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INTRODUCTION

1.1 PRELIMINARY COMMENTS

The phrase "energy methods" in the present study refers to methods that make use of the total potential energy (i.e., strain energy and potential energy due to applied loads) of a system to obtain values of an unknown displacement or force, at a specific point of the system. These include Castigliano's theorems, unit-dummy-load and unit-dummy-displacement methods, and Betti's and Maxwell's theorems. These methods are often limited to the (exact) determination of generalized displacements or forces at fixed points in the structure; in most cases, they cannot be used to determine the complete solution (i.e., displacements and/or forces) as a function of position in the structure. The phrase "variational methods," on the other hand, refers to methods that make use of the variational principles, such as the principles of virtual work and the principle of minimum total potential energy, to determine approximate solutions as continuous functions of position in a body. In the classical sense, a *variational principle* has to do with the minimization or finding stationary values of a functional with respect to a set of undetermined parameters introduced in the assumed solution. The functional represents the total energy of the system in solid and structural mechanics problems, and in other problems it is simply an integral representation of the governing equations. In all cases, the functional includes all the intrinsic features of the problem, such as the governing equations, boundary and/or initial conditions, and constraint conditions.

1.2 THE ROLE OF ENERGY METHODS AND VARIATIONAL PRINCIPLES

Variational principles have always played an important role in mechanics. Variational formulations can be useful in three related ways. First, many problems of mechanics are posed in terms of finding the extremum (i.e., minima or maxima) and thus, by their nature, can be formulated in terms of variational statements. Second, there are problems that can be formulated by other means, such as by vector mechanics (e.g., Newton's laws), but these can also be formulated by means of variational principles. Third, variational formulations form a powerful basis for obtaining approximate solutions to practical problems, many of which are intractable otherwise. The principle of minimum total potential energy, for example, can be regarded as a substitute for the equations of equilibrium of an elastic body, as well as a basis for the development of displacement finite element models that can be used to determine approximate displacement and stress fields in the body. Variational formulations can also serve to unify diverse fields, suggest new theories, and provide a powerful means for studying the existence and uniqueness of solutions to problems. In many cases they can also be used to establish upper and/or lower bounds on approximate solutions.

1.3 SOME HISTORICAL COMMENTS

In modern times, the term "variational formulation" applies to a wide spectrum of concepts having to do with weak, generalized, or direct variational formulations of boundary- and initial-value problems. Still, many of the essential features of variational methods remain the same as they were over 200 years ago when the first notions of variational calculus began to be formulated.

Although Archimedes (287–212 B.C.) is generally credited with the first to use work arguments in his study of levers, the most primitive ideas of variational theory (the minimum hypothesis) are present in the writings of the Greek philosopher Aristotle (384–322 B.C.), to be revived again by the Italian mathematician/engineer Galileo (1564–1642), and finally formulated into a Principle of Least Time by the French mathematician Fermat (1601–1665). The phrase *virtual velocities* was used by Jean Bernoulli in 1717 in his letter to Varignon (1654–1722). The development of early variational calculus, by which we mean the classical problems associated with minimizing certain functionals, had to await the works of Newton (1642–1727) and Leibniz (1646–1716). The earliest applications of such variational ideas included the classical *isoperimetric problem* of finding among closed curves of given length the one that encloses the greatest area, and Newton's problem of determining the solid of revolution of "minimum resistance." In 1696, Jean Bernoulli proposed the problem of the *brachistochrone*: among all curves connecting two points, find the curve traversed in the shortest time by a particle under the influence of gravity. It stood as a challenge to the mathematicians of their day to solve the problem using the rudimentary tools of analysis then available to them or whatever new ones they were capable of developing. Solutions to this problem were presented by some of the greatest mathematicians of the time: Leibniz, Jean Bernoulli's older brother Jacques Bernoulli, L'Hôpital, and Newton.

The first step toward developing a general method for solving variational problems was given by the Swiss genius Leonhard Euler (1707–1783) in 1732 when he presented a "general solution of the isoperimetric problem," although Maupertuis is credited with having put forward a law of minimal property of potential energy for stable equilibrium in his *Mémoires de l'Académie des Sciences* in 1740. It was in Euler's 1732 work and subsequent publication of the principle of least action (in his book *Methodus inveniendi lineas curvas* ...) in 1744 by Euler that variational concepts found a welcome and permanent home in mechanics. He developed all ideas surrounding the principle of minimum potential energy in his work on the *Elastica*, and he demonstrated the relationship between his variational equations and those governing the flexure and buckling of thin rods.

