^3 \gg 1$, with fluctuations of the order of $\sqrt{n}$. Hence, the fluctuation of the interaction potential of the test charged particle in a plasma is

$$\Delta U \sim \sqrt{n}\frac{e^2}{r_\text{D}} \sim e^2 N_0^{1/2} r_\text{D}^{1/2} \,. \tag{1.16}$$

Since $\Delta U \gg U$, the distribution function for the interaction potentials yields $\overline{U} = 0$ and has the form

$$f(U)\,dU = (2\pi \Delta U^2)^{-1/2} \exp\left[\frac{-U^2}{2\Delta U^2}\right] , \tag{1.17}$$

where $f(U)\,dU$ is the probability that the interaction potential lies in the interval from U to $U + dU$. The squared deviation of the distribution (1.17) is

$$\Delta U^2 = \overline{U^2} = 2\int_0^\infty (e\varphi)^2 N_0 4\pi r^2 dr = 4\pi N_0 e^4 r_{\mathrm{D}} = \overline{U}T \gg (\overline{U})^2 ,$$

where the factor 2 takes into account the presence of charged particles of the opposite sign, and N_0 is the mean number density of charged particles of one sign. Thus, we have for an ideal plasma

$$\overline{U} \ll \Delta U \ll T .$$

1.1.8
Microfields in an Ideal Plasma

The charged plasma particles, electrons and ions, create electric fields in a plasma that act on atoms and molecules in a plasma. Although the average field at each point is zero, at this time electric fields are of importance. Guided by the action of these fields on atoms, for a corresponding time range one can average the fields from electrons, that is, the action of electrons on atoms as a result of their collision. Fields from ions at a given point are determined according to their spatial distribution. We will determine below the distribution function $P(E)$ for electric fields at a given point for a given time for a random spatial distribution of ions, so by definition $P(E)\,dE$ is the probability that the electric field strength at a given point ranges from E to $E + dE$, and we use an isotropic form of the distribution over the electric field strengths. We take into account the electric field strength from the ith ion at a distance $\mathbf{r}_i$ from it is

$$\mathbf{E}_i = \frac{e\mathbf{r}_i}{r_i^3} ,$$

and the total electric field of all the ions is

$$\mathbf{E} = \sum_i \mathbf{E}_i .$$

Evidently, we include in this sum ions located at a distance $r_i \ll r_{\mathrm{D}}$ from a test point, that is, we assume that the main contribution to this sum is from ions located close to a test point. In addition, we take a random distribution of ions in space.

We first find the tail of the distribution function that corresponds to large fields. The probability of the nearest ion being located in a distance range from r to $r + dr$ is

$$P(E)\,dE = 4\pi r^2 dr N_{\mathrm{i}} ,$$

where N_i is the number density of ions, and the electric field strength from an individual ion is $E = e/r^2$. From this we obtain

$$P(E)\,dE = \frac{2\pi e^{3/2} N_\mathrm{i}\,dE}{E^{5/2}}\,, \qquad \sqrt{\frac{e}{E}} \ll N_\mathrm{i}^{-1/3}\,. \tag{1.18}$$

From this one can estimate a typical value E_0 of the electric field strength:

$$E_0 \sim e N_\mathrm{i}^{2/3}\,. \tag{1.19}$$

This value corresponds to the electric field strength at the mean distance between ions.

We now determine the distribution function $P(\mathbf{E})$ on the basis of the standard method with the characteristic function $F(\mathbf{g})$, that is,

$$F(\mathbf{g}) = \int \exp(i\mathbf{E}\mathbf{g})\,P(\mathbf{E})\,d\mathbf{E}\,, \tag{1.20}$$

where $\mathbf{g}$ is a three-dimensional variable. Correspondingly, the inverse transformation gives

$$P(\mathbf{E}) = \frac{1}{(2\pi)^3} \int \exp(-i\mathbf{E}\mathbf{g})\,F(\mathbf{g})\,d\mathbf{g}\,. \tag{1.21}$$

Introducing the probability $p(\mathbf{E}_i)$ of a given electric field strength created by the ith ion, we have

$$P(\mathbf{E}) = \prod_i \int p(\mathbf{E}_i)\,d\mathbf{E}_i\,.$$

Evidently, the probability $p(\mathbf{E}_i)$ is identical for different ions, and the characteristic function is

$$F(\mathbf{g}) = \prod_i f_\mathrm{i}(\mathbf{g})\,, \quad f_\mathrm{i}(\mathbf{g}) = \int \exp(i\mathbf{E}_i\mathbf{g})\,p(\mathbf{E}_i)\,d\mathbf{E}_i\,. \tag{1.22}$$

Let n ions ($n \gg 1$) be located in a large volume Ω, so

$$p(\mathbf{E}_i)\,d\mathbf{E}_i = \frac{d\mathbf{r}_i}{\Omega}\,,$$

and $d\mathbf{r}_i$ is the volume element where the ith ion is located with the origin of the frame of reference that is at a test point. For the partial characteristic function and using $p(\mathbf{E}_i)\,d\mathbf{E}_i = d\mathbf{r}_i/\Omega$ this gives

$$f_\mathrm{i}(\mathbf{g}) = \frac{1}{\Omega} \int \exp(i\mathbf{E}_i\mathbf{g})\,d\mathbf{r}_i = 1 + \frac{1}{\Omega} \int \left[\exp(i\mathbf{E}_i\mathbf{g}) - 1\right] d\mathbf{r}_i\,,$$

and this gives

$$\ln f_\mathrm{i}(\mathbf{g}) = \frac{1}{\Omega} \int \left[\exp(i\mathbf{E}_i(\mathbf{r}_i)\mathbf{g}) - 1\right] d\mathbf{r}_i\,.$$

From this, because of the identity of $f_\mathrm{i}(\mathbf{g})$, we obtain

$$\ln F(\mathbf{g}) = \sum_i \ln f_\mathrm{i}(\mathbf{g}) = n \ln f_\mathrm{i}(\mathbf{g}) = N_\mathrm{i} \int \left[\exp(i\mathbf{E}_i\mathbf{g}) - 1\right] d\mathbf{r}_i \ ,$$

where $N_\mathrm{i} = n/\Omega$ is the number density of ions. In accordance with (1.21), from this we obtain for the distribution function $P(\mathbf{E})$ for the electric field strengths in a plasma

$$P(\mathbf{E}) = \frac{1}{(2\pi)^3} \int \exp\left\{-i\mathbf{E}\mathbf{g} + N_\mathrm{i} \int \left[\exp(i\mathbf{E}_i\mathbf{g}) - 1\right] d\mathbf{r}_i\right\} d\mathbf{g} \ .$$

Since the electric field strength from an individual ion is $\mathbf{E}_i = e\mathbf{r}_i/r_i^3$, we have

$$\int \left[\exp(i\mathbf{E}_i\mathbf{g}) - 1\right] d\mathbf{r}_i = \int \left[\exp\left(i\frac{eg\cos\theta_i}{r_i^2} - 1\right)\right] \cdot 2\pi d\cos\theta_i r_i^2 dr_i =$$

$$-4\pi(eg)^{3/2} \int_0^\infty \xi^2 d\xi \left(1 - \frac{\sin\xi}{\xi}\right) = -\frac{8\pi(eg)^{3/2}}{15} \ ,$$

where $\xi = eg/r_i^2$. Hence, the distribution function for electric field strengths takes the form [24]

$$P(E) = \int \exp\left[-\frac{8\pi(eg)^{3/2}N_\mathrm{i}}{15} + i\mathbf{E}\mathbf{g}\right] \frac{d\mathbf{g}}{(2\pi)^3} = \frac{1}{E_0} H(z) \ ,$$

$$z = \frac{E}{E_0} \ , \quad E_0 = 2\pi e \left(\frac{4N_\mathrm{i}}{15}\right)^{2/3} \ . \tag{1.23}$$

As is seen, the typical electric field strength E_0 corresponds to that of (1.19), and the Holtzmark function $H(z)$ is given by

$$H(z) = \frac{2}{\pi z} \int_0^\infty x \sin x \exp\left[-\left(\frac{x}{z}\right)^{3/2}\right] dx \ . \tag{1.24}$$

The Holtzmark function has the following expressions in the limiting cases

$$H(z) = \frac{4}{3\pi} z^2 \ , \quad z \ll 1 \ , \quad H(z) = \frac{15}{8}\sqrt{\frac{2}{\pi}} z^{-5/2} \ , \quad z \gg 1 \tag{1.25}$$

and the case of large electric field strengths $z \gg 1$ corresponds to (1.18). The Holtzmark function is represented in Figure 1.5, and its maximum corresponds to a typical electric field strength $E_\mathrm{max} \approx 1.6 E_0$ that is created at average distances between ions. The Holtzmark function may be constructed in a simple manner from its limiting expressions as

$$h(z) = \frac{4z^2}{3\pi\left[1 + \left(\frac{z}{z_0}\right)^{9/2}\right]} \ , \quad z_0 = 1.32 \ . \tag{1.26}$$

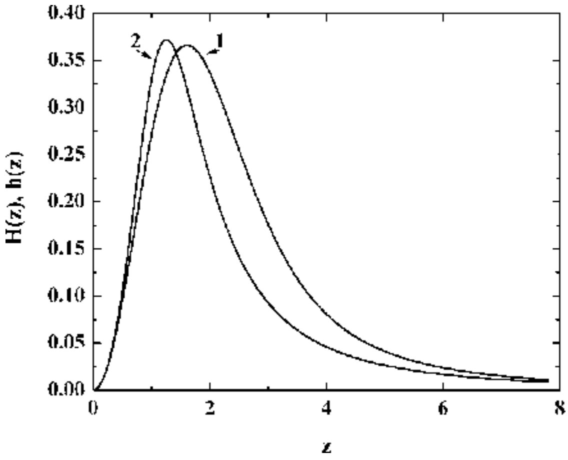

Figure 1.5 The Holzmark function $H(z)$ according to (1.24) (1) and its approximation (1.26) (2).

The approximation $h(z)$ is compared with the Holtzmark function in Figure 1.5.

One can expand these results for a plasma containing multicharged ions. If the ion charge is Z, it is necessary to change e to Ze, which for the typical electric strength E_0 in (1.23) gives

$$E_0 = 2\pi Z e \left(\frac{4N_\mathrm{i}}{15}\right)^{2/3} = 2.60 Z e N_\mathrm{i}^{2/3} . \tag{1.27}$$

Note that microfields in a plasma may be created by both electrons and ions. In this consideration we assume electrons to be distributed uniformly in space. This means that the observation time τ for the ion distribution exceeds significantly a typical time of approximately r/v_e for electron displacement at a distance r that determines this part of the distribution function (v_e is a typical electron velocity). For the maximum of the distribution function ($r \sim N_\mathrm{i}^{-1/3}$) this criterion gives

$$\tau \gg m_\mathrm{e}^{1/2} T_\mathrm{e}^{-1/2} N_\mathrm{i}^{-1/3} , \tag{1.28}$$

where m_e is the electron mass and T_e is a typical electron energy. Simultaneously, the observation time is small compared with a typical time for establishment of a uniform ion distribution, that is, for typical electric field strengths ($E \sim E_0$):

$$\tau \ll M^{1/2} T_\mathrm{i}^{-1/2} N_\mathrm{i}^{-1/3} , \tag{1.29}$$

where M is the ion mass and T_i is a typical electron energy. In particular, for a gas discharge helium plasma with typical parameters $N_\mathrm{i} \sim 10^{12}\,\mathrm{cm}^{-3}$, $T_\mathrm{e} \approx 1\,\mathrm{eV}$, and $T_\mathrm{i} = 400\,\mathrm{K}$ this formula gives $5 \times 10^{-10}\,\mathrm{s} \gg \tau \gg 10^{-12}\,\mathrm{s}$.

