1
The Mathematical Methods of Electrodynamics

1.1
Vector and Tensor Algebra

1.1.1
The Definition of a Tensor and Tensor Operations

In three-dimensional space, select a rectangular and rectilinear (Cartesian[1]) system of coordinates x_1, x_2, x_3. Regard the space as *Euclidean*. This means that all axioms of Euclidean geometry[2] and their consequences considered in school courses on mathematics are valid in it. For instance, the square of the distance between two close points is given by the following expression:

$$\mathrm{d}l^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \ .$$

Along with the original system of coordinates, consider some other systems of common origin yet rotated with respect to the original one (Figure 1.1).

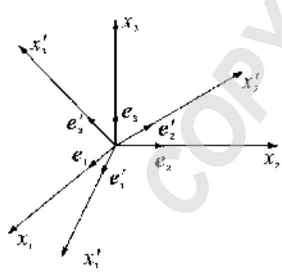

Figure 1.1 The rotation of the Cartesian system of coordinates.

1) René Descartes (Renatus Cartesius) (1596–1650) was a French philosopher and mathematician, the founder of the coordinates method. He introduced a large number of mathematical concepts and notations used even now.

2) Euclid (lived in the third century BC) was an ancient Greek scientist, "the father of geometry." His mathematical treatise entitled *Elements* is the best known. Euclid studied various aspects of geometric optics.

Foundations of Classical and Quantum Electrodynamics, First Edition. Igor N. Toptygin.

A *scalar or invariant* is a quantity that does not change when the system of coordinates is rotated, that is, it is the same in either the original or the rotated system of coordinates

$$S' = S = \text{inv} \,. \tag{1.1}$$

For instance, $\mathrm{d}l^2 = \mathrm{d}l'^2 = \text{inv}$.

In three-dimensional space, a *vector* is the titality of three quantities V_α ($\alpha = 1, 2, 3$) defined in all coordinate systems and transformed according to the following rule:

$$V'_\alpha = a_{\alpha\beta} V_\beta \tag{1.2}$$

(summing of elements over the repeated symbol β, from 1 to 3 is assumed). Here V_β are the projections of the vector on an axis of the original system of coordinates, V'_α are the projections of the vector on an axis of the rotated system, and $a_{\alpha\beta}$ are the coefficients of the transformation, which are the cosines of the angles between the β axis of the original system and the α axis of the rotated system. They may be written through the single vectors (orts) of the coordinate axes:

$$a_{\alpha\beta} = \boldsymbol{e}'_\alpha \cdot \boldsymbol{e}_\beta \,. \tag{1.3}$$

In three-dimensional space, a *tensor of rank 2* is a nine-component quantity $T_{\alpha\beta}$ (each index varies independently assuming three values: 1, 2, 3) which is defined in every system of coordinates and, when a coordinate system is rotated, is transformed as the products of the components of the two vectors $A_\alpha V_\beta$, in the following way:

$$T'_{\alpha\beta} = a_{\alpha\mu} a_{\beta\nu} T_{\mu\nu} \,. \tag{1.4}$$

In three-dimensional space, a tensor of rank s is a 3^s-component quantity $T_{\lambda\ldots\nu}$ that is transformed as the product of s components of vectors:

$$T'_{\beta\ldots\kappa} = a_{\beta\mu} \ldots a_{\kappa\sigma} T_{\mu\ldots\sigma} \,. \tag{1.5}$$

Scalars and vectors may be regarded as tensors of rank 0 and 1, respectively.

Rotation matrix $\widehat{a}$ has the following properties:

1. Orthogonality

$$a_{\alpha\mu} a_{\beta\mu} = \delta_{\alpha\beta} \,, \quad a_{\alpha\mu} a_{\alpha\nu} = \delta_{\mu\nu} \,, \tag{1.6}$$

where

$$\delta_{\alpha\beta} = 1 \quad \text{if} \quad \alpha = \beta \quad \text{and} \quad \delta_{\alpha\beta} = 0 \quad \text{if} \quad \alpha \neq \beta \tag{1.7}$$

is *Kronecker symbol*[3];

3) Leopold Kronecker (1823–1891) was a German mathematician, a specialist in algebra and theory of numbers.

2. The determinant of a rotation matrix equals 1:

$$\det \widehat{a} \equiv |\widehat{a}| = 1 \, . \tag{1.8}$$

3. The product of two rotation matrices

$$\widehat{c} = \widehat{a}\widehat{g} \, , \quad c_{\alpha\beta} = a_{\alpha\mu} g_{\mu\beta} \tag{1.9}$$

describes the evolution of a system resulting from two consecutive rotations, first with matrix $\widehat{g}$ and then with matrix $\widehat{a}$.[4] In the general case, rotation matrices are noncommutative, that is,

$$\widehat{a}\widehat{g} \neq \widehat{g}\widehat{a} \, . \tag{1.10}$$

As follows from property 1, a reverse matrix defined by the relation

$$\widehat{a}^{-1}\widehat{a} = \widehat{a}\widehat{a}^{-1} = \widehat{1} \quad \text{or} \quad a^{-1}_{\alpha\mu} a_{\mu\beta} = a_{\alpha\mu} a^{-1}_{\mu\beta} = \delta_{\alpha} \tag{1.11}$$

results from the original matrix when the latter is transposed, that is, its columns are substituted for lines and vice versa:

$$\widehat{a}^{-1} = \widehat{\tilde{a}} \, , \quad a^{-1}_{\alpha\beta} = \tilde{a}_{\alpha\beta} = a_{\beta\alpha} \, . \tag{1.12}$$

The reverse transformation (1.2) looks like this:

$$V_{\beta} = a^{-1}_{\beta\alpha} V'_{\alpha} \, . \tag{1.13}$$

All vectors are transformed identically according to rule (1.2) when a coordinate system is rotated. But they may behave in one of two ways if a system of coordinates is inverted, that is,

$$x'_{\alpha} = -x_{\alpha} \, . \tag{1.14}$$

Here the transformation matrix is $a_{\alpha\beta} = -\delta_{\alpha\beta}$. Vectors whose components, just like coordinates x_a, change their signs during inversions are called *polar* (regular, real) vectors. Vectors whose components do not change sign as the result of inversions of coordinate systems are called *axial* vectors or *pseudovectors* (an angular velocity, a cross-product of two polar vectors $\boldsymbol{A} \times \boldsymbol{B}$, etc.) This definition also includes tensors of arbitrary rank s: when the inversion of coordinates occurs, the components of *polar* (regular) tensors acquire a factor of $(-1)^s$ and the components of *pseudotensors* acquire a factor of $(-1)^{s+1}$.

The *sum* of two tensors of the same rank produces a third tensor of the same rank with components

$$Q_{\alpha\beta} = T_{\alpha\beta} + P_{\alpha\beta} \, . \tag{1.15}$$

4) The family all rotation operations forms makes a group of three-dimensional rotations. See Gel'fand *et al.* (1958).

The *direct products* of the components of two tensors (without summing) constitute a tensor whose rank equals the sum of the ranks of the factors, for instance,

$$Q_{\alpha\beta\lambda} = T_{\alpha\beta} V_\lambda \,, \tag{1.16}$$

where $Q_{\alpha\beta\lambda}$ is a tensor of rank 3.

The *contraction of a tensor* means the formation of a new tensor whose components are produced by the selection of components with two identical symbols and, further, their summing. For instance, $Q_{\alpha\beta\beta} = A_\alpha$ is a vector and $Q_{\alpha\beta\alpha} = B_\beta$ is another vector. Contraction decreases the rank of the tensor by 2, for instance,

$$S = T_{\alpha\alpha} = \text{inv} \tag{1.17}$$

is a scalar.

When an equality between tensors is written, the rule of the same tensor dimensionality must be observed: only tensors of the same rank may be equated. This means that the number of free symbols (over which no summation is done) must be the same in the first and second members of an equality. The number of pairs of "mute" symbols (those over which summing is done) may be any on the right and on the left.

Tensors may be *symmetric (antisymmetric)* with respect to a pair of indices α and β if their components satisfy the conditions

$$Q_{\alpha\beta\mu} = Q_{\beta\alpha\mu} \quad (Q_{\alpha\beta\mu} = -Q_{\beta\alpha\mu}) \,. \tag{1.18}$$

Tensor components may be either real or complex numbers. In the latter case, the concepts of Hermitian[5)] and anti-Hermitian tensors play an important role. The definition of a Hermitian tensor is as follows:

$$T^h_{\alpha\beta} = T^{h*}_{\beta\alpha} \,, \tag{1.19}$$

where the asterisk indicates complex conjugation. The definition of an anti-Hermitian tensor is as follows:

$$T^{ah}_{\alpha\beta} = -T^{ah*}_{\beta\alpha} \,. \tag{1.20}$$

In applications, invariant unit tensors $\delta_{\alpha\beta}$ and $e_{\alpha\beta\lambda}$ are very important. The former is a symmetric polar tensor whose components coincide with the Kronecker symbol (1.7), whereas the latter is antisymmetric over any pair of indices, and its components are determined by the following conditions:

$$\text{(a)} \quad e_{123} = 1 \,, \quad e_{\alpha\beta\lambda} = -e_{\beta\alpha\lambda} = -e_{\alpha\lambda\beta} = e_{\lambda\alpha\beta} = e_{\beta\lambda\alpha} = -e_{\lambda\beta\alpha} \,. \tag{1.21}$$

5) Charles Hermite (1822–1901) was a French mathematician, the author of works on classical analysis, algebra, and theory of numbers.

It is called the Levi-Civita tensor.[6] Both tensors, transforming during rotations according to rule (1.7), are peculiar in that their components have the same values in all coordinate systems:

$$\delta'_{\alpha\beta} = \delta_{\alpha\beta} , \quad e'_{\alpha\beta\lambda} = e_{\alpha\beta\lambda} . \tag{1.22}$$

Problems

1.1. Prove equality (1.8). What is the determinant of the transformation matrix if rotation is accompanied by the inversion of the coordinate axes?

1.2. Prove the equalities $\delta'_{\alpha\beta} = \delta_{\alpha\beta}$ and $e'_{\alpha\mu\nu} = e_{\alpha\mu\nu}$ for an arbitrary rotation of a coordinate system.

1.3. Write down the rule of transformation for the components of a pseudotensor of rank s that would be valid not just for the rotation but also for the mirror reflections of the coordinate axes.

1.4. Represent an arbitrary tensor of rank 2 $T_{\alpha\beta}$ as the sum of a symmetric tensor ($S_{\alpha\beta} = S_{\beta\alpha}$) and an antisymmetric tensor ($A_{\alpha\beta} = -A_{\beta\alpha}$). Make sure that this representation is unique.

1.5. Represent an arbitrary complex tensor of rank 2 $T_{\alpha\beta}$ as the sum of a Hermitian tensor ($S^h_{\alpha\beta} = S^{h*}_{\beta\alpha}$) and an anti-Hermitian tensor ($A^h_{\alpha\beta} = -A^{h*}_{\beta\alpha}$). Make sure that this representation is unique.

1.6. Show that

1. the contraction of a symmetric tensor and an antisymmetric tensor equals zero: $S_{\alpha\beta} A_{\alpha\beta} = 0$.
2. the contraction of two Hermitian tensors or two anti-Hermitian tensors of rank 2 is a real number.
3. the contraction of a Hermitian tensor and an anti-Hermitian tensor of rank 2 is a purely imaginary number.

1.7. Show that the symmetry of a tensor is a property that is invariant with respect to rotations, that is, a tensor that is symmetric (antisymmetric) over a pair of indices in a certain system of reference remains symmetric (antisymmetric) over these indices in every system rotated with respect to the original one.

1.8. Using rules (1.2)–(1.6) of tensor transformation, show that

1. A_α is a vector (pseudovector) if $A_\alpha B_\alpha = \text{inv}$ and B_α is a vector (pseudovector).
2. A_α is a vector if $A_\alpha = T_{\alpha\beta} B_\beta$ in any system of coordinates and $T_{\alpha\beta}$ is a tensor of rank 2, and B_β is a vector;
3. $T_{\alpha\alpha} = \text{inv}$, where $T_{\alpha\beta}$ is a tensor of rank 2.

6) Tullio Levi-Civita (1873–1941) was an Italian mathematician who contributed to the development of tensor analysis.

4. $\varepsilon_{\alpha\beta}$ is a tensor of rank 2 if A_α and B_α are vectors and $A_\alpha = \varepsilon_{\alpha\beta} B_\beta$ in all systems of coordinates. What is $\varepsilon_{\alpha\beta}$ if A_α is a vector and B_α is a pseudovector? What is $\varepsilon_{\alpha\beta}$ if A_α and B_α are both pseudovectors?
5. $A_{\alpha\beta\lambda} B_{\alpha\beta}$ is a vector if $A_{\alpha\beta\lambda}$ and $B_{\alpha\beta}$ are tensors of ranks 3 and 2, respectively.
6. $T_{\alpha\beta} P_{\alpha\beta}$ is a pseudoscalar if $T_{\alpha\beta}$ and $P_{\alpha\beta}$ are a tensor and a pseudotensor of rank 2, respectively.

1.9. Show the rule of the transformation of an aggregate of volumetric integrals $T_{\alpha\beta} = \int x_\alpha x_\beta \mathrm{d}V$ in the cases of rotation and mirror reflection (x_α, x_β are Cartesian coordinates).

1.10. Show that the components of an antisymmetric tensor of rank 2 $A_{\alpha\beta} = -A_{\beta\alpha}$ (either polar or axial) may be identified by the components of a certain vector C_α (either axial or polar) because they are transformed in the same way in the case of rotation or reflection. In this case, C_α is called the vector dual to tensor $A_{\alpha\beta}$.

1.11. Prove the following equalities:

$$[\boldsymbol{A} \times \boldsymbol{B}]_\alpha = e_{\alpha\beta\lambda} A_\beta B_\lambda \,,$$

$$[\boldsymbol{A} \times \boldsymbol{B}] \cdot \boldsymbol{C} = e_{\alpha\beta\lambda} A_\alpha B_\beta C_\lambda = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \,. \tag{1.23}$$

How are the vector, the dual vector, and the mixed products transformed in the cases of rotation and reflection if all three vectors are polar?

1.12. Show that if the respective components of two vectors are proportional in a certain system of coordinates, then they are also proportional in any other system of coordinates. Vectors such as these are called parallel vectors.

1.13. The area of an elementary parallelogram constructed on the small vectors $\mathrm{d}\boldsymbol{r}$ and $\mathrm{d}\boldsymbol{r}'$ is represented by vector $\mathrm{d}\boldsymbol{S}$ directed along a normal to the plane of the parallelogram and, by the absolute value, is equal to its area. Write down $\mathrm{d}S_\alpha$ in tensor notation.

1.14. Write down, in tensor notation, the volume $\mathrm{d}V$ of the elementary parallelepiped constructed on the small vectors $\mathrm{d}\boldsymbol{r}, \mathrm{d}\boldsymbol{r}', \mathrm{d}\boldsymbol{r}''$. How is it transformed in the cases of rotation and reflection?

1.15. Prove the identities

$$\begin{aligned}
(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) + (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C}) &= 0 \,, \\
(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) + (\boldsymbol{B} \times \boldsymbol{C}) \cdot (\boldsymbol{A} \times \boldsymbol{D}) + (\boldsymbol{C} \times \boldsymbol{A}) \cdot (\boldsymbol{B} \times \boldsymbol{D}) &= 0 \,, \\
\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A}) + \boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) &= 0 \,, \\
(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) - (\boldsymbol{A} \cdot [\boldsymbol{B} \times \boldsymbol{D}])\boldsymbol{C} + (\boldsymbol{A} \cdot [\boldsymbol{B} \times \boldsymbol{C}])\boldsymbol{D} &= 0 \,, \\
(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) - (\boldsymbol{A} \cdot [\boldsymbol{C} \times \boldsymbol{D}])\boldsymbol{B} + (\boldsymbol{B} \cdot [\boldsymbol{C} \times \boldsymbol{D}])\boldsymbol{A} &= 0 \,.
\end{aligned} \tag{1.24}$$

1.16. In a spherical system of coordinates, the two directions $\boldsymbol{n}$ and n' are determined by the angles ϑ, α and ϑ', α'. Find the cosine of the angle θ between them.

1.17. In certain cases, it may be more convenient to consider the complex cyclic components

$$A_{\pm 1} = \frac{\mp A_x \pm i A_y}{\sqrt{2}} , \quad A_0 = A_z , \tag{1.25}$$

of the vector $\boldsymbol{A}$ instead of its Cartesian components. Express the scalar and vector products of two vectors through their cyclic components. Also, express the cyclic components of the radius vector through spherical functions.[7]

1.18. Write down the matrix $\widehat{g}$ of the transformation of the components of a vector in the case of the rotation of the Cartesian system of coordinates around the Ox_3 axis by angle α.

1.19. Form the matrices of the transformation of basic orts when changing from Cartesian to spherical coordinates and back and from Cartesian to cylindrical coordinates and back.

1.20. Find the matrix $\widehat{g}$ of the transformation of the components of a vector in the case of the rotation of the coordinate axes determined by the Euler angles[8] α_1, θ, and α_2 (Figure 1.2) by mutually multiplying matrices corresponding to rotation around the Ox_3 axis by angle α_1, around the line of nodes ON by angle θ, and around the Ox_3' axis by angle α_2.

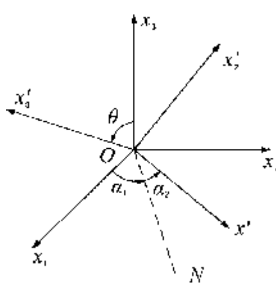

Figure 1.2 The specification of the rotation of Cartesian axes by Euler angles α_1, θ, α_2.

1.21. Find the matrix $\widehat{D}(\alpha_1 \theta \alpha_2)$ used for transforming the cyclic components of vector (1.25) when rotating the system of coordinates. The rotation is determined by the Euler angles α_1, θ, and α_2 (Figure 1.2).

7) The definition of spherical functions is given in Section 1.3; see the answer to Problem 1.118*

8) Leonard Euler (1707–1783) was an outstanding mathematician, astronomer, and physicist who astonished his contemporaries by his efficiency, and range of interests. He was born and studied in Switzerland, but for most of his life worked at the Saint Petersburg Academy of Sciences. Pierre Laplace called him the teacher of all mathematicians of the second half of the eighteenth century.

1.22. Show that the matrix of an infinitesimal rotation of a coordinate system may be written as $\hat{a} = 1 + \hat{\varepsilon}$, where $\hat{\varepsilon}$ is an antisymmetric matrix ($\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$). Find the geometric meaning of $\varepsilon_{\alpha\beta}$.

1.23. Show that the representation of a small rotation by vector $\delta\boldsymbol{\varphi}$ used in the solution of the previous problem is only possible in relation to quantities of the first order of smallness. In the next order, the vector of the resulting rotation is not equal to the sum of the vectors of individual rotations and the relevant matrices do not commute.

1.1.2
The Principal Values and Invariants of a Symmetric Tensor of Rank 2

The selection of a system of coordinates wherein a certain tensor has the simplest structure is of great practical importance. Consider the selection of such a system for a symmetric tensor of rank 2.

If vector $\boldsymbol{n}$ satisfies the condition

$$S_{\alpha\beta} n_\beta = S n_\alpha \,, \quad \alpha, \beta = 1, 2, 3 \,, \tag{1.26}$$

where S is a certain scalar, then the direction that is determined by vector $\boldsymbol{n}$ is called *the principal direction* of the tensor, vector $\boldsymbol{n}$ is called *the proper vector* of the tensor, and S is called its *principal value*.

Example 1.1

Reducing a real ($S_{\alpha\beta} = S^*_{\alpha\beta}$) symmetric ($S_{\alpha\beta} = S_{\beta\alpha}$) tensor of rank 2 to diagonal form means finding such a system of axes wherein only the diagonal components of the tensor are not equal to zero. Specify a way of calculation of the principal values and the principal directions of such tensor.

Solution. Use the system of algebraic equations (1.26) to find the proper vectors and principal values of the tensor in question. Normalize the proper vectors to 1: $n^*_\alpha n_\alpha = 1$. The equations (1.26) and the properties of the tensor $S_{\alpha\beta}$ show us that the proper values of S are real scalars: $S = n^*_\alpha S_{\alpha\beta} n_\beta = S^*$. They follow from the condition of equality to zero of the determinant of the system (1.26):

$$|S_{\alpha\beta} - S\delta_{\alpha\beta}| = 0 \,. \tag{1.27}$$

This is a cubic algebraic equation whose solution, in relation to S, includes three real roots: $S^{(1)}$, $S^{(2)}$, $S^{(3)}$. In the general case, they are different from each other, although multiple roots ($S^{(1)} = S^{(2)} \neq S^{(3)}$ or $S^{(1)} = S^{(2)} = S^{(3)}$) are possible. Here, the bracketed indices are not tensor symbols!

In the case of different roots, inserting the values found for S, one by one, in the system in (1.26) results in two projections of each of the proper vectors $n^{(1)}_\alpha \neq n^{(2)}_\alpha \neq n^{(3)}_\alpha$ through the third one, which is determined by the condition of normalization. All the proper vectors are real because the coefficients of (1.26) are real. They are mutually perpendicular, which follows from the same system of equations: $(S^{(1)} - S^{(2)})(\boldsymbol{n}^{(1)} \cdot \boldsymbol{n}^{(2)}) = 0$. The same goes for the other two pairs. Regarding the proper vectors as the orts of the system of coordinates (they determine the principal axes of the tensor), use (1.26) to find the form of the tensor in this system of axes:

$$\widehat{S}' = \begin{pmatrix} S^{(1)} & 0 & 0 \\ 0 & S^{(2)} & 0 \\ 0 & 0 & S^{(3)} \end{pmatrix} . \tag{1.28}$$

In the case of two repeated roots, $S^{(1)} = S^{(2)}$, the proper vectors $\boldsymbol{n}^{(1)}$ and $\boldsymbol{n}^{(2)}$ are determined ambiguously, that is, any pair of mutually perpendicular directions may be selected in the plane perpendicular to $\boldsymbol{n}^{(3)}$. If all three roots are the same, then any three mutually perpendicular directions may be regarded as the principal axes. □

Problems

1.24. Is it possible to reduce an arbitrary real tensor of rank 2 ($T_{\alpha\beta} \neq T_{\beta\alpha}$) to the diagonal form by rotating its system of coordinates in physical three-dimensional space? What about a Hermitian tensor of rank 2 ($T^h_{\alpha\beta} = T^{h*}_{\beta\alpha}$)?

1.25. Write down a real symmetric tensor of rank 2 $S_{\alpha\beta}$ in an arbitrary system of coordinates through its principal values $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ and the orts $n^{(i)}_\alpha$ of the principal axes.

1.26. Using the characteristic (1.27), compile the invariants relative to rotation from the components of an arbitrary tensor of rank 2 $T_{\alpha\beta}$.

1.27. Using the theorem for the expansion of the determinant in the elements of a row or a column, find the components of the inverse tensor $T^{-1}_{\alpha\beta}$. Its definition coincides with that of (1.11) for the inverse matrix. Indicate the condition of the existence of an inverse tensor.

1.28. Prove the identities

$$e_{\alpha\beta\gamma} e_{\alpha\beta\gamma} = 6 ,$$

$$e_{\alpha\beta\gamma} e_{\alpha\beta\sigma} = 2\delta_{\gamma\sigma} ,$$

$$e_{\alpha\beta\gamma} e_{\alpha\nu\sigma} = \delta_{\beta\nu}\delta_{\gamma\sigma} - \delta_{\beta\sigma}\delta_{\gamma\nu} = \begin{vmatrix} \delta_{\beta\nu} & \delta_{\gamma\nu} \\ \delta_{\beta\sigma} & \delta_{\gamma\sigma} \end{vmatrix} ,$$

$$e_{\alpha\beta\gamma} e_{\mu\nu\sigma} = \delta_{\alpha\mu}\delta_{\beta\nu}\delta_{\gamma\sigma} + \delta_{\alpha\nu}\delta_{\beta\sigma}\delta_{\gamma\mu} + \delta_{\alpha\sigma}\delta_{\beta\mu}\delta_{\gamma\nu}$$
$$-\delta_{\alpha\nu}\delta_{\beta\mu}\delta_{\gamma\sigma} - \delta_{\alpha\mu}\delta_{\beta\sigma}\delta_{\gamma\nu} - \delta_{\alpha\sigma}\delta_{\beta\nu}\delta_{\gamma\mu}$$

$$= \begin{vmatrix} \delta_{\alpha\mu} & \delta_{\beta\mu} & \delta_{\gamma\mu} \\ \delta_{\alpha\nu} & \delta_{\beta\nu} & \delta_{\gamma\nu} \\ \delta_{\alpha\sigma} & \delta_{\beta\sigma} & \delta_{\gamma\sigma} \end{vmatrix} .$$

Using the third identity, prove the formula of vector algebra

$$\boldsymbol{A} \times [\boldsymbol{B} \times \boldsymbol{C}] = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}) .$$

1.29. Write down the following in the invariant vector form:

1. $e_{\alpha\beta\gamma} e_{\alpha\sigma\kappa} e_{\gamma\nu\varepsilon} e_{\kappa\omega\varepsilon} A_\beta A_\sigma B_\nu C_\omega$,
2. $e_{\alpha\beta\gamma} e_{\rho\sigma\kappa} e_{\gamma\nu\varepsilon} e_{\kappa\omega\varepsilon} A_\sigma A_\beta B_\rho B_\alpha C_\omega C_\nu$.

1.30. Prove the identity

$$T_{\alpha\beta} A_\alpha B_\beta - T_{\alpha\beta} A_\beta B_\alpha = 2\boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) ,$$

where $T_{\alpha\beta}$ is an arbitrary tensor of rank 2, $\boldsymbol{A}$ and $\boldsymbol{B}$ are vectors, and $\boldsymbol{C}$ is the vector of the dual antisymmetric part of the tensor $T_{\alpha\beta}$.

1.31. Present the product $(\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}))(\boldsymbol{A}' \cdot (\boldsymbol{B}' \times \boldsymbol{C}'))$ as the sum of members that contain only the scalar products of the vectors.

Hint: Apply the theorem for the multiplication of determinants or use the pseudotensor $e_{\alpha\beta\gamma}$.

1.32. Show that the only vector whose components are the same in all systems of coordinates is a null vector, that any tensor of rank 3 whose components are the same in all systems of coordinates is proportional to $e_{\alpha\beta\gamma}$, and that any tensor of rank 4 whose components are the same in all systems of coordinates is proportional to $(\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu})$.

1.33. Regard $\boldsymbol{n}$ as a unit vector whose directions in space are equiprobable. Find the mean values of its components and their products – n_α, $n_\alpha n_\beta$, $n_\alpha n_\beta n_\gamma$, $n_\alpha n_\beta n_\gamma n_\nu$ – using the transformational properties of the quantities sought.

1.34. Find the average values for all directions of the expressions $(\boldsymbol{a} \cdot \boldsymbol{n})^2$, $(\boldsymbol{a} \cdot \boldsymbol{n})(\boldsymbol{b} \cdot \boldsymbol{n})$, $(\boldsymbol{a} \cdot \boldsymbol{n})\boldsymbol{n}$, $(\boldsymbol{a} \times \boldsymbol{n})^2$, $(\boldsymbol{a} \times \boldsymbol{n}) \cdot (\boldsymbol{b} \times \boldsymbol{n})$, $(\boldsymbol{a} \cdot \boldsymbol{n})(\boldsymbol{b} \cdot \boldsymbol{n})(\boldsymbol{c} \cdot \boldsymbol{n})(\boldsymbol{d} \cdot \boldsymbol{n})$, if $\boldsymbol{n}$ is a unt vector whose all directions are equiprobable and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ are constant vectors.

Hint: Use the results obtained in the previous problem.

1.35. Write down all possible invariants of polar vectors $\boldsymbol{n}$, and $\boldsymbol{n}'$ and pseudovector $\boldsymbol{l}$.

1.36. What independent pseudoscalars may be made of two polar vectors $\boldsymbol{n}$ and $\boldsymbol{n}'$ and one pseudovector $\boldsymbol{l}$? What independent pseudoscalars may be made of three polar vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$?

1.1.3
Covariant and Contravariant Components

In physics, many problems require nonorthogonal and curvilinear systems of coordinates be used so that the relations between the old and new coordinates are nonlinear and different from (1.2). The transition to new coordinates may not come down to just the simple and obvious rotation of axes. One of the most important areas where such a mathematical apparatus needs to be used is special and, especially, general relativity.

Closing this section, we will come up with the definition of tensors with respect to overall transformations of coordinates and consider their basic properties in three-dimensional Euclidean space. This is appropriate because in three-dimensional space the meaning of many concepts and relation is more obvious and transparent than in four-dimensional space–time of the relativistic theory. We will begin by immersing ourselves in these issues by considering a case that is half way between Cartesian rectangular coordinates and common coordinates when the coordinate axes of the reference frame are still rectilinear but become nonorthogonal (oblique or affine coordinates).

