CHAPTER 1

BEAMS IN BENDING

This book deals with the extension, bending, and torsion of bars, especially thin-walled members. Although computational approaches for the analysis and design of bars are emphasized, traditional analytical solutions are included.

We begin with a study of the bending of beams, starting with a brief review of some of the fundamental concepts of the theory of linear elasticity. The theory of beams in bending is then treated from a strength-of-materials point of view. Both topics are treated more thoroughly in Pilkey and Wunderlich (1994). Atanackovic and Guran (2000), Boresi and Chong (1987), Gould (1994), Love (1944), and Sokolnikoff (1956) contain a full account of the theory of elasticity. References such as these should be consulted for the derivation of theory-of-elasticity relationships that are not derived in this chapter. Gere (2001), Oden and Ripperger (1981), Rivello (1969), and Uugural and Fenster (1981) may be consulted for a detailed development of beam theory.

1.1 REVIEW OF LINEAR ELASTICITY

The equations of elasticity for a three-dimensional body contain 15 unknown functions: six stresses, six strains, and three displacements. These functions satisfy three equations of equilibrium, six strain–displacement relations, and six stress–strain equations.

1.1.1 Kinematical Strain–Displacement Equations

The displacement vector **u** at a point in a solid has the three components $u_x(x, y, z)$, $u_y(x, y, z)$, $u_z(x, y, z)$ which are mutually orthogonal in a Cartesian coordinate system and are taken to be positive in the direction of the positive coordinate axes. In vector notation,

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^{\mathrm{T}} \tag{1.1}$$

Designate the normal strains by ϵ_x, ϵ_y, and ϵ_z and the shear strains are $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$. The shear strains are symmetric (i.e., $\gamma_{ij} = \gamma_{ji}$). In matrix notation

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \epsilon_x & \epsilon_y & \epsilon_z & \gamma_{xy} & \gamma_{xz} & \gamma_{yz} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \epsilon_x & \epsilon_y & \epsilon_z & 2\epsilon_{xy} & 2\epsilon_{xz} & 2\epsilon_{yz} \end{bmatrix}^{\mathrm{T}} \tag{1.2}$$

As indicated, $\gamma_{ik} = 2\epsilon_{ik}$, where γ_{ik} is sometimes called the engineering shear strain and ϵ_{ik} the theory of elasticity shear strain.

The linearized strain–displacement relations, which form the *Cauchy strain tensor*, are

$$\begin{aligned} \epsilon_x &= \frac{\partial u_x}{\partial x} & \epsilon_y &= \frac{\partial u_y}{\partial y} & \epsilon_z &= \frac{\partial u_z}{\partial z} \\ \gamma_{xy} &= \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} & \gamma_{xz} &= \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} & \gamma_{yz} &= \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \end{aligned} \tag{1.3}$$

In matrix form Eq. (1.3) can be written as

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \\ \partial_y & \partial_x & 0 \\ \partial_z & 0 & \partial_x \\ 0 & \partial_z & \partial_y \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{1.4}$$

or

$$\boldsymbol{\epsilon} \quad = \quad \mathbf{D} \quad \mathbf{u}$$

with the differential operator matrix

$$\mathbf{D} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \\ \partial_y & \partial_x & 0 \\ \partial_z & 0 & \partial_x \\ 0 & \partial_z & \partial_y \end{bmatrix} \tag{1.5}$$

Six strain components are required to characterize the state of strain at a point and are derived from the three displacement functions $u_x,\ u_y,\ u_z$. The displacement field must be continuous and single valued, because it is being assumed that the body remains continuous after deformations have taken place. The six strain–displacement equations will not possess a single-valued solution for the three displacements if the strains are arbitrarily prescribed. Thus, the calculated displacements could possess tears, cracks, gaps, or overlaps, none of which should occur in practice. It appears as though the strains should not be independent and that they should be required to satisfy special conditions. To find relationships between the strains, differentiate the expression for the shear strain γ_{xy} with respect to x and y,

$$\frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \frac{\partial^2}{\partial x\, \partial y}\frac{\partial u_x}{\partial y} + \frac{\partial^2}{\partial x\, \partial y}\frac{\partial u_y}{\partial x} \tag{1.6}$$

According to the calculus, a single-valued continuous function f satisfies the condition

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x} \tag{1.7}$$

With the assistance of Eq. (1.7), Eq. (1.6) may be rewritten, using the strain–displacement relations, as

$$\frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} \tag{1.8}$$

showing that the three strain components γ_{xy}, ϵ_x, ϵ_y are not independent functions. Similar considerations that eliminate the displacements from the strain–displacement relations lead to five additional relations among the strains. These six relationships,

$$\begin{aligned}
2\frac{\partial^2 \epsilon_x}{\partial y\, \partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)\\
2\frac{\partial^2 \epsilon_y}{\partial x\, \partial z} &= \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right)\\
2\frac{\partial^2 \epsilon_z}{\partial x\, \partial y} &= \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y}\right)\\
\frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} &= \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2}\\
\frac{\partial^2 \gamma_{xz}}{\partial x\, \partial z} &= \frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2}\\
\frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z} &= \frac{\partial^2 \epsilon_z}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial z^2}
\end{aligned} \tag{1.9}$$

are known as the strain *compatibility conditions* or *integrability conditions*. Although there are six conditions, only three are independent.