In addition, it is necessary to account for screening of the ion field in a plasma, which requires us to change the electric field strength E in (1.23) to $E \exp(-r/r_\mathrm{D})$

in accordance with (1.8). But in this case only electrons partake in screening of ion fields and hence the Debye–Hückel radius is given by

$$r_\mathrm{D} = \sqrt{\frac{T_\mathrm{e}}{4\pi N_0 e^2}}$$

instead of (1.9). For an ideal plasma (1.3) this effect is weak for the maximum of the distribution function $E \sim E_0$.

1.1.9
Beam Plasma

We considered above a quasineutral plasma as a widespread plasma type. In reality, plasma boundaries (called plasma sheaths, or double layers) contain a nonneutral plasma. Plasma properties in this region depend on processes that occur there. If charged particles are generated by a metallic surface or charged particles recombine on walls, an intermediate layer of nonneutral plasma arises between the plasma and the surface. A nonneutral plasma also occurs if charged particles are collected in certain regions or traps, and if they are transported through space by the action of external fields in the form of beams. Thus, a nonneutral plasma is a specific physical object [25, 26], so the primary characteristic of a nonneutral plasma arises from the strong fields created by the particle charge, and that restricts the plasma density. As an illustration, we shall consider a classic example of a nonneutral plasma formed near a hot cathode.

We start from a simple estimation. Let a beam of electrons have number density $N_\mathrm{e} \sim 10^{10}\,\mathrm{cm}^{-3}$ (the minimum boundary of the electron number density for a glow discharge) and beam radius $\rho_0 \sim 1\,\mathrm{cm}$. Then from the Poisson equation (1.4) we have the following estimation for the electric voltage φ between the beam center and its boundary:

$$\varphi \sim 4\pi e N_\mathrm{e} \rho_0^2 \sim 20\,\mathrm{keV}\,.$$

This shows the necessity of using specific electric optics to conserve this plasma.

One can generate an electron beam in a simple way by heating a metallic surface in a vacuum; this will cause the surface to emit an electron flux as a result of thermoemission. Using electric fields allows us to accelerate electrons and remove them from the surface in the form of a beam. But the parameters of this beam can be limited by internal electric fields that arise from electron charges. We can find the properties of such a beam, created between two flat plates a distance L apart, with an electric potential U_0 between them.

The electron current density i is constant in the gap because electrons are not produced nor do they recombine in the gap. This gives

$$i = e N_\mathrm{e}(x) v_\mathrm{e}(x) = \mathrm{const}\,,$$

where x is the distance from the cathode, N_e is the electron number density, $v_\mathrm{e} = \sqrt{2e\varphi(x)/m_\mathrm{e}}$ is the electron velocity, and the electric potential is zero at the cathode surface, that is, $\varphi(0) = 0$. An electron charge creates an electric field that slows the electrons. We can analyze this relationship. The electric field strength $E = -d\varphi/dx$ satisfies the Poisson equation

$$\frac{dE}{dx} = -4\pi e N_e(x) = -4\pi i \sqrt{\frac{m_e}{2e\varphi}} \, .$$

Multiplication of this equation by $E = -d\varphi/dx$ provides an integrating factor that makes a simple integration possible. We obtain

$$E^2 = E_0^2 + 16\pi i \sqrt{\frac{m_e \varphi}{2e}} \, , \tag{1.30}$$

where $E_0 = E(0)$.

We need to establish the boundary condition on the cathode. We consider the regime where the current density of the beam is small compared with the electron current density of thermoemission. This means that most of the emitted electrons return to the metallic surface, and the external electric field does not significantly alter the equilibrium between the emitted electrons and the surface. Then the boundary condition on the cathode is the same as in the absence of the external electric field, so $E(0) = 0$. Formula (1.30) leads to the distribution of the electric potential in the gap, given by

$$\varphi(x) = \left(9\pi i \sqrt{\frac{m_e}{2e}}\right)^{2/3} x^{4/3} \, .$$

This can be inverted to obtain the connection between the electron current density and the parameters of the gap [27–30]:

$$i = \frac{2}{9\pi} \sqrt{\frac{e}{2m_e}} \frac{\varphi_0^{3/2} \, L^2}{} . \tag{1.31}$$

This dependence is known as the three-halves power law. This describes the behavior of a nonneutral plasma that is formed in the space between two plane electrodes with different voltages. This voltage difference in this case is determined by the charge that is created by charged atomic particles [29–31] and has a universal character; in particular, in a magnetron discharge [32, 33], where electrons are magnetized and hence reproduction of a charge in a magnetron discharge results from ion flux to the cathode with energy of hundreds of electronvolts that creates secondary electrons. Along with this, metal atoms are sputtered, which determines the applications of this gas discharge.

We consider one more example of transport of the flux of charged particles through a vacuum: an electron beam of radius a that is fixed by a longitudinal magnetic field inside a cylindrical metal tube of a radius ρ_0 [34]. According to the Gauss theorem, the electric field strength E at a distance ρ from the beam center outside the beam surface is

$$E = \frac{4\pi N_e}{2\pi\rho} = \frac{2I}{\rho e v_e} ,$$

where N_e is the number of electrons per unit beam length, I is the electron current, and v_e is the electron velocity in the beam. On the basis of the equation $E = -d\varphi/dx$ for the voltage difference between the beam and metal tube, this gives

$$\varphi = \frac{2I}{ev_e} \ln \frac{\rho_0}{a},$$

where we assume $\rho_0 \gg a$ and ignore the potential variation inside the electron current. Hence, when the electron beam enters a space inside the tube through a grid or when it intersects a surface with the electric potential of the walls, each electron loses energy $e\varphi$. Let the initial energy of electrons in the beam be E, so the energy becomes $E - e\varphi$ after entry into the tube. This corresponds to the electron velocity

$$v_e = \sqrt{\frac{2(E - e\varphi)}{m_e}}$$

or to the electron current

$$I = \frac{ev_e\varphi}{2\ln(\rho/a)} = \frac{\varphi\sqrt{(2e/m_e)(E - e\varphi)}}{2\ln(\rho/a)} .$$

From this we find the maximum current that is possible under given conditions,

$$I_B = \frac{1}{3}\sqrt{\frac{2}{3m_e}} \cdot \frac{E^{3/2}}{\ln(\rho/a)} , \tag{1.32}$$

which is called the Bursian current [35] and corresponds to the beam voltage with respect to the walls:

$$\varphi_{\max} = \frac{2E}{3e} .$$

The formula for the maximum electric current has the form of the three-halves power law and may be written in the form

$$I_B = I_0 \frac{E^{3/2}}{\ln(\rho/a)} , \tag{1.33}$$

and if the electron energy is expressed in electronvolts, we have [34] $I_0 = 12.7$ μA.

In this case of propagation of the electron beam through a vacuum, the electric potential brakes the electron beam because of the noncompensated charge in the electron beam [34, 36, 37], which is similar to the previous case of electron thermoemission and propagation between plane electrodes. One can expect that neutralization of the beam by introduction of the ion component in the beam region will remove the current limit. Nevertheless, in the latter case of beam propagation inside a cylindrical tube the limiting current is conserved, but its value increases and is $3\sqrt{3}I_B$ [34, 37–39].

1.2
Statistics of Atomic Particles in Excited and Weakly Ionized Gases

1.2.1
Distribution Function of a System of Identical Particles

The subject of this book is a weakly ionized gas that consists of a large number of atoms or molecules and a small admixture of electrons and ions. Each of its components is a system of many weakly interacting identical particles, and our first task is to represent the distribution of particles in these systems for various parameters, and apply to a system of many identical particles the laws of statistical physics. In this case we deal with the distribution function of free particles $f(x)$ over a parameter x, so for a macroscopic uniform system of particles $f(x)dx$ is the number density of particles with the value of this parameter between x and $x + dx$. Within the framework of statistical physics [40, 41], we expound the distribution function for each parameter twofold using the ergodic theorem [42, 43]. In the first case, we deal with an ensemble of a large number of particles, and the distribution function determines the average number particles with a value of this parameter in an indicated range. In the second case, one test particle is observed, and a given parameter of this particle varies in time as a result of interaction of this particle with other particles. Then the distribution function characterizes the relative time when a given parameter is found in an indicated range.

We start from the energy distribution for weakly interacting particles in a closed system. In an ensemble of a large number of particles, each of the particles is in one of a set of states described by the quantum numbers i. The goal is to find the average number of particles that are found in one of these states. For example, if we have a gas of molecules, the problem is to find the molecular distribution of its vibrational and rotational states. We shall consider problems of this type below. Consider a closed system of N particles, so this number does not change with time, as well as the total energy of particles. Denoting the number of particles in the ith state by n_i, we have the condition of conservation of the total number of particles in the form

$$N = \sum_i n_i \, . \tag{1.34}$$

Since the ensemble of particles is closed, that is, it does not exchange by energy with other systems, we require conservation of the total energy E of the particles, where

$$E = \sum_i \varepsilon_i n_i \, , \tag{1.35}$$

where ε_i is the energy of a particle in the ith state. In the course of evolution of the system, an individual particle can change its state, but the average number of particles in each state stays essentially the same. Such a state of a closed system is called a state of thermodynamic equilibrium.

Transitions of individual particles between states result from collisions with other particles. We denote by $W(n_1, n_2, \ldots, n_i \ldots)$ the probability that n_1 particles are found in the first state, n_2 particles are found in the second state, n_i particles are found in the ith state, and so on. We wish to calculate the number of possible realizations of this distribution. First, we take the n_1 particles for the first state from the total number of particles N. There are

$$C_N^{n_1} = \frac{N!}{(N - n_1)! n_1!}$$

ways to do this. Next, we select n_2 particles corresponding to the second state from the remaining $N - n_1$ particles. This can be done in $C_{N-n_1}^{n_2}$ ways. Continuing this operation, we find the probability distribution to be

$$W(n_1, n_2, \ldots n_i, \ldots) = \text{const} \cdot N! \Big/ \prod_i (n_i!) \,, \tag{1.36}$$

where const is a normalization constant. By this formula, Boltzmann introduced the equipartition law [44–47], which is the basis of statistical mechanics. We note the contradiction between the description of an ensemble of identical particles within the framework of statistical mechanics and the dynamical description [48, 49]. The principle of detailed balance is valid in the dynamical description of this ensemble, which means that reverse of time $t \to -t$ requires the development of this particle ensemble in the inverse direction, that is, particles move strictly along the same trajectories in the inverse direction. The equipartition law requires the development of the ensemble to its equilibrium state. This contradiction will be removed if the trajectories of particles are became slightly uncertain.

1.2.2
The Boltzmann Distribution

Let us determine the most probable number of particles $\overline{n}_i$ that are to be found in a state i assuming a large number $\overline{n}_i \gg 1$ of particles in each state and requiring that the probability W as well as its logarithm have maxima at $n_i = \overline{n}_i$. We then introduce $dn_i = n_i - \overline{n}_i$, assume that $\overline{n}_i \gg dn_i \gg 1$, and expand the value $\ln W$ over the interval dn_i near the maximum of this value. Using the relation

$$\ln n! = \ln\left(\prod_{m=1}^{n} m\right) \approx \int_1^n dx \ln x \,,$$

we have $(d/dn)(\ln n!) \approx \ln n$.