Example 1.2

Three noncoplanar and nonorthogonal unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are selected as the basic vectors in a three-dimensional Euclidean space. Three systems of rectilinear lines passing through every point of the space and parallel to the basic vectors are the coordinate lines. Build a mutual basis $\boldsymbol{e}^1$, $\boldsymbol{e}^2$, $\boldsymbol{e}^3$ which, by definition, is connected to the original basis by the following relations:

$$\boldsymbol{e}^\alpha \cdot \boldsymbol{e}_\beta = \delta^\alpha_\beta = \begin{cases} 0\,, & \alpha \neq \beta\,; \\ 1\,, & \alpha = \beta\,. \end{cases} \tag{1.29}$$

Will the vectors of the mutual basis be unit vectors?

Expand an arbitrary vector $\boldsymbol{A}$ (including also the radius vector $\boldsymbol{r}$) in vectors $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}^\beta$ of the original and mutual bases. Show the geometric meaning of its components in both cases (in the first case, they are called *contravariant* and are labeled with upper indices, A^1, A^2, A^3. In the second case, they are *covariant*, and are labeled with lower indices, A_1, A_2, A_3).

Solution. In accordance with (1.29), $\boldsymbol{e}^1$ must be perpendicular to $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. Look for it in the form of $\boldsymbol{e}^1 = k\boldsymbol{e}_2 \times \boldsymbol{e}_3$ and, from the condition of normalization $\boldsymbol{e}^1 \cdot \boldsymbol{e}_1 = 1$, find

$$k = \frac{1}{\underline{V}} = \frac{1}{\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)}\,,$$

where $k^{-1} = \underline{V}$ is the volume of the parallelepiped built on the vectors of the original basis. $\underline{V} > 0$ if the right-hand system of coordinates is selected. Therefore,

$$e^{\alpha} = \frac{e_{\beta} \times e_{\gamma}}{\underline{V}} , \tag{1.30}$$

where α, β, and γ form a cyclic permutation. Radius vector r and any other vectors are expanded in basic vectors in the usual way:

$$r = x_1 e^1 + x_2 e^2 + x_3 e^3 = x^1 e_1 + x^2 e_2 + x^3 e_3 . \tag{1.31}$$

Multiplying the first equality, in a scalar way, by e_{α}, we find

$$x_{\alpha} = e_{\alpha} \cdot r . \tag{1.32}$$

Therefore, the geometric meaning of the covariant components is revealed by projecting the radius vector, in the usual way, by lowering perpendiculars from the end of the vector onto the coordinate axes. When this has been done, the directions of the contravariant basic vectors, by which the covariant components of the vector are multiplied, do not coincide with the directions of the coordinate axes (Figure 1.3) and have no unit lengths. For instance, if vector e_3 is orthogonal to e_1 and e_2 and the angle between the latter is ϕ, then $|e^1| = |e^2| = 1/\sin\phi$ and the length of the hypotenuse $OB = |x_1 e^1| = x_1/\sin\phi > x_1$. However, the length of the leg $OC = x_1$. As follows from (1.31) and Figure 1.3, the contravariant components result from projecting the vector onto the coordinate axes with segments parallel to the axes. For them, a representation identical to (1.32) is valid:

$$x^{\alpha} = e^{\alpha} \cdot r . \tag{1.33}$$

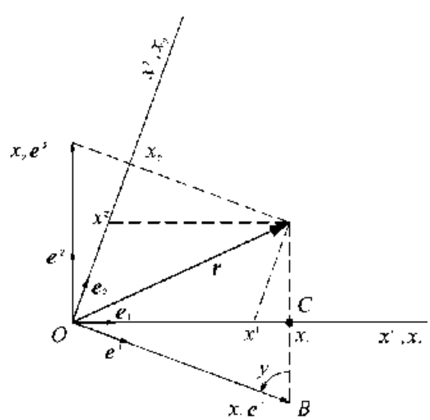

Figure 1.3 The clarification of the geometric meaning of the covariant and contravariant components of a vector.

□

Example 1.3

Determine the nine-component quantities:

$$g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta \ , \quad g^{\alpha\beta} = \boldsymbol{e}^\alpha \cdot \boldsymbol{e}^\beta \ , \tag{1.34}$$

where $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}^\beta$ are the basic vectors of the original and mutual nonorthogonal bases, introduced in Example 1.2. The values $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are called the covariant and contravariant components of a *metric tensor*.

Prove the following relations that connect the covariant and contravariant components of an arbitrary vector (the rules of rasing and lowering indices):

$$\text{(i)} \quad A_\alpha = g_{\alpha\beta} A^\beta \ ; \quad \text{(ii)} \quad A^\alpha = g^{\alpha\beta} A_\beta \ ; \quad \text{(iii)} \quad g_{\alpha\beta} g^{\beta\gamma} = g_\alpha^\gamma \equiv \delta_\alpha^\gamma \ . \tag{1.35}$$

Here, δ_α^γ is a Kronecker symbol.

Find the determinants of a covariant and a contravariant metric tensor and express them through the volumes $\underline{V}$ and $\overline{V}$ of parallelepipeds built on the vectors of the original and mutual bases.

Solution. The expression below follows from expansion (1.31):

$$\boldsymbol{A} = A_\beta \boldsymbol{e}^\beta = A^\beta \boldsymbol{e}_\beta \ .$$

Multiplying it, in a scalar way, by $\boldsymbol{e}_\alpha$ and using the definitions of mutual basis (1.30) and metric tensor (1.34), we get the first expression in (1.35); multiplying this expansion, in a scalar way, by $\boldsymbol{e}^\alpha$, we get the second expression in (1.35); and inserting the second expression in (1.35) in the first expression in (1.35), we get the third expression in (1.35).

If we label $g = |g_{\alpha\beta}|$ and use definition (1.34) and the formula from the first task in Problem 1.29, we find the following:

$$\begin{aligned} g &= \begin{vmatrix} \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 & \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 & \boldsymbol{e}_1 \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_1 & \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 & \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 & \boldsymbol{e}_3 \cdot \boldsymbol{e}_2 & \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 \end{vmatrix} \\ &= \begin{bmatrix} (\boldsymbol{e}_1 \cdot \boldsymbol{e}_1)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_3) + (\boldsymbol{e}_2 \cdot \boldsymbol{e}_1)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_3) \\ +(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_3) - (\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1) \\ -(\boldsymbol{e}_2 \cdot \boldsymbol{e}_3)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_1) - (\boldsymbol{e}_3 \cdot \boldsymbol{e}_3)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_1) \end{bmatrix} \\ &= e_{\alpha\beta\gamma} (\boldsymbol{e}_1)_\alpha (\boldsymbol{e}_2)_\beta (\boldsymbol{e}_3)_\gamma \, e_{\mu\nu\sigma} (\boldsymbol{e}_1)_\mu (\boldsymbol{e}_2)_\nu (\boldsymbol{e}_3)_\sigma \\ &= [\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)]^2 = \underline{V}^2 > 0 \ . \end{aligned}$$

In the same way, we get $|g^{\alpha\beta}| = \overline{V}^2$. As follows from (1.35), $|g^{\alpha\beta}| g = 1$; therefore, $|g^{\alpha\beta}| = g^{-1} = \underline{V}^{-2} > 0$ and $\overline{V} = \underline{V}^{-1}$. □

Problems

1.37. When we transition from one oblique rectilinear system of coordinates to another, the basic vectors $\boldsymbol{e}_\alpha$ determining the directions of the coordinate axes are transformed in accordance with the following law:

$$\boldsymbol{e}'_\alpha = a_\alpha{}^\beta \boldsymbol{e}_\beta \, , \tag{1.36}$$

where $a_\alpha{}^\beta$ is the transformation matrix.[9]

1. Express its elements through the scalar products of the basic vectors of the original and transformed systems.
2. Build the reverse transformation matrix.
3. Show that the same matrices define the transformations of the vectors of the mutual basis.
4. Find the rules of the transformation of the covariant and contravariant components of an arbitrary vector.
5. Find the rules of the transformation of the covariant and contravariant components of a metric tensor.

1.38. Show the laws of the transformation of the vectors of the original and mutual bases in the case of the mirror reflection of the system of coordinates.

1.39. Express the scalar product of two vectors in three different forms: through the covariant and contravariant components and through both of them. Prove its invariance with respect to the transformations (1.36) of the coordinate system. Express, in various forms, the square of the distance $\mathrm{d}l^2$ between two close points.

1.40. Write down the vector product of two vectors $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$ in terms the covariant and contravariant components of the factors.

1.41. Write down the cosine of the angle between vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in terms of their covariant and contravariant components.

1.1.4
Tensors in Curvilinear and Nonorthogonal Systems of Coordinates

We will now consider arbitrary transformations in the case of a transition from a Cartesian to a certain curvilinear and, generally speaking, nonorthogonal system of coordinates or between curvilinear and nonorthogonal systems of coordinates (Borisenko and Tarapov, 1966, Section 2.8). The connection between the coordinates x^α and x'^β $(\alpha, \beta = 1, 2, 3)$ of two coordinate systems described by certain general form relations is

$$x^\alpha = f^\alpha(x'^1, x'^2, x'^3) \tag{1.37}$$

9) The transformation in question is not necessarily limited to the rotation of the oblique system as a whole. It may change the angles between the axes and coordinates scales.

(we will now indicate coordinate numbers with upper indices). The linear homogeneous function $f^\alpha(x'^1, x'^2, x'^3)$ with constant coefficients corresponds to the affine transformation (1.36). The rotation of the orthogonal rectilinear coordinate system is determined by the orthogonal matrix of coefficients with a unit determinant.

So that (1.37) can be solved with respect to x'^β and the reverse transformation $x'^\beta = \varphi^\beta(x^1, x^2, x^3)$ can be found, the functional determinant J must be different from zero,

$$J = \left| \frac{\partial x^\alpha}{\partial x'^\beta} \right| \neq 0 , \tag{1.38}$$

which hereafter will be presumed. The differentials of the coordinates are transformed in accordance with

$$\mathrm{d}x^\alpha = \frac{\partial x^\alpha}{\partial x'^\beta} \mathrm{d}x'^\beta , \tag{1.39}$$

where the coefficients of the transformation $\partial x^\alpha / \partial x'^\beta$, in the general case, become the functions of the coordinates. The connection between the differentials remains linear, as in the case of affine transformations, which, generally speaking, is not the case for the connection between the coordinates themselves. Although (1.37) describes the transition from the orthogonal Cartesian system of coordinates x^α to an arbitrary system q^β (to make things clearer, we hereafter will label curvilinear coordinates as q), we will write the square of the distance between close points with the use of (1.39) as

$$\mathrm{d}l^2 = \delta_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta = g_{\mu\nu} \mathrm{d}q^\mu \mathrm{d}q^\nu , \tag{1.40}$$

where the values

$$g_{\mu\nu}(q) = \frac{\partial x^\alpha}{\partial q^\mu} \frac{\partial x^\beta}{\partial q^\nu} \delta_{\alpha\beta} , \quad g_{\mu\nu} = g_{\nu\mu} \tag{1.41}$$

are called the *covariant components of the metric tensor,* and its *contravariant components* $g^{\mu\nu} = g^{\nu\mu}$ are determined by the conditions

$$g^{\alpha\nu} g_{\nu\mu} = g_{\mu\nu} g^{\nu\alpha} = \delta^\alpha_\mu , \tag{1.42}$$

which means that the tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ are mutually inverse. Because the coefficients of transformation (1.39) satisfy the relation

$$\frac{\partial x^\alpha}{\partial q^\beta} \frac{\partial q^\beta}{\partial x^\nu} = \frac{\partial x^\alpha}{\partial x^\nu} = \delta^\alpha_\nu , \tag{1.43}$$

the contravariant components of the metric tensor may be written as[10]

$$g^{\alpha\beta} = \frac{\partial q^\alpha}{\partial x^\sigma} \frac{\partial q^\beta}{\partial x^\kappa} \delta^{\sigma\kappa} . \tag{1.44}$$

10) Tensors $\delta_{\mu\nu}$, $\delta^{\mu\nu}$, and δ^μ_ν correspond to the rectilinear Cartesian system of coordinates, their contravariant and covariant components coincide with each other, and the location of the symbols is indifferent.

The latter relations, just like (1.41), may be regarded as the rule of the transformation of the metric tensor from Cartesian coordinates ($\delta^{\sigma\kappa}$) to arbitrary curvilinear coordinates q^α. It is easy to see that the same rule applies to the transformation of the metric tensor from a curvilinear system q^α to another curvilinear system q'^β:

$$g'^{\kappa\sigma} = \frac{\partial q'^\kappa}{\partial x^\mu}\frac{\partial q'^\sigma}{\partial x^\nu}\delta^{\mu\nu} = \frac{\partial q'^\kappa}{\partial q^\alpha}\frac{\partial q'^\sigma}{\partial q^\beta}g^{\alpha\beta}\,, \tag{1.45}$$

where $g^{\alpha\beta}$ is defined in accordance with (1.44).

One can easily make sure that the relations written above mostly repeat the formulas obtained when considering the oblique-angled (affine) system of coordinates, being their generalizations, in a certain way. For instance, multiplying both parts of (1.39) by the Cartesian orts $\boldsymbol{e}_\alpha^{(D)}$ and relabeling x'^β as q^β, we get the increase of the radius vector

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{e}_\alpha^{(D)}\mathrm{d}x^\alpha = \frac{\partial x^\alpha}{\partial q^\beta}\boldsymbol{e}_\alpha^{(D)}\mathrm{d}q^\beta = \boldsymbol{e}_\beta \mathrm{d}q^\beta\,.$$

This means that the basic vectors $\boldsymbol{e}_\beta$ of the curvilinear system (not unit in the general case) may be written as

$$\boldsymbol{e}_\beta = \frac{\partial x^\alpha}{\partial q^\beta}\boldsymbol{e}_\alpha^{(D)}\,. \tag{1.46}$$

The right-hand side of the latter equality includes Cartesian orthogonal unit vectors. As follows from (1.46), the connection between the basic vectors of the curvilinear systems of coordinates q'^μ and q^β looks the same way as (1.46):

$$\boldsymbol{e}'_\beta = \frac{\partial q^\alpha}{\partial q'^\beta}\boldsymbol{e}_\alpha\,. \tag{1.47}$$

Further on, we will define the vectors of the mutual basis $\boldsymbol{e}^\beta$ of the curvilinear system. As follows from (1.46) and the conditions in (1.29),

$$\boldsymbol{e}^\alpha \cdot \boldsymbol{e}_\beta = \frac{\partial x^\mu}{\partial q^\beta}\boldsymbol{e}^\alpha \cdot \boldsymbol{e}^\mu_{(D)} = \delta^\alpha_\beta\,, \tag{1.48}$$

which means that

$$\boldsymbol{e}^\alpha = \frac{\partial q^\alpha}{\partial x^\nu}\boldsymbol{e}^\nu_{(D)} \tag{1.49}$$

(we use the equality of the lower and upper symbols for Cartesian vectors). Finally, considering (1.41) and (1.44), we see that the relations in (1.34) remain valid for curvilinear coordinates,

$$g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta\,, \quad g^{\alpha\beta} = \boldsymbol{e}^\alpha \cdot \boldsymbol{e}^\beta\,, \quad g^\beta_\alpha = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}^\beta = \delta^\beta_\alpha\,, \tag{1.50}$$

as do the rules of raising and lowering indices (1.35).

We now will give a definition of tensor, as it relates to the general transformations of coordinates.

A *tensor of rank 2* in the three-dimensional space is a nine-component quantity whose contravariant components are transformed as products of the differentials of coordinates, that is, in accordance with the following:

$$T^{\alpha\beta} = \frac{\partial q^\alpha}{\partial q'^\mu}\frac{\partial q^\beta}{\partial q'^\nu}T'^{\mu\nu} \quad \text{or} \quad T'^{\mu\nu} = \frac{\partial q'^\mu}{\partial q^\alpha}\frac{\partial q'^\nu}{\partial q^\beta}T^{\alpha\beta} \,. \tag{1.51}$$

This definition is directly generalized to include tensors of any rank. For instance, scalar S is not transformed, whereas the covariant components of a tensor of rank 1 (vector) are transformed in accordance with

$$A_\alpha = \frac{\partial q'^\beta}{\partial q^\alpha}A_\beta \,. \tag{1.52}$$

The fundamental difference between the above definition of a tensor and the previous ones (for the cases of rotation and affine transformation) is that now the transformation coefficients depend on the locations. This means that the definition of a tensor is of a local nature. For instance, the products of the components of vectors located at different points $q^\alpha \neq p^\alpha$, that is, $A^\alpha(q)B^\beta(p)$, do not form a tensor.

Unlike Cartesian coordinates, the totality of arbitrary curvilinear coordinates q^α, $\alpha = 1, 2, 3$, does not form a vector because the coordinates do not comply with rule of transformation (1.51). Most significantly, these peculiarities manifest themselves in differentiating and integrating tensor operations, which are considered in Section 1.2.

The covariant components of a tensor of any rank are produced from the contravariant ones by the metric tensor as per (1.35). In the general case, the mixed tensor depends on the place, first or second, occupied by the upper and lower symbols, that is, $T_\alpha{}^\beta \neq T^\beta{}_\alpha$. The contraction operation, decreasing the rank of any tensor by 2, is defined as summation over one upper and one lower indices, for instance,

$$A_\alpha B^\alpha = A'_\beta B'^\beta = \text{inv} \,, \quad T_{\alpha\beta}{}^\beta = C_\alpha \tag{1.53}$$

– the covariant vector, and so on.

Problems

1.42. Express the components of a metric tensor through the components of the orthogonal Cartesian orts $\boldsymbol{e}_\alpha^{(D)} = \boldsymbol{e}^\alpha_{(D)}$, $\alpha = 1, 2, 3$ specified in a certain curvilinear system of coordinates.

1.43. Show that the functional determinant (1.39) is expressed through the determinant of a metric tensor $g = |g_{\mu\nu}| : \; J = \sqrt{g}$.

Hint: Following from equality (1.42), express the determinant g through the determinants of the matrices found in the second member of the equality.

1.44. Write down the square of the length of the vector A^2 and the cosine of the angle between two vectors in a arbitrary curvilinear system of coordinates.

1.45. Transform the antisymmetric unit tensor $e^{\alpha\beta\gamma}$ in an curvilinear system of coordinates.

1.46. The metric tensor $g_{\alpha\beta}$ determining the square of the small element of length in curvilinear nonorthogonal coordinates, in accordance with formulas (1.41), is known. Three curvilinear coordinate lines may be drawn through each point of the space, only one coordinate q^1, q^2, or q^3 changing along each of these lines, whereas the other two remain constant.

1. Find the connection between the element of length of a coordinate line and the differential of the respective coordinate.
2. Indicate the three basic vectors tangent to the coordinate curves at the specified point.
3. Find the cosines of the angles between the coordinate curves at that point.
4. Indicate the properties the metric tensor must have to make the curvilinear system orthogonal.

1.47. Write down the covariant and contravariant components of a metric tensor for a spherical and a cylindrical system of coordinates (see the drawing in the solution of Problem 1.18). Also, write down the vectors of the covariant and contravariant bases, expressing them through the basic orts considered in Problem 1.18.

1.48. Show that the volume element in curvilinear coordinates has the following form:

$$\mathrm{d}V = \sqrt{g}\,\mathrm{d}q^1\mathrm{d}q^2\mathrm{d}q^3\,, \tag{1.54}$$

where g is the determinant of a metric tensor. Find the volume element in spherical and cylindrical coordinates.

Hint: The volume element sought is the volume of an oblique-angled parallelepiped built on the elementary lengths $\mathrm{d}l^1$, $\mathrm{d}l^2$, and $\mathrm{d}l^3$ of the curvilinear coordinate axes. It may be found with the use of the results obtained in Problems 1.40 and 1.46.

Recommended literature:
Borisenko and Tarapov (1966); Arfken (1970); Rashevskii (1953); Lee (1965); Mathews and Walker (1964). See also Ugarov (1997, Addendum I).

1.2 Vector and Tensor Calculus

Scalar or vector functions representing the distribution of various physical quantities in three-dimensional space are sometimes called the fields of those quantities. This is how one may speak of fields of temperatures $T(x, y, z)$ or pressures $p(x, y, z)$ in the atmosphere, the fields of speeds in moving fluids or gases $\boldsymbol{u}(x, y, z)$, the electromagnetic vector field, and so on. Derivatives and integrals from such scalar and vector functions have certain common mathematical properties, which are very important for physical applications. One should become familiar and comfortable with these properties in advance. Only then, may such areas of physics as the theory of electromagnetic phenomena, the mechanics of fluids, gases, and solid bodies, quantum physics, and quantum field theory be successfully learned and fully understood.

1.2.1 Gradient and Directional Derivative. Vector Lines

We encounter the concept of the gradient of a scalar function in classical mechanics when learning about the properties of potential forces. Let us say there is a differentiable function $U(x, y, z)$ whose partial derivatives are equal to the components of the vector of the force $\boldsymbol{F}(x, y, z)$, which, in this case, is called a *potential*:

$$F_x = -\frac{\partial U}{\partial x}\,, \quad F_y = -\frac{\partial U}{\partial y}\,, \quad F_z = -\frac{\partial U}{\partial z}\,, \quad \text{or} \quad \boldsymbol{F} = -\nabla U(x, y, z)\,, \tag{1.55}$$

where

$$\nabla = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z} = \boldsymbol{e}_\alpha \frac{\partial}{\partial x_\alpha} \tag{1.56}$$

is *Hamilton's operator*[11] (nabla).

$$\text{grad}\, U(x, y, z) \equiv \nabla U(x, y, z) = \boldsymbol{e}_x \frac{\partial U}{\partial x} + \boldsymbol{e}_y \frac{\partial U}{\partial x} + \boldsymbol{e}_z \frac{\partial U}{\partial z} \tag{1.57}$$

is called the *gradient* of the scalar function $U(x, y, z)$. The necessary and sufficient conditions for the representation of the vector as a scalar function come in the form of equalities:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}\,, \quad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}\,, \quad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}\,. \tag{1.58}$$

They follow from the equality of cross-derivatives, for example,

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x}\,.$$

So far, we have been using only Cartesian coordinates. A generalization to include oblique nonorthogonal coordinates will be made in the closing part of this section (also see Problem 1.50 and later).

11) William Rowan Hamilton (1805–1865) was an outstanding Irish mathematician and physicist. He was engaged in mechanics and optics, and created the mathematical apparatus that, after many decades, became the basis of quantum mechanics and quantum field theory.

It is important to understand that a gradient is always directed toward increasing U, along a normal to the surface of the constant value of the scalar field $U(x, y, z) = \text{const}$. This follows from our obtaining, when differentiating the latter equality, $\mathrm{d}\boldsymbol{r} \cdot \nabla U = 0$. Since $\mathrm{d}\boldsymbol{r}$ is here a tangent to the surface $U = \text{const}$, the gradient is perpendicular to that surface.

Example 1.4

Show that the derivative of the scalar function, along the direction determined by the unit vector $\boldsymbol{l}$, is equal to the projection of the gradient onto that direction:

$$\frac{\partial U}{\partial l} = \text{grad}_l U \equiv (\boldsymbol{l} \cdot \nabla) U . \tag{1.59}$$

Solution. Label the derivative, along the specified direction $\boldsymbol{l}$, as $\partial U/\partial l$. When displaced from the point with radius vector $\boldsymbol{r}$ to a distance s along the direction $\boldsymbol{l}$, the function will take the value of $U(x + l_x s, y + l_y s, z + l_z s)$. The derivative in the specified direction is the derivative at distance s:

$$\begin{aligned}\frac{\partial U}{\partial l} &= \frac{\partial}{\partial s} U(x + l_x s, y + l_y s, z + l_z s)|_{s=0} = \frac{\partial U}{\partial x} l_x + \frac{\partial U}{\partial y} l_y + \frac{\partial U}{\partial z} l_z \\ &= (\boldsymbol{l} \cdot \nabla) U(\boldsymbol{r}) .\end{aligned}$$

□

Expression (1.58) also makes sense when applied to an arbitrary vector $\boldsymbol{A}(x, y, z)$: the quantity $(\boldsymbol{l} \cdot \nabla)\boldsymbol{A}(x, y, z)$ is a *derivative of vector* $\boldsymbol{A}$ *in direction* $\boldsymbol{l}$. This follows from the condition that the operator $(\boldsymbol{l} \cdot \nabla)$ must be applied to every projection of $\boldsymbol{A}$ and will produce the required derivatives, whereas their combination must be construed as a derivative of the whole vector in the specified direction.

A vivid conception of the structure of the vector field $\boldsymbol{A}$ is provided by *vector lines*.[12] These are lines tangents to which, at any point, indicate the direction of vector $\boldsymbol{A}$ at that point. It is easy to write a system of equations in order to find the vector lines of the specified field $\boldsymbol{A}(x, y, z)$. The condition of the small element $\mathrm{d}\boldsymbol{l} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ being parallel to the vector line and vector $\boldsymbol{A}$ may be written as $\boldsymbol{A} \times \mathrm{d}\boldsymbol{l} = 0$. Having written this vector equality in projections on the respective axes, we get differential equations for two families of surfaces whose intersect lines are exactly the vector lines sought.

For instance, using Cartesian coordinates, we will have

$$\frac{\mathrm{d}x}{A_x(x, y, z)} = \frac{\mathrm{d}y}{A_y(x, y, z)} = \frac{\mathrm{d}z}{A_z(x, y, z)} . \tag{1.60}$$

12) If $\boldsymbol{A}$ is a vector of a force, the lines are called force lines. Sometimes, the term “force lines” is applied to any vector regardless of its physical meaning.

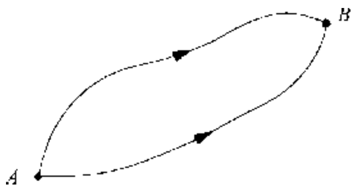

Figure 1.4 The independence of work done by a potential force from the shape of the path of a material point.

The vector lines of any potential vector are perpendicular to the equipotential surfaces $U(x, y, z) = \text{const}$. This follows from the properties of the gradient of a scalar function.

The loop integral of the scalar product of a potential vector and the vector element of the length of the loop has an important property:

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_A^B (F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z) , \tag{1.61}$$

where the vector $\mathrm{d}\boldsymbol{s}$ has constituents $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$, that is, the differentials of the coordinates are not independent and are just increments *along the loop*. Such integrals express work done by the force $\boldsymbol{F}$ on a material point moving along a specified trajectory from A to B and many other physical quantities. If the vector is a potential vector, then

$$F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z = -\frac{\partial U}{\partial x}\mathrm{d}x - \frac{\partial U}{\partial y}\mathrm{d}y - \frac{\partial U}{\partial z}\mathrm{d}z = -\mathrm{d}U \tag{1.62}$$

is the complete differential of the function $U(x, y, z)$. The computation of the integral gives us

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = -\int_A^B \mathrm{d}U = U_A - U_B , \tag{1.63}$$

where $\mathrm{d}U$ is the increase of the function along the small segment $\mathrm{d}\boldsymbol{s}$ and $\int_A^B \mathrm{d}U$ is the full increase along the distance AB.

In this case, integration along the loop does not depend on the form of the curve, and only depends on the start and end points of the integration (Figure 1.4).

Integrating along a closed loop (Figure 1.5), we get the following:

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = U_A - U_B , \quad \int_B^A \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = U_B - U_A ,$$

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} + \int_B^A \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = 0 . \tag{1.64}$$

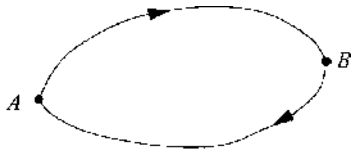

Figure 1.5 Diagram for the computation of the circulation of a vector along a closed loop.

Closed-loop integration over $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$ is called *the circulation of vector* $\boldsymbol{F}$ along the loop. The circulation of a *potential vector* along any closed loop equals zero (however, an arbitrary vector has no such property!).

It is important, however, to note that the condition of the representation of a vector as (1.55) is necessary but not sufficient for equalities (1.63) and (1.64) to be valid. It is also necessary for the potential function $U(\boldsymbol{r})$ to be the unambiguous function of a point. Otherwise, for instance, after the circulation of the loop and return to point A, the potential U may take a different value, and equality (1.64) will be no longer valid.

Problems

1.49. Show that when a Cartesian system of coordinates is rotated, Hamilton's operator (∇) (1.56) is transformed in accordance with rule (1.2) of vector transformation.

1.50. Find the potential energy that corresponds to the force $F_x(x, y) = x + y$, $F_y(x, y) = x - y^2$. Find the work R done by this force between points (0,0) and (a, b).

1.51. Show that in cylindrical and spherical systems of coordinates , Hamilton's operator ∇ is expressed, respectively, as

1. $$\nabla = \boldsymbol{e}_\rho \frac{\partial}{\partial \rho} + \boldsymbol{e}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \boldsymbol{e}_z \frac{\partial}{\partial z} , \tag{1.65}$$

2. $$\nabla = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\vartheta \frac{1}{r} \frac{\partial}{\partial \vartheta} + \boldsymbol{e}_\varphi \frac{1}{r \sin\vartheta} \frac{\partial}{\partial \varphi} . \tag{1.66}$$

For that purpose, consider the elementary lengths in the directions of the respective coordinate orts and use formula (1.59), which connects the gradient with the directional derivative.