A great impetus to the development of variational mechanics began in the writings of Lagrange (1736–1813), first in his correspondence with Euler. Euler worked intensely in developing Lagrange's method, but delayed publishing his results until Lagrange's works were published in 1760 and 1761. Lagrange used d'Alembert's principle to convert dynamics to statics and then used the principle of virtual displacements to derive his famous equations governing the laws of dynamics in terms of kinetic and potential energy. Euler's work, together with Lagrange's *Mécanique analytique* of 1788, laid down the basis for the variational theory of dynamical systems. Further generalizations appeared in the fundamental work of Hamilton in 1834. Collectively, all these works have had a monumental impact on virtually every branch of mechanics.

A more solid mathematical basis for variational theory began to be developed in the eighteenth and early nineteenth century. Necessary conditions for the existence of "minimizing curves" of certain functionals were studied during this period, and we find among contributors of that era the familiar names of Legendre, Jacobi, and Weierstrass. Legendre gave criteria for distinguishing between maxima and minima in 1786, without considering criteria for existence, and Jacobi gave sufficient conditions for existence of extrema in 1837. A more rigorous theory of existence of extrema was put together by Weierstrass, who, with Erdmann, established in 1865 conditions on extrema for variational problems involving corner behavior.

During the last half of the nineteenth century, the use of variational ideas was widespread among leaders in theoretical mechanics. We mention the works of Kirchhoff on plate theory, Lamé, Green, and Kelvin on elasticity, and the works of Betti, Maxwell, Castigliano, Menabrea, and Engesser for discrete structural systems. Lamé was the first in 1852 to prove a work equation, named after his colleague Clapeyron, for deformable bodies. Lamé's equation was used by Maxwell [1] for the solution of redundant frame works using the unit-dummy-load technique. In 1875 Castigliano published an extremum version of this technique, but attributed the idea to Menabrea. A generalization of Castigliano's work is due to Engesser [2].

Among prominent contributors to the subject near the end of the nineteenth century and in the early years of the twentieth century, particularly in the area of variational methods of approximation and their applications to physical problems, were Rayleigh [3], Ritz [4], and Galerkin [5]. Modern variational principles began in the 1950s with the works of Hellinger [6] and Reissner [7,8] on mixed variational principles for elasticity problems. A variety of generalizations of classical variational principles have appeared, and we shall not describe them here.

In closing this section, we note that a short historical account of early variational methods in mechanics can be found in the book of Lanczos [9] and a brief review of certain aspects of the subject as it stood in the early 1950s can be found in the book of Truesdell and Toupin [10]; additional information can be found in Smith's history of mathematics [11] and in the historical treatises on mechanics by Mach [12], Dugas [13], and Timoshenko [14]. Reference to much of the relevant contemporary literature can be found in the books by Washizu [15] and Oden and Reddy [16]. Additional historical papers and textbooks on variational methods are listed at the end of this chapter (see [17–56]).

1.4 PRESENT STUDY

The objective of the present study is to introduce energy methods and variational principles of solid and structural mechanics and to illustrate their use in the derivation and solution of the equations of applied mechanics, including plane elasticity, beams, frames, and plates. Of course, variational formulations and methods presented in this book are also applicable to problems outside solid mechanics. To equip the reader with the necessary mathematical tools and background from the theory of elasticity that are useful in the sequel, a review of vectors, matrices, tensors, and governing equations of elasticity are provided in the next two chapters. To keep the scope of the book within reasonable limits, only linear problems are considered. Although stability and vibration problems are introduced via examples and exercises, a detailed study of these topics is omitted.

In the following chapter we summarize the algebra and calculus of vectors and tensors. In Chapter 3 we give a brief review of the equations of solid mechanics, and in Chapter 4 we present the concepts of work and energy, energy principles, and Castigliano's theorems of structural mechanics. In Chapter 5 we present principles of virtual work, potential energy, and complementary energy. Chapter 6 is dedicated to Hamilton's principle for dynamical systems, and in Chapter 7 we introduce the Ritz, Galerkin, and weighted-residual methods. In Chapter 8, applications of variational methods to the formulation of plate bending theories and their solution by variational methods are presented. For the sake of completeness and comparison, analytical solutions of bending, vibration, and buckling of circular and rectangular plates are also presented. An introduction to the finite element method and its application to displacement finite element models of beams and plates is discussed in Chapter 9. The final chapter, Chapter 10, is devoted to the discussion of mixed variational principles, and mixed finite element models of beams and plates. To keep the scope of the book within reasonable limits, theory and analysis of shells is not included.

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