On the basis of this relation, we obtain from (1.36)

$$\ln W(n_1, n_2, \ldots, n_i \ldots) = \ln W(\overline{n}_1, \overline{n}_2, \ldots \overline{n}_i, \ldots) - \sum_i \ln \overline{n}_i dn_i - \sum_i dn_i^2/(2\overline{n}_i) \,. \tag{1.37}$$

The condition that this quantity is maximal is

$$\sum_i \ln \overline{n}_i dn_i = 0 \,. \tag{1.38}$$

In addition to this equation, we take into account the relations following from (1.34) and (1.35) which account for the conservation of the total number of particles and their total energy:

$$\sum_i dn_i = 0 \,, \tag{1.39}$$

$$\sum_i \varepsilon_i dn_i = 0 \,. \tag{1.40}$$

Equations (1.38), (1.39), and (1.40) allow us to determine the average number of particles in a given state. Multiplying (1.39) by $-\ln C$ and (1.40) by $1/T$, where C and T are characteristic parameters of this system, and summing the resulting equations, we have

$$\sum_i \left(\ln \overline{n}_i - \ln C + \frac{\varepsilon_i}{T}\right) dn_i = 0 \,.$$

Because this equation is fulfilled for any dn_i, one can require that the expression in the parentheses is equal to zero. This leads to the following expression for the most probable number of particles in a given state:

$$\overline{n}_i = C \exp\left(-\frac{\varepsilon_i}{T}\right) \,. \tag{1.41}$$

This formula is the Boltzmann distribution.

We now determine the physical nature of C and T in (1.41) that follows from the additional equations (1.34) and (1.35). From (1.34) we have $C \sum_i \exp(-\varepsilon_i/T) = N$. This means that the value C is the normalization constant. The energy parameter T is the particle temperature and characterizes the average energy of a particle. Below we express this parameter in energy units and hence we will not use the dimensioned proportionality factor – the Boltzmann constant $k = 1.38 \times 10^{-16}$ erg/K – as is often done. Thus, the Kelvin is the energy unit, equal to 1.38×10^{-16} erg (see Appendix B).

We can prove that at large $\overline{n}_i$ the probability of observing a significant deviation from $\overline{n}_i$ is small. According to (1.37) and (1.38) the probability W near its maxima is

$$W(n_1, n_2, \ldots n_i \ldots) = W(\overline{n}_1, \overline{n}_2, \ldots \overline{n}_i, \ldots) \exp\left[-\sum_i \frac{(n_i - \overline{n}_i)^2}{2\overline{n}_i}\right] \,.$$

From this it follows that a significant shift of n_i from its average value $\overline{n}_i$ is $|n_i - \overline{n}_i| \sim 1/(\overline{n}_i)^{1/2}$. Since $\overline{n}_i \gg 1$, the relative shift of the number of particles in one state is small $(|n_i - \overline{n}_i|/\overline{n}_i \sim (\overline{n}_i)^{3/2}$. Thus, the observed number of particles in a given state differs little from its average value.

1.2.3
Statistical Weight of a State and Distributions of Particles in Gases

Above we used subscript i to refer to one particle state, whereas below we consider a general case where i characterizes a set of degenerate states. We introduce the statistical weight g_i of a state that is a number of degenerate states i. For example, a diatomic molecule in a rotational state with the rotational quantum number J has the statistical weight $g_i = 2J + 1$, equal to the number of momentum projections on the molecular axis. Including accounting for the statistical weight, formula (1.41) takes the form

$$\overline{n_j} = Cg_j \exp\left(-\frac{\varepsilon_j}{T}\right) , \tag{1.42}$$

where C is the normalization factor and the subscript j refers to a group of states. In particular, this formula gives the relation between the number densities N_0 and N_j of particles in the ground and excited states, respectively:

$$N_j = N_0 \frac{g_j}{g_0} \exp\left(-\frac{\varepsilon_j}{T}\right) , \tag{1.43}$$

where ε_j is the excitation energy and g_0 and g_j are the statistical weights of the ground and excited states.

We now determine the statistical weight of states in a continuous spectrum. The wave function of a free particle with momentum p_x moving along the x-axis is given by $\exp(i p_x x/\hbar)$ to within an arbitrary factor if the particle is moving in the positive direction and by $\exp(-i p_x x/\hbar)$ if the particle is moving in the negative direction. (The quantity $\hbar = 1.054 \times 10^{-27}$ erg s is the Planck constant h divided by 2π.) Suppose that the particle is in a potential well with infinitely high walls. The particle can move freely in the region $0 < x < L$ and the wave function at the walls goes to zero. To construct a wave function that corresponds to free motion inside the well and goes to zero at the walls, we superpose the basic free-particle solutions, so $\psi = C_1 \exp(i p_x x/\hbar) + C_2 \exp(-i p_x x/\hbar)$. From the boundary condition $\psi(0) = 0$ it follows that $\psi = C \sin(p_x x/\hbar)$, and from the second boundary condition $\psi(L) = 0$ we obtain $p_x L/\hbar = \pi n$, where n is an integer. This procedure thus yields the allowed quantum energies for a particle moving in a rectangular well with infinitely high walls.

From this it follows that the number of states for a particle with momentum in the range from p_x to $p_x + dp_x$ is given by $dn = L dp_x/(2\pi\hbar)$, where we take into account the two directions of the particle momentum. For a spatial interval dx, the number of particle states is

$$dn = \frac{dp_x dx}{2\pi\hbar} .$$

Generalizing to the three-dimensional case, we obtain

$$dn = \frac{dp_x dx}{2\pi\hbar} \frac{dp_y dy}{2\pi\hbar} \frac{dp_z dz}{2\pi\hbar} = \frac{d\mathbf{p} d\mathbf{r}}{(2\pi\hbar)^3} . \tag{1.44}$$

Here and below we use the notation $d\mathbf{p} = dp_x dp_y dp_z$ and $d\mathbf{r} = dx dy dz$. The quantity $d\mathbf{p}d\mathbf{r}$ is called a differential element of the phase space, and the number of states in (1.44) is the statistical weight of the continuous spectrum of states because it is the number of states for an element of the phase space.

Let us consider now some cases of the Boltzmann distribution of particles. First we analyze the distribution of diatomic molecules in vibrational and rotational states. The excitation energy for the vth vibrational level of the molecule is given by $\hbar\omega v$ if the molecule is modeled by a harmonic oscillator. Here $\hbar\omega$ is the energy difference between neighboring vibrational levels. On the basis of (1.42), we have

$$N_v = N_0 \exp\left(-\frac{\hbar\omega v}{T}\right) , \qquad (1.45)$$

where N_0 is the number density of molecules in the ground vibrational state. Because the total number density of molecules is

$$N = \sum_{v=0}^{\infty} N_v = N_0 \sum_{v=0}^{\infty} \exp\left(-\frac{\hbar\omega v}{T}\right) = \frac{N_0}{\left[1 - \exp\left(-\frac{\hbar\omega}{T}\right)\right]} ,$$

the number density of excited molecules is

$$N_v = N \frac{\exp(-\hbar\omega v/T)}{[1 - \exp(-\hbar\omega/T)]} . \qquad (1.46)$$

The excitation energy of the rotational state with angular momentum J is given by $BJ(J+1)$, where B is the rotational constant of the molecule, and the statistical weight of this state is $2J + 1$. On the basis of the normalization condition $\sum_J N_{vJ} = N_v$ and assuming $B \ll T$ (as is usually done), the number density of molecules in a given vibrational–rotational state is

$$N_{vJ} = N_v(2J+1)\frac{B}{T}\exp\left[-\frac{BJ(J+1)}{T}\right] . \qquad (1.47)$$

As an example of the particle distribution in an external field, let us consider the distribution of particles in the gravitational field. In this case (1.42) gives $N(x) \sim \exp(-U/T)$, where U is the potential energy of the particle in the external field. For the gravitational field we have $U = mgz$, where m is the mass of the molecule, g is the free-fall acceleration, and z is the altitude above the Earth's surface. Formula (1.42) takes the form of the barometric distribution, and then has the form

$$N(z) = N(0)\exp\left(-\frac{mgh}{T}\right) , \quad l = \frac{T}{mg} , \qquad (1.48)$$

where $N(z)$ is the number density of molecules at altitude z. For atmospheric air near the Earth's surface at room temperature we have $l \approx 9$ km, that is, atmospheric pressure falls noticeably at altitudes of a few kilometers.

1.2.4
The Maxwell Distribution

We now consider the velocity distribution of free particles. This distribution is the end result of energy-changing collisions of the particles. The Boltzmann formula (1.42) provides the necessary information. In the one-dimensional case, the particle energy is $mv_x^2/2$, and the statistical weight of this state is proportional to dv_x, that is, to the number of particles $n(v_x)$ whose velocity is in the interval from v_x to $v_x + dv_x$. Formula (1.42) then yields

$$N(v_x)dv_x = C\exp\left(-\frac{mv_x^2}{2T}\right)dv_x \ ,$$

where C is the normalization factor. Correspondingly, in the three-dimensional case we have

$$N(v)d\mathbf{v} = C\left(-\frac{mv^2}{2T}\right)d\mathbf{v} \ ,$$

where the vector $\mathbf{v}$ has components v_x, v_y, and v_z, $d\mathbf{v} = dv_x dv_y dv_z$, and the kinetic energy of the particle $mv^2/2$ is the sum of the kinetic energies for all the directions of motion. In particular, for the number density of particles $N(\mathbf{v})$ we have, after using the normalization condition,

$$N(\mathbf{v}) = N\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left(-\frac{mv^2}{2T}\right) \ , \tag{1.49}$$

where N is the total number density of particles. When we introduce the function $\varphi(v_x) \sim N(v_x)$, normalized such that

$$\int\limits_{-\infty}^{\infty}\varphi(v_x)dv_x = 1 \ , \quad \varphi(v_x) = \left(\frac{m}{2\pi T}\right)^{1/2}\exp\left(-\frac{mv_x^2}{2T}\right) \ , \tag{1.50}$$

then

$$N(\mathbf{v}) = N\varphi(v_x)\varphi(v_y)\varphi(v_z) = N\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left(-\frac{mv^2}{2T}\right) \ . \tag{1.51}$$

These particle velocity distributions are called Maxwell distributions [50–52]. The average kinetic energy of particles following from (1.49) is

$$\frac{1}{2}m\overline{v^2} = \frac{1}{2}m\overline{v_x^2} + \frac{1}{2}m\overline{v_y^2} + \frac{1}{2}m\overline{v_z^2} = \frac{3}{2}m\overline{v_x^2} \ .$$

This is to be combined with the result

$$\begin{aligned}\frac{1}{2}m\overline{v_x^2} &= \frac{\int_{-\infty}^{\infty}(mv_x^2/2)\exp\left[-mv_x^2/(2T)\right]dv_x}{\int_{-\infty}^{\infty}\exp\left[-mv_x^2/(2T)\right]dv_x} \\ &= -\frac{d}{d\,(1/T)}\ln\left\{\int\limits_{-\infty}^{\infty}\exp\left[-mv_x^2/(2T)\right]dv_x\right\} \\ &= -\frac{d}{d\,(1/T)}\ln(aT^{1/2}) = \frac{T}{2} \ ,\end{aligned}$$

where the constant a does not depend on the temperature. Thus, the particle kinetic energy per degree of freedom is $T/2$, and correspondingly the average particle kinetic energy in the three-dimensional space is $m\overline{v^2}/2 = 3T/2$. These relations can be used as the definition of the temperature.

1.2.5
The Saha Distribution

We considered above the distributions of gas particles in bound or free states. Now we analyze the specific distribution for plasma systems that contain both bound and free electron states. We must examine the equilibrium between continuous and discrete electron states. This equilibrium is maintained by the processes

$$A^+ + e \leftrightarrow A\,,$$

where e is the electron, A^+ is the ion, and A is the atom. We consider a quasineutral plasma in which the electron and ion number densities are the same.

Consider an ionized gas in a volume Ω and denote the average number of electrons, ions, and atoms in this volume by n_e, n_i, and n_a, respectively (note that $n_\mathrm{e} = n_i$ here). Relation (1.43) gives the ratio between the number density of free and bound states of electrons as

$$\frac{n_i}{n_\mathrm{a}} = \frac{g_\mathrm{e} g_\mathrm{i}}{g_\mathrm{a}} \frac{1}{(2\pi\hbar)^3} \int \exp\left(-\frac{J + p^2/2m_\mathrm{e}}{T}\right) d\mathbf{p}d\mathbf{r}\,.$$

Here $g_\mathrm{e} = 2$, g_i, and g_a are the statistical weights of electrons, ions, and atoms corresponding to their electronic states, J is the atomic ionization potential, and p is the free electron momentum, so $J + p^2/(2m_\mathrm{e})$ is the energy for transition from the ground state of the atom to a given state of a free electron. It is assumed that atoms are found only in the ground state.