1.52. Use Cartesian spherical and cylindrical coordinates (see (1.56), (1.65), and (1.66)) to find $\mathrm{grad}\,(\boldsymbol{l} \cdot \boldsymbol{r})$, $(\boldsymbol{l} \cdot \nabla)\boldsymbol{r}$, where $\boldsymbol{r}$ is a radius vector and $\boldsymbol{l}$ is a constant vector.

1.53. Show that

$$\mathrm{grad}\, f(r) = \frac{\mathrm{d}f}{\mathrm{d}r} \frac{\boldsymbol{r}}{r} .$$

1.54. Write down a system of equations determining the vector lines in cylindrical and spherical coordinates, respectively.

1.55. Find

$$\operatorname{grad}\frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3}\,, \quad \boldsymbol{p} = \text{const}\,.$$

1.56. Use spherical coordinates to draw a family of lines tangent to vector

$$\boldsymbol{E} = \frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{p}}{r^3}\,, \quad \boldsymbol{p} = \text{const}\,.$$

1.57. Write down the cyclic components of a gradient in spherical coordinates. Find the definition of the cyclic components in the situation in Problem 1.17.

1.2.2
Divergence and Curl. Integral Theorems

Now, we will consider the effect of the ∇ operator on an arbitrary vector $\boldsymbol{A}$. As is known, two vectors may produce two types of products: a scalar

$$\operatorname{div}\boldsymbol{A} \equiv \nabla\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial A_\alpha}{\partial x_\alpha} \tag{1.67}$$

and a vector

$$\operatorname{curl}\boldsymbol{A} \equiv \nabla\times\boldsymbol{A} = \boldsymbol{e}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \boldsymbol{e}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \boldsymbol{e}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\,. \tag{1.68}$$

Both of these quantities are extremely important for vector calculus and are called the *divergence* (scalar!) and the *curl* (vector!). The left-hand-side members of the equalities contain the respective lettering. The right-hand-side members contain their explicit expressions *in Cartesian coordinates only*. For you to better realize their mathematical and physical meanings, we give other definitions of these important quantities, less formal and more obvious, if somewhat more complex. Yet the latter disadvantage is also an advantage in that the definitions in questions, unlike (1.67) and (1.68), do not depend on the selection of a system of coordinates. We will begin with divergence.

Select point M where you would like to define the divergence of vector field $\boldsymbol{A}(\boldsymbol{r})$. Surround that point with a closed smooth surface, enclosing a certain volume ΔV and find at every point of the surface an outside normal $\boldsymbol{n}$. We will call the product $\boldsymbol{n}\,\mathrm{d}S$ the vector element of the surface. The integral over the closed surface $\oint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$ produces the flux of the vector $\boldsymbol{A}$ through the surface S. Now, we will define divergence in a way different from (1.67):

$$\operatorname{div}\boldsymbol{A}(\boldsymbol{r}) = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\oint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}\,. \tag{1.69}$$

It is presumed here that the volume ΔV shrinks into point M. The little circle on the integral sign means a closed surface.

Example 1.5

Make sure that the definitions (1.67) and (1.69) are equivalent when Cartesian coordinates are used. In order to do that, select volume $\Delta V = \mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ forming a small rectangular parallelepiped with edges $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ and find the boundary (1.69).

Solution. Making use of the smallness of the ribs of the parallelepiped, write down the approximate expression for the surface integral:

$$\begin{aligned}\oint_S \boldsymbol{A} \cdot d\boldsymbol{S} &\approx [A_x(x + dx, \bar{y}, \bar{z}) - A_x(x, \bar{y}, \bar{z})]dydz \\ &\quad + [A_y(\bar{x}, y + dy, \bar{z}) - A_y(\bar{x}, y, \bar{z})]dxdz \\ &\quad + [A_z(\bar{x}, \bar{y}, z + dz) - A_z(\bar{x}, \bar{y}, z)]dxdy \\ &\approx \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dV\,.\end{aligned}$$

The mean value theorem was used when evaluating the integrals over the six separate edges, the quantities $\bar{x}$, $\bar{y}$, and $\bar{z}$ being the values of the coordinates at a certain point of a respective edge. Also considered was the fact that normals are directed oppositely at the opposite edges and that when the volume shrinks to point M, all the coordinates take the values they must have at that point. Using the latter result, make sure that the definition of divergence (1.69), when Cartesian coordinates are used, leads to formula (1.67). □

This means that the divergence at a certain point is other than zero if there is a nonzero vector flux through a closed surface surrounding the point in question. Inside the surface, there must be a source of a vector field that creates the flux. This is to say that divergence characterizes the density of field sources.

The above method of computing an integral over a small surface may be used to obtain explicit expressions of divergence in the most often used systems of coordinates, such as spherical, cylindrical, and so on. The shape of the volume should be selected each time so that one of the coordinates remains constant on each of its side surfaces.

Example 1.6

On the basis of the definition of divergence (1.67), produce a relation connecting the integral from div $\boldsymbol{A}$ over a certain volume with vector flux $\boldsymbol{A}$ through the surface bounding the volume in question.

Solution. Select any finite volume V bounded by a smooth closed surface S. Divide it into small cells ΔV_i, each bounded by a respective surface ΔS_i. The surfaces bounding the cells adjacent to the outside surface S will partially coincide with S. All other portions of the surfaces S_i will be shared by pairs of adjacent cells. Making use of the smallness of each cell, use relation (1.69), giving it an approximate form:

$$(\operatorname{div} \boldsymbol{A})_i \Delta V_i \approx \oint_{S_i} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S}_i \, . \tag{1.70}$$

Now sum the first and second members of the latter approximate equality over i and pass to a limit, reducing the volume of each cell to zero and expanding the number of cells to infinity. The first member of the equality will now become an integral over the full volume V of divergence $\boldsymbol{A}$: $\int_V \operatorname{div} \boldsymbol{A} \mathrm{d}V$. In the second member of the equality, the integrals over the inner portions of the surface will cancel each other, the outer normals to each pair of adjacent cells being oppositely directed. Only the integral over the outside surface S bounding the full volume V remains. As a result, you will have an exact integral relation,

$$\int_V \operatorname{div} \boldsymbol{A} \mathrm{d}V = \oint_S \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} \, , \tag{1.71}$$

called the *Gauss–Ostrogradskii theorem*[13] (in Western literature, the name Ostrogradskii is omitted).

The Gauss–Ostrogradskii theorem is applicable to any tensor of rank $s \geq 1$, for instance,

$$\int_V \frac{\partial T_{\alpha\beta\mu}}{\partial x_\mu} \mathrm{d}V = \oint_S T_{\alpha\beta\mu} \mathrm{d}S_\mu \tag{1.72}$$

(for the proof, refer to Problem 1.70*). □

The curl of a vector field allows a definition similar to that of divergence (1.69). At point M, specify a unit vector $\boldsymbol{n}$, that is, a direction. Make up a small flat area ΔS containing a point M and perpendicular to $\boldsymbol{n}$. Then define the direction of tracing the loop l that bounds the area, coordinated with the direction $\boldsymbol{n}$ as per the right-screw rule. The projection of the rotor onto direction $\boldsymbol{n}$ at point M is defined as follows:

$$\operatorname{curl}_n \boldsymbol{A} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_l \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \, , \tag{1.73}$$

where the integral represents the circulation of the vector $\boldsymbol{A}$ along the closed loop l.

13) Carl Friedrich Gauss (1777–1855) was an outstanding German mathematician, astronomer, and physicist. Mikhail Ostrogradskii (1801–1862) was a Russian mathematician known for his works in mathematical physics, theoretical mechanics, and probability theory.

Example 1.7

Make sure that the definitions of (1.68) and (1.73) are equivalent when Cartesian coordinates are used. For that purpose, find the projections of the curl on Cartesian axes using (1.73) and by selecting a rectangular area with sides parallel to the coordinate axes.

Solution. Direct $\boldsymbol{n}$ along the Oz axis, select a rectangular area $\Delta S = \mathrm{d}S = \mathrm{d}x\mathrm{d}y$, and use, as in the previous integral calculation, the mean value theorem to get the following:

$$\oint_l \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l} \approx [A_y(x+\mathrm{d}x,\bar{y},z) - A_y(x,\bar{y},z)]\mathrm{d}y$$
$$+ [A_x(\bar{x},y,z) - A_y(\bar{x},y+\mathrm{d}y,z)]\mathrm{d}x$$
$$\approx \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathrm{d}S \ .$$

After inserting this result into (1.73) and passing to a limit, we get the exact expression for $\mathrm{curl}_z \boldsymbol{A}$ in Cartesian coordinates, coinciding with (1.68). In the same way, one may find other projections of the curl. The curl will be other than zero if the lines of vector $\boldsymbol{A}$ curved, having either closed or spiral configurations. □

Example 1.8

Using the definition of the curl (1.73), find the integral relation that connects the circulation of any vector along a closed loop with the curl flux of that vector through a nonclosed surface bounded by that loop.

Solution. Find an arbitrary three-dimensional nonclosed surface S bounded by loop l and, at every point of the surface, find normal $\boldsymbol{n}$. Divide the surface into small portions ΔS_i, each bounded by loop l_i. On the basis of (1.73), an approximate value may be written for every such area:

$$\mathrm{curl}_n \boldsymbol{A}\Delta S_i \approx \oint_{l_i} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}_i \ . \tag{1.74}$$

After summing the two members of the approximate equality over i and passing to a limit of the infinitely small areas, we get the exact equality (*Stokes theorem*[14]):

$$\int_S \mathrm{curl}\,\boldsymbol{A}\cdot \mathrm{d}\boldsymbol{S} = \oint_l \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l} \ . \tag{1.75}$$

14) George Gabriel Stokes (1819–1903) was an Irish physicist and mathematician.

An integral over the outer loop that bounds area S remains in the second member. All integrals over inner loops are canceled. Stokes theorem connects the integral over the curl flux through the surface with the circulation of the vector along the loop that bounds that surface. □

1.2.3
Solenoidal and Potential (Curl-less) Vectors

Let us say that vector field $\boldsymbol{H}(\boldsymbol{r})$, over the whole space, satisfies the condition

$$\operatorname{div} \boldsymbol{H} = 0 \tag{1.76}$$

(in this case, vector $\boldsymbol{H}$ is called a *solenoidal* vector). This, for instance, is a property of a magnetic field. It is possible to prove (we will, for now, abstain from doing that) that condition (1.76) is necessary and sufficient for vector $\boldsymbol{H}$ to be represented as the curl of another vector $\boldsymbol{A}(\boldsymbol{r})$:

$$\boldsymbol{H} = \operatorname{curl} \boldsymbol{A} . \tag{1.77}$$

Using the rules of vector differentiation, we can easily make sure that condition (1.76) is satisfied whatever the value of $\boldsymbol{A}$ is:

$$\operatorname{div} \boldsymbol{H} = \nabla \cdot \boldsymbol{H} = \nabla \cdot [\nabla \times \boldsymbol{A}] = [\nabla \times \nabla] \cdot \boldsymbol{A} = 0 .$$

As noted previously, a *potential vector* is a vector that may be represented as the gradient of a certain scalar function:

$$\boldsymbol{E}(\boldsymbol{r}) = -\operatorname{grad} U(\boldsymbol{r}) \equiv -\nabla U(\boldsymbol{r}) . \tag{1.78}$$

The necessary and sufficient conditions of the potentiality of a vector are expressed by equalities of the kind in (1.58), which, in their vector form, give the following:

$$\operatorname{curl} \boldsymbol{E} = 0 . \tag{1.79}$$

Using the definition of the potential vector (1.78) and expressing the curl operation through the ∇ operator, we make sure that equality (1.79) is equally valid for any $U(\boldsymbol{r})$ functions that have second derivatives.

1.2.4
Differential Operations of Second Order

Differential operations of second order appear when the ∇ operator is applied to expressions of the kind ∇U, $\nabla \cdot \boldsymbol{A}$, and $\nabla \times \boldsymbol{A}$ that already contain this operator. Using the rules of vector algebra, we find that, in Cartesian coordinates, the *Laplace* *operator*[15)]

$$\nabla \cdot \nabla U(\boldsymbol{r}) = (\nabla \cdot \nabla)\, U(\boldsymbol{r}) = \nabla^2 U(\boldsymbol{r}) = \Delta\, U(\boldsymbol{r}) \,, \tag{1.80}$$

$\Delta = \nabla^2$, has the following form:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \,. \tag{1.81}$$

This is a very important operator used in just about all problems when complex physical phenomena have to be described in the language of mathematics.

Further,

$$\nabla\nabla \cdot \boldsymbol{A} \equiv \nabla(\nabla \cdot \boldsymbol{A}) = \operatorname{grad}\operatorname{div} \boldsymbol{A} \,. \tag{1.82}$$

Even though such a combination of derivatives is hardly rare, no more compact letter notation has been devised for it.

The last operation of this kind is called a double vortex. It is transformed with the use of the following vector algebra formula (one should remember to place the differentiable vector function to the right of any operators that may affect it):

$$\operatorname{curl}\operatorname{curl} \boldsymbol{A} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \operatorname{grad}\operatorname{div} \boldsymbol{A} - \Delta \boldsymbol{A} \,. \tag{1.83}$$

We see, therefore, that all the differential operations involving scalar and vector functions are expressed through the ∇ operator.

Problems

1.58. Show that div $\boldsymbol{A}$ (1.67) and the Laplace operator (1.81) are invariant with respect to rotations of Cartesian systems of coordinates and that curl $\boldsymbol{A}$ (1.68) is transformed as an antisymmetric tensor of rank 2 or as a vector that is dual to it.

1.59. Find $\nabla \cdot \boldsymbol{r}$, $\nabla \times \boldsymbol{r}$, $\nabla \cdot [\boldsymbol{\omega} \times \boldsymbol{r}]$, and $\nabla \times [\boldsymbol{\omega} \times \boldsymbol{r}]$, where $\boldsymbol{\omega}$ is a constant vector.

1.60. Find

$$\boldsymbol{H} = \operatorname{curl} \frac{(\boldsymbol{m} \times \boldsymbol{r})}{r^3} \,, \quad \boldsymbol{m} = \text{const} \,.$$

Build vector lines for vector $\boldsymbol{H}$ (draw a picture).

1.61. Using the rules of vector algebra and calculus and without making projections onto the coordinate axes, prove the following important identities frequently used in practical calculations:

$$\operatorname{grad}(\varphi\psi) = \varphi \operatorname{grad} \psi + \psi \operatorname{grad} \varphi \,, \tag{1.84}$$

15) Pierre Simon Laplace (1749–1827) was a French astronomer, mathematician, and physicist who actively expressed the ideas of mechanistic determinism; he was an atheist. His many scientific achievements were outstanding. Laplace repeatedly changed his politics, remaining in favor in republican France as well as in France under the rule of Napoleon Bonaparte and the restored Bourbons.

$$\operatorname{div}(\varphi \boldsymbol{A}) = \varphi \operatorname{div} \boldsymbol{A} + \boldsymbol{A} \cdot \operatorname{grad} \varphi \,, \tag{1.85}$$

$$\operatorname{curl}(\varphi \boldsymbol{A}) = \varphi \operatorname{curl} \boldsymbol{A} - \boldsymbol{A} \times \operatorname{grad} \varphi \,, \tag{1.86}$$

$$\operatorname{div}(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \operatorname{curl} \boldsymbol{A} - \boldsymbol{A} \cdot \operatorname{curl} \boldsymbol{B} \,, \tag{1.87}$$

$$\operatorname{curl}(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A} \operatorname{div} \boldsymbol{B} - \boldsymbol{B} \operatorname{div} \boldsymbol{A} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} \,, \tag{1.88}$$

$$\operatorname{grad}(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times \operatorname{curl} \boldsymbol{B} + \boldsymbol{B} \times \operatorname{curl} \boldsymbol{A} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} \,. \tag{1.89}$$

Here, φ and ψ are the scalar and $\boldsymbol{A}$, $\boldsymbol{B}$ vector functions of the coordinates.

1.62. Prove the following identities:

$$\boldsymbol{C} \cdot \operatorname{grad}(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \cdot (\boldsymbol{C} \cdot \nabla)\boldsymbol{B} + \boldsymbol{B} \cdot (\boldsymbol{C} \cdot \nabla)\boldsymbol{A} \,, \tag{1.90}$$

$$(\boldsymbol{C} \cdot \nabla)(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A} \times (\boldsymbol{C} \cdot \nabla)\boldsymbol{B} - \boldsymbol{B} \times (\boldsymbol{C} \cdot \boldsymbol{A} \,, \tag{1.91}$$

$$(\nabla \cdot \boldsymbol{A})\boldsymbol{B} = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{B} \operatorname{div} \boldsymbol{A} \,, \tag{1.92}$$

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot \operatorname{curl} \boldsymbol{C} = \boldsymbol{B} \cdot (\boldsymbol{A} \cdot \nabla)\boldsymbol{C} - \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \nabla)\boldsymbol{C} \,, \tag{1.93}$$

$$(\boldsymbol{A} \times \nabla) \times \boldsymbol{B} = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A} \times \operatorname{curl} \boldsymbol{B} - \boldsymbol{A} \operatorname{div} \boldsymbol{B} \,, \tag{1.94}$$

$$(\nabla \times \boldsymbol{A}) \times \boldsymbol{B} = -\boldsymbol{A} \operatorname{div} \boldsymbol{B} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A} \times \operatorname{curl} \boldsymbol{B} + \boldsymbol{B} \times \operatorname{curl} \boldsymbol{A} \,. \tag{1.95}$$

1.63. Find $\operatorname{grad} \varphi(r)$, $\operatorname{div} \varphi(r)\boldsymbol{r}$, $\operatorname{curl} \varphi(r)\boldsymbol{r}$, and $(\boldsymbol{l} \cdot \nabla)\varphi(r)\boldsymbol{r}$.

1.64. Find a function $\varphi(r)$ that satisfies the condition $\operatorname{div} \varphi(r)\boldsymbol{r} = 0$.

1.65. Find the divergences and curls of the following vectors:

$$(\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{b}, (\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{r} \,, \quad \varphi(r)(\boldsymbol{a} \times \boldsymbol{r}) \,, \quad \boldsymbol{r} \times (\boldsymbol{a} \times \boldsymbol{r}) \,,$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors.

1.66. Find $\operatorname{grad} \boldsymbol{r} \cdot \boldsymbol{A}(r)$, $\operatorname{grad} \boldsymbol{A}(r) \cdot \boldsymbol{B}(r)$, $\operatorname{div} \varphi(r)\boldsymbol{A}(r)$, $\operatorname{curl} \varphi(r)\boldsymbol{A}(r)$, and $(\boldsymbol{l} \cdot \nabla)\varphi(r)\boldsymbol{A}(r)$.

1.67. Prove that

$$(\boldsymbol{A} \cdot \nabla)\boldsymbol{A} = -\boldsymbol{A} \times \operatorname{curl} \boldsymbol{A} \quad \text{if} \quad \boldsymbol{A}^2 = \text{const} \,.$$

1.68. Transform the integral over volume $\int_V (\operatorname{grad} \varphi \cdot \operatorname{curl} \boldsymbol{A}) \mathrm{d}V$ into the integral over the surface.

1.69. Express the integrals over the closed surface $\oint_S \boldsymbol{r}(\boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S})$ and $\oint_S (\boldsymbol{a} \cdot \boldsymbol{r}) \mathrm{d}\boldsymbol{S}$ in terms of the volume bounded by that surface. Here $\boldsymbol{a}$ is a constant vector.

Hint: Multiply each integral by the arbitrary constant vector $\boldsymbol{b}$ and use the Gauss–Ostrogradskii theorem

1.70*. Transform the integrals over a closed surface

$$\oint \boldsymbol{n}\varphi \mathrm{d}S, \oint (\boldsymbol{n}\times \boldsymbol{A})\mathrm{d}S\ , \quad \oint (\boldsymbol{n}\cdot \boldsymbol{b})\boldsymbol{A}\mathrm{d}S\ , \quad \oint T_{\alpha\beta}(\boldsymbol{r})\boldsymbol{n}_\beta \mathrm{d}S$$

into integrals over the volume bounded by that surface. Here $\boldsymbol{b}$ is a constant vector and $\boldsymbol{n}$ is the ort of the normal.

1.71. Using one of the identities proven in the previous problem, formulate the Archimedean law by summing pressures applied to the elements of the surface of a submerged body.

1.72*. Prove the identity

$$\int_V (\boldsymbol{A}\cdot \mathrm{curl\,curl}\,\boldsymbol{B} - \boldsymbol{B}\cdot \mathrm{curl\,curl}\,\boldsymbol{A})\mathrm{d}V = \oint_S (\boldsymbol{B}\times \mathrm{curl}\,\boldsymbol{A} - \boldsymbol{A}\times \mathrm{curl}\,\boldsymbol{B})\cdot \mathrm{d}\boldsymbol{S}\ . \tag{1.96}$$

1.73. Inside volume V, vector $\boldsymbol{A}$ satisfies the condition div $\boldsymbol{A} = 0$ and at the boundary of the volume (surface S) the condition $A_n = 0$. Prove that $\int_V \boldsymbol{A}\mathrm{d}V = 0$.

1.74*. Prove that

$$\mathrm{div}_{\boldsymbol{r}} \int_V \frac{\boldsymbol{A}(\boldsymbol{r}')\mathrm{d}V'}{|\boldsymbol{r}-\boldsymbol{r}'|} = 0\ ,$$

where $\boldsymbol{A}(\boldsymbol{r})$ is the vector defined in the previous problem.

1.75. Prove the Green's identities[16)]

$$\int_V (\varphi\Delta\psi + \nabla\varphi\nabla\psi)\mathrm{d}V = \oint_S \varphi\nabla\psi\cdot \mathrm{d}\boldsymbol{S}\ , \tag{1.97}$$

$$\int_V (\varphi\Delta\psi - \psi\Delta\varphi)\mathrm{d}V = \oint_S (\varphi\nabla\psi - \psi\nabla\varphi)\cdot \mathrm{d}\boldsymbol{S}\ , \tag{1.98}$$

where φ an ψ are scalar differentiable functions.

1.76. Transform the integral over the closed loop $\oint_l u\mathrm{d}f$ into the integral over a surface bounded by that loop.

1.77*. Prove the integral identities

$$\oint_l \varphi \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{n}\times \mathrm{grad}\,\varphi)\mathrm{d}S\ , \tag{1.99}$$

16) George Green (1793–1841) was an English mathematician and physicist who introduced the concept of potential and contributed to the development of the theory of electrical and magnetic phenomena.

$$\oint_l (\mathrm{d}\boldsymbol{l} \times \boldsymbol{A}) = \int_S ((\boldsymbol{n} \times \nabla) \times \boldsymbol{A}) \mathrm{d}S \ , \tag{1.100}$$

$$\oint_l \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{A} = \int_S (\boldsymbol{n} \times \nabla) \cdot \boldsymbol{A} \mathrm{d}S \ . \tag{1.101}$$

Here $\boldsymbol{n}$ is the ort of the normal to the surface, φ and $\boldsymbol{A}$ are functions of the coordinates, l is a closed loop, and S is a nonclosed surface bounded by that loop. These identities may be regarded as special cases of the generalized Stokes theorem

$$\oint_l (\ldots) \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{n} \times \nabla)(\ldots) \mathrm{d}S \ , \tag{1.102}$$

where the symbol $(\ldots)$ labels a tensor of any rank.

1.78. Show that if the scalar function ψ is a solution of the Helmholtz equation[17] $\triangle\psi + k^2\psi = 0$ and $\boldsymbol{a}$ is a certain constant vector, then the vector functions $\boldsymbol{L} = \nabla\psi$, $\boldsymbol{M} = \nabla \times (\boldsymbol{a}\psi)$, and $\boldsymbol{N} = \nabla \times \boldsymbol{M}$ satisfy the Helmholtz vector equation $\triangle\boldsymbol{A} + k^2\boldsymbol{A} = 0$.

1.2.5 Differentiating in Curvilinear Coordinates

Unlike in Cartesian rectangular coordinates, when we use curvilinear nonorthogonal coordinates $q^\alpha (\alpha = 1, 2, 3)$, x^β $(\beta = 1, 2, 3)$, the derivative over coordinates from a tensor of rank $s \geq 1$ does not produce any tensor, which we will see later. This is due to the local nature of the definition of the tensor (1.51) applicable to a certain point. In the meantime, a derivative is defined through the difference of the values of two vectors at close but still different points. In order to define a *covariant derivative* from a tensor of any rank, that is, such a differential operation that increases the rank of a tensor by one, we will, for simplicity, consider a tensor of rank 1 (vector) and expand it in basic vectors of the curvilinear system of coordinates in question:

$$\boldsymbol{A} = A^\mu \boldsymbol{e}_\mu = A_\mu \boldsymbol{e}^\mu \ . \tag{1.103}$$

Differentiate the equalities in (1.103) and form the covariant derivatives:

$$A_{\mu;\alpha} \equiv \boldsymbol{e}_\mu \cdot \frac{\partial \boldsymbol{A}}{\partial q^\alpha} = \frac{\partial A_\mu}{\partial q^\alpha} + A_\nu \boldsymbol{e}_\mu \cdot \frac{\partial \boldsymbol{e}^\nu}{\partial q^\alpha} \ , \tag{1.104}$$

$$A^\mu_{;\alpha} \equiv \boldsymbol{e}^\mu \cdot \frac{\partial \boldsymbol{A}}{\partial q^\alpha} = \frac{\partial A^\mu}{\partial q^\alpha} + A^\nu \boldsymbol{e}^\mu \cdot \frac{\partial \boldsymbol{e}_\nu}{\partial q^\alpha} \ . \tag{1.105}$$

17) Herman Ludwig Ferdinand Helmholtz (1821–1894) was a German physicist, mathematician, physiologist, and psychologist.

The first members of the equalities use the notation commonly accepted for covariant derivatives of covariant and contravariant vector components, respectively. The sign of the identity is followed by their definitions. The second members include derivatives of the components of the vector and basic vectors. In curvilinear systems of coordinates, unlike in Cartesian coordinates, derivatives of basic vectors are not equal to zero.

Differentiating equality (1.48) over the coordinate, we find that

$$\boldsymbol{e}_\mu \cdot \frac{\partial \boldsymbol{e}^\nu}{\partial q^\alpha} = -\boldsymbol{e}^\nu \cdot \frac{\partial \boldsymbol{e}_\mu}{\partial q^\alpha} . \tag{1.106}$$

Now, add the *Christoffel symbols* of the second kind to our consideration:[18)]

$$\Gamma^\nu_{\mu\alpha} = \boldsymbol{e}^\nu \cdot \frac{\partial \boldsymbol{e}_\mu}{\partial q^\alpha} . \tag{1.107}$$

They allow us to write covariant derivatives in a more compact form:

$$A_{\mu;\alpha} = \frac{\partial A_\mu}{\partial q^\alpha} - A_\nu \Gamma^\nu_{\mu\alpha} , \quad A^\mu_{;\alpha} = \frac{\partial A^\mu}{\partial q^\alpha} - A^\nu \Gamma^\mu_{\nu\alpha} . \tag{1.108}$$

Christoffel symbols are not tensors since they do not satisfy the applicable rules of transformation. They are symmetric as to the two lower symbols: $\Gamma^\nu_{\mu\alpha} = \Gamma^\nu_{\alpha\mu}$. The latter property follows from the representation of basic vectors (1.46):

$$\frac{\partial \boldsymbol{e}_\mu}{\partial q^\alpha} = \frac{\partial \boldsymbol{e}_\alpha}{\partial q^\mu} . \tag{1.109}$$

The rules (1.108) of computing a covariant derivative of a tensor of rank 1 are generalized, in an obvious way, to include tensor T of any rank. Besides the derivative over the coordinate from the tensor in question, one needs to add as many terms with a plus sign as the tensor has upper symbols and as many terms with a minus sign as the tensor has lower symbols.

Example 1.9

Express the Christoffel symbols (1.107) through the components of metric tensor $g_{\mu\nu}$.

Solution. The definition (1.107) of Christoffel symbols allows us to write the following relation:

$$\boldsymbol{e}_\nu \Gamma^\nu_{\mu\alpha} = \frac{\partial \boldsymbol{e}_\mu}{\partial q^\alpha} . \tag{1.110}$$

It follows from the equality $(\boldsymbol{e}_\nu)_\lambda (\boldsymbol{e}^\nu)^\sigma = \delta^\sigma_\lambda$, which follows from the representations of basic vectors (1.46) and (1.49).

18) Elwin Bruno Christoffel (1829–1900) was a German mathematician.

If we use the relation,

$$e^{\nu}\Gamma_{\nu,\mu\alpha} = \frac{\partial \boldsymbol{e}_{\mu}}{\partial q^{\alpha}} , \tag{1.111}$$

Also consider Christoffel symbols of the first kind, $\Gamma_{\nu,\mu\alpha}$

As follows from (1.110) and (1.111), Christoffel symbols of the first and second kinds may be regarded as the coefficients of the expansion of the quantity $\partial \boldsymbol{e}_{\mu}/\partial q^{\alpha}$ in vectors of covariant and contravariant bases.

Using (1.48), we find from (1.111) that

$$\Gamma_{\nu,\mu\alpha} = \boldsymbol{e}_{\nu} \cdot \frac{\partial \boldsymbol{e}_{\mu}}{\partial q^{\alpha}} . \tag{1.112}$$

Multiplying (1.110), in a scalar way, by $\boldsymbol{e}_{\lambda}$ and (1.111) by $\boldsymbol{e}^{\lambda}$ and using (1.50), we find the connection between Christoffel symbols of the first and second kinds:

$$\Gamma_{\nu,\mu\alpha} = g_{\nu\lambda}\Gamma^{\nu}_{\mu\alpha} , \quad \Gamma^{\nu}_{\mu\alpha} = g^{\nu\lambda}\Gamma_{\nu,\mu\alpha} . \tag{1.113}$$

Then, sequentially using the symmetry of the two symbols separated by a comma and relations (1.109), (1.112), and (1.113), find the following:

$$\begin{aligned}\Gamma_{\nu,\mu\alpha} &= \frac{1}{2}\left(\boldsymbol{e}_{\nu} \cdot \frac{\partial \boldsymbol{e}_{\mu}}{\partial q^{\alpha}} + \boldsymbol{e}_{\nu} \cdot \frac{\partial \boldsymbol{e}_{\alpha}}{\partial q^{\mu}}\right) \\ &= \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial q^{\alpha}} + \frac{\partial g_{\alpha\nu}}{\partial q^{\mu}} - \boldsymbol{e}_{\mu} \cdot \frac{\partial \boldsymbol{e}_{\nu}}{\partial q^{\alpha}} - \boldsymbol{e}_{\alpha} \cdot \frac{\partial \boldsymbol{e}_{\nu}}{\partial q^{\mu}}\right) \\ &= \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial q^{\alpha}} + \frac{\partial g_{\alpha\nu}}{\partial q^{\mu}} - \frac{\partial g_{\alpha\mu}}{\partial q^{\nu}}\right) ,\end{aligned} \tag{1.114}$$

$$\Gamma^{\nu}_{\mu\alpha} = \frac{1}{2}g^{\nu\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial q^{\alpha}} + \frac{\partial g_{\alpha\lambda}}{\partial q^{\mu}} - \frac{\partial g_{\alpha\mu}}{\partial q^{\nu}}\right) . \tag{1.115}$$

□

Example 1.10

Find the rules of the transformation of Christoffel symbols of the first and second kinds when they are transferred to another curvilinear coordinate system.

Solution. Do the sequential computations

$$\begin{aligned}\Gamma'^{\nu}_{\mu\alpha} &= \boldsymbol{e}'^{\nu} \cdot \frac{\partial \boldsymbol{e}'_{\mu}}{\partial q'^{\alpha}} = \frac{\partial q'^{\nu}}{\partial q^{\beta}}\boldsymbol{e}^{\beta} \cdot \frac{\partial}{\partial q'^{\alpha}}\frac{\partial q^{\sigma}}{\partial q'^{\mu}}\boldsymbol{e}_{\sigma} \\ &= \frac{\partial q'^{\nu}}{\partial q^{\beta}}\frac{\partial q^{\lambda}}{\partial q'^{\alpha}}\frac{\partial q^{\sigma}}{\partial q'^{\mu}}\boldsymbol{e}^{\beta} \cdot \frac{\partial \boldsymbol{e}_{\sigma}}{\partial q^{\lambda}} + \frac{\partial q'^{\nu}}{\partial q^{\beta}}\frac{\partial q^{\lambda}}{\partial q'^{\alpha}}\frac{\partial q'^{\kappa}}{\partial q^{\lambda}}\frac{\partial^2 q^{\sigma}}{\partial q'^{\kappa}\partial q'^{\mu}}\boldsymbol{e}^{\beta} \cdot \boldsymbol{e}_{\sigma}\end{aligned}$$

$$= \frac{\partial q'^{\nu}}{\partial q^{\beta}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \Gamma^{\beta}_{\lambda\sigma} + \frac{\partial q'^{\nu}}{\partial q^{\beta}} \frac{\partial^2 q^{\beta}}{\partial q'^{\alpha} \partial q'^{\mu}} , \tag{1.116}$$

$$\begin{aligned} \Gamma'_{\nu,\mu\alpha} &= \boldsymbol{e}'_{\nu} \cdot \frac{\partial \boldsymbol{e}'_{\mu}}{\partial q'^{\alpha}} = \frac{\partial q^{\beta}}{\partial q'^{\nu}} \boldsymbol{e}_{\beta} \cdot \frac{\partial}{\partial q'^{\alpha}} \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \boldsymbol{e}_{\sigma} \\ &= \frac{\partial q^{\beta}}{\partial q'^{\nu}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \boldsymbol{e}_{\beta} \cdot \frac{\partial \boldsymbol{e}_{\sigma}}{\partial q^{\lambda}} + \frac{\partial q^{\beta}}{\partial q'^{\nu}} \frac{\partial^2 q^{\sigma}}{\partial q'^{\alpha} \partial q'^{\mu}} \boldsymbol{e}_{\beta} \cdot \boldsymbol{e}_{\sigma} \\ &= \frac{\partial q^{\beta}}{\partial q'^{\nu}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \Gamma_{\beta,\lambda\sigma} + \frac{\partial q^{\beta}}{\partial q'^{\nu}} \frac{\partial^2 q^{\beta}}{\partial q'^{\alpha} \partial q'^{\mu}} g_{\beta\sigma} . \end{aligned} \tag{1.117}$$

Only the first terms in the second members of the resulting expressions conform to the rules of the transformation of tensors. The second terms violate the said rules, which means that *Christoffel symbols are not tensors.* □

Example 1.11

Prove that the covariant derivatives of the vectors $A_{\nu;\alpha}$ and $A^{\nu}_{;\alpha}$ are transformed as covariant and mixed tensors, respectively, of rank 2.

Solution. Using the definition of covariant derivative (1.104) and the rule of transformation (1.116), sequentially find the following:

$$\begin{aligned} A'_{\mu;\alpha} &= \frac{\partial A'_{\mu}}{\partial q'^{\alpha}} - \Gamma'^{\nu}_{\mu\alpha} A'_{\nu} \\ &= \frac{\partial}{\partial q^{\lambda}} \left(\frac{\partial q^{\sigma}}{\partial q'^{\mu}} A_{\sigma} \right) \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \\ &\quad - \left(\frac{\partial q'^{\nu}}{\partial q^{\beta}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \Gamma^{\beta}_{\lambda\sigma} + \frac{\partial q'^{\nu}}{\partial q^{\beta}} \frac{\partial^2 q^{\beta}}{\partial q'^{\alpha} \partial q'^{\mu}} \right) \frac{\partial q^{\kappa}}{\partial q'^{\nu}} A_{\kappa} \\ &= \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} \left(\frac{\partial A_{\sigma}}{\partial q^{\lambda}} - \Gamma^{\beta}_{\lambda\sigma} A_{\beta} \right) = \frac{\partial q^{\sigma}}{\partial q'^{\mu}} \frac{\partial q^{\lambda}}{\partial q'^{\alpha}} A_{\sigma;\lambda} . \end{aligned} \tag{1.118}$$

It has been proven that the quantity in question is transformed as a covariant tensor of rank 2. When considering the second tensor, one must use the following equality:

$$\frac{\partial q'^{\nu}}{\partial q^{\beta}} \frac{\partial^2 q^{\beta}}{\partial q'^{\alpha} \partial q'^{\mu}} = \frac{\partial q^{\beta}}{\partial q'^{\alpha}} \frac{\partial q^{\lambda}}{\partial q'^{\mu}} \frac{\partial^2 q'^{\nu}}{\partial q^{\beta} \partial q^{\lambda}} . \tag{1.119}$$

It follows from differentiating over the coordinate of an equality such as (1.43). □

Problems

1.79. Show that a derivative of a coordinate of the scalar (gradient) $\partial S/\partial q^\mu = S_{;\mu}$ is a covariant vector.

1.80. Show that a covariant curl coincides with a proper curl:

$$A_{\mu;\nu} - A_{\nu;\mu} = \frac{\partial A_\mu}{\partial q^\nu} - \frac{\partial A_\nu}{\partial q^\mu} .$$

1.81*. Show that the covariant divergence of a covariant vector (scalar) may be written as

$$A^\mu_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\mu} \left(\sqrt{g} A^\mu\right) . \tag{1.120}$$

1.82. In curvilinear coordinates, write down the Laplace operator influencing a scalar function.

1.83. Write down covariant divergence $T^{\mu\nu}{}_{;\mu}$ for any tensor of rank 2.

1.84. Do the same for the antisymmetric tensor $A^{\mu\nu} = -A^{\nu\mu}$.

1.85. Prove the following relation for the covariant components of the antisymmetric tensor $A_{\mu\nu} = -A_{\nu\mu}$:

$$A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = \frac{\partial A_{\mu\nu}}{\partial q^\lambda} + \frac{\partial A_{\lambda\mu}}{\partial q^\nu} + \frac{\partial A_{\nu\lambda}}{\partial q^\mu} .$$

1.86. Find the covariant derivatives of the metric tensor $g_{\mu\nu;\lambda}$ and $g^{\nu\mu}{}_{;\lambda}$.

1.87. Prove the identity $\partial g_{\mu\nu}/\partial q^\lambda = \Gamma_{\mu,\nu\lambda} + \Gamma_{\nu,\mu\lambda}$.

1.2.6 Orthogonal Curvilinear Coordinates

Orthogonal curvilinear coordinates in which $g_{\mu\nu} = 0$ while $\mu \neq \nu$ (see Problem 1.46) are practically used very frequently. In those cases, the following notation is used: $g_{\mu\nu} = h^2_\mu(q)\delta_{\mu\nu}$ (no summing over μ is necessary). The element of length is written as

$$\mathrm{d}l^2 = g_{\mu\nu}\mathrm{d}q^\mu \mathrm{d}q^\nu = h_1^2(\mathrm{d}q^1)^2 + h_2^2(\mathrm{d}q^2)^2 + h_3^2(\mathrm{d}q^3)^2 , \tag{1.121}$$

where, in accordance with (1.46), values h_μ (*Lamé coefficients*)[19] have the following form:

$$h_\mu = \sqrt{\left(\frac{\partial x}{\partial q^\mu}\right)^2 + \left(\frac{\partial y}{\partial q^\mu}\right)^2 + \left(\frac{\partial z}{\partial q^\mu}\right)^2} . \tag{1.122}$$

19) Gabriel Lamé (1795–1870) was a French mathematician and engineer who conducted research in mathematical physics and the theory of elasticity.

Since $\sqrt{g} = h_1 h_2 h_3$, the invariant volume element (1.54) assumes the following form:

$$\mathrm{d}V = h_1 h_2 h_3 \mathrm{d}q^1 \mathrm{d}q^2 \mathrm{d}q^3 \,. \tag{1.123}$$

The characteristic peculiarity of an orthogonal basis is that the vectors of the original and mutual bases have the same directions but different sizes and physical dimensions (because the coordinates x^α and q^β may have different dimensions). This is why the dimensions of different components of the same vector, expanded in the vectors of those bases, may also be different, which creates a certain inconvenience when physical problems are being solved. This is why the introduction of an orthogonal basis of unit vectors $\boldsymbol{e}_{\alpha *}, \boldsymbol{e}_{\alpha *} \cdot \boldsymbol{e}_{\beta *} = \delta_{\alpha\beta}$ is useful (we will label them with lower indices and an asterisk) and through which, in accordance with (1.50), the covariant and contravariant bases will be expressed in the following way:

$$\boldsymbol{e}_\beta = h_\beta \boldsymbol{e}_{\beta *} \,, \quad \boldsymbol{e}^\beta = \frac{1}{h_\beta} \boldsymbol{e}_{\beta *} \,. \tag{1.124}$$

The expansion of an arbitrary vector $\boldsymbol{A}$ in orts $\boldsymbol{e}_{\beta *}$ assumes the following form:

$$\boldsymbol{A} = A_{1*} \boldsymbol{e}_{1*} + A_{2*} \boldsymbol{e}_{2*} + A_{3*} \boldsymbol{e}_{3*} \,, \tag{1.125}$$

where the "physical" components of the vector $A_{\mu *}$ now have the same dimensionality matching that of $\boldsymbol{A}$, that is, the physical quantity in question, and are connected to its covariant and contravariant components by the following relations:

$$A_{\beta *} = \frac{A_\mu}{h_\mu} = A^\mu h_\mu \,. \tag{1.126}$$

Since the use of the basis $\boldsymbol{e}_{\beta *}$ is convenient, hereafter we will use that basis everywhere, omitting the asterisk.

Using relations (1.120)–(1.126), and also (1.25), write down the principal operations of differentiation in orthogonal curvilinear coordinates:

$$\operatorname{grad} S = \frac{1}{h_1} \frac{\partial S}{\partial q^1} \boldsymbol{e}_1 + \frac{1}{h_2} \frac{\partial S}{\partial q^2} \boldsymbol{e}_2 + \frac{1}{h_3} \frac{\partial S}{\partial q^3} \boldsymbol{e}_3 \,; \tag{1.127}$$

$$\operatorname{div} \boldsymbol{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q^1} (h_2 h_3 A_1) + \frac{\partial}{\partial q^2} (h_1 h_3 A_2) + \frac{\partial}{\partial q^3} (h_1 h_2 A_3) \right] ; \tag{1.128}$$

$$\Delta S = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q^1} \left(\frac{h_2 h_3}{h_1} \frac{\partial S}{\partial q^1} \right) + \frac{\partial}{\partial q^2} \left(\frac{h_1 h_3}{h_2} \frac{\partial S}{\partial q^2} \right) + \frac{\partial}{\partial q^3} \left(\frac{h_1 h_2}{h_3} \frac{\partial S}{\partial q^3} \right) \right] ; \tag{1.129}$$

$$\begin{aligned}\operatorname{curl}\boldsymbol{A} &= \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial q^2}(h_3 A_3) - \frac{\partial}{\partial q^3}(h_2 A_2)\right]\boldsymbol{e}_1 \\ &+ \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial q^3}(h_1 A_1) - \frac{\partial}{\partial q^1}(h_3 A_3)\right]\boldsymbol{e}_2 \\ &+ \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial q^1}(h_2 A_2) - \frac{\partial}{\partial q^2}(h_1 A_1)\right]\boldsymbol{e}_3 \,. \end{aligned} \tag{1.130}$$

Problems

1.88. From the common expressions (1.27)–(1.29), derive the basic differential operations below in the (r, α, z) cylindrical coordinate system where $x = r\cos\alpha$, $y = r\sin\alpha$, and $z = z$:

$$\operatorname{grad} S = \frac{\partial S}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial S}{\partial \alpha}\boldsymbol{e}_\alpha + \frac{\partial S}{\partial z}\boldsymbol{e}_z \,; \tag{1.131}$$

$$\operatorname{div}\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\alpha}{\partial \alpha} + \frac{\partial A_z}{\partial z} \,; \tag{1.132}$$

$$\Delta S = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 S}{\partial \alpha^2} + \frac{\partial^2 S}{\partial z^2} \,; \tag{1.133}$$

$$\begin{aligned}\operatorname{curl}\boldsymbol{A} &= \left[\frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}\right]\boldsymbol{e}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\boldsymbol{e}_\alpha \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\alpha) - \frac{\partial A_r}{\partial \alpha}\right]\boldsymbol{e}_z \,. \end{aligned} \tag{1.134}$$

1.89. Do the same for the (r, ϑ, α) spherical coordinate system where $x = r\sin\vartheta\cos\alpha$, $y = r\sin\vartheta\sin\alpha$, and $z = r\cos\vartheta$:

$$\operatorname{grad} S = \frac{\partial S}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial S}{\partial \vartheta}\boldsymbol{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial S}{\partial \alpha}\boldsymbol{e}_\alpha \,; \tag{1.135}$$

$$\operatorname{div}\boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}(A_\vartheta \sin\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial A_\alpha}{\partial \alpha} \,; \tag{1.136}$$

$$\Delta S = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial S}{\partial \vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 S}{\partial \alpha^2} \,; \tag{1.137}$$

$$\begin{aligned}\operatorname{curl}\boldsymbol{A} &= \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial \vartheta}(A_\alpha \sin\vartheta) - \frac{\partial A_\vartheta}{\partial \alpha}\right]\boldsymbol{e}_r \\ &+ \left[\frac{1}{r\sin\vartheta}\frac{\partial A_r}{\partial \alpha} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\alpha)\right]\boldsymbol{e}_\vartheta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\vartheta) - \frac{\partial A_r}{\partial \vartheta}\right]\boldsymbol{e}_\alpha \,. \end{aligned} \tag{1.138}$$

1.90*. Use identity (1.83) to write the projections of the vector $\triangle \boldsymbol{A}$ onto the axes of a cylindrical coordinate system.

1.91*. Do the same for a spherical coordinate system.

1.92. Find the general form solution of Laplace's equation for a scalar function that depends only on (i) r, (ii) α, and (iii) z (cylindrical coordinates).

1.93. Find the general form solution of Laplace's equation for a scalar function that depends only on (i) r, (ii) ϑ, and (iii) α (spherical coordinates).

Note In Problems 1.94*–1.98*, examples of curvilinear orthogonal systems of coordinates are considered. These systems are more complex than cylindrical and spherical systems. For more information, see Arfken (1970) and Stratton (1948)

1.94*. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (a > b > c)$$

represents an ellipsoid with semiaxes a, b, and c. The equations

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} = 1\,, \quad \xi \geq -c^2\,,$$

$$\frac{x^2}{a^2+\eta} + \frac{y^2}{b^2+\eta} + \frac{z^2}{c^2+\eta} = 1\,, \quad -c^2 \geq \eta \geq -b^2\,,$$

$$\frac{x^2}{a^2+\zeta} + \frac{y^2}{b^2+\zeta} + \frac{z^2}{c^2+\zeta} = 1\,, \quad -b^2 \geq \zeta \geq -a^2$$

represent an ellipsoid and one-sheet and two-sheet hyperboloids confocal with the first ellipsoid, respectively. Each point of the space is crossed by a surface characterized by values ξ, η, and ζ. ξ, η, and ζ are called ellipsoidal coordinates of the point x, y, z. Find the formulas of transformation of ellipsoidal to Cartesian coordinates. Make sure that an ellipsoidal system of coordinates is orthogonal. Find the Lamé coefficients and Laplace's operator in ellipsoidal coordinates.

1.95*. When $a = b > c$, the ellipsoidal coordinate system (see the previous problem) degenerates to become a so-called flattened spheroidal coordinate system. When this happens, the coordinate ζ becomes constant, equals $-a^2$, and must be replaced by another coordinate. To serve as such, an azimuthal angle α on the surface xy is selected. The coordinates ξ and η are found from the following equations:

$$\frac{r^2}{a^2+\xi} + \frac{z^2}{c^2+\xi} = 1\,, \quad \xi \geq -c^2\,,$$

$$\frac{r^2}{a^2+\eta} + \frac{z^2}{c^2+\eta} = 1\,, \quad -c^2 \geq \eta \geq -a^2\,,$$

where $r^2 = x^2 + y^2$.

Surfaces $\xi = \text{const}$ are flattened ellipsoids of rotation around the Oz axis. Surfaces $\eta = \text{const}$ are one-sheet hyperboloids of rotation confocal with them (Figure 1.6).

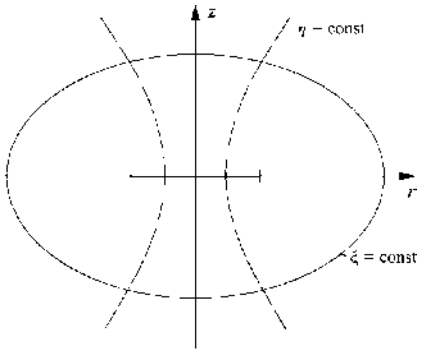

Figure 1.6 A flattened spheroidal system of coordinates.

Find the expressions for r and z in flattened spheroidal coordinates, the Lamé coefficients, and Laplace's operator in those coordinates.

1.96*. An extended spheroidal system of coordinates is derived from an ellipsoidal one (Problem 1.94*) when $a > b = c$. When this happens, the coordinate η becomes constant and must be replaced with an azimuthal angle α marked off on the yz surface by the Oy axis. The coordinates ξ and ζ are found from the following equations:

$$\frac{x^2}{a^2+\xi} + \frac{r^2}{b^2+\xi} = 1\,, \quad \xi \ge -b^2\,,$$

$$\frac{x^2}{a^2+\zeta} + \frac{r^2}{b^2+\zeta} = 1\,, \quad -b^2 \ge \zeta \ge -a^2\,,$$

where $r^2 = y^2 + z^2$.

The surfaces of the constants ξ and η are extended ellipsoids and two-sheet hyperboloids of rotation (Figure 1.7). Express the quantities x and r through ξ and ζ. Find the Lamé coefficients and Laplace's operator in the variables ξ, ζ, and α.

1.97*. Bispherical cooordinates ξ, η, and α are connected to Cartesian coordinates by the following relations:

$$x = \frac{a\sin\eta\cos\alpha}{\cosh\xi - \cos\eta}\,, \quad y = \frac{a\sin\eta\sin\alpha}{\cosh\xi - \cos\eta}\,, \quad z = \frac{a\sinh\xi}{\cosh\xi - \cos\eta}\,,$$

where a is a constant parameter, $-\infty < \xi < \infty$, $0 < \eta < \pi$, and $0 < \alpha < 2\pi$.

Show that the coordinate surfaces $\xi = \text{const}$ are spheres,

$$x^2 + y^2 + (z - a\coth\xi)^2 = \left(\frac{a}{\sinh\xi}\right)^2\,,$$

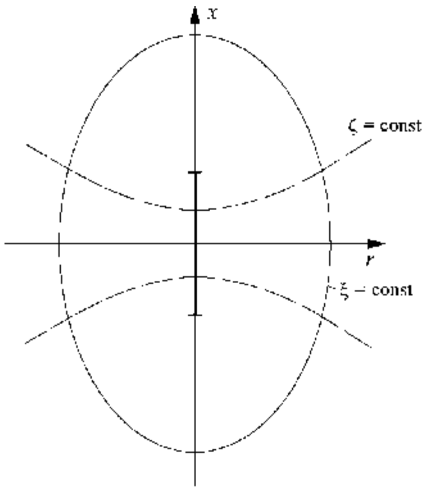

Figure 1.7 An extended spheroidal system of coordinates.

the surfaces $\eta = \text{const}$ are spindle-shaped surfaces of rotation around the Oz axis, whose equation is

$$\left(\sqrt{x^2+y^2} - a\cot\eta\right)^2 + z^2 = \left(\frac{a}{\sin\eta}\right)^2 ,$$

and surfaces $\alpha = \text{const}$ are half-planes diverging from the Oz axis (Figure 1.8). Make sure that these coordinate surfaces are orthogonal with respect to each other. Find the Lamé coefficients and Laplace's operator.

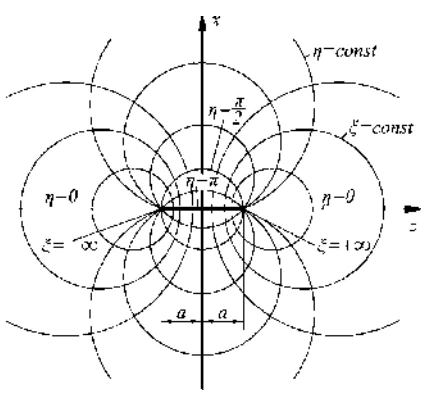

Figure 1.8 A bispherical system of coordinates.

1.98*. The toroidal coordinates ρ, ξ, and α make up an orthogonal system and are connected to Cartesian coordinates by the following relations:

$$x = \frac{a\sinh\rho\cos\alpha}{\cosh\rho - \cos\xi} , \quad y = \frac{a\sinh\rho\sin\alpha}{\cosh\rho - \cos\xi} , \quad z = \frac{a\sin\xi}{\cosh\rho - \cos\xi} ,$$

where a is a constant parameter, $-\infty < \rho < \infty$, $-\pi < \xi < \pi$, and $0 < \alpha < \pi$.

Show that $\rho = \ln(r_1/r_2)$ (see Figure 1.9, displaying the surfaces $\alpha = \text{const}$ and $\alpha + \pi = \text{const}$) and the quantity ξ is the angle between r_1 and r_2 ($\xi > 0$ if $z > 0$ and $\xi < 0$ if $z < 0$). What is the form of the coordinate surfaces $\rho = \text{const}$ and $\xi = \text{const}$? Find the Lamé coefficients.

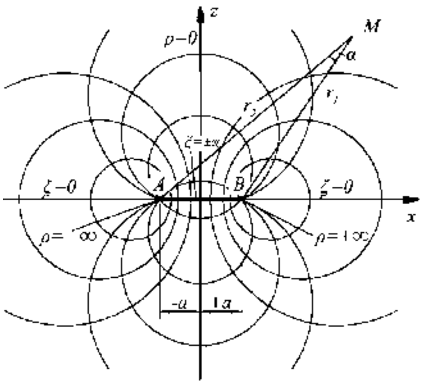

Figure 1.9 A toroidal system of coordinates.

Suggested literature:
Borisenko and Tarapov (1966); Weinberg (1972); Arfken (1970); Mathews and Walker (1964); Lee (1965); Rashevskii (1953); Morse and Feshbach (1953); Stratton (1948); Madelung (1957)

1.3 The Special Functions of Mathematical Physics

1.3.1 Cylindrical Functions

Cylindrical functions are used when solving many specific problems. Of these, the *Bessel functions* are the most commonly used. They may be obtained by expanding a purposely selected exponent (*generating function*) in a power series over u:

$$\exp\left\{\frac{x}{2}\left(u - \frac{1}{u}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(x) u^n . \tag{1.139}$$

The coefficients $J_n(x)$ of this expansion are called Bessel functions[20] of the first kind and order n. The representation of a Bessel function as a power series may be

20) Friedrich Wilhelm Bessel (1784–1846) was a German astronomer, land surveyor, and mathematician.

obtained from the power series for exponents:

$$\exp\left(\frac{xu}{2}\right)\exp\left(-\frac{x}{2u}\right) = \sum_{r=0}^{\infty}\left(\frac{x}{2}\right)^r \frac{u^r}{r!}\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^s\frac{u^{-s}}{s!}\,. \tag{1.140}$$

These expansions are valid for any (including complex) values of x and u, which is due to the unboundedness of the radius of convergence of an exponent. Changing to summing over $n = r - s(-\infty < n < \infty)$, we get, from (1.140)

$$\begin{aligned}\exp\left\{\frac{x}{2}\left(u-\frac{1}{u}\right)\right\} &= \sum_{n=-\infty}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}u^n \\ &= \sum_{n=-\infty}^{\infty} J_n(x)u^n\,,\end{aligned} \tag{1.141}$$

wherefrom it follows that

$$J_n(x) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}\,. \tag{1.142}$$

The use of this representation for $J_n(x)$ is expedient when $n \geq 0$. When $n < 0$, the following may be written instead of (1.142):

$$J_n(x) = \sum_{s=|n|}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} = \sum_{s=0}^{\infty}\frac{(-1)^{s+|n|}}{s!(|n|+s)!}\left(\frac{x}{2}\right)^{|n|+2s}u^n. \tag{1.143}$$

This is because when $s + n < 0$, $(s + n)! \to \infty$. As a result, we get a simple dependence between the Bessel functions of the whole positive and negative orders:

$$J_{-n}(x) = (-1)^n J_n(x)\,. \tag{1.144}$$

Example 1.12

Obtain recurrent relations between Bessel functions of various orders by differentiating equality (1.139) over u and over x, comparing the second and first members of the equality.

Solution. Differentiating (1.139) over u, we get

$$\begin{aligned}\frac{x}{2}\left(1+\frac{1}{u^2}\right)\exp\left[\frac{x}{2}\left(u-\frac{1}{u}\right)\right] &= \frac{x}{2}\left(1+\frac{1}{u^2}\right)\sum_{n=-\infty}^{\infty}J_n(x)u^n \\ &= \sum_{n=-\infty}^{\infty} nJ_n(x)u^{n-1}\,.\end{aligned}$$

Equating the coefficients of u^{n-1} in second and first members of the latter equality, we find the following:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) . \tag{1.145}$$

Differentiating (1.139) over x, we get in the same way

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) . \tag{1.146}$$

These recurrent relations may be rewritten as follows, in other forms:

$$J_{n\pm 1} = \frac{n}{x} J_n(x) \mp J'(x) ; \quad J_{n\mp 1} = \pm x^{\mp n} \frac{\mathrm{d}}{\mathrm{d}x}[x^{\pm n} J_n(x)] . \tag{1.147}$$

Specifically,

$$J_1(x) = -J_0'(x) . \tag{1.148}$$

□

Example 1.13

Obtain representations of the Bessel function as integrals from exponential and trigonometric functions. For that purpose, use the substitution $u = \exp(i\varphi)$ in expansion (1.139).