Integration of this expression over electron momenta yields

$$\frac{n_i}{n_\mathrm{a}} = \frac{g_\mathrm{e} g_\mathrm{i}}{g_\mathrm{a}} \left(\frac{m_\mathrm{e} T}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{J}{T}\right) \int d\mathbf{r}\,.$$

Integrating over the volume, we take into account that transposition of the states of any pair of electrons does not change the state of the electron system. Therefore $\int d\mathbf{r} = \Omega/n_\mathrm{e}$. Introducing the number densities of electrons $N_\mathrm{e} = n_\mathrm{e}/\Omega$, ions $N_\mathrm{i} = n_i/\Omega$, and atoms $N_\mathrm{a} = n_\mathrm{a}/\Omega$, we deduce that

$$\frac{N_\mathrm{e} N_\mathrm{i}}{N_\mathrm{a}} = \frac{g_\mathrm{e} g_\mathrm{i}}{g_\mathrm{a}} \left(\frac{m_\mathrm{e} T}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{J}{T}\right)\,. \tag{1.52}$$

This result is called the Saha distribution [53].

One can write the Saha distribution in the form of the Boltzmann distribution (1.42) as

$$\frac{N_\mathrm{i}}{N_\mathrm{a}} = \frac{g_\mathrm{c}}{g_\mathrm{a}} \exp\left(-\frac{J}{T}\right)\,, \quad g_\mathrm{c} = \frac{g_\mathrm{e} g_\mathrm{i}}{N_\mathrm{e}} \cdot \left(\frac{m_\mathrm{e} T}{2\pi\hbar^2}\right)^{3/2}\,, \tag{1.53}$$

where g_c is the effective statistical weight of the electron continuous spectrum. For an ideal plasma, this statistical weight is rather large because the electron number density N_e is small compared with a typical atom number density. This leads to the conclusion that a remarkable degree of ionization takes place at relatively small temperatures, $T \ll J$. However, the probability of atomic excitation is very small at these temperatures; that is, the number density of excited atoms is small. Hence, at these temperatures, atoms are either in the ground state or ionized.

1.2.6
Dissociative Equilibrium in Molecular Gases

Equilibrium between atoms and molecules in a molecular gas is maintained by the processes

$$X + Y \leftrightarrow XY .$$

This equilibrium bears analogy to the equilibrium between discrete and continuous atomic states corresponding to bound and free states of atoms. We can find the relation between the equilibrium number densities of atoms and molecules in this case by analogy with the Saha distribution. On the basis of (1.52), one can express the relationship between the number densities of atoms and molecules in the ground state as [41]

$$\frac{N_X N_Y}{N_{XY}(v = 0, J = 0)} = \frac{g_X g_Y}{g_{XY}} \left(\frac{\mu T}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{D}{T}\right) , \tag{1.54}$$

where g_X, g_Y, and g_{XY} are the statistical weights of atoms and molecules with respect to their electron states, μ is the reduced mass of atoms X and Y, and D is the dissociation energy of the molecule.

In contrast to the ionization equilibrium of atoms, in this case most molecules are found in excited states. Using (1.46) and (1.47), which connect the number density of molecules in the ground state to their total number density, we can transform (1.54) to the form

$$\frac{N_X N_Y}{N_{XY}} = \frac{g_X g_Y}{g_{XY}} \left(\frac{\mu T}{2\pi\hbar^2}\right)^{3/2} \frac{B}{T} \left[1 - \exp\left(-\frac{\hbar\omega}{T}\right)\right] \exp\left(-\frac{D}{T}\right) , \tag{1.55}$$

where N_{XY} is the total molecular number density.

1.2.7
Laws of Blackbody Radiation

An ionized gas contains excited atoms or molecules that emit radiation, so it is necessary to examine how the gas interacts with radiation. If the interaction of radiation with a gas is strong, the distance that an individual photon travels before being absorbed is relatively small. Then we deal with so-called equilibrium radiation [54]. Radiation in a vessel whose walls are at a temperature T will be absorbed and emitted by the walls, and these processes establish the equilibrium between the radiation and the walls of the vessel. This radiation is called blackbody radiation.

To calculate the average number of photons in a particular state, we use the fact that photons obey Bose–Einstein statistics. Therefore, the presence of a photon in a given state does not depend on whether other photons with this energy are also in this state. Then according to the Boltzmann formula (1.43), the relative probability that n photons with energy $\hbar\omega$ are found in a given state is $\exp(-\hbar\omega n/T)$. Thus, the mean number of photons in this state is

$$\overline{n_\omega} = \frac{\sum_n n\exp(-\hbar\omega n/T)}{\sum_n \exp(-\hbar\omega n/T)} = \frac{1}{\left[\exp(\hbar\omega/T)-1\right]} \,. \tag{1.56}$$

This formula is called the Planck distribution [55].

We introduce the spectral radiation density U_ω as the radiation energy per unit time and volume in a unit frequency range. We shall show below how this quantity can be determined. The radiation energy in the frequency interval from ω to $\omega + d\omega$ is $\Omega\, U_\omega d\omega$, where Ω is the volume of a region where the radiation is located. Alternatively, this quantity can be expressed as $2\hbar\omega n_\omega \Omega\, d\boldsymbol{k}/(2\pi)^3$, where the factor 2 accounts for the two polarizations of an electromagnetic wave, $\boldsymbol{k}$ is the photon wave number, n_ω is the number of photons in a single state, and $\Omega\, d\boldsymbol{k}/(2\pi)^3$ is the number of states in an element of the phase space. Using the dispersion relation $\omega = ck$ between the frequency ω and wave vector $\boldsymbol{k}$ of the photon (c is light velocity), the equivalence of these two aspects of the same quantity yields

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3} n_\omega \,. \tag{1.57}$$

When the Planck distribution (1.56) is inserted into (1.57), we obtain the Planck radiation formula:

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3\left[\exp(\hbar\omega/T)-1\right]} \,. \tag{1.58}$$

In the limiting case $\hbar\omega \ll T$, this result transforms to the Rayleigh–Jeans formula [56, 57]:

$$U_\omega = \frac{\omega^2 T}{\pi^2 c^3} \,, \quad \hbar\omega \ll T \,. \tag{1.59}$$

This expression corresponds to the classical limit, and hence does not contain the Planck constant. The opposite limit yields the Wien formula [58]:

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\exp\left(-\frac{\hbar\omega}{T}\right) \,, \quad \hbar\omega \gg T \,. \tag{1.60}$$

We shall now apply (1.58) to find the radiative flux emitted by a blackbody surface. It may be defined as the flux of radiation coming from a hole in a cavity with perfectly absorbing walls when this cavity contains the blackbody radiation. The blackbody surface emits an isotropic flux $c U_\omega$, and hence the photon flux at a frequency ω outside the blackbody boundary is

$$j_\omega = \int \frac{d\Theta}{4\pi} c U_\omega = \frac{c U_\omega}{4} = \frac{\hbar\omega^3}{4\pi^2 c^2 \left[\exp(\hbar\omega/T) - 1\right]} , \tag{1.61}$$

where the element of solid angle is $d\Theta = d\varphi d(\cos\theta)$ and we assumed that the photon flux is directed perpendicular to the blackbody surface. From this we obtain the Stefan–Boltzmann formula for the total radiative flux leaving the emitting surface:

$$J = \int_0^\infty j_\omega d\omega = \frac{c}{4} \int_0^\infty U_\omega d\omega = \sigma T^4 , \tag{1.62}$$

where the Stefan–Boltzmann constant σ is given by

$$\sigma = \frac{1}{4\pi^2 c^2 \hbar^3} \int_0^\infty \frac{1}{e^x - 1} x^3 dx = \frac{\pi^2}{60 c^2 \hbar^3} = 5.67 \times 10^{-12} \frac{\mathrm{W}}{\mathrm{cm}^2\,\mathrm{K}^4} .$$

One can evaluate the functional dependence of the radiation flux (1.62) in a simple way on the basis of the dimensional analysis. The result must depend on the following parameters: T (the radiation temperature), $\hbar$ (the Planck constant), and c (the light velocity). From these parameters one can compose only one combination that has the dimension of a flux; it is $J \sim T^4 \hbar^{-3} c^{-2}$, consistent with (1.62).

1.2.8
Ionization Equilibrium in a Plasma with Particles

Plasma properties can be influenced by the presence in the plasma of small particles on a variety of size scales, including atomic and molecular clusters. We shall refer to all such small particles as aerosols, and the plasma containing them as an aerosol plasma, although the size of the particles can have an influence on the plasma properties. One such plasma property is the ionization equilibrium. The presence of small particles in a hot gas may alter the ionization equilibrium because the energy for an electron to bind to a surface is smaller than the ionization potential of atoms constituting this surface. For example, the copper ionization potential is 7.73 eV, whereas the copper work function, the energy for an electron to bind to a copper surface, is 4.40 eV. The corresponding values are 7.58 and 4.3 eV for silver, and 3.89 and 1.81 eV for cesium. Thus, the presence of such particles in a hot vapor alters the equilibrium density of charged particles. We assume in the following that electrons in a hot gas or vapor result from small particles only. Our goal is to determine the equilibrium charge of these particles and the number density of electrons in a plasma.

For simplicity, we assume all particles to be spherical and to have the same radius r_0. This radius is taken to be sufficiently large such that

$$r_0 \gg \frac{e^2}{T} \,. \tag{1.63}$$

This criterion allows us to consider a particle as bulk matter, rather than needing to describe its microscopic properties. The addition of a single electron to the aerosol particle makes only a slight difference to the electric potential of the particle. We use this property to write the relationship between the number densities of particles n_Z and n_{Z+1} that possess charges Z and $Z+1$, respectively. By analogy with the Saha distribution (1.52), we have

$$\frac{n_Z N_\mathrm{e}}{n_{Z+1}} = 2\left(\frac{m_\mathrm{e} T}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{W_Z}{T}\right) \,, \tag{1.64}$$

where W_Z is the electron binding energy for the particle with charge Z, N_e is the electron number density, and the factor 2 accounts for the electron statistical weight (two spin projections). The electron binding energy of a charged particle W_Z is the sum of the electron binding energy for the neutral particle W_0 of a given material and the potential energy of the charged particle. Using the electric potential for a particle of charge $Z + 1/2$ (the average between Z and $Z+1$), we have

$$W_Z = W_0 + \left(Z + \frac{1}{2}\right)\frac{e^2}{r_0} \,.$$

Substituting this into (1.64) transforms it to the form

$$\frac{n_Z N_\mathrm{e}}{n_{Z+1}} = 2\left(\frac{m_\mathrm{e} T}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{W_0}{T} - \frac{\left(Z+\frac{1}{2}\right)e^2}{r_0 T}\right) \,. \tag{1.65}$$

This relation gives the charge distribution of the particles. If the average charge is large, this distribution is sharp. Specifically, introducing n_0, the number density of neutral particles, (1.65) leads to

$$n_Z = n_{Z-1} A \exp\left(-\frac{Ze^2}{r_0 T}\right) = n_0 A^Z \exp\left(-\frac{Z^2 e^2}{2 r_0 T}\right) \,,$$

where $A = (2/N_\mathrm{e}) \cdot [m_\mathrm{e} T/(2\pi\hbar^2)]^{3/2} \exp(-W_0/T)$. For charges that are close to the average, this relationship is conveniently written in the form

$$n_Z = n_{\overline{Z}} \exp\left(-\frac{(Z-\overline{Z})^2}{2\Delta Z^2}\right) \,,$$

where $\Delta Z^2 = r_0 T/e^2 \gg 1$ because of (1.63). The average charge of the particles follows from the relation $\overline{Z} e^2/(r_0 T) = \ln A$, which gives

$$\overline{Z} = \frac{r_0 T}{e^2} \ln\left[\frac{2}{N_\mathrm{e}}\left(\frac{m_\mathrm{e} T}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{W_0}{T}\right)\right] \,. \tag{1.66}$$