Solution. The substitution leads to the expansion

$$\exp(ix \sin\varphi) = \sum_{n=-\infty}^{\infty} J_n(x) \exp(in\varphi) . \tag{1.149}$$

Use the periodicity of the functions $\sin\varphi$ and $\exp(in\varphi)$ and also the easily verifiable equality

$$\int_{\alpha}^{\alpha+2\pi} \exp(i(n-m)\varphi)\mathrm{d}\varphi = 2\pi\delta_{mn} ,$$

where m is an integer and α is any real number. Multiplying both parts of (1.149) by $\exp(-im\varphi)$ and integrating over φ, we get the representation of the Bessel function:

$$J_m(x) = \frac{1}{2\pi} \int_{\alpha}^{\alpha+2\pi} \exp(ix\sin\varphi - im\varphi)\mathrm{d}\varphi$$
$$= \frac{(-i)^m}{2\pi} \int_{\alpha}^{\alpha+2\pi} \exp(ix\cos\varphi - im\varphi)\mathrm{d}\varphi . \tag{1.150}$$

□

Example 1.14

Presume that certain functions of $Z_\nu(x)$, different, generally speaking, from Bessel functions (1.142), satisfy the recurrent relations (1.145)–(1.148) when the value of $n = \nu$ is arbitrary and complex. Produce a differential equation of the second order whose solution is $Z_\nu(x)$.

Solution. Differentiate the second equality (1.147) over x and add a term equaling zero to it (replacing $n \to \nu$, $J_n \to Z_\nu$):

$$\begin{aligned} Z'_{\nu-1} &= [x^{-\nu}(x^\nu Z_\nu)']' + \frac{1}{x}\left(Z'_\nu + \frac{\nu}{x}Z_\nu - Z_{\nu-1}\right) \\ &= Z''_\nu + \frac{\nu+1}{x}Z'_\nu - \frac{1}{x}Z_{\nu-1}\,. \end{aligned}$$

Once again, add a term equaling zero to the second member:

$$\begin{aligned} Z'_{\nu-1} &= Z''_\nu + \frac{\nu+1}{x}Z'_\nu - \frac{1}{x}Z_{\nu-1} + \frac{n}{x}\left[Z_{\nu-1} - \frac{\nu}{x}Z_\nu - Z'_\nu\right] \\ &= Z''_\nu + \frac{1}{x}Z'_\nu - \frac{n^2}{x^2}Z_\nu + \frac{\nu-1}{x}Z_{\nu-1}\,. \end{aligned}$$

Finally, from the second equality in (1.147), if we make the replacement $n+1 \to \nu$, $J_n \to Z_\nu$, we find

$$Z_\nu = \frac{\nu-1}{x}Z_{\nu-1} - Z'_{\nu-1}\,.$$

Excepting $Z'_{\nu-1}$ from the latter two equalities, we get the *Bessel equation*, satisfied by the function $Z_\nu(x)$:

$$Z''_\nu + \frac{1}{x}Z'_\nu + \left(1 - \frac{\nu^2}{x^2}\right)Z_\nu = 0\,. \tag{1.151}$$

□

This or a similar equation appears when solving many physical problems. Below, we will briefly summarize the basic information concerning the solutions of this equation. The generation of the necessary formulas is shown in special mathematical texts (Arfken, 1970; Nikiforov and Uvarov, 1988; Gradshtein and Ryzhik, 2007; Lee, 1965; Mathews and Walker, 1964; Abramovitz and Stegun, 1965; Vilenkin, 1988).

A solution of (1.151), limited when $\mathrm{Re}\,\nu \geq 0$, called a Bessel function of the first order when $x \to 0$, may be represented as a power series, which is a generalization of (1.142):

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(\nu+s+1)}\left(\frac{z}{2}\right)^{2s}\,. \tag{1.152}$$

The independent variable is labeled z, because the series remains valid whatever the values of ν and throughout the complex plane z, except for the slit along the negative part of the real axis.

Another linearly independent solution, when $\nu \neq n = 0, \pm 1, \ldots$, may be $J_{-\nu}(x)$. When n is an integer, there is a linear connection (1.144) between the two solutions shown. This is why the Bessel function of the second kind (the same as Neumann's function[21]) or Weber's function[22]) is selected as the second solution:

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} . \tag{1.153}$$

This solution has a finite bound when $\nu \to n$.

Also, Bessel functions of the third order, also called Hankel functions[23]), may be selected as two linearly independent solutions:

$$H_\nu^{(1)}(z) = J_\nu(z) + i\,Y_\nu(z) ; \quad H_\nu^{(2)}(z) = J_\nu(z) - i\,Y_\nu(z) . \tag{1.154}$$

All these functions are solutions of Bessel's equation. The functions Y_ν and $H_\nu^{(1,2)}$ have singularities when $z \to 0$. All these solutions satisfy the recurrent relations (1.145)–(1.147) (with the replacement of $n \to \nu$, $J_n \to Z_\nu$).

The asymptotic values are as follows: when $z \to 0$,

$$J_\nu(z) \approx \frac{z^\nu}{2^\nu \Gamma(\nu+1)} , \quad \nu \neq -1, -2, \ldots , \tag{1.155}$$

$$Y_0(z) \approx -i H_0^{(1)}(z) \approx i H_0^{(2)}(z) \approx \frac{2}{\pi} \ln z , \tag{1.156}$$

$$Y_\nu(z) \approx -i H_\nu^{(1)}(z) \approx i H_\nu^{(2)}(z) \approx -\frac{\Gamma(\nu)}{\pi}\left(\frac{z}{2}\right)^{-\nu} , \quad \mathrm{Re}\nu > 0 , \tag{1.157}$$

and when $|z| \to \infty$ and ν is arbitrary,

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) , \quad |\arg z| < \pi , \tag{1.158}$$

$$Y_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) , \quad |\arg z| < \pi , \tag{1.159}$$

$$H_\nu^{(1)}(z) \approx \sqrt{\frac{2}{\pi z}} \exp\left[i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] , \quad -\pi < \arg z < 2\pi , \tag{1.160}$$

$$H_\nu^{(2)}(z) \approx \sqrt{\frac{2}{\pi z}} \exp\left[-i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] , \quad -2\pi < \arg z < \pi . \tag{1.161}$$

21) Karl Gottfried Neumann (1832–1925) was a German mathematician.
22) Heinrich Weber (1842–1913) was a German mathematician.
23) Hermann Hankel (1839–1873) was a German mathematician and a historian of mathematics.

Cylindrical functions of purely imaginary arguments are called *modified Bessel functions*. The second of them is also called the Macdonald function.[24] They are described by the following relations:

$$I_\nu(z) = e^{-i\pi\nu/2} J_\nu(iz) \,, \tag{1.162}$$

$$K_\nu(z) = \frac{\pi}{2} e^{i\pi(\nu+1)/2} H_\nu^{(1)}(iz) \tag{1.163}$$

or

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{s=0}^{\infty} \frac{(z/2)^{2s}}{s!\Gamma(\nu+s+1)} \,, \quad K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} \,. \tag{1.164}$$

These functions take real values when ν and $z > 0$ are real. Recurrent relation and differentiation formulas are produced from (1.145)–(1.147), (1.162), and (1.163). For instance,

$$I_0'(z) = I_1(z) \,, \quad K_0'(z) = -K_1(z) \,. \tag{1.165}$$

Modified Bessel functions satisfy the equation

$$W_\nu'' + \frac{1}{z} W_\nu' - \left(1 + \frac{\nu^2}{z^2}\right) W_\nu = 0 \,. \tag{1.166}$$

The asymptotic values are as follows: when $z \to 0$,

$$I_\nu(z) \approx \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \,, \quad \nu \neq -1, -2 \ldots \,, \tag{1.167}$$

$$K_0(z) \approx -\ln z \,, \quad K_\nu(z) \approx \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} , \ \mathrm{Re}\nu > 0 \,, \tag{1.168}$$

and when $|z| \to \infty$,

$$I_\nu(z) \approx \frac{1}{\sqrt{2\pi z}} e^z \,, \quad |\arg z| < \frac{\pi}{2} \,, \tag{1.169}$$

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z}, \quad |\arg z| < \frac{3\pi}{2} \,. \tag{1.170}$$

The spherical functions of Bessel, Hankel, and Weber often appear when problems are solved in spherical coordinates. They are of half-integer order and are described by the following equalities:

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x) \,; \quad h_l^{(1,2)} = \sqrt{\frac{\pi}{2x}} H_{l+\frac{1}{2}}^{(1,2)}(x) \,;$$
$$y_l(x) = \sqrt{\frac{\pi}{2x}} Y_{l+\frac{1}{2}}(x) \,. \tag{1.171}$$

24) Hector Munro Macdonald (1865–1935) was a Scottish physicist and mathematician.

All these functions (their common symbol is $z_l(x)$) satisfy the following equation:

$$z_l'' + \frac{2}{x} z_l' + \left[1 - \frac{l(l+1)}{x^2}\right] z_l = 0 \,.$$

When x is small,

$$j_l(x) \approx \frac{x^l}{1 \cdot 3 \cdots (2l+1)} \,, \quad h_l^{(1,2)} \approx \mp \frac{x^{-l-1}}{1 \cdot 3 \cdots (2l-1)} \,. \tag{1.172}$$

When x is large,

$$j_l(x) \approx \frac{1}{x} \cos\left[x - \frac{(l+1)\pi}{2}\right] \,,$$
$$h_l^{(1,2)}(x) \approx \frac{1}{x} \exp\left\{\pm\left[x - \frac{(l+1)\pi}{2}\right]\right\} \,. \tag{1.173}$$

Problems

1.99. Compute the indefinite integrals
$\int x^{\nu} Z_{\nu-1}(x) \mathrm{d}x$ and $\int x^{-\nu} Z_{\nu+1}(x) \mathrm{d}x$.

1.100. Compute the definite integrals
$\int_0^\infty J_1(x) \mathrm{d}x$, $\int_0^\infty J_2(x) x^{-1} \mathrm{d}x$, and $\int_0^\infty J_n(x) x^{-n} \mathrm{d}x$.

1.101. Prove the equality of the integrals
$\int_0^\infty J_n(x) \mathrm{d}x = \int_0^\infty J_n(x) \mathrm{d}x$, $n = 0, 1, \ldots$

1.102. Obtain the integral representation

$$J_0(x) = \frac{2}{\pi} \int_0^1 \frac{\cos ux}{\sqrt{1-u^2}} \mathrm{d}u \,.$$

Hint: Perform the substitution $u = \sin\varphi$.

1.103*. Compute the integrals

$$\int_0^{\pi/2} J_0(x \cos\varphi) \cos\varphi \mathrm{d}\varphi = \frac{\sin x}{x} \quad \text{and} \quad \int_0^{\pi/2} J_1(x \cos\varphi) \mathrm{d}\varphi = \frac{1 - \cos x}{x} \,.$$

Hint: You may use expansion in power series.

1.104. Produce the formulas

$$J_n(x) = (-1)^k x^n \left(\frac{\mathrm{d}}{x\mathrm{d}x}\right)^k \left(x^{k-n} J_{n-k}(x)\right)$$
$$= (-1)^n x^n \left(\frac{\mathrm{d}}{x\mathrm{d}x}\right)^n J_0(x) \,.$$

1.105. Produce recurrent relations for the modified Bessel functions:

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_\nu(z) \; ; \quad I_{\nu-1}(z) + I_{\nu+1}(z) = 2 I'_\nu(z) \; ; \tag{1.174}$$

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z) \; ;$$
$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2 K'_\nu(z) \; . \tag{1.175}$$

1.106*. Show that

$$J_0(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) = \sum_{n=-\infty}^{\infty} J_n(r_1) J_n(r_2) \exp(i n \vartheta) \; ,$$

where ϑ is the angle between vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

1.107*.

1. Write an equation satisfied by the function $u(x) = J_n(ax)$.
2. Compute the integral ($b \neq a$)

$$\int_0^1 x J_n(ax) J_n(bx) \mathrm{d}x = \frac{a J'_n(a) J_n(b) - b J'_n(b) J_n(a)}{b^2 - a^2} \; . \tag{1.176}$$

3. $a \neq b$ are the roots of the equation $J_n(x) = 0$, that is, $J_n(a) = J_n(b) = 0$. Show that

$$\int_0^1 x J_n(ax) J_n(bx) \mathrm{d}x = 0 \quad \text{and} \quad \int_0^1 x J_n^2(ax) \mathrm{d}x = \frac{1}{2} [J'_n(a)]^2 \; . \tag{1.177}$$

Note The first equality (1.177) expresses the property called *the orthogonality of Bessel functions weighted by x.*

1.108*. Produce "summation theorems" for Bessel functions:

$$\sum_{k=-\infty}^{\infty} J_{n-k}(x) J_k(y) = J_n(x+y) \; , \quad n = 0, 1, 2, \ldots \; ; \tag{1.178}$$

$$\sum_{k=-\infty}^{\infty} (-1)^k J_{n-k}(x) J_k(x) = 0 \; , \quad n = 1, 2, \ldots \; ; \tag{1.179}$$

$$J_0(x) J_0(y) + 2 \sum_{k=1}^{\infty} (-1)^k J_k(x) J_k(y) = J_0(x+y) \; , \quad n = 1, 2, \ldots \; ; \tag{1.180}$$

$$J_0^2(x) + 2 \sum_{k=1}^{\infty} (-1)^k J_k^2(x) = J_0(2x) \; . \tag{1.181}$$

1.3.2
Spherical Functions and Legendre Polynomials

Spherical functions and Legendre polynomials are widely used in many fields of physics, especially in electrodynamics and quantum mechanics. The generating function for Legendre polynomials[25] $P_l(\cos\vartheta)$ is the reverse distance between two points with radius vectors $\boldsymbol{a}$ and $\boldsymbol{r}$, the angle between them equaling ϑ:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|} = \frac{1}{\sqrt{r^2+a^2-2ar\cos\vartheta}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l P_l(\cos\vartheta)\,, \qquad \frac{a}{r}<1\,. \tag{1.182}$$

Designating $x = \cos\vartheta$ and $u = a/r$ and using binomial expansion, which, for the negative exponents is conveniently written as

$$(1-\alpha)^{-q} = \sum_{n=0}^{\infty}\frac{\Gamma(n+q)}{n!\Gamma(q)}\alpha^n\,, \quad |\alpha|<1\,,$$

and the binomial expansion for $\alpha^n = (2ux-u^2)^n$, we get a double sum:

$$(1-2ux+u^2)^{-1/2} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\Gamma(n+1/2)}{\Gamma(1/2)k!(n-k)!}(-1)^k(2x)^{n-k}u^{n+k}\,.$$

Begin summing over k and $n+k=l\geq 0$, which will result in the rearrangement of the terms of the series. In this case, this rearrangement is valid because the infinite series is absolutely convergent, which will be shown below. As the result, we have

$$\begin{aligned}(1-2ux+u^2)^{-1/2} &= \sum_{l=0}^{\infty}\sum_{k=0}^{l}\frac{\Gamma(l-k+1/2)}{\Gamma(1/2)k!(l-2k)!}(-1)^k(2x)^{l-2k}u^l\\ &= \sum_{l=0}^{\infty}P_l(x)u^l\,,\end{aligned}$$

where

$$\begin{aligned}P_l(x) &= \sum_{k=0}^{l}\frac{\Gamma(l-k+1/2)}{\Gamma(1/2)k!(l-2k)!}(-1)^k(2x)^{l-2k}\\ &= \sum_{k=0}^{l}(-1)^k\frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!}x^{l-2k}\,.\end{aligned} \tag{1.183}$$

In the latter two equalities, the sum over k is actually limited to the value of the integer part of $l/2$ because the infinite factorial of the negative integer in the denominator will eliminate all terms with $l-2k<0$.

25) Adrien-Marie Legendre (1752–1833) was a French mathematician.

Example 1.15

Find the values of the polynomials $P_l(1)$, $P_l(-1)$, and $P_l(0)$ by assigning particular values to the angle ϑ in (1.182) and using the binomial expansion.

Solution. If we assume that $\cos\vartheta = 1$, we find, from (1.182), that

$$\frac{1}{1-u} = \sum_{l=0}^{\infty} u^l = \sum_{l=0}^{\infty} P_l(1) u^l$$

and, therefore, $P_l(1) = 1$ whatever the values of l, and $P_0 = 1$ when $0 \le \vartheta \le \pi$. Similarly, we get the following:

$$P_l(-1) = (-1)^l \,, \quad P_{2l}(0) = (-1)^l \frac{(2l-1)!!}{2^l l!} \,, \quad l \ge 1 \,;$$
$$P_{2l+1}(0) = 0 \,, \quad l \ge 0 \,. \tag{1.184}$$

□

Example 1.16

Acquire limits of the values of Legendre polynomials $|P_l(\cos\vartheta)| \le 1$ by analyzing the expansion of the generating function (1.182) in series over $\cos m\vartheta$.

Solution. Sequentially obtain the following from the generating function:

$$\begin{aligned} 1 - 2u\cos\vartheta + u^2)^{-1/2} &= (1 - ue^{i\vartheta})^{-1/2}(1 - ue^{i\vartheta})^{-1/2} \\ &= \left\{1 + \frac{1}{2}ue^{i\vartheta} + \frac{3}{8}u^2 e^{2i\vartheta} + \cdots\right\} \\ &\quad \times \left\{1 + \frac{1}{2}ue^{-i\vartheta} + \frac{3}{8}u^2 e^{-2i\vartheta} + \cdots\right\} \\ &= \sum_{l=0}^{\infty} P_l(\cos\vartheta) u^l \,, \end{aligned}$$

where $P_l(\cos\vartheta) = \sum_{k=0}^{l} a_k \cos k\vartheta$. Coefficients a_k are selected from the values in braces and, importantly, they are all not negative: $a_k \ge 0$. In this case, the sum $\sum a_k \cos k\vartheta$ is maximal when $\vartheta = 0$, which corresponds to $P_l(1) = 1$. Therefore, $|P_l(\cos\vartheta)| \le 1$. □

The estimate we have made allows us to establish that the series (1.182), when $a/r < 1$, is absolutely convergent, that is, what converges is the series $\sum_{l=0}^{\infty} |P_l(\cos\vartheta)|(a/r)^l$. This follows from the established inequality and the fact that the dominating series $\sum_{l=0}^{\infty}(a/r)^l$ is knowingly convergent when $a/r < 1$. It represents the sum of the elements of a decreasing geometric progression.

Expansion (1.183) may yield a more compact representation of the Legendre polynomials if the following transforms are done sequentially:

$$\begin{aligned} P_l(x) &= \sum_{k=0}^{l}(-1)^k \frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!} x^{l-2k} \\ &= \sum_{k=0}^{l} \frac{(-1)^k}{2^l k!(l-k)!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l x^{2l-2k} \\ &= \frac{1}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l \sum_{k=0}^{l} \frac{(-1)^k l!}{k!(l-k)!} x^{2l-2k} = \frac{1}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l (x^2-1)^l \,. \end{aligned} \tag{1.185}$$

The latter expression for Legendre polynomials is called the *Rodrigues formula*.[26]

Example 1.17

Using the Rodrigues formula, produce recurrent relations between the Legendre polynomials:

$$P_l'(x) = x P_{l-1}'(x) + l P_{l-1}(x) \,; \tag{1.186}$$

$$\begin{aligned} (1-x^2) P_l''(x) = 2(l+1) P_{l+1}'(x) - 2(l+2)x \\ \times P_l'(x) - (l+1)(l+2) P_l(x) \,. \end{aligned} \tag{1.187}$$

Using the said relations, obtain a differential equation of the second order satisfied by $P_l(x)$.

Solution. Use the Leibniz formula[27] to find the derivative of order n from the product of the following functions:

$$(fg)^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(n-k)} g^{(k)} \,.$$

26) Benjamin Olinde Rodrigues (1794–1851) was a French mathematician and economist.

27) Gottfried Wilhelm Leibniz (1646–1716) was a German philosopher, jurist, and historian as well as a mathematician, physicist, and inventor. He was one of the founders of classical mathematical analysis.

Compute

$$P_l'(x) = \frac{1}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l+1}(x^2-1)^l = \frac{2l}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l}\left[x(x^2-1)^{l-1}\right]$$
$$= \frac{2l}{2^l l!}\left[x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l}(x^2-1)^{l-1} + l\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l-1}(x^2-1)^{l-1}\right].$$

Expression (1.186) follows from this equality and the Rodrigues formula. Obtain (1.187) using the following similar relation:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l+2}(x^2-1)^{l+1} = (x^2-1)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l+2}(x^2-1)^l$$
$$+ 2(l+2)x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l+1}(x^2-1)^l$$
$$+ (l+1)(l+2)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l}(x^2-1)^l .$$

Obviously, the two recurrent relations obtained produce the differential *equation of Legendre* that has the following form:

$$(1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0 . \quad (1.188)$$

The second linearly independent solution of Legendre's equation has singularities when $x = \pm 1$. □

Example 1.18

The adjoint Legendre polynomials are described by the expression

$$P_l^m(x) = (1-x^2)^{m/2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m}P_l(x)$$
$$= \frac{(1-x^2)^{m/2}}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l+m}(x^2-1)^l , \quad -l \le m \le l . \quad (1.189)$$

Obtain a differential equation satisfied by the adjoint Legendre polynomials.

Solution. When $m > 0$, differentiate the two parts of Legendre's equation (1.188) m times and get an equation of the form

$$(1-x^2)F'' - 2(m+1)xF' + (l-m)(l+m+1)F = 0$$

for the function

$$F(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m}P_l(x) = (1-x^2)^{-m/2}P_l^m(x) .$$

After inserting the derivatives in the equation obtained,

$$F'(x) = (1-x^2)^{-m/2}\left[\frac{dP_l^m}{dx} + \frac{mxP_l^m}{1-x^2}\right],$$

$$F''(x) = (1-x^2)^{-m/2}$$
$$\times\left[\frac{d^2P_l^m}{dx^2} + \frac{2mx}{1-x^2}\frac{dP_l^m}{dx} + \frac{mP_l^m}{1-x^2} + \frac{m(m+2)x^2P_l^m}{(1-x^2)^2}\right],$$

find the required equation:

$$(1-x^2)\frac{d^2P_l^m}{dx^2} - 2x\frac{dP_l^m}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m = 0\,. \tag{1.190}$$

Since the equation is not sensitive to the sign of m, $P_l^{-m}(x)$ and $P_l^m(x)$ may differ only in the factor independent of x (see Problem 1.116*). □

Example 1.19

Use (1.190) to prove the orthogonality of the adjoint Legendre polynomials where symbols m are the same and symbols l are different.

Solution. Write down (1.190) in the form

$$\frac{d}{dx}(1-x^2)\frac{dP_l^m}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m = 0$$

and another similar equation for $P_{l'}^m$. Further, multiply the first equation by $P_{l'}^m$ and the second one by P_l^m, deduct two equations term by term, and integrate over x. This gives us

$$[l(l+1) - l'(l'+1)]\int_{-1}^{1} P_{l'}^m(x)P_l^m(x)dx = 0\,,$$

wherefrom we obtain the orthogonality of

$$\int_{-1}^{1} P_{l'}^m(x)P_l^m(x)dx = 0\,,\quad l' \neq l\,. \tag{1.191}$$

□

Example 1.20

The spherical Legendre function $Y_{lm}(\vartheta, \varphi)$ is described as follows:

$$Y_{lm}(\vartheta, \varphi) = C_{lm} P_l^m(\vartheta) e^{im\varphi} , \tag{1.192}$$

where $P_l^m(\vartheta)$ is an adjoint Legendre polynomial, expressed through trigonometric functions and C_{lm} is the normalization factor. Find the law of transition of this function when the coordinate system is inverted. Make sure that the Legendre spherical functions are orthogonal as to their indices when integrated over the whole spatial angle and write, in an explicit form, the condition of their normalization per unit.

Solution. When the coordinate system is inverted (see Section 1.1), the polar angles are transformed as per the rule $\vartheta \to \pi - \vartheta, \varphi \to \pi + \varphi$, $\cos\vartheta \to -\cos\vartheta$, $e^{im\varphi} \to (-1)^m e^{im\varphi}$. On the basis of the definition of $P_l^m(x)$ (1.189) and (1.191), find

$$Y_{lm}(\vartheta, \varphi) \to Y_{lm}(\pi - \vartheta, \pi + \varphi) = (-1)^l Y_{lm}(\vartheta, \varphi) . \tag{1.193}$$

Integrating over the whole spatial angle means that the boundaries of the angle measurement are $0 \le \vartheta \le \pi, 0 \le \varphi \le 2\pi$. Integrating over φ ensures the orthogonality of m-indexed spherical functions:

$$\int_0^{2\pi} e^{i(m-m')\varphi} d\varphi = 2\pi\delta_{mm'} .$$

Orthogonality over index l is ensured by the adjoint Legendre polynomials (see Example 1.19). The condition of orthogonality and per-unit normalization is as follows:

$$\int_0^{\pi} \sin\vartheta d\vartheta \int_0^{2\pi} d\varphi\, Y^*_{l'm'}(\vartheta, \varphi)\, Y_{lm}(\vartheta, \varphi) = \delta_{ll'}\delta_{mm'} . \tag{1.194}$$

The normalization factor is found as per the following condition:

$$2\pi|C_{lm}|^2 \int_{-1}^{1} [P_l^m(x)]^2 dx = 1 .$$

For the computation of the latter integral, see Problem 1.118*. □

Here is a rather useful relation called *the summation of spherical functions theorem.* Assume that it is probable, which it actually is. If θ is an angle between two vectors (r, ϑ, φ) and $(r', \vartheta', \varphi)$, that is,

$$\cos\theta = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi - \varphi') ,$$

then

$$P_l(\cos\theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\vartheta,\varphi) Y_{lm}(\vartheta',\varphi') \,. \tag{1.195}$$

The derivation of this expansion may be found in Arfken (1970). The method of the theory of group representations is described in detail in Vilenkin (1988) and Gel'fand *et al.* (1958). See also Abramovitz and Stegun (1965), Gradshtein and Ryzhik (2007), Kolokolov *et al.* (2000), and Madelung (1957).

Problems

1.109. Show that when $x = \cos\vartheta$, Legendre's equation assumes the following form:

$$\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\sin\vartheta\frac{\mathrm{d}P_l}{\mathrm{d}\vartheta} + l(l+1)P_l = 0 \,. \tag{1.196}$$

1.110. Obtain the recurrent relations

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x) \,,$$

$$(2l+1)P_l(x) = P'_{l+1}(x) - P'_{l-1}(x) \,,$$

where $l = 1, 2, \ldots$

For that purpose, you may use the Rodrigues formula and the method used in Example 1.12 when considering Bessel functions.

1.111. Using the recurrent relations, find the first five Legendre polynomials.

1.112*. Using the Rodrigues formula, prove the orthogonality of Legendre polynomials with various values of l and find the normalization integral:

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\mathrm{d}x = \frac{2}{2l+1}\delta_{ll'} \,. \tag{1.197}$$

Hint: Express the normalization integral through the Euler beta function.

1.113. Using the generating function for Legendre polynomials, obtain the expansion

$$\frac{1-u^2}{(1-2ux+u^2)^{3/2}} = \sum_{l=0}^{\infty}(2l+1)P_l(x)u^l \,.$$

1.114. Using the results from Example 1.17, obtain the second Legendre polynomial in the form $P_2 = \sum_{k=0}^{2} a_k \cos k\vartheta$.

1.115. Write down (1.190) for adjoint Legendre polynomials in spherical coordinates.

1.116*. Using formula (1.189), show that

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) . \tag{1.198}$$

Hint: Apply the Leibniz formula to the product $(x-1)^l(x+1)^l$ (see Example 1.17).

1.117. Write down, in explicit form, Legendre polynomials P_l^m for $l = 0, 1, 2, 3$.

1.118*. Find the normalization coefficient C_{lm} introduced in Example 1.20. Write down, in explicit form, Legendre's spherical function.

1.119. Write down an equation satisfied by Legendre's spherical function $Y_{lm}(\vartheta, \varphi)$.

1.3.3
Dirac Delta Function

We encounter the concept of the Dirac delta function[28] when trying to describe the charge density $\rho(\boldsymbol{r})$ of a point particle. If a particle with charge e is at the origin, then, obviously, the function $\rho(\boldsymbol{r})$ must have the following properties:

$$\rho(\boldsymbol{r}) = 0 \quad \text{if} \quad r \neq 0 . \tag{1.199}$$

Yet when $r \to 0$, the density of $\rho(\boldsymbol{r})$ must increase fast enough for

$$\int\limits_{\Delta V} \rho(\boldsymbol{r}) \mathrm{d}V = e , \tag{1.200}$$

that is, for an integral over any volume ΔV, containing the point where the particle in question is located, to have the final value that equals the charge e.

Having written $\rho(\boldsymbol{r}) = e\delta(\boldsymbol{r})$, we get, from (1.199) and (1.200), the conditions determining the three-dimensional delta function:

$$\delta(\boldsymbol{r}) = 0 , \quad r \neq 0 ; \quad \delta(\boldsymbol{r}) \to \infty , \quad r \to 0 ; \tag{1.201}$$

$$\int\limits_{\Delta V} \delta(\boldsymbol{r}) \mathrm{d}V = 1 . \tag{1.202}$$

The one-dimensional delta function is described by similar relations:

$$\delta(x) = 0 , \quad x \neq 0 ; \quad \delta(x) \to \infty , \quad x \to 0 ; \quad \int\limits_{\Delta} \delta(x) \mathrm{d}x = 1 , \tag{1.203}$$

28) Paul Adrien Maurice Dirac (1902–1984) was an outstanding English theoretical physicist, a Nobel Prize recipient, one of the founders of quantum mechanics, and the creator of the first quantum field theory (quantum electrodynamics). He formulated the relativist quantum equation for electrons and other leptons and introduced the concept of antiparticles (see Chapter 6).

where Δ is the segment of the x axis that contains the point $x = 0$.

The delta function belongs to the class of singular generalized functions. It acquires its exact meaning under an integral. Consider the integral of the product of the delta function and any continuous and bounded function $f(x)$:

$$\int_{x_1}^{x_2} \delta(x) f(x) \mathrm{d}x \ ,$$

where $x_1 < 0$ and $x_2 > 0$. Since $\delta(x) = 0$ when $x \neq 0$, then only the small neighborhood ϵ of the point $x = 0$ where $f(x)$ is constant, equaling $f(0)$, makes a contribution to the integral:

$$\int_{x_1}^{x_2} \delta(x) f(x) \mathrm{d}x = f(0) \ . \tag{1.204}$$

Further, having replaced variable x with $x-a$ in the argument of the delta function, retracing the previous reasoning, we find the following:

$$\int_{x_1}^{x_2} \delta(x - a) f(x) \mathrm{d}x = f(a) \ , \tag{1.205}$$

if the interval (x_1, x_2) contains the point $x = a$.

Equalities (1.203) and (1.204) show that $\delta(x)$ is an even function of its argument:

$$\delta(x) = \delta(-x) \ . \tag{1.206}$$

Using the latter property and inserting the variable $|\alpha| x = y$, make sure that the relation

$$\int_{x_1}^{x_2} \delta(\alpha x) f(x) \mathrm{d}x = \frac{1}{|\alpha|} f(0) \tag{1.207}$$

is valid. Finally, consider the integral

$$\int_{x_1}^{x_2} \delta(g(x)) f(x) \mathrm{d}x \ ,$$

where a certain smooth function $g(x)$ is in the argument of the delta function. Only points where $g(x) = 0$, that is, the real roots of the function $g(x)$, contribute to the integral. Having labeled them as a_i, we may write

$$\int_{x_1}^{x_2} \delta(g(x)) f(x) \mathrm{d}x = \sum_i \int_{a_i - \epsilon}^{a_i + \epsilon} \delta(g(x)) f(x) \mathrm{d}x \ ,$$

where ϵ is a small number. If $f(x)$ is continuous, then $f(x)$ in the segment $[a_i - \epsilon, a_i + \epsilon]$ may be replaced by $f(a_i)$ and $g(x)$ approximated with the first member of the expansion $g(x) = g'(a_i)(x - a_i)$. As the result, using (1.207), we get

$$\int_{x_1}^{x_2} \delta(g(x)) f(x) \mathrm{d}x = \sum_i \frac{1}{|g'(a_i)|} f(a_i) . \tag{1.208}$$

This property of the delta function may be written as a symbolic equality:

$$\delta(g(x)) = \sum_i \frac{1}{|g'(a_i)|} \delta(x - a_i) . \tag{1.209}$$

If $g'(a_i) = 0$, that is, a_i is a multiple root, relations (1.208) and (1.209) become meaningless. Similarly, the product $\delta(x) f(x)$ is meaningless if the function $f(x)$ has a singularity when $x = 0$.

The derivative from the delta function may also be found. Its exact meaning is in the formula

$$\int_{x_1}^{x_2} f(x) \frac{\partial \delta(x - a)}{\partial x} \mathrm{d}x = -\frac{\partial f(a)}{\partial a} , \tag{1.210}$$

which is produced by integrating by parts. Derivatives of higher orders are found in a similar way:

$$\int_{x_1}^{x_2} f(x) \delta^{(n)}(x - a) \mathrm{d}x = (-1)^n f^{(n)}(a) . \tag{1.211}$$

The function $\delta(x)$ may be regarded as a derivative from the Heaviside step (or staircase) function[29] $\Theta(x)$. This follows from the obvious relation

$$\int_{x_1}^{x} \delta(x) \mathrm{d}x = \Theta(x) = \begin{cases} 1, & x > 0 , \\ \frac{1}{2}, & x = 0 , \\ 0, & x < 0 , \end{cases} \tag{1.212}$$

where the lower bound of integration x_1 is any negative number. Differentiating this equality, we get the following:

$$\Theta'(x) = \delta(x) . \tag{1.213}$$

In equality (1.212), when the bound of integration coincides with the point where the argument of the delta function is reduced to zero, we use *half* of the value of

29) Oliver Heaviside (1850–1925) was an English physicist, engineer, and mathematician. He developed the basics of operational and vector calculus in their present state. For instance, Heaviside introduced the concept of ort, the name "nabla" for Hamilton's operator (∇), and the in-bold notation for labeling vectors (Prokhorov, Yu.,V. (1988) Mathematical Encyclopaedic Dictionary, Sovietskaya Enciklopediya).

the smooth function $f(x) = 1$, that is, we use the integration rule:

$$\int_{x_1}^{a} f(x)\delta(x-a)\mathrm{d}x = \frac{1}{2} f(a)\,. \tag{1.205'}$$

This rule agrees with property (1.206), which is the evenness of the delta function.

The three-dimensional delta function may be regarded as the product of three one-dimensional delta functions:

$$\delta(\boldsymbol{r} - \boldsymbol{a}) = \delta(x - a_x)\delta(y - a_y)\delta(z - a_z)\,. \tag{1.214}$$

This is why all the above properties of one-dimensional delta functions are easily generalized to include the case of three dimensions.

1.3.4 Certain Representations of the Delta Function

One may obtain a visual representation of the delta function and its derivatives by looking at the diagram of a certain continuous function $\delta_\epsilon(x-a)$, such as $\int_\Delta \delta_\epsilon(x-a)\mathrm{d}x = 1$. The parameter ϵ characterizes the width of the interval within which the function in question is other than zero (Figure 1.10).

The delta function and its derivatives are defined as the limits

$$\delta(x-a) = \lim_{\epsilon \to 0} \delta_\epsilon(x-a)\,, \qquad \frac{\partial \delta(x-a)}{\partial x} = \lim_{\epsilon \to 0} \frac{\partial \delta_\epsilon(x-a)}{\partial x}\,,$$

and so on.

Many nonsingular functions depending on a parameter when it has certain limiting values assume the properties of the delta function. The most often used such representations of the delta function areas follows:

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon^2 + x^2} = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left(\frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon} \right)\;; \tag{1.215}$$

$$\delta(x) = \frac{1}{\pi} \lim_{K \to \infty} \left(\frac{\sin Kx}{x} \right)\;; \tag{1.216}$$

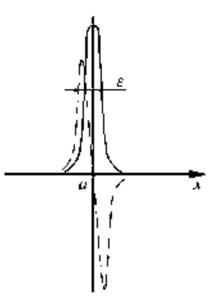

Figure 1.10 The visualization of the delta function and its first derivative.

$$\delta(x) = \frac{1}{\pi} \lim_{K \to \infty} \left(\frac{\sin^2 Kx}{Kx^2} \right) ; \tag{1.217}$$

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi \epsilon}} e^{-x^2/\epsilon} . \tag{1.218}$$

Example (1.216) yields the following representations:

$$\delta(x) = \frac{1}{2\pi} \lim_{K \to \infty} \int_{-K}^{K} e^{ikx} \mathrm{d}k = \frac{1}{\pi} \lim_{K \to \infty} \int_{0}^{K} \cos kx \mathrm{d}k . \tag{1.219}$$

They may be regarded as expansions of the delta function in a Fourier integral.[30)] Sometimes, formulas (1.219) are written without the sign for passage to the limit when integrating over infinite limits.

It is easy to make sure that any of the representations (1.215)–(1.219) agrees with all the properties of (1.203)–(1.207) and the definition (1.210) of a derivative from the delta function. *When computing integrals with delta functions with the use of representations such as ((1.215))–((1.219)), one should pass to the limit after integrating.* For instance, when using (1.216), we have

$$\int_{-a}^{b} \delta(x) f(x) \mathrm{d}x = \frac{1}{\pi} \lim_{K \to \infty} \int_{-aK}^{bK} f\left(\frac{y}{K}\right) \frac{\sin y}{y} \mathrm{d}y = f(0) , \tag{1.220}$$

and the limit (1.216) per se does not exist.

1.3.5 The Representation of the Delta Function through Loop Integrals in a Complex Plane

We will now use Cauchy's[31)] integral formula:

$$\frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - a} \mathrm{d}z = f(a) , \tag{1.221}$$

where $f(z)$ is a function without singularities either within the area bounded by the closed loop C or on the loop itself, in the plane of the complex variable z, integration over which is done counterclockwise. As follows from the comparison of (1.221) with (1.205), the quantity

$$\frac{1}{2\pi i} \frac{1}{z - a}$$

may be regarded as a representation of $\delta(z - a)$ if we agree to integrate over the closed loop that surrounds point $z = a$, within which, just as on the loop itself,

30) For Fourier integrals, see Section 1.3.8.

31) Augustin-Louis Cauchy (1789–1857) was an outstanding French mathematician and physicist. Unlike Laplace, Cauchy was a catholic and a royalist.

there are no other singularities of the subintegral expression. For instance, the loop C may be a circle of small radius.

In applications, one frequently encounters an integral over a proper axis:

$$\int_{x_1}^{x_2} \frac{f(x)}{x-a} \mathrm{d}x \,,$$

where $f(x)$ has no singularities on the segment $[x_1, x_2]$, whereas the limits x_1, x_2 may be infinite. Such an integral, when a is real, has no particular value because the subintegral expression has a pole on the path of integration. Computing this integral requires additional information, that is, the rule of circumventing the special point must be indicated. Usually, the circumvention rule is established on the basis of physical arguments:

$$\int_{C_{\mathrm{Re}}+C_r} \frac{f(x)}{x-a} \mathrm{d}x = \int_{C_r} \frac{f(x)}{x-a} \mathrm{d}x + \int_{C_{\mathrm{Re}}} \frac{f(x)}{x-a} \mathrm{d}x \,.$$

This means that the integral in the first member of the above relation, where integration is done over the whole loop, may be represented (see Figure 1.11) as the sum of two integrals. In the first one of these, integration is done over either the top or the bottom semicircle of a small radius C_r, whereas in the second one, it is done over the remaining part of the loop running along the proper axis (this part of the loop is labeled with the symbol C_{Re}).

The integral over the semicircle of radius $\epsilon \to 0$ gives half of the remainder (with a minus sign for the upper loop, as in Figure 1.11, because the pole is circumvented clockwise):

$$\int_{C_r} \frac{f(x)}{x-a} \mathrm{d}x = -i\pi f(a) \,.$$

The computation of the integral over the proper axis, with the excepted main point, is done so as to find its principal value:

$$\int_{x_1}^{x_2} \frac{f(x)}{x-a} \mathrm{d}x = \lim_{\epsilon \to 0} \left\{ \int_{x_1}^{a-\epsilon} \frac{f(x)}{x-a} \mathrm{d}x + \int_{a+\epsilon}^{x_2} \frac{f(x)}{x-a} \mathrm{d}x \right\} \equiv \mathcal{P} \int_{x_1}^{x_2} \frac{f(x)}{x-a} \mathrm{d}x \,.$$

When we circumvent the pole along the lower semicircle, the sign of half-remainder changes. As a result, we get the following rule of computing integrals (Sokhotskii

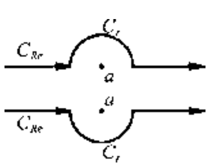

Figure 1.11 The contours of the rounding of poles in the plane of a complex variable.

formulas)[32]:

$$\frac{1}{x-a} = \mp i\pi\delta(x-a) + \mathcal{P}\frac{1}{x-a} \ . \qquad (1.222)$$

The symbol $\mathcal{P}$ represents the principal value (the upper sign is for the upper loop and the lower sign is for the lower one; see Figure 1.11).

Instead of deforming the path of integration, one may slightly displace the pole away from the proper axis. This is done by adding a small imaginary part to the number a: $a \to a \mp i\epsilon$, $\epsilon \to 0$. This kind of substitution will give the following form to identity (1.222):

$$\lim_{\epsilon\to 0} \frac{1}{x-a \pm i\epsilon} = \mp i\pi\delta(x-a) + \frac{\mathcal{P}}{x-a} \ . \qquad (1.223)$$

Both identities, (1.222) and (1.223), have a symbolic (operator) character and must be understood in the way that the integration of their second and first members with any continuous function gives the same result.

Having separated, in the first member of equality (1.223), the real part of the complex expression from the imaginary one, we get the representation (1.215) for $\delta(x-a)$ (with the substitution $x \to x-a$) and for the principal value

$$\frac{\mathcal{P}}{x-a} = \lim_{\epsilon\to 0} \frac{x-a}{(x-a)^2+\epsilon^2} \ . \qquad (1.224)$$

For the rigorous mathematical theory of generalized functions, please see Vladimirov (2002). The applied aspects are described in Zel'dovich and Myshkis (1972). See also Kolokolov *et al.* (2000).

Problems

1.120. Compute the integrals

$$\int_{-2}^{3} (x^2-x-5)\delta(-3x)\mathrm{d}x \ , \quad \int_{-10}^{-3} (x+3)\delta(x+5)\mathrm{d}x \ , \quad \int_{0}^{5} (x+5)\delta(x+5)\mathrm{d}x,$$

and

$$\int_{-\infty}^{\infty} \exp(\alpha x)\delta(x^2+x-2)\mathrm{d}x \ , \quad \alpha = \text{const} \ .$$

1.121. Simplify the expressions $(x-a)\delta(x-a)$, $f(x)\delta(x-a)$, and $(3x^3-7x)\delta(2x^2-6x-4)$.

32) Julian Sokhotskii (1842–1927) was a Russian mathematician who contributed to the development of the theory of functions of complex variables.

1.122. Prove that representations (1.215), (1.217), and (1.218) describe the delta function. For that purpose, compute the integrals of the form $\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x$ from the continuous function $f(x)$, substituting the second member of the respective representation for $\delta(x)$, and then make sure that, after proceeding to the limit, the said integrals produce $f(0)$.

1.123. Write the three-dimensional delta functions $\delta(\boldsymbol{r})$ and $\delta(\boldsymbol{r}-\boldsymbol{a})$ in cylindrical coordinates, where $\boldsymbol{a} = (a_\perp, \alpha_0, a_z)$ is a constant vector given by its cylindrical coordinates.

1.124. Do the same in spherical coordinates $\boldsymbol{a} = (a, \vartheta_0,\ \alpha_0)$.

1.125. Using the delta function, write down the first derivative from the discontinuous function:

$$f(x) = \begin{cases} x^3\,, & \text{if } x < 1\,, \\ 2\,, & \text{if } x = 1\,, \\ x^2 + 2\,, & \text{if } x > 1\,. \end{cases}$$

1.126. The function $f(x)$ has jump discontinuities (finite jumps) at points a_i, $i = 1, 2, \ldots, n$. Write down its first derivative through the delta function.

1.127. Find the rule for computing the integral from the product $f(x)x^n\delta^{(m)}(x)$, where $f(x)$ is the function differentiated (in the classical sense) when $x = 0$, $\delta^{(m)}(x)$ is the mth derivative of the delta function, and n is a positive integer.

1.128. Show that the function $G(|\boldsymbol{r}-\boldsymbol{r}'|) = 1/|\boldsymbol{r}-\boldsymbol{r}'|$ satisfies a Poisson equation with a delta-like second member:

$$\Delta G(|\boldsymbol{r}-\boldsymbol{r}'|) = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')\,. \tag{1.225}$$

1.3.6
Expansion in Total Systems of Orthogonal and Normalized Functions. General Considerations

Let us say there is a certain system of linearly independent functions $\varphi(x, \lambda_n) \equiv \varphi_n(x)$, generally complex valued, defined over a certain interval $[a, b]$ of a real variable x and dependent on the real parameter λ that takes a discrete series of values: $\lambda_1, \lambda_2, \ldots$

Such systems of functions often appear when solving ordinary differential equations or equations in partial derivatives with appropriate boundary conditions, and the number of functions in them is, usually, infinitely large: $n = 0, 1, \ldots$ Let us say the functions have the following properties:

1. They are normalized to unity, that is,

$$\int_a^b |\varphi_n(x)|^2 \mathrm{d}x = 1\,. \tag{1.226}$$

2. They are mutually orthogonal, that is,

$$\int \varphi_m^*(x)\varphi_n(x)\mathrm{d}x = 0 \quad \text{at} \quad m \neq n \,. \tag{1.227}$$

Here, the asterisk marks a complex conjugate. Such systems are called *orthonormalized*, and equalities (1.226) and (1.227) may be written similarly with the use of the Kronecker delta symbol:

$$\int_a^b \varphi_m^*(x)\varphi_n(x)\mathrm{d}x = \delta_{mn} \,. \tag{1.228}$$

Now, we will consider an arbitrary function $f(x)$ with integrable square. That is, a function for which the integral $\int_a^b |f(x)|^2\mathrm{d}x$ is finite. In the case of the finite interval $[a, b]$, this condition will be satisfied by any piecewise continuous function with a limited number of finite jumps within this interval. Now, we will find out how possible the expansion of such a function is in a series over functions $\varphi_n(x)$. For that purpose, we will, firstly, approximate the function in question as a linear superposition that includes n basic functions:

$$f(x) = \sum_{k=0}^{n} c_n\varphi_k(x) + R_n(x) \,, \tag{1.229}$$

where the remainder of the series is labeled $R_n(x)$. We will select the coefficients c_n of the superposition so as to ensure the smallest approximation error. Our measure of error will be the quantity

$$G_n = \int_a^b |R_n(x)|^2\mathrm{d}x = \int_a^b \left| f(x) - \sum_{k=0}^{n} c_n\varphi_k(x) \right|^2 \mathrm{d}x \,. \tag{1.230}$$

Opening the square of the module and using the condition of orthonormality (1.228), we will have

$$\begin{aligned} G_n = {} & \int_a^b |f(x)|^2\mathrm{d}x - \sum_{k=0}^{n} c_k \int_a^b f^*(x)\varphi_k(x)\mathrm{d}x \\ & - \sum_{k=0}^{n} c_k^* \int_a^b f(x)\varphi_k^*(x)\mathrm{d}x + \sum_{k=0}^{n} c_k^* c_k \,. \end{aligned} \tag{1.231}$$

The necessary condition of the minimum quantity G_n, regarded as the function of coefficients c_k, gives us

$$c_k = \int_a^b f(x)\varphi_k^*(x)\mathrm{d}x \,, \tag{1.232}$$

and the expansion error assumes the form of

$$G_n = \int_a^b |f(x)|^2 dx - \sum_{k=0}^{n} |c_k|^2 . \tag{1.233}$$

Since $G_n \geq 0$ by definition, the inequality

$$\sum_{k=0}^{n} |c_k|^2 \leq \int_a^b |f(x)|^2 dx \tag{1.234}$$

is valid whatever the value of n. If the equality

$$\lim_{n\to\infty} G_n = 0 \tag{1.235}$$

is valid for any function with integrable square at the limit, or in another form

$$\int_a^b |f(x)|^2 dx = \sum_{k=o}^{\infty} |c_k|^2 \tag{1.236}$$

(Parseval's identity)[33], then the system of functions $\varphi_n(x), n = 0, 1, \ldots$ is called *complete* or *closed*. These terms mean that no other functions linearly independent of $\varphi_n(x)$ and orthogonal to them exist: any function of the series in question is expandable in a series:

$$f(x) = \sum_{k=0}^{\infty} c_k \varphi_k(x) , \tag{1.237}$$

where expansion coefficients are given by formula (1.232). We note that the above conditions ensure the convergence "on the average" of series (1.237), that is, the reduction of integral (1.230) to zero. This means that the convergence of the series on the function $f(x)$ in question may be disrupted at certain points whose number is finite. If the system of functions $\varphi_n(x)$ is orthonormalized but not complete, then, instead of Parseval's identity (1.236), *Bessel's inequality* becomes valid:

$$\sum_{k=0}^{\infty} |c_k|^2 \leq \int_a^b |f(x)|^2 dx . \tag{1.238}$$

Example 1.21

Show that a complete system of orthonormalized functions satisfies the following relation:

$$\sum_{k=0}^{\infty} \varphi_k^*(x')\varphi_k(x) = \sum_{k=0}^{\infty} \varphi_k(x')\varphi_k^*(x) = \delta(x - x') , \tag{1.239}$$

33) Marc-Antoine Parseval (1755–1836) was a French mathematician.

which may be regarded as one more, different from (1.236), form of the condition of completeness (closeness).

Solution. Having inserted the expansion coefficients from (1.232) into (1.237) and changed the order of the operations of summation and integration, we will have

$$f(x) = \int_a^b dx' f(x') \sum_{k=0}^{\infty} \varphi_k^*(x')\varphi_k(x) = \int_a^b K(x, x') f(x')\mathrm{d}x' , \tag{1.240}$$

where

$$K(x, x') = \sum_{k=0}^{\infty} \varphi_k^*(x')\varphi_k(x) . \tag{1.241}$$

Since equality (1.240) must be valid for any function $f(x)$ of a large class, then the nucleus $K(x, x')$ of the integral transformation (11.15) must have the properties of a delta function. Having computed the expansion coefficients $\delta(x - x')$ for the system of functions $\varphi_k(x)$ as per (1.232), we may make sure that this is the case[34]:

$$c_n = \int \delta(x - x')\varphi_n^*(x)\mathrm{d}x = \varphi_n^*(x') .$$

Therefore, equality (1.239) is true and is the expansion of the delta function in functions $\varphi_k(x)$. □

In certain physical problems, especially in quantum mechanics, a complete system includes not just a discrete series of functions $\varphi_n(x)$ but also functions $\varphi(x, \lambda)$ dependent on the parameter λ, which assumes continuous values from a certain interval, or just functions with a continuous parameter. In cases such as that, the expansion of any function includes both the sum and the integral over the continuous values of λ or just the integral, whereas the condition of completeness assumes the following form:

$$\delta(x - x') = \sum_{k=0}^{\infty} \varphi_k^*(x')\varphi_k(x) + \int \varphi^*(x', \lambda)\varphi(x, \lambda)\mathrm{d}\lambda . \tag{1.242}$$

1.3.7
Fourier Series

The proof of the completeness of specific systems of functions is a nontrivial mathematical problem whose solutions may be found in particular texts[35] (see,

34) Here we leave the class of proper functions with integrable square and use generalized functions.
35) Jean Baptiste Joseph Fourier (1768–1830) was a French mathematician who worked on problems of mathematical physics, especially the theory of heat conduction.

e.g., Sneddon, 1951; Arfken, 1970; Lee, 1965; Tolstov, 1976). The class of complete orthonormalized systems includes the Legendre's system of spherical functions $Y_{lm}(\vartheta, \varphi), l = 0, 1, \ldots, m = -l, -l+1, \ldots, l-1, l$ considered above. Any bounded function "on the surface of a sphere" that is dependent on angles ϑ and φ may be expanded in such functions. When there is no dependence on φ, complete systems on a sphere are formed by Legendre polynomials $P_l(\cos\vartheta)$.

One of the most widely used and complete systems of functions, orthonormalized over the interval $[-\pi, +\pi]$, is the trigonometric system:

$$\frac{1}{\sqrt{2\pi}}\,, \quad \frac{\cos n\tau}{\sqrt{\pi}}\,, \quad \frac{\sin n\tau}{\sqrt{\pi}}\,, \qquad n = 1, 2, \ldots \tag{1.243}$$

The orthonormality of this system of functions may be easily verified directly. The expansion of a certain function in a series over trigonometric functions forms its *Fourier series*. However, sometimes, a general expansion (1.237) over any complete orthonormalized system of functions is also called a Fourier series (in a wider sense).

Because the trigonometric functions (1.243) are periodic, a function being expanded will be represented by the Fourier series, whatever the values of τ, only if it is periodic and has the same period 2π, that is, $f(\tau) = f(\tau + 2n\pi)$, $n = \pm1, \pm2, \ldots$, or if it is specified within the finite segment $b - a = 2L > 0$. In the latter case, in (1.243), the variable τ to $\pi x/L$ must be replaced and the reference point of the coordinate x shifted to the center of the interval $[a, b]$, that is, $x' = x - a - L, -L \le x' \le +L$ is introduced. The function in question, if a Fourier series is set for it, will be expanded, in this case periodically, to the whole proper Ox axis. A nonperiodic function specified over an infinite interval will be correctly represented by a Fourier series only at the final segment $2L$. For it to be represented over the whole Ox axis, the Fourier integral (see below) must be used.

If a Fourier series represents a function that has jump discontinuities (finite jumps), it will, at the point of a jump $x = x_0$, converge on the half sum of the values of the function located on both sides of the jump:

$$\sum_{k=0}^{\infty} c_n \varphi_n(x_0) = \frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)]\,. \tag{1.244}$$

Example 1.22

Write down the Fourier expansion over interval $[-L, +L]$, selecting, as a complete system of functions,[36] exponents with imaginary index $\exp(in\pi x/L), n = 0, \pm1, \ldots$

36) The completeness of the system follows from that previously used. The functions $\sin n\tau$ and $\cos n\tau$ are linearly expressed through $\exp(in\tau)$. This is why the notation of the exponents signifies another form of trigonometric series.

Solution. Make sure that the components in question are mutually orthogonal over the interval $[-L, +L]$:

$$\int_{-L}^{L} \exp\left\{\frac{i(m-n)\pi x}{L}\right\} \mathrm{d}x = 2L\delta_{mn} .$$

Write down the required expansion as

$$f(x) = \sum_{n=-\infty}^{\infty} F_n \exp\left\{\frac{in\pi x}{L}\right\} . \tag{1.245}$$

In order to find the expansion coefficients F_n, multiply both members of (1.245) by $\exp(im\pi x/L)$ and integrate over the interval in question. Owing to the orthogonality of the exponents, after integration in the sum over n, only one member with $n = m$ is left. This will allow you to find the coefficients of the Fourier series:

$$F_m = \frac{1}{2L}\int_{-L}^{L} f(x) \exp\left\{\frac{im\pi x}{L}\right\} \mathrm{d}x . \tag{1.246}$$

As follows from (1.246), if $f(x)$ is a real function, then the Fourier coefficients (1.246), being, in the general case, complex quantities, satisfy the condition $F_{-n} = F_n^*$. This condition ensures the reality of the sum of the series (1.245). □

The Fourier expansion, obviously, may be generalized to include the case of functions that depend on several variables.

Problems

1.129. Expand the periodic function specified within the interval $[-\pi, +\pi]$ in the Fourier series under the conditions $f(x) = x$ for $0 \leq x \leq \pi$ and $f(-x) = f(x)$.

1.130. Do the same for the function under the conditions $f(x) = a$ at $0 \leq x \leq \pi$ and $f(-x) = -f(x)$.

1.131. Expand the periodic function specified within the interval $[-L, +L]$ in the Fourier series under the conditions $f(x) = a$ when $0 \leq x < L/2$ and $f(x) = 0$ when $L/2 < x \leq L$ and $f(-x) = f(x)$.

1.3.8
Fourier Integral

We will now consider a system of functions dependent on the real parameter λ, which takes a continuous series of values:

$$\varphi(x, \lambda) = \frac{1}{\sqrt{2\pi}} e^{i\lambda x} , \quad -\infty < \lambda < \infty . \tag{1.247}$$

These functions are determinate and bounded whatever the real values of the coordinate x may be, that is, within the infinite interval $-\infty < x < \infty$. Using the representation (1.219) of the delta function, we compute the integral

$$\int_{-\infty}^{\infty} \varphi(x,\lambda)\varphi^*(x',\lambda)\mathrm{d}\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, e^{i\lambda(x-x')} = \delta(x-x') \,.$$

The resulting relation coincides with (1.242) (when there are no discrete values of λ) and evidences the completeness of the system of functions $\varphi(x,\lambda)$. This is why any function of a rather large class, defined over the whole proper Ox axis, may be expanded in functions $\varphi(x,\lambda)$:

$$f(x) = \int_{-\infty}^{\infty} F(\lambda)\varphi(x,\lambda)\mathrm{d}\lambda = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}\mathrm{d}\lambda \,. \tag{1.248}$$

The function $F(\lambda)$ is called the *Fourier image* of the original function $f(x)$ or its *Fourier amplitude*. It may be found in the same way as the Fourier series coefficients were found in Example 1.22: by multiplying both members of equality (1.248) by $\varphi^*(x,\mu)$ and integrating over the coordinate x. Changing the order of integration, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\varphi^*(x,\mu)\mathrm{d}x &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, F(\lambda)\int_{-\infty}^{\infty} e^{ix(\lambda-\mu)}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \mathrm{d}\lambda\, F(\lambda)\delta(\lambda-\mu) = F(\mu) \,. \end{aligned} \tag{1.249}$$

This is the equality that allows us to find the Fourier amplitude of the specified function $f(x)$.

The direct and inverted Fourier transforms are often written more easily in their asymmetric form:

$$f(x) = \int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}\frac{\mathrm{d}\lambda}{2\pi} \,, \quad F(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\mathrm{d}x \,. \tag{1.250}$$

The reality of the Fourier integral is ensured by the relation

$$F(-\lambda) = F^*(\lambda) \tag{1.251}$$

when λ and $f(x)$ are real.

Expansion in the Fourier integral is easily generalized for the case of several dimensions. For instance, in three-dimensional space, the Fourier transform may be written as

$$f(\boldsymbol{r}) = \int F(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \frac{d^3\boldsymbol{k}}{(2\pi)^3} , \quad F(\boldsymbol{k}) = \int f(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d^3\boldsymbol{r} . \tag{1.252}$$

In both integrals, integration is done over the whole space.