This result must be combined with the condition for plasma quasineutrality

$$N_e = \overline{Z} n \ ,$$

where n is the total number density of particles. Combining these equations to remove the electron number density, we find the average charge of the particles to be

$$\overline{Z} = \frac{r_0 T}{e^2} \left\{ \ln \left[\frac{2}{\overline{Z} n} \left(\frac{m_e T}{2\pi\hbar^2} \right)^{3/2} \right] - \frac{W_0}{T} \right\} \ . \tag{1.67}$$

1.2.9
Thermoemission of Electrons

For high temperatures or large particle size, $Ze^2/(r_0 T)$ becomes small. For example, at $r_0 = 10\,\mu\text{m}$ and $T = 2000\,\text{K}$, the very large particle charge of $Z = 1200$ would be required to make this parameter become unity. If the parameter is small, then it follows from (1.66) that

$$N_e = 2 \left(\frac{m_e T}{2\pi\hbar^2} \right)^{3/2} \exp\left(-\frac{W_0}{T} \right) \ . \tag{1.68}$$

This formula describes the equilibrium density of electrons above a flat surface if the electric potential of the particle is small compared with a typical thermal energy. Therefore, the conditions near the particle and far from it are identical. Then the average particle charge is determined from the relation $N_e = \overline{Z} n$, where the number densities of electrons and of particles are both known.

Equation (1.68) allows us to obtain a simple expression for the electron current from the surface of a hot cathode. In the case of equilibrium between electrons and a hot surface, the electron current from the surface is equal to the current toward it. Assuming unit probability for electron attachment to the surface upon their contact, we obtain that the electron current density i toward the surface is equal to the electron current density from the surface:

$$i = e \left(\frac{T}{2\pi m_e} \right)^{1/2} N_e = \frac{e m_e T^2}{4\pi^2 \hbar^3} \exp\left(-\frac{W_0}{T} \right) \ . \tag{1.69}$$

This result is known as the Richardson–Dushman formula [61–65] describing the electron current density for electron emission by a hot surface. This type of emission is called thermoemission of electrons. For the analysis of gas discharge problems, it is convenient to rewrite the Richardson–Dushman formula (1.69) for the thermoemission current density in the form

$$i = A_R T^2 \exp\left(-\frac{W}{T} \right) \ , \tag{1.70}$$

wherein the Richardson parameter A_R, according to (1.69), has the value [66] $60\,\text{A}/(\text{cm}^2\,\text{K}^2)$. Table 1.5 contains the parameters in (1.69) for real metals. In this table, W is the metal work function, T_b is the metal boiling point, and i_b is the current density at the boiling point.

Table 1.5 Parameters in (1.70) for electron thermoemission from metals [59, 60], A_R is expressed in square meters per square centimeter per square Kelvin.

Material	A_R	W, eV	Material	A_R	W, eV	Material	A_R	W, eV
Ba	60	2.11	Feα	26	4.5	Pt	33	5.32
Be	300	3.75	Feγ	1.5	4.21	Si	8	3.6
Cs	160	1.81	Mo	55	4.15	Ta	120	4.25
C	15	4.5	Ni	30	4.61	Th	70	3.38
Cr	120	3.90	Nb	120	4.19	Ti	60	3.86
Co	41	4.41	Os	1100	5.93	U	60	3.27
Cu	120	4.57	Pa	60	4.9	W	60	4.54
Hf	22	3.60	Pd	60	4.99	Y	100	3.27
Ir	120	5.27	Re	720	4.7	Zr	330	4.12

1.2.10
The Treanor Effect

A weakly ionized gas can be regarded as a system of weakly interacting atomic particles. This system can be divided into subsystems, and in the first approximation each subsystem can be considered as an independent closed system. The next approximation, taking into account a weak interaction between subsystems, makes it possible to establish connections among subsystem parameters. There are a variety of ways in which this decomposition can be done, with the selection depending on the nature of the problem. Often it is convenient to divide an ionized gas into atomic and electronic subsystems. Energy exchange in electron–atom collisions is slight owing to the large difference in their masses, so equilibrium within the atomic and electronic subsystems is established separately. If a weakly ionized gas is located in an external electric or electromagnetic field, which acts on electrons mostly, the electron and atom temperatures may be different. This means that both atoms and electrons can be characterized by Maxwell distributions for their translational energies, but with different mean energies.

Another example of weakly interacting subsystems that we shall consider below relates to a molecular gas in which exchanges of vibrational energy between colliding molecules have a resonant character. This is a more effective process than collisions of molecules with transitions of the vibrational energy to excitations of rotational and translational degrees of freedom. If vibrational and translational degrees of freedom are excited or are cooled in a different way, then different vibrational and translational temperatures will exist in such a molecular gas. This situation occurs in gas discharge molecular lasers, where vibrational degrees of freedom are excited selectively, and also in gas dynamical lasers, where a rapid cooling of translational degrees of freedom occurs as a result of gas expansion. The same effect occurs in shock waves and as a result of gas expansion after a nozzle. There is thus a wide variety of situations where a molecular gas is characterized by different vibrational and translational temperatures. However, the resonant character of exchange of vibrational excitation takes place only for weakly excited molecules. At moderate excitations, the resonant character is lost because of molecular anharmonicity. This leads to a particular type of distribution of molecular states that we shall now analyze.

We consider a nonequilibrium gas consisting of diatomic molecules where the translational temperature T differs from the vibrational temperature T_ν. The equilibrium between vibrational states is maintained by resonant exchange of vibrational excitations in collisions of molecules, as expressed by

$$M(v_1) + M(v_2) \leftrightarrow M(v_1') + M(v_2') \ , \tag{1.71}$$

where the quantities in the parentheses are vibrational quantum numbers. Assuming molecules to be harmonic oscillators, we obtain from this the condition

$$v_1 + v_2 = v_1' + v_2' \ .$$

The excitation energy of a vibrational level is

$$E_v = \hbar\omega(v + 1/2) - \hbar\omega x_e(v + 1/2)^2 \ ,$$

where ω is the harmonic oscillator frequency and x_e is the anharmonicity parameter. The second term of this expression is related to the establishment of equilibrium in the case being considered, where translational and vibrational temperatures are different. Specifically, the equilibrium condition leads to the relation

$$N(v_1)N(v_2)k\left(v_1, v_2 \to v_1', v_2'\right) = N(v_1')N(v_2')k(v_1', v_2' \to v_1, v_2) \ ,$$

where $N(v)$ is the number density of molecules in a given vibrational state and $k(v_1, v_2 \to v_1', v_2')$ is the rate constant for a given transition. Because these transitions are governed by the translational temperature, the equilibrium condition gives

$$k\left(v_1, v_2 \to v_1', v_2'\right) = k\left(v_1', v_2' \to v_1, v_2\right)\exp(\Delta E/T) \ ,$$

where $\Delta E = \Delta E(v_1) + \Delta E(v_2) - \Delta E(v_1') - \Delta E(v_2')$ is the difference of the energies for a given transition, where $\Delta E(v) = -\hbar\omega x_e(v + 1/2)^2$. From this, one finds the number density of excited molecules to be [67, 68]

$$N(v) = N_0 \exp\left[-\frac{\hbar\omega v}{T_\nu} + \frac{\hbar\omega x_e v(v+1)}{T}\right] \ , \tag{1.72}$$

where N_0 is the number density of molecules in the ground vibrational state. This formula is often called the Treanor distribution.

Formula (1.72) gives a nonmonotonic population of vibrational levels as a function of the vibrational quantum number. Assuming the minimum of this function to correspond to large vibrational numbers, we have for the position of the minimum

$$v_{\min} = \frac{1}{2x_e}\frac{T}{T_\nu} \gg 1 \ , \tag{1.73}$$

Table 1.6 Parameter $1/(2x_e)$ in (1.73).

Molecule	**H_2**	**OH**	**CO**	**N_2**	**NO**	**O_2**
$1/(2x_e)$	18	22	82	84	68	66

and the minimum number density of excited molecules is given by

$$N(v_{\min}) = N_0 \exp\left(-\frac{\hbar\omega\, v_{\min}}{2T_v}\right) .$$

Table 1.6 contains the values of the first factor in (1.73) for some molecules. The effect considered is remarkable at $v \sim 10$ in terms of the distinction between vibrational and translational temperatures.

Thus, the special feature of the Treanor effect is that at high vibrational excitations, collisions of molecules with transfer of vibrational excitation energy to translational energy become effective. This causes a mixing of vibrational and translational subsystems, and invalidates the Boltzmann distribution for excited states as a function of vibrational temperature. Note that the model employed is not valid for very large excitations because of the vibrational relaxation processes.

1.2.11
Normal Distribution

A commonly encountered case in plasma physics as well as in other types of physics is one in which a variable changes by small increments, each change occurring randomly, and the distribution of the variable after many steps is studied. Examples of this are the diffusive motion of a particle, or the energy distribution of electrons in a gas. This energy distribution as it occurs in a plasma results from elastic collisions of electrons with atoms, with each collision between an electron and an atom leading to an energy exchange between them that is small because of the large difference in their masses. Thus, in a general statement of this problem, we seek the probability that some variable z has a given value after $n \gg 1$ steps, if the distribution for each step is random and its parameters are given.

Let the function $f(z, n)$ be the probability that the variable has a given value after n steps, with $\varphi(z_k)dz_k$ the probability that after the kth step the change of the variable lies in the interval between z_k and $z_k + dz_k$. Since the functions $f(z)$ and $\varphi(z)$ are probabilities, they are normalized by the condition

$$\int_{-\infty}^{\infty} f(z, n)dz = \int_{-\infty}^{\infty} \varphi(z)dz = 1 .$$

By definition of the above functions we have

$$f(z, n) = \int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_n \prod_{k=1}^{n} \varphi(z_k) , \quad z = \sum_{k=1}^{n} z_k . \tag{1.74}$$

We introduce the Fourier transforms

$$G(p) = \int\limits_{-\infty}^{\infty} f(z)\exp(-ipz)dz\,, \quad g(p) = \int\limits_{-\infty}^{\infty} \varphi(z)\exp(-ipz)dz\,, \tag{1.75}$$

which can be inverted to give

$$f(z) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} G(p)\exp(ipz)dp, \quad \varphi(z) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} g(p)\exp(ipz)dp\,.$$

Equation (1.75) yields

$$g(0) = \int\limits_{-\infty}^{\infty} \varphi(z)dz = 1\,, \quad g'(0) = -i\int\limits_{-\infty}^{\infty} z\varphi(z)dz = -i\overline{z_k}\,, \quad g''(0) = -\overline{z_k^2}\,, \tag{1.76}$$

where $\overline{z_k}$ and $\overline{z_k^2}$ are the mean shift and the mean square shift of the variable after one step. From (1.74) and (1.76) it follows that

$$G(p) = \int\limits_{-\infty}^{\infty} \exp\left(-ip\sum_{k=1}^{n} z_k\right)\prod_{k=1}^{n}\varphi(z_k)dz_k = g^n(p)\,,$$

and hence

$$f(z) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} g^n(p)\exp(ipz)dp = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\exp(n\ln g + ipz)dp\,.$$

Since $n \gg 1$, the integral converges at small p. Expanding $\ln g$ in a power series of p, we have

$$\ln g = \ln\left(1 - i\overline{z_k}p - \overline{z_k^2}p^2/2\right) = -i\overline{z_k}p - \overline{z_k^2}p^2/2 + \left(\overline{z_k}\right)^2 p^2/2\,,$$

which gives

$$f(z) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\exp\left[ip(n\overline{z_k} - z) - n\overline{z_k^2}p^2/2\right]dp$$
$$= \frac{1}{\sqrt{2\pi\Delta^2}}\exp\left[-\frac{(z-\overline{z})^2}{2\Delta^2}\right]\,. \tag{1.77}$$

In this expression, $\overline{z} = n\overline{z_k}$ is the mean shift of the variable for n steps, $\overline{z^2} = n\overline{z_k^2}$, and $\Delta^2 = \overline{z^2} - (\overline{z})^2$ is the root-mean-square deviation of this value. Formula (1.77) is called the normal distribution, or the Gaussian distribution. It is valid if the principal contribution to the integral in (1.76) comes from small values of p, that is, if $\overline{z_k}p \ll 1$ and $\overline{z_k^2}p^2 \ll 1$. Because this integral is determined by a range $n\overline{z_k^2}p^2 \sim 1$, the Gaussian distribution is valid for a large number of steps $n \gg 1$.