Example 1.23

Obtain the expansion in the Fourier integral for an infinite interval $-\infty < x < \infty$ by the way of passage to the limit $L \to \infty$ in formulas (1.245) and (1.246).

Solution. When $L \to \infty$, the adjacent members summed as per (1.245) are almost equal. This is why summation may be replaced by integration over $dn = (L/\pi)d\lambda$ within $-\infty, +\infty$. By labeling $\lim_{L\to\infty} 2LF_n$ through $F(\lambda)$, we get, from (1.245) and (1.246), relation (1.250). □

Besides the already mentioned sources, for more information about expansion in systems of functions, series, and integrals, see Arfken (1970), Sneddon (1951), Madelung (1957), and Tolstov (1976).

Problems

1.132. Express the Fourier image of the derivative $f'(x)$ through the Fourier image $F(\lambda)$ of the function $f(x)$. It is presumed that the integral $\int_{-\infty}^{\infty} |f(x)|dx$ is convergent.

1.133. Do the same for the function $f(ax)\exp(ibx)$.

1.134. Find the Fourier image of the function $f(x) = (1 + x^2)^{-1}$.

Hint: Regarding x as a complex variable, close the path of integration with an arc of an infinite radius and use the residue theorem.

1.135. Find the Fourier image of the function $\exp(-\alpha^2 x^2)$.

1.136. Find the three-dimensional Fourier image of the function $f(r) = \exp(-\alpha^2 r^2)$.

1.137*. Find the three-dimensional Fourier image of the function $G(r) = r^{-1}$.

1.138*. Expand the plane wave $\exp(ikr\cos\theta)$ in series over Legendre polynomials $P_l(\cos\theta)$. Find the expansion coefficients, using the orthogonality of Legendre polynomials.

1.139. Assume that the directions of the vectors $\boldsymbol{k}$ and $\boldsymbol{r}$ are specified in a spherical coordinate system by the angles (θ, ϕ) and ϑ, φ, respectively. Expand the plane wave $\exp(i\boldsymbol{k}\cdot\boldsymbol{r})$ in a series over spherical Legendre functions.

Hint: Use the summation theorem for spherical functions.

1.140*. Prove the identity

$$\frac{1}{\sqrt{r_\perp^2 + z^2}} = \int_0^\infty e^{-k|z|} J_0(k r_\perp) \mathrm{d}k \ ,$$

where $r_\perp$ and z are cylindrical coordinates.

1.4
Answers and Solutions

1.1 As follows from (1.6), which is the result of definition (1.3), $|\widehat{a}|^2 = 1$ whatever the angles of rotation are, that is, $|\widehat{a}| = \pm 1$. However, when the angle of rotation equals zero (identical transformation), $|\widehat{a}| = 1$. Since the elements of the rotation matrix are continuous functions of angles, the last value is preserved for all values of the rotation angles. When the axes are inverted, $|\widehat{g}| = -1$. The product of the matrices $\widehat{a}\widehat{g} = \widehat{g}\widehat{a}$, for which $|\widehat{a}\widehat{g}| = |\widehat{a}||\widehat{g}| = -1$, corresponds to the rotation accompanied by the reflection of the axes. Transformations with determinant $+1$ are called proper and transformations with determinant -1 are called nonproper.

1.3

$$P'_{\alpha\beta\cdots\kappa} = |\widehat{a}| a_{\alpha\mu} a_{\beta\nu} \cdots a_{\kappa\sigma} P_{\mu\nu\cdots\sigma} \ . \tag{1.253}$$

Here $|\widehat{a}|$ is the determinant of the transformation matrix. When the three axes are inverted, the transformation matrix $a_{\alpha\beta} = -\delta_{\alpha\beta}$, and that is why $|\widehat{a}| = -1$ and $P'_{\alpha\beta\cdots\kappa} = (-1)^{s+1} P_{\alpha\beta\cdots\kappa}$, in keeping with the definition of a pseudotensor of rank s. Formula (1.5) correctly describes transformations of a polar tensor during rotations and reflections but does not describe reflections of a pseudotensor (even though it correctly describes its rotations).

The rule of the transformation of the asymmetric tensor of rank 3 $e_{\alpha\beta\gamma}$ that describes rotations and reflections must also contain the determinant $|\widehat{a}|$. In the absence of the determinant, the components of the tensor would change their sign during reflections.

1.4

$$T_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) + \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) \ . \tag{1.254}$$

1.5

$$T_{\alpha\beta} = T^h_{\alpha\beta} + T^{ah}_{\alpha\beta} \ , \quad \text{where} \quad T^h_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} + T^*_{\beta\alpha}) \ ,$$

$$T^{ah}_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} - T^*_{\beta\alpha}) \ . \tag{1.255}$$

1.9 $T_{\alpha\beta}$ form a polar tensor of rank 2.

1.10

$$C_\alpha = \frac{1}{2} e_{\alpha\beta\gamma} A_{\beta\gamma} \,, \tag{1.256}$$

that is, $C_1 = A_{23} = -A_{32}$, $C_2 = A_{31} = -A_{13}$, and $C_3 = A_{12} = -A_{21}$.

1.11 $[\boldsymbol{A} \times \boldsymbol{B}]$ is a pseudovector or a polar antisymmetric tensor of rank 2 dual to it: $A_\beta B_\gamma - A_\gamma B_\beta$.$[\boldsymbol{A} \times \boldsymbol{B}] \times \boldsymbol{C}$ is a polar vector and $[\boldsymbol{A} \times \boldsymbol{B}] \cdot \boldsymbol{C}$ is a pseudoscalar.

1.13

$$\mathrm{d}S_\alpha = e_{\alpha\beta\gamma} \mathrm{d}x_\beta \mathrm{d}x'_\gamma = \frac{1}{2} e_{\alpha\beta\gamma} \mathrm{d}S_{\beta\gamma} \,, \tag{1.257}$$

where $\mathrm{d}S_{\beta\gamma} = \mathrm{d}x_\beta \mathrm{d}x'_\gamma - \mathrm{d}x_\gamma \mathrm{d}x'_\beta$ is the projection of the area of the parallelogram onto the coordinate plane $x_\beta x_\gamma$.

1.14

$$\mathrm{d}V = [\mathrm{d}\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}'] \cdot \mathrm{d}\boldsymbol{r}'' = e_{\alpha\beta\gamma} \mathrm{d}x_\alpha \mathrm{d}x'_\beta \mathrm{d}x''_\gamma \,. \tag{1.258}$$

The element of volume is a pseudoscalar. When $\mathrm{d}\boldsymbol{r} = \boldsymbol{e}_1 \mathrm{d}x_1$, $\mathrm{d}\boldsymbol{r}' = \boldsymbol{e}_2 \mathrm{d}x_2$, and $\mathrm{d}\boldsymbol{r}'' = \boldsymbol{e}_3 \mathrm{d}x_3$, we get the usual expression for an element of volume in Cartesian coordinates: $\mathrm{d}V = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$.

1.16

$$\cos\theta = \cos\vartheta \cos\vartheta' + \sin\vartheta \sin\vartheta' \cos(\alpha - \alpha') \,. \tag{1.259}$$

1.17 $(\boldsymbol{A} \times \boldsymbol{B})_0 = i(A_{-1}B_{+1} - A_{+1}B_{-1})$, $(\boldsymbol{A} \times \boldsymbol{B})_{\pm 1} = \pm(A_0 B_{\pm 1} - A_{\pm 1} B_0)$, $\boldsymbol{A} \cdot \boldsymbol{B} = \sum_{\mu=-1}^{+1} (-1)^\mu A_{-\mu} B_\mu$, $r_\mu = r(4\pi/3)^{1/2}(-1)^\mu Y_{1\mu}(\vartheta, \alpha)$.

1.18

$$\widehat{g} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{1.260}$$

1.19 When changing from Cartesian unit vectors to spherical ones (see Figure 1.12a), we have $e'_\mu = a_{\mu\beta} e_\beta$, where e_β $(\beta = 1, 2, 3)$ are Cartesian and e'_μ $(\mu = r, \vartheta, \alpha)$ are spherical unit vectors.

$$\widehat{a} = \begin{pmatrix} \sin\vartheta\cos\alpha & \sin\vartheta\sin\alpha & \cos\vartheta \\ \cos\vartheta\sin\alpha & \cos\vartheta\cos\alpha & -\sin\vartheta \\ -\sin\alpha & \cos\alpha & 0 \end{pmatrix} \,,$$

$$\widehat{a}^{-1} = \begin{pmatrix} \sin\vartheta\cos\alpha & \cos\vartheta\cos\alpha & -\sin\alpha \\ \sin\vartheta\sin\alpha & \cos\vartheta\sin\alpha & \cos\alpha \\ \cos\vartheta & -\sin\vartheta & 0 \end{pmatrix} \,.$$

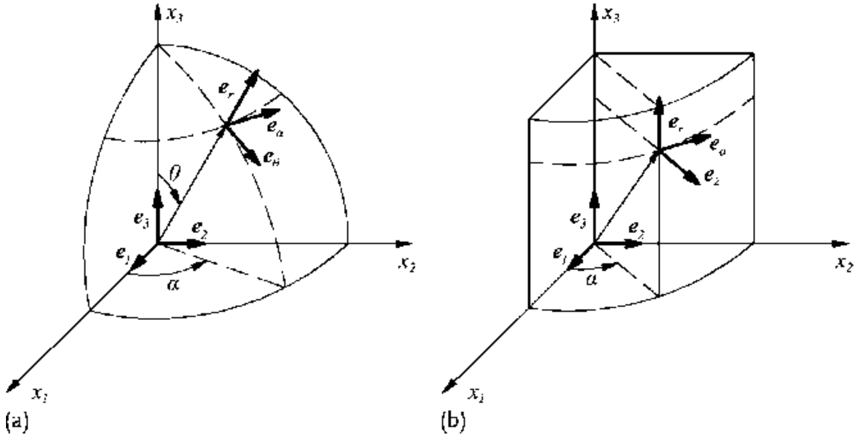

Figure 1.12 Changing from Cartesian to spherical orts (a), and changing from Cartesian to cylindrical orts (b).

When changing from Cartesian unit vectors to cylindrical ones e_r, e_α, e_z (see Figure 1.12b), we have

$$\widehat{a} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widehat{a}^{-1} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.20 Using the results obtained in the previous problem, we get

$$\widehat{g}(\alpha_1\theta\,\alpha_2) = \widehat{g}(\alpha_2)\widehat{g}(\theta)\widehat{g}(\alpha_1) =$$
$$\begin{pmatrix} \cos\alpha_1\cos\alpha_2 - \cos\theta\sin\alpha_1\sin\alpha_2; & \sin\alpha_1\cos\alpha_2 + \cos\theta\cos\alpha_1\sin\alpha_2; & \sin\theta\sin\alpha_2 \\ -\cos\alpha_1\sin\alpha_2 - \cos\theta\sin\alpha_1\cos\alpha_2; & -\sin\alpha_1\sin\alpha_2 + \cos\theta\cos\alpha_1\cos\alpha_2; & \sin\theta\cos\alpha_2 \\ \sin\alpha_1\sin\theta & -\sin\theta\cos\alpha_1 & \cos\theta \end{pmatrix}. \tag{1.261}$$

1.21

$$\widehat{D}(\alpha_1\theta\,\alpha_2) =$$
$$\begin{pmatrix} (1/2)(1+\cos\theta)e^{i(\alpha_1+\alpha_2)}; & -(i/\sqrt{2})\sin\theta\, e^{i\alpha_2}; & -(1/2)(1-\cos\theta)e^{i(\alpha_2-\alpha_1)} \\ -(i/\sqrt{2})\sin\theta\, e^{i\alpha_1}; & \cos\theta; & -(i/\sqrt{2})\sin\theta\, e^{-i\alpha_1} \\ -(1/2)(1-\cos\theta)e^{i(\alpha_1-\alpha_2)}; & -(i/\sqrt{2})\sin\theta\, e^{-i\alpha_2}; & (1/2)(1+\cos\theta)e^{-i(\alpha_1+\alpha_2)} \end{pmatrix}.$$

1.22 The zero angle rotation matrix equals 1 (identical transformation) and when rotation is by a small angle, $|\varepsilon_{\alpha\beta}| \ll 1$. To prove the antisymmetry of $\widehat{\varepsilon}$, we will use the invariance of $r^2 = \delta_{\alpha\beta}x_\alpha x_\beta$ relative to the rotation. Since $x'_\alpha = x_\alpha + \varepsilon_{\alpha\beta}x_\beta$, we have $r'^2 = r^2 + 2\varepsilon_{\alpha\beta}x_\alpha x_\beta$ to small quantities of the first order. As follows from the invariance of r^2, $\varepsilon_{\alpha\beta}x_\alpha x_\beta = 0$ when x_α are arbitrary, which is possible only when $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$.

Now, we introduce a vector with components $\delta\varphi_\alpha = (1/2)e_{\alpha\beta\gamma}\varepsilon_{\beta\gamma}$. Then, $\boldsymbol{r}' = \boldsymbol{r} + \delta\boldsymbol{\varphi} \times \boldsymbol{r}$, which shows that $\delta\boldsymbol{\varphi}$ is the vector of an infinitely small rotation whose direction indicates the axis of rotation and its size indicates the angle of rotation.

1.23 If rotations are specified by small vectors $\delta\boldsymbol{\varphi}_1$ and $\delta\boldsymbol{\varphi}_2$, then, after the second rotation,

$$\boldsymbol{r}'' = \boldsymbol{r}' + \delta\boldsymbol{\varphi}_2 \times \boldsymbol{r}' = \boldsymbol{r} + (\delta\boldsymbol{\varphi}_1 + \delta\boldsymbol{\varphi}_2) \times \boldsymbol{r} + \delta\boldsymbol{\varphi}_2 \times (\delta\boldsymbol{\varphi}_1 \times \boldsymbol{r}) .$$

The vector of the resulting rotation $\delta\boldsymbol{\varphi} = \delta\boldsymbol{\varphi}_1 + \delta\boldsymbol{\varphi}_2$ may be introduced only if the last member of the second order is disregarded.

In the general case, the noncommutative character of the rotation matrices is shown by expression (1.261): the completion of rotation in the sequence $\alpha_2, \theta, \alpha_1$, inverse with respect to the matrix it is written for (1.261) corresponds to the substitution $\alpha_1 \to \alpha_2$. In this event, the form of the matrix will change if $\theta \neq 0$. The case of $\theta = 0$ is in keeping with rotation by angles α_1 and α_2 around the same Ox_3 axis and such rotations are commutative.

1.24 Any tensor of rank 2 may be written as $T_{\alpha\beta} = S_{\alpha\beta} + A_{\alpha\beta}$, whereas any Hermitian tensor may be written as $T^h_{\alpha\beta} = S_{\alpha\beta} + iA_{\alpha\beta}$, where $S_{\alpha\beta}$ and $A_{\alpha\beta}$ are symmetric and antisymmetric real tensors. The antisymmetric tensor $A_{\alpha\beta}$ is equivalent to a vector (see Problem 1.10) which will not be reduced to zero by any rotations. This is why only the real symmetric part of the any tensor of rank 2 may be diagonalized.

1.25

$$S_{\alpha\beta} = S^{(1)} n^{(1)}_\alpha n^{(1)}_\beta + S^{(2)} n^{(2)}_\alpha n^{(2)}_\beta + S^{(3)} n^{(3)}_\alpha n^{(3)}_\beta . \tag{1.262}$$

1.26 Computing a determinant (1.27) while keeping in mind that the principal values of tensor $S^{(i)}$ may be invariant only if such are the coefficients of an algebraic cubic equation, we find three invariants:

$$I_1 = S_{11} + S_{22} + S_{33} = S^{(1)} + S^{(2)} + S^{(3)}, \tag{1.263}$$

$$I_2 = D_{11} + D_{22} + D_{33} = S^{(1)} S^{(2)} + S^{(1)} S^{(3)} + S^{(2)} S^{(3)}, \tag{1.264}$$

$$I_3 = D = S^{(1)} S^{(2)} S^{(3)} , \tag{1.265}$$

where $D = |\widehat{S}|$ is the determinant of the tensor and $D_{\alpha\beta}$ are the algebraic cofactors of the determinant. Expressions in the second members of the equalities follow from Viète's theorem[37] about the connection between the coefficients of a cubic equation with its roots. The result is valid for any tensor of rank 2.

37) François Viète (1540–1603) was an French mathematician, and a lawyer by trade.

1.27 The row and column expansions of the determinant $D = |\widehat{T}|$ are written, respectively, as

$$T_{\alpha\beta} D_{\gamma\beta} = D\delta_{\alpha\gamma} \ , \quad D_{\gamma\alpha} T_{\gamma\beta} = D\delta_{\alpha\beta} \ ,$$

where $D_{\gamma\alpha} = (-1)^{\alpha+\gamma} \Delta_{\gamma\alpha}$ is an algebraic cofactor, and $\Delta_{\gamma\alpha}$ is the minor determinant D, that is, the determinant remaining after the elimination in it of the γ row and the α column. In accordance with the results obtained in the previous problem, since D is invariant and since $\delta_{\alpha\beta}$ is a tensor, then the algebraic cofactors $D_{\gamma\alpha}$ also form a tensor. Relations

$$T^{-1}_{\alpha\beta} = \frac{D_{\beta\alpha}}{D} \tag{1.266}$$

form a tensor inverse to $\widehat{T}$. To make the inverse tensor possible, it is necessary and sufficient that $D = |\widehat{T}| \neq 0$.

1.29

1. $A^2(B \cdot C) + (A \cdot B)(A \cdot C)$.
2. $[(A \times B) \times C] \cdot [(A' \times B') \times C']$.

1.31

$$\begin{aligned} &(A \cdot A')(B \cdot B')(C \cdot C') + (A \cdot B')(B \cdot C')(C \cdot A') \\ &+ (B \cdot A')(C \cdot B')(A \cdot C') - (A \cdot C')(C \cdot A')(B \cdot B') \\ &- (A \cdot B')(B \cdot A')(C \cdot C') - (B \cdot C')(C \cdot B')(A \cdot A') \ . \end{aligned}$$

1.32 Now, we will present our proof for a vector and tensor of rank 2.

1. In accordance with the situation in the problem, at any rotation $A'_\alpha = A_\alpha$, that is, $A'_x = A_x$, $A'_y = A_y$, and $A'_z = A_z$. Rotating the coordinate system around the Oz axis by angle π, we get $A'_x = -A_x$, $A'_y = -A_y$, and $A'_z = A_z$. These equalities are compatible with the previous ones only if $A_x = A_y = 0$. Performing rotation around the Ox axis by angle π, we will similarly prove that $A_z = 0$, that is, vector $A = 0$.
2. Any tensor of rank 2 may be represented as the sum of a symmetric tensor and an antisymmetric tensor: $T_{\alpha\beta} = S_{\alpha\beta} + A_{\alpha\beta}$. An antisymmetric tensor is equivalent to a certain pseudovector and, in accordance with what was proven above, its components do not depend on the reference frame only if they are equal to zero. So we will consider a symmetric tensor $S_{\alpha\beta}$.

We will select a coordinate system where the symmetric tensor has a diagonal form $S^{(\alpha)}\delta_{\alpha\beta}$. If $S^{(\alpha)}$ are not equal to each another, then the components of the tensor depend on the selection of the axis, that is, what digit (1, 2, or 3) denotes the selected one. Only when $S^{(1} = S^{(2)} = S^{(3)} = S$ do the components of the tensor $S\delta_{\alpha\beta}$ not depend on the selection of the axis.

1.33

$$\overline{n_\alpha} = 0\,, \overline{n_\alpha n_\beta} = \frac{1}{3}\delta_{\alpha\beta}, \overline{n_\alpha n_\beta n_\gamma} = 0\,,$$
$$\overline{n_\alpha n_\beta n_\gamma n_\mu} = \frac{1}{15}(\delta_{\alpha\beta}\delta_{\gamma\nu} + \delta_{\alpha\gamma}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\gamma})\,.$$

1.34

$$\frac{a^2}{3}, \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{3}, \frac{\boldsymbol{a}}{3}, \frac{2a^2}{3}, \frac{2\boldsymbol{a}\cdot\boldsymbol{b}}{3}, \frac{(\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{c}\cdot\boldsymbol{d}) + (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) + (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})}{15}\,.$$

1.35 $n^2, n'^2, l^2, \boldsymbol{n}\cdot\boldsymbol{n}', (\boldsymbol{n}\times\boldsymbol{n}')\cdot\boldsymbol{l}, (\boldsymbol{n}\cdot\boldsymbol{l})^2, (\boldsymbol{n}'\cdot\boldsymbol{l})^2, (\boldsymbol{n}\cdot\boldsymbol{l})(\boldsymbol{n}'\cdot\boldsymbol{l})$.

1.36 $\boldsymbol{n}\cdot\boldsymbol{l}, \boldsymbol{n}'\cdot\boldsymbol{l}, \boldsymbol{n}_1\cdot(\boldsymbol{n}_2\times\boldsymbol{n}_3)$.

1.37

1. $a_\alpha{}^\beta = \boldsymbol{e}'_\alpha\cdot\boldsymbol{e}^\beta$.

2. If $\boldsymbol{e}_\beta = (\widehat{a}^{-1})_\beta{}^\alpha$, then $(\widehat{a}^{-1})_\beta{}^\alpha = \boldsymbol{e}'^\alpha\cdot\boldsymbol{e}_\beta = a^\alpha{}_\beta \neq a_\alpha{}^\beta$; in accordance with the definition of the inverse matrix $a_\alpha{}^\beta a^\gamma{}_\beta = \delta^\gamma_\alpha$, $a_\beta{}^\alpha a^\beta{}_\gamma = \delta^\alpha_\gamma = \delta^\gamma_\alpha$.

3. $\boldsymbol{e}'^\alpha = a^\alpha{}_\beta\boldsymbol{e}^\beta\,, \quad \boldsymbol{e}^\alpha = a_\beta{}^\alpha\boldsymbol{e}'^\beta$.
4. $A'_\beta = a_\beta{}^\alpha A_\alpha\,, \quad A'^\beta = a^\beta{}_\alpha A^\alpha\,; \quad A_\alpha = a^\beta{}_\alpha A'_\beta\,, \quad A^\alpha = a_\beta{}^\alpha A'^\beta$.

5. $g'_{\alpha\beta} = a_\alpha{}^\gamma a_\beta{}^\mu g_{\gamma\mu}\,, \quad g'^{\alpha\beta} = a^\alpha{}_\gamma a^\beta{}_\mu g^{\gamma\mu}\,, \quad g_{\alpha\beta} = a^\gamma{}_\alpha a^\mu{}_\beta g'_{\gamma\mu}\,,$
$g^{\alpha\beta} = a_\gamma{}^\alpha a_\mu{}^\beta g'^{\gamma\mu}$.

The formulas in answer 4 are generalized directly for the cases of the transformation of covariant, contravariant, and mixed components of tensors of any rank.

1.38 When the systems of coordinates are inverted, the components of the vectors of both bases change their signs: $\boldsymbol{e}'_\alpha = -\boldsymbol{e}_\alpha$, $\boldsymbol{e}'^\alpha = -\boldsymbol{e}^\alpha$, $\alpha = 1, 2, 3$.

1.39

$$\boldsymbol{A}\cdot\boldsymbol{B} = g^{\alpha\beta}A_\alpha B_\beta = g_{\alpha\beta}A^\alpha B^\beta = A^\alpha B_\alpha = A_\alpha B^\alpha = \text{inv}\,. \tag{1.267}$$

$$\mathrm{d}l^2 = \mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} = g^{\alpha\beta}\mathrm{d}x_\alpha\mathrm{d}x_\beta = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta = \mathrm{d}x^\alpha\mathrm{d}x_\alpha = \text{inv}\,. \tag{1.268}$$

Note: In all cases when covariant and contravariant components do not coincide, the operation of tensor contraction must be done as a summation assumed over one upper and one lower index. Any tensor summed over two upper and two lower symbols is not a tensor of any rank.

1.40

$$C_\alpha = \sqrt{g}(A^\beta B^\gamma - A^\gamma B^\beta)\,, \quad C^\alpha = (1/\sqrt{g})(A_\beta B_\gamma - A_\gamma B_\beta)\,, \tag{1.269}$$

where $g = |\widehat{g}|$ and numbers α, β, γ form a circular permutation 1, 2, 3.

The formulas shown may be regarded as the generalization of expression (1.23) for the case of an oblique basis. Having written (1.269) in the form of

$$C_\alpha = E_{\alpha\beta\gamma} A^\beta B^\gamma\,, \quad C^\alpha = E^{\alpha\beta\gamma} A_\beta B_\gamma\,, \tag{1.270}$$

we find a representation for an antisymmetric tensor of rank 3 in the oblique basis:

$$E_{\alpha\beta\gamma} = \sqrt{g} e_{\alpha\beta\gamma}\,, \quad E^{\alpha\beta\gamma} = \frac{1}{\sqrt{g}} e^{\alpha\beta\gamma}\,, \tag{1.271}$$

where $e_{\alpha\beta\gamma}$ and $e^{\alpha\beta\gamma}$, related to the orthogonal basis, are similar and determined by conditions (1.21). It is easy to verify that $E^{\alpha\beta\gamma}$ is produced from $E_{\alpha\beta\gamma}$ (and vice versa) in accordance with the rule of upping and lowering indices as per (1.35).

1.41 $\cos\theta = \frac{A_\alpha B^\alpha}{(A_\beta A^\beta B_\gamma B^\gamma)^{1/2}}$.

1.42 $g_{\mu\nu} = \left(\boldsymbol{e}^\alpha_{(D)}\right)_\mu \left(\boldsymbol{e}^{(D)}_\alpha\right)_\nu$, $g^{\mu\nu} = \left(\boldsymbol{e}^\alpha_{(D)}\right)^\mu \left(\boldsymbol{e}^{(D)}_\alpha\right)^\nu$, $g^\mu_\nu = \left(\boldsymbol{e}^\alpha_{(D)}\right)^\mu \left(\boldsymbol{e}^{(D)}_\alpha\right)_\nu = \delta^\mu_\nu$.

1.44

$$\boldsymbol{A}^2 = g_{\alpha\beta} A^\alpha A^\beta = A^\alpha A_\alpha\,; \quad \cos\theta = \frac{A_\alpha B^\alpha}{\left(\boldsymbol{A}^2 \boldsymbol{B}^2\right)^{1/2}}\,,$$

in complete analogy with the result obtained in Problem 1.41 for an affine system.

1.45 In accordance with the common rules (1.51), we have

$$E^{\mu\nu\lambda} = \frac{\partial q^\mu}{\partial x^\alpha} \frac{\partial q^\nu}{\partial x^\beta} \frac{\partial q^\lambda}{\partial x^\gamma} e^{\alpha\beta\gamma}\,,$$

wherefrom the antisymmetry of $E^{\mu\nu\lambda}$ over any pair of symbols follows. This allows us to write $E^{\alpha\beta\gamma} = S e^{\alpha\beta\gamma}$, where S is a certain scalar. To define it, we consider a special case and get

$$E^{123} = \frac{\partial q^1}{\partial x^\alpha} \frac{\partial q^2}{\partial x^\beta} \frac{\partial q^3}{\partial x^\gamma} e^{\alpha\beta\gamma} = \left|\frac{\partial q^\lambda}{\partial x^\alpha}\right| = J^{-1} = g^{-1/2}\,,$$

where (1.39) and (1.53) are used. Therefore, $S = g^{-1/2}$, and so we have obtained, in another way, the second formula (1.271).

1.46

1. $\mathrm{d}l_{(1)} = \sqrt{g_{11}}\mathrm{d}q^1$, $\mathrm{d}l_{(2)} = \sqrt{g_{22}}\mathrm{d}q^2$, $\mathrm{d}l_{(3)} = \sqrt{g_{33}}\mathrm{d}q^3$.

2. The covariant basis (1.46) is such a vector.
3. Using the result obtained in Problem 1.44, we find

$$\cos\vartheta_{12} = \frac{g_{12}}{\sqrt{g_{11}g_{22}}}\,, \quad \cos\vartheta_{13} = \frac{g_{13}}{\sqrt{g_{11}g_{33}}}\,, \quad \cos\vartheta_{23} = \frac{g_{23}}{\sqrt{g_{22}g_{33}}}\,.$$

4. For the curvilinear coordinate system to be orthogonal, it is necessary and sufficient that the equalities $g_{12} = g_{23} = g_{13} = 0$ be valid in every point of space.

1.47 For a spherical system $g_{rr} = 1$, $g_{\vartheta\vartheta} = r^2$, $g_{\alpha\alpha} = r^2\sin^2\vartheta$, $g_{r\vartheta} = g_{r\alpha} = g_{\vartheta\alpha} = 0$; $g^{rr} = 1$, $g^{\vartheta\vartheta} = r^{-2}$, $g^{\alpha\alpha} = r^{-2}\sin^{-2}\vartheta$, $g^{r\vartheta} = g^{r\alpha} = g^{\vartheta\alpha} = 0$; $\boldsymbol{e}_r = \boldsymbol{e}^r = \boldsymbol{e}_{r*}$, $\boldsymbol{e}_\vartheta = r^2\boldsymbol{e}^\vartheta = r\boldsymbol{e}_r$, $\boldsymbol{e}_\alpha = r^2\sin^2\vartheta\,\boldsymbol{e}^\alpha = r\sin\vartheta\,\boldsymbol{e}_{\alpha*}$.