1.3
Rarefied and Dense Plasmas

1.3.1
Criteria for an Ideal Plasma

Many varieties of plasmas are known, with most of them composed of weakly interacting particles (see Figures 1.1–1.3). These plasma systems are analogues of gaseous systems of neutral particles. When they experience weak interactions, the system of charged particles is an ideal plasma, with the criterion for plasma ideality expressed by (1.3) between plasma parameters. The small parameter of the theory, the plasma parameter, has the form [69]

$$\gamma = \frac{N_e e^6}{T_e^3} , \tag{1.78}$$

and the condition to have an ideal plasma is $\gamma \ll 1$. Specific values for γ can be estimated on the basis of fundamental physical considerations.

The first criterion for an ideal plasma refers to the condition that the mean interaction energy of a plasma particle with its neighbors must be small compared with its kinetic energy $3T/2$. (We assume in the following that electrons and other plasma particles have the same temperature.) The electric potential arising from a charged plasma particle is given by (1.10), and close to the nucleus it is

$$\varphi = \frac{q}{r} - \frac{q}{r_D} , \quad r \ll r_D ,$$

where q is the particle charge, r is the distance from this particle, and r_D is the Debye–Hückel radius. The first term in this expression is the particle potential in a vacuum, and the second term is the electric potential that is created by the neighboring plasma particles. This means that the average interaction energy of a particle of charge e with other plasma particles is e^2/r_D, and the criterion for an ideal plasma ($e^2/r_D \ll 3T/2$) leads to the value

$$\gamma \ll \frac{9}{32\pi} = 0.09 . \tag{1.79}$$

A second criterion follows from the condition that many charged particles are located in the sphere of radius r_D, that is, $4\pi r_D^3 N_e/3 \gg 1$. This gives the value

$$\gamma \ll \frac{1}{96\pi} = 0.003 . \tag{1.80}$$

Criteria (1.79) and (1.80) both contain identical combinations of parameters, but yield different numerical values. Criterion (1.79) is preferred, because it encompasses a larger region in which the ratio of the potential energy of a particle to its kinetic energy can be a small parameter, and thus is useful for expansion of plasma parameters. Nevertheless, Figures 1.1 and 1.2 display criteria (1.79) and (1.80) as boundaries between an ideal plasma and a dense plasma.

The quantity selected to characterize a plasma may not be γ, but may instead may be some function of γ. Often one uses the coupling constant Γ of the plasma, introduced as the ratio of the Coulomb interaction potential of a charged particle with its nearest neighbors to the thermal energy,

$$\Gamma = \frac{e^2}{r_\mathrm{W} T} = \left(\frac{4\pi\gamma}{3}\right)^{1/3} , \tag{1.81}$$

where $r_\mathrm{W} = (4\pi N_\mathrm{e}/3)^{-1/3}$ is the Wigner–Seitz radius. A dense plasma is one with $\Gamma \gg 1$, and is called a strongly coupled plasma. The condition for the plasma to be ideal is $\Gamma \ll 1$, which implies that

$$\gamma \ll \frac{3}{4\pi} = 0.2 . \tag{1.82}$$

We can express plasma properties as a function of the plasma coupling constant Γ. The ratio of the average interaction energy of a charged plasma particle with other particles to its mean kinetic energy has the form

$$\frac{e^2}{r_\mathrm{D}} \cdot \frac{2}{3T} = \frac{4}{3}\sqrt{2\pi\gamma} = \frac{(2\Gamma)^{3/2}}{2^{1/2}} .$$

The number of electrons in a sphere of radius r_D is

$$N_\mathrm{D} = \frac{4\pi r_\mathrm{D}^3 N_\mathrm{e}}{3} = \frac{1}{6\sqrt{8\pi\gamma}} = \frac{1}{(6\Gamma)^{3/2}} = \frac{0.07}{\Gamma^{3/2}} ,$$

and the criterion $N_\mathrm{D} \gg 1$ of an ideal plasma gives $\Gamma \ll 0.2$.

Note that the plasma under consideration consists of electrons and ions with a single charge. For a plasma with multicharged ions, the ideality criterion (1.5) is violated at low ion density. In particular, for interaction between ions after changing in criterion (1.5) e to Ze, where Z is the ion charge, we obtain the plasma parameter (1.78) in the form

$$\gamma = \frac{Z^6 e^6 N_\mathrm{i}}{T^3} .$$

In particular, for a dusty plasma with typical parameters [70] $Z = 10^3$, the number density of particles $N_\mathrm{p} = 10^3\,\mathrm{cm}^{-3}$, and $T = 400\,\mathrm{K}$ we have for the plasma parameter $\gamma \sim 10^5$, that is, in this example one can consider the plasma to have strong coupling.

1.3.2
Conditions for Ideal Equilibrium Plasmas

We expect a plasma to fail to be an ideal plasma as the plasma density increases, but if a plasma is prepared from a gas and ionization equilibrium is supported between electrons and atoms, this plasma is ideal. Below we convince ourselves of this with a simple example. For this goal we take a weakly ionized gas at a low temperature and find the dependence of the plasma parameter γ defined by (1.78) on the plasma density when there is equilibrium between charged and neutral plasma particles (ionization equilibrium). From the Saha distribution (1.52) for the electron number density N_e and the atom number density N_a in a quasineutral plasma, we obtain the relation

$$\frac{N_e^2}{N_a} = g\left(\frac{m_e T_e}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{J}{T_e}\right) ,$$

where $g = g_e g_i/g_a$, where g_e, g_i, and g_a are the statistical weights of electrons, ions, and atoms, respectively, T_e is the electron temperature, and J is the atomic ionization potential. We take the total number density of nuclei, $N = N_e + N_a (N_e = N_i)$, as a parameter, and determine the dependence $\gamma(N)$.

Let us write the Saha distribution in the form

$$\gamma^2 = \left(N - \frac{\gamma T_e^3}{e^6}\right)\frac{C}{T_e^{9/2}} \cdot \exp\left(-\frac{J}{T_e}\right) ,$$

where $C = g m_e^{3/2} e^{12}/[\hbar^3 (2\pi)^{3/2}]$. Concentrating our attention on plasmas with the maximum departure from ideal plasma conditions, we choose the plasma temperature at a given N such that γ is maximal. The condition $d\gamma/dT_e = 0$ leads to the expressions

$$N = g\frac{3T(J - 3T_e/2)}{(J - 9T_e/2)^2}\left(\frac{m_e T_e}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{J}{T_e}\right) , \tag{1.83}$$

$$N_e = N\left(\frac{J - 9T_e/2}{J - 3T_e/2}\right) . \tag{1.84}$$

The maximum values of γ are located at $T_e \leq 2J/9$, and, in the limit of large N, the temperature approaches $2J/9$. In this limit the degree of plasma ionization goes to zero as N increases, and the plasma parameter γ increases with the increase of N. On the basis of the above equations, we analyze the limiting case where there is basic violation of the conditions for an ideal plasma. We take into account that (1.83) and (1.84) are valid for an ideal plasma. The maximum values of γ at a given large N lead to the expressions for the temperature T_e and the degree of ionization N_e/N:

$$\frac{(T_e - T_0)^2}{T_e^2} = \frac{4g}{9N}\left(\frac{m_e T_0}{2\pi\hbar^2}\right)^{3/2} e^{-9/2} , \tag{1.85}$$

$$\frac{N_e}{N} = \frac{3}{2}\frac{(T_e - T_0)}{T_0} , \tag{1.86}$$

where T_0 is defined as $T_0 = 2J/9$. It is evident that the plasma parameter behaves as $\gamma \sim \left(N a_0^3\right)^{1/2}$ in the limit of large N, and the coupling constant of the plasma is estimated to be $\Gamma \sim \left(N a_0^3\right)^{1/6}$, where a_0 is the Bohr radius. In particular, in the limit of large N, these expressions for a hydrogen plasma become

$$\frac{N_e}{N} = \frac{5.1 \times 10^{-3}}{(N a_0^3)^{1/2}} , \quad \gamma = 3.7\left(N a_0^3\right)^{1/2} , \quad \Gamma = 2.5(N a_0^3)^{1/6} , \tag{1.87}$$

with validity constrained by $N a_0^3 \ll 1$. From this, it follows for the hydrogen plasma that $\Gamma = 1$ at $N a_0^3 = 0.004$.

One can conclude from these results that the degree of plasma ionization decreases with an increase of the plasma density. This means that departure from the ideal nature of the plasma is accompanied by an increase in the number density of neutral particles. Thus, interactions involving neutral atomic particles are of importance for the properties of a strongly coupled plasma. One can consider the above results from another standpoint. We have an equilibrium plasma that is characterized by the electron temperature T_e and the electron number density N_e, and the relation between these parameters is determined by the Saha relation (1.52). In addition to this, one can obtain a given electron temperature T_e from the energy balance because the specific energy of a plasma is $(3T_e + J)N_e$, where we assume identical temperatures of electrons and ions. From this it is seen that one can obtain an equilibrium plasma with different values of the plasma parameter γ depending on the plasma energy.

1.3.3
Instability of Two-Component Strongly Coupled Plasmas

From the above analysis it follows that a plasma prepared from a gas is ideal if ionization equilibrium between electrons and atoms is supported and external fields are absent. However, a dense plasma without neutral particles can be created for a short time under nonequilibrium conditions. For example, let us ionize an atomic gas by laser radiation as a result of atom photoionization if the photon energy is close to the atomic ionization potential. For example, if an atomic gas at room temperature under a pressure of 6 Torr is ionized fully such that the electron temperature of a forming plasma is 1000 K, the plasma parameter (1.78) for this plasma $\gamma = 1$. Our goal is to estimate the lifetime of such a plasma and compare it with actual times for plasma generation.

Under these conditions the energy $(3T_e + J)N_e$ related to unit volume of this plasma will be conserved in the course of plasma evolution, but the number density of electrons will be decreased as a result of electron–ion recombination, and released energy will go into electron heating. This leads to a decrease of the plasma parameter (1.78), that is, this plasma becomes ideal. In evaluating the rate of this process, we take into account the conservation of the specific energy $(3T_e + J)N_e$ of this plasma, and assuming the thermal energy of electrons to be small compared with the atomic ionization potential J $(T_e \ll J)$, we obtain from this the following connection between the rates of change of the electron temperature and number density in the course of plasma relaxation:

$$\frac{dT_e}{dt} = -\frac{J}{3}\frac{d\ln N_e}{dt} .$$

This leads to the following equation for evolution of the plasma parameter (1.78):

$$\frac{d\gamma}{dt} = \gamma \cdot \frac{J}{T_e}\frac{d\ln N_e}{dt} . \tag{1.88}$$

We now analyze the character of plasma relaxation to ionization equilibrium in the first stage when the gas is fully ionized, and one can ignore ionization of atoms. Then relaxation results from recombination of electrons and ions according to the scheme

$$2e + A^{+} \to e + A^{*} \to e + A \,, \tag{1.89}$$

and the balance equation for the electron number density is

$$\frac{dN_\mathrm{e}}{dt} = -KN_\mathrm{e}^2 N_\mathrm{i} \,, \tag{1.90}$$

where for a quasineutral plasma the number densities of electrons and ions are identical $N_\mathrm{e} = N_\mathrm{i}$, and K is the recombination coefficient for the three body process (1.89). Subsequently, we analyze this process in detail, but now we find the rate constant K from the dimension considerations, according to which it has dimensions centimeters to the sixth power per second and includes the parameters e^2, m_e, and T_e ($T_\mathrm{e} \ll J$). Then the recombination coefficient is

$$K = \frac{ae^{10}}{T_\mathrm{e}^{9/2} m_\mathrm{e}^{1/2}} \,, \tag{1.91}$$

where the numerical coefficient $a \sim 1$.