For a cylindrical system $g_{rr} = g_{zz} = 1$, $g_{\alpha\alpha} = r^2$, $g_{r\alpha} = g_{rz} = g_{\alpha z} = 0$; $g^{rr} = g^{zz} = 1$, $g^{\alpha\alpha} = r^{-2}$, $g^{r\alpha} = g^{rz} = g^{\alpha z} = 0$; $\boldsymbol{e}_r = \boldsymbol{e}^r = \boldsymbol{e}_{r*}$, $\boldsymbol{e}_\alpha = r^2\boldsymbol{e}^\alpha = r\boldsymbol{e}_{\alpha*}$, $\boldsymbol{e}_z = \boldsymbol{e}^z = \boldsymbol{e}_{z*}$.

The asterisks mark basic unit orts introduced in Problem 1.18. Covariant and contravariant basic vectors have different dimensions and are different in length. Their lengths are, generally speaking, not unity.

1.50 $R = \frac{a^2}{2} + ab - \frac{b^3}{3}$.

1.52 $\boldsymbol{l}, \boldsymbol{l}$.

1.54

$$\frac{dr}{A_r} = \frac{r\,d\alpha}{A_\alpha} = \frac{dz}{A_z}\,, \quad \frac{dr}{A_r} = \frac{r\,d\vartheta}{A_\vartheta} = \frac{r\sin\vartheta\,d\alpha}{A_\alpha}\,.$$

1.55 Because the gradient contains the first derivatives over the coordinates, the following notation may be used:

$$\operatorname{grad}\frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3} = \frac{\operatorname{grad}(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3} + (\boldsymbol{p}\cdot\boldsymbol{r})\operatorname{grad}\frac{1}{r^3}\,.$$

Using the results obtained in Problems (1.52) and (1.53), we finally get the following:

$$\operatorname{grad}\frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3} = -\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} + \frac{\boldsymbol{p}}{r^3}\,.$$

1.56 Direct the polar axis along the vector $\boldsymbol{p}$ and project the vector $\boldsymbol{E}$ onto the orts of spherical coordinates:

$$E_r = \frac{2p\cos\vartheta}{r^3}\,, \quad E_\vartheta = \frac{p\sin\vartheta}{r^3}\,, \quad E_\varphi = 0\,.$$

In spherical coordinates, vector lines are found with the following system of equations:

$$\frac{dr}{E_r} = \frac{r d\vartheta}{E_\vartheta} = \frac{r\sin\vartheta\,d\varphi}{E_\varphi}\,.$$

The reduction of the E_φ component to zero means that the differential $\mathrm{d}\varphi = 0$, that is, $\varphi = \text{const}$ must also become zero and, therefore, all the vector lines lie within the planes passing through the vector $\boldsymbol{p}$. Inserting the nonzero projections of $\boldsymbol{H}$ into the only remaining equation and eliminating common factors, we get a first-order differential equation with separable variables: $\mathrm{d}r/r = 2\cot\vartheta\,\mathrm{d}\vartheta$. The termwise integration of the first and second members gives $\ln r - \ln r_0 = 2\ln\sin\vartheta$ or $r(\vartheta) = r_0 \sin^2\vartheta$, where r_0 is the constant of integration that has the meaning of the distance between the vector line and the origin in the plane perpendicular to the vector $\boldsymbol{p}$.

1.57

$$\nabla_{\pm 1} = \mp\frac{1}{\sqrt{2}} e^{\pm i\alpha}\left(\sin\vartheta\frac{\partial}{\partial r} + \frac{\cos\vartheta}{r}\frac{\partial}{\partial\vartheta} \pm \frac{i}{r\sin\vartheta}\frac{\partial}{\partial\alpha}\right),$$

$$\nabla_0 = \cos\vartheta\frac{\partial}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial}{\partial\vartheta}\,.$$

1.59 3, 0, 0, $2\boldsymbol{\omega}$.

1.60

$$\boldsymbol{H} = \frac{3(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{m}}{r^3}\,.$$

1.63 $\frac{\varphi'\boldsymbol{r}}{r}$, $3\varphi + r\varphi'$, 0, $\boldsymbol{l}\varphi + \boldsymbol{r}(\boldsymbol{l}\cdot\boldsymbol{r})\frac{\varphi'}{r}$.

1.64 $\varphi(r) = \frac{\text{const}}{r^3}$.

1.65 $(\boldsymbol{a}\cdot\boldsymbol{b})$, $\boldsymbol{a}\times\boldsymbol{b}$; $4(\boldsymbol{a}\cdot\boldsymbol{r})$, $\boldsymbol{a}\times\boldsymbol{r}$; 0, $(2\varphi + r\varphi')\boldsymbol{a} - \boldsymbol{r}(\boldsymbol{a}\cdot\boldsymbol{r})\frac{\varphi'}{r}$; $-2(\boldsymbol{a}\cdot\boldsymbol{r})$, $3(\boldsymbol{r}\times\boldsymbol{a})$.

1.66 $\boldsymbol{A} + (\boldsymbol{r}\cdot\boldsymbol{A}')\frac{\boldsymbol{r}}{r}$, $(\boldsymbol{A}'\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\boldsymbol{B}')\frac{\boldsymbol{r}}{r}$, $\frac{\varphi'}{r}(\boldsymbol{r}\cdot\boldsymbol{A}) + \frac{\varphi}{r}(\boldsymbol{r}\cdot\boldsymbol{A}')$, $\frac{\varphi'}{r}(\boldsymbol{r}\times\boldsymbol{A}) + \frac{\varphi}{r}(\boldsymbol{r}\times\boldsymbol{A}')$, $\frac{\boldsymbol{l}\cdot\boldsymbol{r}}{r}(\varphi'\boldsymbol{A} + \varphi\boldsymbol{A}')$.

1.68

$$\int\limits_V (\operatorname{grad}\varphi\cdot\operatorname{curl}\boldsymbol{A})\mathrm{d}V = \oint\limits_S (\boldsymbol{A}\times\operatorname{grad}\varphi)\cdot\mathrm{d}\boldsymbol{S} = \oint\limits_S \varphi\operatorname{curl}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}\,.$$

1.69 $\boldsymbol{a}V$, $\boldsymbol{a}V$.

1.70* Use the method of the scalar multiplication of a constant vector by each of the integrals in question to find the following relations:

$$\oint\limits_S \boldsymbol{n}\varphi\mathrm{d}S = \int\limits_V \operatorname{grad}\varphi\,\mathrm{d}V\,; \tag{1.272}$$

$$\oint_S (\boldsymbol{n} \times \boldsymbol{A}) \mathrm{d}S = \int_V \operatorname{curl} \boldsymbol{A} \,\mathrm{d}V \;; \tag{1.273}$$

$$\oint_S (\boldsymbol{n} \cdot \boldsymbol{b}) \boldsymbol{A} \mathrm{d}S = \int_V (\boldsymbol{b} \cdot \nabla) \boldsymbol{A} \,\mathrm{d}V \;; \tag{1.274}$$

$$\oint_S T_{\alpha\beta} \boldsymbol{n}_\beta \mathrm{d}S = \int_V \frac{\partial T_{\alpha\beta}}{\partial x_\beta} \mathrm{d}V \;. \tag{1.275}$$

All these relations may be regarded as generalizations of the Gauss–Ostrogradskii theorem:

$$\oint_S \boldsymbol{n}(\ldots) \mathrm{d}S = \int_V \nabla(\ldots) \mathrm{d}V \;, \tag{1.276}$$

where the $(\ldots)$ symbol denotes a tensor of any rank.

1.76 $\int_S (\nabla u \times \nabla f) \cdot \mathrm{d}\boldsymbol{S}$.

1.81* As per the common rule (1.105), covariant divergence is expressed as follows:

$$\Gamma^\mu_{\mu\alpha} = \frac{1}{2} g^{\mu\lambda} \left(\frac{\partial g_{\mu\lambda}}{\partial q^\alpha} + \frac{\partial g_{\alpha\lambda}}{\partial q^\mu} - \frac{\partial g_{\alpha\mu}}{\partial q^\lambda} \right) = \frac{1}{2} g^{\mu\lambda} \frac{\partial g_{\mu\lambda}}{\partial q^\alpha} \;,$$

that is,

$$A^\mu_{;\mu} = \frac{\partial A^\mu}{\partial q^\mu} + \frac{1}{2} g^{\mu\lambda} \frac{\partial g_{\mu\lambda}}{\partial q^\alpha} A^\alpha \;. \tag{1}$$

Now consider the determinant $g = |g_{\mu\nu}|$. Its differential equals the sum of the differentials of all its elements multiplied by the respective algebraic cofactors: $\mathrm{d}g = D^{\mu\nu} \mathrm{d}g_{\mu\nu}$, where $D^{\mu\nu} = (-1)^{\mu+\nu} \Delta^{\mu\nu}$, $\Delta^{\mu\nu}$ is the minor. On the other hand, the algebraic cofactors are expressed through the components of the inverse tensor, that is, $g^{\mu\nu}$: $D^{\mu\nu} = g g^{\mu\nu}$ (see Problem 1.27). In the end, we have

$$\mathrm{d}g = g g^{\mu\nu} \mathrm{d}g_{\mu\nu} \;, \quad g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial q^\lambda} = \frac{1}{g} \frac{\partial g}{\partial q^\lambda} \;.$$

Having inserted the latter quantity in (1), we find the expression specified in the condition for the problem.

1.82

$$\Delta S = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\mu} \left(\sqrt{g} g^{\mu\nu} \frac{\partial S}{\partial q^\nu} \right) \;. \tag{1.277}$$

1.83

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\mu} \left(\sqrt{g} T^{\mu\nu}\right) + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} \; .$$

1.84

$$A^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\nu} \left(\sqrt{g} A^{\mu\nu}\right) \; .$$

1.86

$$g_{\mu\nu;\lambda} = g^{\mu\nu}{}_{;\lambda} = 0 \; .$$

1.90*

$$\begin{aligned} (\Delta \boldsymbol{A})_r &= \Delta A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\alpha}{\partial \alpha}, \\ (\Delta \boldsymbol{A})_\alpha &= \Delta A_\alpha - \frac{A_\alpha}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \alpha}, \\ (\Delta \boldsymbol{A})_z &= \Delta A_z \; . \end{aligned} \tag{1.278}$$

1.91*

$$\begin{aligned} (\Delta \boldsymbol{A})_r &= \Delta A_r - \frac{2}{r^2} A_r - \frac{2}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}(A_\vartheta \sin\vartheta) - \frac{2}{r^2 \sin\vartheta}\frac{\partial A_\alpha}{\partial\alpha}, \\ (\Delta \boldsymbol{A})_\vartheta &= \Delta A_\vartheta - \frac{A_\vartheta}{r^2 \sin^2\vartheta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\vartheta} - \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial A_\alpha}{\partial\alpha}, \\ (\Delta \boldsymbol{A})_\alpha &= \Delta A_\alpha - \frac{A_\alpha}{r^2\sin^2\vartheta} + \frac{2}{r^2\sin\vartheta}\frac{\partial A_r}{\partial\alpha} + \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial A_\vartheta}{\partial\alpha} \; . \end{aligned} \tag{1.279}$$

1.92 (i) $A + B\ln r$; (ii) $A + B\alpha$; (iii) $A + Bz$.

1.93 (i) $A + \frac{B}{r}$; (ii) $A + B\ln\tan(\frac{\vartheta}{2})$; (iii) $A + B\alpha$.

1.94*

$$x = \pm\left[\frac{(\xi + a^2)(\eta + a^2)(\zeta + a^2)}{(b^2 - a^2)(c^2 - a^2)}\right]^{1/2},$$

$$y = \pm\left[\frac{(\xi + b^2)(\eta + b^2)(\zeta + b^2)}{(c^2 - b^2)(a^2 - b^2)}\right]^{1/2},$$

$$z = \pm\left[\frac{(\xi + c^2)(\eta + c^2)(\zeta + c^2)}{(a^2 - c^2)(b^2 - c^2)}\right]^{1/2};$$

$$h_1 = \frac{\sqrt{(\xi - \eta)(\xi - \zeta)}}{2R_\xi}, \quad h_2 = \frac{\sqrt{(\eta - \zeta)(\eta - \xi)}}{2R_\eta},$$

$$h_3 = \frac{\sqrt{(\zeta - \xi)(\zeta - \eta)}}{2R_\zeta};$$

$$\Delta = \frac{4}{(\xi - \eta)(\xi - \zeta)(\eta - \zeta)} \left[(\eta - \zeta) R_\xi \frac{\partial}{\partial \xi} \left(R_\xi \frac{\partial}{\partial \xi} \right) \right.$$
$$\left. + (\zeta - \xi) R_\eta \frac{\partial}{\partial \eta} \left(R_\eta \frac{\partial}{\partial \eta} \right) + (\xi - \eta) R_\zeta \frac{\partial}{\partial \zeta} \left(R_\zeta \frac{\partial}{\partial \zeta} \right) \right],$$

where $R_u = \sqrt{(u + a^2)(u + b^2)(u + c^2)}$. The formulas for x, y, and z show that there are eight threes: x, y, z for every three values of ξ, η, ζ. One may make sure that the ellipsoidal system of coordinates is orthogonal by finding gradients $\nabla\xi, \nabla\eta, \nabla\zeta$ and then the scalar products $\nabla\xi \cdot \nabla\eta$ and so on, all of which turn out to be equal to zero. The gradients may be found directly from the equations determining ellipsoidal coordinates (see the condition for the problem), resulting in a gradient from both members of each of these equations.

1.95*

$$z = \pm \left[\frac{(\xi + c^2)(\eta + c^2)}{c^2 - a^2} \right]^{1/2}, \quad r = \left[\frac{(\xi + a^2)(\eta + a^2)}{a^2 - c^2} \right]^{1/2};$$

$$h_1 = \frac{\sqrt{\xi - \eta}}{2R_\xi}, \quad h_2 = \frac{\sqrt{\xi - \eta}}{2R_\eta}, \quad h_3 = r,$$

where

$$R_\xi = \sqrt{(\xi + a^2)(\xi + c^2)}, \quad R_\eta = \sqrt{(\eta + a^2)(-\eta - c^2)};$$

$$\Delta = \frac{4}{\xi - \eta} \left[R_\xi \frac{\partial}{\partial \xi} \left(R_\xi \frac{\partial}{\partial \xi} \right) + R_\eta \frac{\partial}{\partial \eta} \left(R_\eta \frac{\partial}{\partial \eta} \right) \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2}.$$

1.96*

$$x = \pm \left[\frac{(\xi + a^2)(\zeta + a^2)}{a^2 - b^2} \right]^{1/2}, \quad r = \left[\frac{(\xi + b^2)(\zeta + b^2)}{b^2 - a^2} \right]^{1/2};$$

$$h_1 = \frac{\sqrt{\xi - \zeta}}{2R_\xi}, \quad h_2 = r, \quad h_3 = \frac{\sqrt{\xi - \zeta}}{2R_\zeta},$$

where

$$R_\xi = \sqrt{(\xi + a^2)(\xi + b^2)}, \quad R_\zeta = \sqrt{(\zeta + a^2)(-\zeta - b^2)};$$

$$\Delta = \frac{4}{\xi - \zeta} \left[R_\xi \frac{\partial}{\partial \xi} \left(R_\xi \frac{\partial}{\partial \xi} \right) + R_\zeta \frac{\partial}{\partial \zeta} \left(R_\zeta \frac{\partial}{\partial \zeta} \right) \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2}.$$

1.97*

$$h_\xi = h_\eta = \frac{a}{\cosh\xi - \cos\eta}\,, \qquad h_\alpha = \frac{a\sin\eta}{\cosh\xi - \cos\eta}\;;$$

$$\Delta = \frac{(\cosh\xi - \cos\eta)^3}{a^2}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{\cosh\xi - \cos\eta}\frac{\partial}{\partial\xi}\right)\right.$$
$$+\frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\frac{\sin\eta}{\cosh\xi - \cos\eta}\frac{\partial}{\partial\eta}\right)$$
$$\left.+\frac{1}{\sin^2\eta(\cosh\xi - \cos\eta)}\frac{\partial^2}{\partial\alpha^2}\right].$$

1.98* The surfaces $\rho =$ const are toroids:

$$(\sqrt{x^2+y^2} - a\coth\rho)^2 + z^2 = \left(\frac{a}{\sinh\rho}\right)^2\;;$$

the surfaces $\xi =$ const are spherical segments:

$$(z - \arctan\xi)^2 + x^2 + y^2 = \left(\frac{a}{\sin\xi}\right)^2;$$

$$h_\rho = h_\xi = \frac{a}{\cosh\rho - \cos\xi}\,, \qquad h_\alpha = \frac{a\sinh\rho}{\cosh\rho - \cos\xi}\,.$$

1.99 $x^\nu Z_\nu(x) + C\,, \quad -x^{-\nu} Z_\nu(x) + C.$

1.100 $1, \frac{1}{2}, \frac{1}{2^n n!}\,.$

1.107*

1. $xu'' + u' + x(a^2 - \frac{n^2}{x^2})u = 0\,.$
2. The integral is computed with the use of the equations for the functions $u(x)$ and $v(x) = J_n(bx)$.
3. The first equality (1.175) directly follows from (1.174); the second one results from passage to the limit.

1.111

$$P_0 = 1\,, \quad P_1 = x\,, \quad P_2 = \frac{1}{2}(3x^2 - 1)\,, \quad P_3 = \frac{1}{2}(5x^3 - 3x)\,,$$
$$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)\,, \quad P_5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)\,.$$

1.112* The integral that should be computed contains the product of the derivatives $\int_{-1}^{1}[(x^2-1)^l]^{(l)}[(x^2-1)^{l'}]^{(l')}\mathrm{d}x$. To make things clear, $l' \le l$. Integrating by parts, l times, we find $(-1)^l \int_{-1}^{1}(x^2-1)^l[(x^2-1)^{l'}]^{(l+l')}\mathrm{d}x$. The second factor under the integral is other than zero only when $l' = l$. In this case, beginning by integrating over the angle ϑ, we get

$$\int_{-1}^{1}(x^2-1)^l[(x^2-1)^l]^{(2l)}\mathrm{d}x = (-1)^l(2l)!2\int_0^{\pi/2}(\sin\vartheta)^{2l+1}\mathrm{d}\vartheta$$
$$= (-1)^l(2l)!B\left(l+1,\frac{1}{2}\right) .$$

The last integral here is expressed through a beta function (see the definition in Abramovitz and Stegun, 1965; Gradshtein and Ryzhik, 2007):

$$B(z,w) \equiv \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = 2\int_0^{\pi/2}(\sin\vartheta)^{2z-1}(\cos\vartheta)^{2w-1}\mathrm{d}\vartheta .$$

Carefully eliminating factorials and gamma functions, we get the result specified in the condition for the problem.

1.114 $P_2 = \frac{1+3\cos 2\vartheta}{4}$.

1.115

$$\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\sin\vartheta\frac{\mathrm{d}P_l^m}{\mathrm{d}\vartheta} + \left[l(l+1) - \frac{m^2}{\sin^2\vartheta}\right]P_l^m = 0 .$$

1.116* Use the Leibniz formula to find

$$(1-x^2)^{m/2}[(x+1)^l(x-1)^l]^{(l+m)}$$
$$= (-1)^{m/2}\sum_{k=0}^{l+m}\frac{(l+m)!l!l!}{k!(l+m-k)!(k-m)!(l-k)!}(x+1)^{k-m/2}(x-1)^{l-k+m/2} ;$$

and

$$(1-x^2)^{-m/2}[(x+1)^l(x-1)^l]^{(l-m)}$$
$$= (-1)^{-m/2}\sum_{s=0}^{l-m}\frac{(l-m)!l!l!}{s!(l-m-s)!(s+m)!(l-s)!}(x+1)^{s+m/2}(x-1)^{l-s-m/2} .$$

In both sums, the limits of summation are, actually, determined by the presence in the denominators of the factorials of negative integers. In the second sum, replace the summation index $s \to k-m$. As a result, in both sums, factors that depend on k become the same. Comparing them with each other, we find the formula specified in the condition for the problem.

1.117

$$P_l^0 = P_l,$$

$$P_1^1 = -2P_1^{-1} = (1-x^2)^{1/2} = \sin\vartheta,$$

$$P_2^1 = -6P_2^{-1} = 3x(1-x^2)^{1/2} = 3\cos\vartheta\sin\vartheta,$$

$$P_2^2 = 24P_2^{-2} = 3(1-x^2) = 3\sin^2\vartheta,$$

$$P_3^1 = -12P_3^{-1} = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2} = \frac{3}{2}(5\cos^2\vartheta-1)\sin\vartheta,$$

$$P_3^2 = 120P_3^{-2} = 15x(1-x^2) = 15\cos\vartheta\sin^2\vartheta,$$

$$P_3^3 = -720P_3^{-3} = 15(1-x^2)^{3/2} = 15\sin^3\vartheta\ .$$

In the general case, adjoint Legendre polynomials contain radicals $(1-x^2)^{1/2}$ and, therefore, strictly speaking, are not polynomials.

1.118* When computing the normalization integral, use formula (1.198) and the method used to solve Problem 1.112*:

$$\int_{-1}^{1}[P_l^m(x)]^2\mathrm{d}x = (-1)^m\frac{(l+m)!}{(l-m)!}\int_{-1}^{1}P_l^m(x)P_l^{-m}(x)\mathrm{d}x$$
$$= \frac{(2l)!(l+m)!}{2^{2l}l!l!(l-m)!}B(l+1,1/2)\ ,$$

$$C_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\ .$$

Expressing the beta function through gamma functions and reducing fractions, we find the normalization factor (down to the phase factor over the module equaling 1, which remains arbitrary).) In the end, we get the normalized spherical Legendre function:

$$Y_{lm}(\vartheta,\varphi)$$
$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(1-\cos^2\vartheta)^{m/2}\left(\frac{\mathrm{d}}{\mathrm{d}\cos\vartheta}\right)^{l+m}(\cos^2\vartheta-1)^l e^{im\varphi}\ .$$

1.119

$$\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial Y_{lm}}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2 Y_{lm}}{\partial\varphi^2} + l(l+1)Y_{lm} = 0\ .$$

1.120 $-\frac{5}{3}$, -2 , 0 , $\exp(\alpha)+\exp(-2\alpha)$.

1.121 $0\,, \quad f(a)\delta(x-a)\,, \quad 82\delta(x-4)+2\delta(x+1)\,.$

1.123

$$\delta(\boldsymbol{r}) = \frac{1}{2\pi r_\perp}\delta(r_\perp)\delta(z)\,,$$
$$\delta(\boldsymbol{r}-\boldsymbol{a}) = \frac{1}{a_\perp}\delta(r_\perp - a_\perp)\delta(\alpha-\alpha_0)\delta(z-a_z)\,.$$

To achieve passage to the limit $\boldsymbol{a} \to 0$, one must not only make values $a_\perp$ and a_z tend to zero, but must also average the second member over the azimuthal angle α_0, since a zero vector has no direction.

1.124

$$\delta(\boldsymbol{r}) = \frac{1}{4\pi r^2}\delta(r)\,, \quad \delta(\boldsymbol{r}-\boldsymbol{a}) = \frac{1}{a^2}\delta(r-a)\delta(\cos\vartheta - \cos\vartheta_0)\delta(\alpha-\alpha_0)\,.$$

1.125

$$f'(x) = g(x) + 2\delta(x-1)\,, \quad \text{where} \quad g(x) = \begin{cases} 3x^2\,, & \text{if } x < 1\,, \\ 2x\,, & \text{if } x > 1\,. \end{cases}$$

1.126

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} + \sum_{k=1}^{n} \Delta f_k \delta(x-a_k)\,,$$

where $\Delta f_k = f(a_k+0) - f(a_k-0)$, $\mathrm{d}f/\mathrm{d}x$ is a proper ("classical") derivative in the areas of the smooth variation of the function.

1.127

$$\frac{(-1)^m m!}{(n-m)!} f^{(m-n)}(0) \quad \text{at} \quad m \ge n\,, \quad 0 \quad \text{at} \quad m < n\,.$$

1.128 When $\boldsymbol{r} \neq \boldsymbol{r}'$, G is a bounded differentiable function and the equation is satisfied, since $\Delta G = 0$. When $\boldsymbol{r} \to \boldsymbol{r}'$, the function has a singularity. To find out the nature of that singularity ΔG when $\boldsymbol{r} \to \boldsymbol{r}'$, integrate (1.225) over the volume of a small sphere of radius $R \to 0$ with its center at the point $\boldsymbol{r} = \boldsymbol{r}'$. Using the Gauss–Ostrogradskii theorem, we get

$$\int_V \Delta G \,\mathrm{d}V = \int_V \operatorname{div}\operatorname{grad}\left(\frac{1}{r}\right)\mathrm{d}V = \oint_S \left(\nabla\frac{1}{r}\right)\cdot \mathrm{d}\boldsymbol{S}$$
$$= -\int \frac{1}{R^2} R^2 \mathrm{d}\Omega = -4\pi\,.$$

The same value is obtained by the integral over the volume in the second member of the equation, which is how it is satisfied.

1.129

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2} \ .$$

1.130

$$f(x) = \frac{4a}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1} \ .$$

1.131

$$f(x) = \frac{a}{2} + \frac{2a}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\cos((2k+1)\pi x/L)}{2k+1} \ .$$

The Fourier series of $f(-x) = f(x)$, which is a function that is even within the interval $[-L, +L]$, contains only cosines. The odd function $f(-x) = -f(x)$ is expandable in sines. A function that has no clear-cut parity contains both sines and cosines in its Fourier expansion.

1.132 $i\lambda F(\lambda)$.

1.133

$$\frac{1}{a} F\left(\frac{\lambda - b}{a}\right) \ .$$

1.134 $\pi \exp(-|\lambda|)$.

1.135

$$\frac{\sqrt{\pi}}{\alpha} \exp\left(-\frac{\lambda^2}{4\alpha^2}\right) \ .$$

1.136

$$\frac{\pi^{3/2}}{\alpha^3} \exp\left(-\frac{k^2}{4\alpha^2}\right) \ ,$$

where $\boldsymbol{k}$ is a radius vector of the three-dimensional space of Fourier variables (see (1.252)).

1.137* $\frac{4\pi}{k^2+\kappa^2}$.

1.138* To compute Fourier integral (1.252), use spherical coordinates and select the Oz axis along the vector $\boldsymbol{k}$. Firstly, doing integration over angles and then over r, we find

$$F(k) = \lim_{R\to\infty} \frac{4\pi}{k^2}[1 - \cos(kR)] \ .$$

Formally, the function in the second member has no limit. However, it is easy to understand that the limit of the cosine may be regarded as effectively equal to zero since, when the inverted Fourier transformation is being done, the member with infinitely oscillating cosine will make zero contribution. As a result, we have $F(k) = 4\pi/k^2$.

1.139 Write down the expansion as

$$\exp(ikr\cos\theta) = \sum_{l=0}^{\infty} u_l(kr) P_l(\cos\theta)$$

and, using the orthogonality of Legendre polynomials (see Problem 1.112*), find an integral representation of the functions u_l sought:

$$u_l(kr) = \frac{2l+1}{2}\int_{-1}^{1} e^{ikrx} P_l(x)\mathrm{d}x \, .$$

Use the Rodrigues formula and integrate l times by parts to get

$$u_l(kr) = \frac{(2l+1)(-ikr)^l}{2^{l+1}l!}\int_{-1}^{1} e^{ikrx}(x^2-1)^l\mathrm{d}x \, .$$

Further, expand the exponent in a power series and integrate this absolutely convergent series termwise. Only terms with even powers of x are left:

$$u_l(kr) = \frac{(2l+1)(-ikr)^l}{2^l l!}\sum_{m=0}^{\infty}\frac{(ikr)^{2m}}{(2m)!}\int_{0}^{1} x^{2m}(x^2-1)^l\mathrm{d}x \, .$$

Finally, the transition to a new integration variable $t = x^2$, $\mathrm{d}x = \mathrm{d}t/2\sqrt{t}$ allows us to express the integral in the latter equality through a beta function:

$$\int_0^1 t^{k-1/2}(1-t)^l dt = B(k+1/2, l+1) = \frac{\Gamma(k+1/2)\Gamma(l+1)}{\Gamma(l+k+3/2)} \, .$$

In the end, bundle all the factors to get the following series:

$$\begin{aligned} u_l(kr) &= i^l(2l+1)\frac{(kr)^l}{1\cdot 3\ldots(2l+1)} \\ &\quad\times\left\{1 - \frac{k^2r^2/2}{1!(2l+3)} + \frac{(k^2r^2/2)^2}{2!(2l+3)(2l+5)} - \ldots\right\} \\ &= i^l(2l+1)j_l(kr) \, , \end{aligned}$$

where $j_l(kr)$ is a spherical Bessel function; see formulas (10.1.2) in Abramovitz and Stegun (1965).

1.140*

$$\exp(i\boldsymbol{k} \cdot \boldsymbol{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr) Y_{lm}^*(\theta, \phi) Y_{lm}(\vartheta, \varphi) .$$