Taking $a = 1$, we obtain the balance equation (1.88) in the form

$$\frac{d\gamma}{dt} = -\gamma \cdot \frac{J}{T_\mathrm{e}} KN_\mathrm{e}^2 = -\gamma^3 \cdot \frac{J\sqrt{T_\mathrm{e}}}{e^2\sqrt{m_\mathrm{e}}} = -\frac{\gamma^3}{\tau} \,, \quad \tau = \frac{e^2}{J}\sqrt{\frac{m_\mathrm{e}}{T_\mathrm{e}}} \,. \tag{1.92}$$

As is seen, although a typical relaxation time τ exceeds the atomic time, the difference between these times is not great. For example, for a hydrogen plasma and $T_\mathrm{e} = 1000\,\mathrm{K}$, τ is about 10^{-15} s, whereas the minimum time for plasma generation is approximately 10^{-14} s for short-pulse lasers, and the minimum time of a laser pulse exceeds a typical ionization relaxation time for this plasma. This means that it is impossible to create a strongly coupled plasma of low temperature in a two-component plasma consisting of electrons and ions only. Hence, the neutral component of a plasma or external fields are of importance to support a plasma with strong coupling, in contrast to an ideal plasma, where many of its properties (such as plasma oscillations and interaction with electromagnetic waves) are independent of its neutral components. Thus, the word "plasma" in its general sense meaning a system of charged particles is not appropriate for a dense plasma of low temperature, whereas it is a suitable description for ideal plasmas that are weakly ionized gases at low temperature.

1.3.4
Special Features of Strongly Coupled Plasmas

The above analysis of a strongly coupled plasma makes it possible to identify the distinguishing features of such an object. We first present a contrast to a weakly coupled plasma. When we apply the term "plasma" to a weakly ionized gas, we concentrate on properties that are determined by charged particles. These properties of an ideal plasma do not depend on the presence of neutral particles even though the density of neutral particles is greatly in excess of the density of charged particles. For example, the presence of neutral atoms in the Earth's ionosphere has no effect on the character of the propagation of electromagnetic waves through it. This explicitly plasma property is qualitatively distinct from such ionosphere properties as thermal capacity and thermal conductivity that are determined by the neutral component only. We can thus use the term "plasma" as a universal description of a wide variety of systems of weakly ionized gases whose electric or electromagnetic properties are determined by charged particles only. This allows us to analyze in a general way the properties of systems that may be very diverse in terms of the properties arising from their neutral components.

The explanation for this situation is to be found in the character of the interactions in a weakly ionized gas. There is a short-range interaction between neutral particles – atoms or molecules – and a long-range Coulomb interaction between charged particles. These interactions produce effects that can be treated independently in a gaseous system. Therefore, if a particular property of this system is determined by a long-range interaction between charged particles, one can ignore the short-range interactions in this system, That is, the presence of neutral particles does not affect these properties of the system. It follows from the nature of this deduction that such a conclusion can be valid only for systems with weak interactions among the constituents; that is, it can apply only to gaseous systems. On the other hand, a strongly coupled plasma is a system with strong interactions among the particles, and these strong interactions make it impossible to divide interactions into independent short-range and long-range types. Therefore, the term "strongly coupled plasma" cannot be employed as a generic description for a wide variety of systems as can be done in the case of an ideal plasma.

We can illustrate this conclusion with some examples of strongly coupled plasmas. Consider a metallic plasma. Taking a metal of a singly-valent element, we assume that all the valence electrons contribute to the conductivity, and that the metallic ions form the metal lattice. Then the electron number density is $N_\mathrm{e} = \rho/m$, where ρ is the density of the metal and m is the atomic mass. For example, in the case of copper, we have $N_\mathrm{e} = 8.4 \times 10^{22}\ \mathrm{cm}^{-3}$ corresponding to room temperature, along with the parameters $\gamma = 1.5 \times 10^7$ and $\Gamma = 400$. Assuming this plasma to be classical, we find that it is a strongly coupled plasma. The other example of a nonideal plasma is an electrolyte, which is a solution with positive and negative ions. The high density of molecules in the solution makes it a strongly coupled plasma. These examples refer to stationary strongly coupled plasmas. The strong action of an intense, short pulse of energy on matter can generate a nonstationary strongly coupled plasma. Plasmas of this type can result from a strong explosion that compresses matter. Another example of this type is the plasma associated with laser fusion, where a target is irradiated by short laser pulses directed onto the target simultaneously from different directions. The energy transferred to the target causes extreme heating and compression, with a dense plasma created as a result.

These examples illustrate the above conclusion for a strongly coupled plasma: namely, that its properties are determined by interactions involving both neutral and charged particles. Different physical objects of this type do not have general properties determined only by the charged particles. Therefore, when one describes a system as a "strongly coupled plasma", the immediate implication is that it is not one of a general class with identical properties as in the case of an ideal plasma.

1.3.5
Quantum Plasmas

We can treat a dense, low-temperature plasma of metals as a degenerate Fermi gas. Because of the low temperature and high density, the system has quantum properties. Let us consider the limit when $T = 0$, and take into consideration the Pauli principle, according to which two electrons cannot be in the same state. With the positive charge distributed uniformly over the plasma volume, this plasma is a degenerate electron gas. At zero temperature the plasma electrons have distinct momenta p in the interval $0 \leq p \leq p_\mathrm{F}$. The Fermi momentum p_F is found from the relation

$$n = 2\int \frac{d\mathbf{p}d\mathbf{r}}{(2\pi\hbar)^3} ,$$

where n is the total number of plasma electrons, the factor 2 accounts for the two possible directions of the electron spin, and $d\mathbf{p}$ and $d\mathbf{r}$ are elements of the electron momentum and the plasma volume. Introducing the electron number density $N_\mathrm{e} = n/\int d\mathbf{r}$, we have $p_\mathrm{F} = (3\pi^2\hbar^3 N_\mathrm{e})^{1/3}$ for the Fermi momentum, and the Fermi energy is

$$\varepsilon_\mathrm{F} = \frac{p_\mathrm{F}^2}{2m_\mathrm{e}} = \frac{(3\pi^2 N_\mathrm{e})^{2/3}\hbar^2}{2m_\mathrm{e}} . \tag{1.93}$$

The parameters of a classical plasma satisfy the relation $\varepsilon_\mathrm{F} \ll T$ or $r_\mathrm{D}^2 N_\mathrm{e}^{1/3} \gg a_0$. We define a quantum plasma to be a charged particle system characterized by the small parameter

$$\eta = \frac{T}{\varepsilon_\mathrm{F}} , \tag{1.94}$$

exactly opposite to the condition for a classical plasma. The Fermi energy is a fundamental parameter of a degenerate electron gas, and it is the parameter that is used for the analysis of a quantum plasma. We introduce the parameter characterizing the ideal nature of a quantum plasma by analogy with the plasma parameter (1.3) as the ratio of the Coulomb interaction energy of electrons to the Fermi energy, or

$$\xi = \frac{e^2/r_\mathrm{W}}{\varepsilon_\mathrm{F}} = \frac{2^{5/3}}{3\pi a_0 N_\mathrm{e}^{1/3}} = \frac{0.337}{a_0 N_\mathrm{e}^{1/3}} , \tag{1.95}$$

where r_W is the Wigner–Seitz radius and a_0 is the Bohr radius. The ideal degenerate electron gas has a high density compared with the characteristic atomic density, or $N_e a_0^3 \gg 1$, that (1.95) is equivalent to $\xi \ll 1$. This means that the greater the electron number density, the more the properties of a degenerate electron gas determine the properties of a quantum plasma. In contrast, the role of the Coulomb interaction between charged particles of the plasma decreases with increase of the electron number density.

We can apply the model of a degenerate electron gas to describe the behavior of electrons in metals. Table 1.7 lists parameters of real metallic plasmas at room temperature. It can be seen that η is small at room temperature, meaning that the metallic plasma has the character of a quantum plasma. But the Coulomb interaction involving electrons and ions of metals is comparable to the exchange interaction potential of electrons, determined by the Pauli principle. Thus, a metallic plasma is a quantum plasma in which the potential of the Coulomb interaction of charged particles and the exchange interaction potential of the electrons have the same order of magnitude. As is seen, in all the cases of a metallic plasma according to the data in Table 1.7 the Coulomb interaction between the nearest electrons exceeds the exchange interaction between electrons, but these values are comparable.

Positive ions of real metals form a crystalline lattice at low temperatures. An important role in these crystals is played by the non-Coulomb interaction of free electrons with ions and bound electrons. Consider a simplified problem where electrons and ions of the metal participate only in the Coulomb interactions between them. The ions of this system form a crystalline lattice, and the energy per coupled pair of charged particles (one electron and ion) is

$$\varepsilon = \frac{3p_F^2}{10m_e} - \kappa e^2 N_e^{1/3} , \tag{1.96}$$

where the first term is the mean electron kinetic energy, the second term is the mean energy of the Coulomb interaction between charged particles, and κ depends on the lattice type. Here we take into account the redistribution of charged particles resulting from their interaction that leads to the attractive character of the mean interaction energy.

Accounting for $p_F \sim N_e^{1/3}$ and optimizing (1.96) for the specific plasma energy, we find that the optimal parameters for the plasma are

$$a_0 N_e^{1/3} = \frac{5\kappa\pi^{4/3}}{3^{5/3}} = 0.174\kappa , \quad \varepsilon_{\min} = -\varepsilon_0\kappa^2 , \tag{1.97}$$

Table 1.7 Parameters of metallic plasmas at room temperature.

Metal	**Li**	**Na**	**Mg**	**Al**	**K**	**Cu**	**Ag**	**Cs**	**Au**	**Hg**
N_e, 10^{22} cm^{-3}	4.6	2.5	8.6	18	1.3	8.4	5.9	0.85	5.9	8.5
η, 10^{-3}	5.5	8.2	3.6	2.2	13	3.7	4.6	17	4.6	3.6
ξ	1.8	2.2	1.4	1.1	2.7	1.4	1.6	3.1	1.6	1.4

where $\varepsilon_0 = 2.5/(3^{5/3}\pi^{4/3})(m_e e^4/\hbar^2) = 2.4\,\mathrm{eV}$. This manipulation shows that the system may have a stable configuration of bound ions and electrons (i.e., $\varepsilon_{\min} < 0$). The stable distribution of charged particles corresponds to $\xi = 1.9/\kappa$. The system so described is called a Wigner crystal. It can be seen that a Wigner crystal, like real metals, is characterized by an electron number density of the order of the typical atom number density a_0^{-3}. Note that the above results are based on simple models which do not account for details of interaction in metals, and hence the results have a qualitative character.

1.3.6
Ideal Electron–Gas and Ion–Gas Systems

The gaseous state condition for a weakly ionized gas relates not only to the interactions among charged particles or among neutral particles as separate groups but also to interactions between charged and neutral particles. The interaction between neutral atomic particles has a short-range character, whereas the interaction of a charged particle with neutral particles may be long range and is stronger than the interaction between neutral particles. Therefore, one can expect a violation of the condition for the gaseous state to occur in the interaction of one charged particle with surrounding particles in a dense gas. That is, in a system of atoms and a single charged particle, where the interaction of the atoms satisfies the gaseous state criterion, the interaction of the charged particle with the atoms does not have gaseous character, that is, the charged particle interacts with many atoms simultaneously. We now consider this phenomenon in detail.

We begin by examining the behavior of electrons in a dense gas. The gaseous character of the interaction between electrons and atoms implies the condition

$$\lambda = (N\sigma)^{-1} \gg \overline{r}\,, \tag{1.98}$$

where λ is the mean free path of an electron in the gas, σ is the cross section for electron–atom scattering, N is the number density of atoms, and $\overline{r}$ is the mean distance between atoms. We take $\overline{r}$ to be the Wigner–Seitz radius, so $\overline{r} = (4\pi N/3)^{-1/3}$. The electron–atom cross section is represented by $\sigma = 4\pi L^2$, where L is the scattering length for slow electrons scattered by atoms.

We can write the condition (1.98) as

$$N \ll N_{\mathrm{cr}}\,, \tag{1.99}$$

where $N_{\mathrm{cr}} = (4\pi L^3\sqrt{3})^{-1}$. Table 1.8 lists values of N_{cr} and the critical pressure $p_{cr} = N_{\mathrm{cr}}T$ for electron interaction at $T = 300\,\mathrm{K}$. It is seen that the gaseous state condition for interaction of electrons with atoms can be violated in a dense gas.

The other gaseous state conditions for the electron–atom interaction in a dense gas require that positions of neighboring atoms should not influence electron–atom scattering. These relationships give

$$L \gg \overline{r}\,, \quad p\overline{r}/\hbar \ll 1\,. \tag{1.100}$$

Table 1.8 Critical parameters for interaction of electrons and ions with a gas.

Gas, vapor	**He**	**Ar**	**Kr**	**Xe**	**Cs**
N_{cr}^{e}, 10^{21} cm^{-3}	200	90	7	1	0.03
p_{cr}^{e}, atm	8000	3000	300	50	1
N_{cr}^{i}, 10^{21} cm^{-3}	6.3	2.8	2.6	2.0	0.12

The first condition in (1.100) gives $N \ll 3\sqrt{3}N_{cr}$, so it is less restrictive than condition (1.99). The second condition in (1.100) has the form

$$N \ll N_1(T)\ , \tag{1.101}$$

where

$$N_1(T) = \frac{3\hbar^3}{4\pi(2m_e T)^{3/2}}\ .$$

This is independent of the identity of the gas. At a temperature $T = 300$ K, we have $N_1 = 2 \times 10^{22}$ cm^{-3}, corresponding to a gas pressure of 700 atm.

For a gas consisting of atoms plus ions arising from those atoms, we have the gaseous state condition $N\sigma^{3/2} \ll 1$, where σ is the cross section of the resonant charge exchange process, and $\overline{r}$ is the mean distance between particles. The charge exchange cross section is relatively large, and therefore the gaseous state condition for ions is violated at relatively small densities of the gas. Table 1.8 lists values of gas densities $N_{cr} = (\pi/2)^{1/2}\sigma^{-3/2}$ at which the criterion to have a gaseous state for interaction of ions and atoms is violated at a gas temperature of 1000 K.

We conclude that electron–atom or ion–atom interactions in a weakly ionized gas can be nonpairwise even when the interaction between neutral particles satisfies the gaseous state condition. Then charged particles interact simultaneously with several neutral particles, and the gaseous character of the interaction is lost.

1.3.7
Decrease of the Atomic Ionization Potential in Plasmas

A dense plasma alters the states of its constituent atoms, and thereby can change the atomic spectrum [71–73]. In particular, spectra of metallic plasmas differ significantly from spectra of isolated metal atoms. The greater the plasma density, the more drastic is the change in atomic parameters, including the ionization potential. Below we estimate the decrease of the atomic ionization potential as a function of the plasma density.

We can consider an atom as an isolated object starting from the principal quantum number n, for which the atomic size r_n is small compared to the average distance between ions as $N_i^{-1/3}$. Since the average size of an excited atom is $r_n \sim a_0 n^2$ (a_0 is the Bohr radius), the ionization potential for this quantum number and for charge Z of the Coulomb center is $J_n \sim Z^2e^2/(a_0n^2)$. From this we have for the decrease ΔJ in the atomic ionization potential in a plasma [74]

$$\Delta J = J_n \sim Ze^2 N_{\mathrm{i}}^{1/3}\,. \tag{1.102}$$

In particular, according to calculations by the method of molecular dynamics, this value is

$$\Delta J = 3.2 Ze^2 N_{\mathrm{i}}^{1/3}\,. \tag{1.103}$$

One can consider the ionization potential decrease for atoms in a plasma to be a result of the action of plasma microfields. Indeed, let us consider an excited electron located in the field of the Coulomb center with charge Z and an external electric field of strength E. Then the interaction potential U between an electron and the Coulomb center in the direction x of the electric field is (see Figure 1.6)

$$U = -\frac{Ze^2}{x} - eEx\,,$$

where the origin of the frame of reference is taken as the Coulomb center. The maximum of this interaction potential $U_{\max}$ gives the decrease of the ionization potential:

$$\Delta J = -U_{\max} = 2\sqrt{eE\cdot Ze^2}\,.$$

In particular, if we use a typical electric field strength in a plasma in accordance with (1.26), $E_0 = 2.6 ZeN_{\mathrm{i}}^{2/3}$, we obtain for the decrease in the ionization potential $\Delta J = 3.2 Ze^2 N_{\mathrm{i}}^{1/3}$ in accordance with (1.103).

Thus, the presence of microfields in a plasma may lead to the disappearance of excited atom states in a plasma. The disappearance of spectral lines was observed for the first time by Lanchos [75, 76] in 1930, and this effect means that the radiative lifetime for an observed atomic transition becomes equal to the time for the

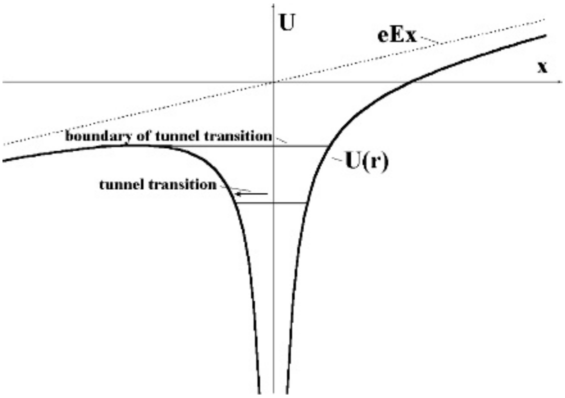

Figure 1.6 The potential energy in the plane of the electric field for an electron located in the field of a Coulomb center and electric field.

Table 1.9 Boundary values of the ion number density $(N_i)_n$ for disappearance of levels with principal quantum number n in a hydrogen-like plasma.

n	2	3	4	6	10
$(N_i)_n$, cm^{-3} for $Z = 1$	4.0×10^{20}	3.5×10^{19}	6.3×10^{18}	5.5×10^{17}	2.6×10^{16}
$(N_i)_n$, cm^{-3} for $Z = 4$	2.6×10^{22}	2.3×10^{21}	4.0×10^{20}	3.5×10^{19}	1.6×10^{18}

tunneling transition of the bound electron in the potential in Figure 1.6. We now consider a stronger action of the electric field when the decay time for an excited atomic level becomes equal to the orbital period for the excited electron. Then the boundary number density of ions $(N_i)_n$ for the disappearance of atomic levels with principal quantum number n follows from the relation

$$\frac{Z^2 e^2}{2 a_0 n^2} = \Delta J$$

and is given by

$$N_i = N_0 \frac{Z^3}{n^6}\,, \quad N_0 = 2.6 \times 10^{22}\,\text{cm}^{-3}\,. \tag{1.104}$$

Table 1.9 contains certain values of the boundary ion number densities for a hydrogen-like plasma. Note that the number density of electrons N_e for a plasma with multicharged ions of charge Z is connected to the ion number density by the relation $N_e = Z N_i$. The values in Table 1.9 characterize the transition from an ideal plasma to a nonideal plasma or a plasma with strong coupling.

1.3.8
Spectrum of Atoms in a Plasma

The above analysis shows that one can divide states of electrons in the field of a Coulomb center in a plasma into three parts, bound electron states, free electron states, and localized electron states. Bound electron states correspond to discrete levels of an excited atom (ion) as it has in a vacuum. The energy ε_n of a state with principal quantum number n is given by

$$\varepsilon_n = -\frac{Z^2 e^2}{2 a_0 n^2}\,.$$

Since the statistical weight of states with principal quantum number n is $2n^2$, the density of states $\rho(\varepsilon)$ for discrete states is

$$\rho(\varepsilon) = \frac{dg}{d\varepsilon} = \frac{4 n^4 a_0}{Z^2 e^2} = \frac{Z^2 e^2}{a_0 \varepsilon^2}\,, \quad \varepsilon \le -\Delta J\,.$$

According to (1.103) at the boundary binding energy of a bound electron $\varepsilon_b = \Delta J$ the density of electron states is

$$\rho(-\varepsilon_b) \approx \frac{1}{10 a_0 e^2 N_i^{2/3}} \,.$$

For free electrons we use (1.53) for the statistical weight of free states with respect to bound ones. Using the Maxwell distribution function (1.49) for free electrons with the electron temperature T_e, we have for the density of free electron states

$$\rho_c(\varepsilon) = \frac{g_e g_i}{N_e} \cdot \frac{m_e^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \cdot \varepsilon^{1/2} \exp\left(-\frac{\varepsilon}{T_e}\right) \,. \tag{1.105}$$

Evidently, this formula, as well as the Maxwell distribution (1.49) for free electrons, holds true if the electron energy exceeds remarkably the energy of interaction with plasma ions. Taking the quantity ΔJ given by (1.103) as the average potential energy for an electron, we define the boundary energy of free electrons as

$$\varepsilon_c = \Delta J = -\varepsilon_b \,.$$

Assuming $\varepsilon_c \ll T_e$, we obtain for the density of electron states in a hydrogen plasma ($Z = 1$)

$$\rho_c(\varepsilon) = \frac{g_e g_i}{N_e} \cdot \frac{m_e^{3/2} (\Delta J)^{1/2}}{\sqrt{2}\pi^2 \hbar^3} \,.$$

Taking $g_e = 2$, $g_i = 1$, we obtain

$$\rho(\varepsilon_c) \approx \frac{0.26}{N_i^{5/6} a_0^{5/2} \varepsilon_0} \,,$$

where ε_0 is the atomic energy $\varepsilon_0 = e^2/a_0 = m_e e^4/\hbar^2$.

We now define an ideal plasma that is characterized by gas properties such that $\rho(\varepsilon_c) \gg \rho(-\varepsilon_b)$, that is,

$$\frac{\rho(\varepsilon_c)}{\rho(-\varepsilon_b)} \approx \frac{2.6}{N_i^{5/6} a_0^{5/2} \varepsilon_0} \,.$$

If we define an ideal plasma such that in an energy range of localized electrons the density of electron states decreases monotonically as the electron energy decreases, we obtain the criterion for an ideal plasma independently of the electron temperature

$$N_i a_0^3 \ll 300 \,, \tag{1.106}$$

that is, the ion density exceeds the atomic density ($N_0 = 6.76 \times 10^{24}\,\mathrm{cm}^{-3}$). This criterion shows that as long as the ion (electron) number density of a plasma is small compared with the atomic one N_0, the transferring range of energies between free and bound electron states, where electrons are bound simultaneously with many ions, is not important.

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