1.1.2 Material Law

The kinematical conditions of Section 1.1.1 are independent of the material of which the body is made. The material is introduced to the formulation through a *material law*, which is a relationship between the stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\epsilon}$. Other names are the *constitutive relations* or the *stress–strain equations*.

Figure 1.1 shows the stress components that define the state of stress in a three-dimensional continuum. The quantities $\sigma_x, \sigma_y,$ and σ_z designate stress components normal to a coordinate plane and $\tau_{xy}, \tau_{xz}, \tau_{yz}, \tau_{yx}, \tau_{zx}$, and τ_{zy} are the shear stress components. In the case of a normal stress, the single subscript indicates that the stress acts on a plane normal to the axis in the subscript direction. For the shear stresses, the first letter of the double subscript denotes that the plane on which the stress acts is normal to the axis in the subscript direction. The second subscript letter designates the coordinate direction in which the stress acts. As a result of the need to satisfy an equilibrium condition of moments, the shear stress components must be symmetric that is,

$$\tau_{xy} = \tau_{yx} \qquad \tau_{xz} = \tau_{zx} \qquad \tau_{yz} = \tau_{zy} \tag{1.10}$$

Then the state of stress at a point is characterized by six components. In matrix form,

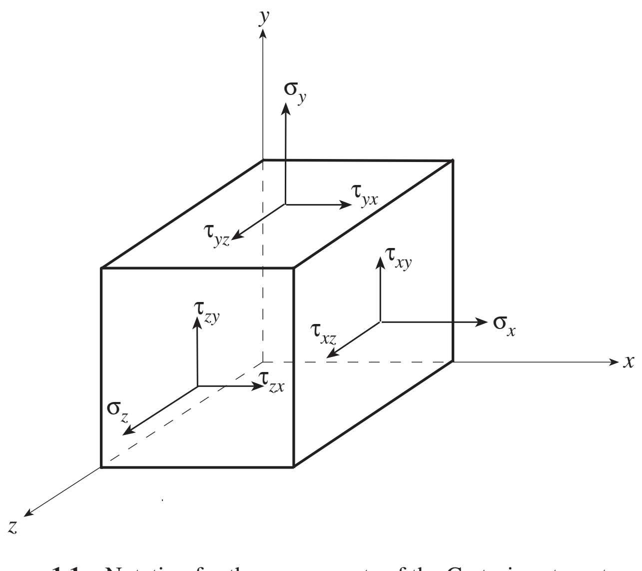


Figure 1.1 Notation for the components of the Cartesian stress tensor.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{xz} & \tau_{yz} \end{bmatrix}^{\mathrm{T}} \tag{1.11}$$

For a solid element as shown in Fig. 1.1, a face with its outward normal along the positive direction of a coordinate axis is defined to be a positive face. A face with its normal in the negative coordinate direction is defined as a negative face. Stress (strain) components on a positive face are positive when acting along a positive coordinate direction. The components shown in Fig. 1.1 are positive. Components on a negative face acting in the negative coordinate direction are defined to be positive.

An *isotropic* material has the same material properties in all directions. If the properties differ in various directions, such as with wood, the material is said to be *anisotropic*. A material is *homogeneous* if it has the same properties at every point. Wood is an example of a homogeneous material that can be anisotropic. A body formed of steel and aluminum portions is an example of a material that is inhomogeneous, but each portion is isotropic.

The stress–strain equations for linearly elastic isotropic materials are

$$\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \\ \epsilon_y &= \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_x + \sigma_z) \\ \epsilon_z &= \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{xz} &= \frac{\tau_{xz}}{G} \\ \gamma_{yz} &= \frac{\tau_{yz}}{G} \end{aligned} \tag{1.12}$$

where E is the *elastic* or *Young's modulus,* ν is *Poisson's ratio*, and G is the *shear modulus*. Only two of these three material properties are independent. The shear modulus is given in terms of E and ν as

$$G = \frac{E}{2(1 + \nu)} \tag{1.13}$$

$$
\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \cdots \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \frac{1}{E} \left[\begin{array}{ccc|ccc} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & \mathbf{0} & \\ -\nu & -\nu & 1 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 2(1+\nu) & 0 & 0 \\ & \mathbf{0} & & 0 & 2(1+\nu) & 0 \\ & & & 0 & 0 & 2(1+\nu) \end{array}\right] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \cdots \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}
$$

$$
\boldsymbol{\epsilon} = \mathbf{E}^{-1} \boldsymbol{\sigma} \tag{1.14}
$$

Stresses may be written as a function of the strains by inverting the six relationships of Eq. (1.12) that express strains in terms of stresses. The result is

$$
\begin{aligned}
\sigma_x &= \lambda e + 2G\epsilon_x \\
\sigma_y &= \lambda e + 2G\epsilon_y \\
\sigma_z &= \lambda e + 2G\epsilon_z \\
\tau_{xy} = G\gamma_{xy} \qquad & \tau_{xz} = G\gamma_{xz} \qquad \tau_{yz} = G\gamma_{yz}
\end{aligned} \tag{1.15}
$$

where e is the change in volume per unit volume, also called the *dilatation*,

$$
e = \epsilon_x + \epsilon_y + \epsilon_z \tag{1.16}
$$

and λ is *Lamé's constant*,

$$
\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{1.17}
$$

The matrix form appears as

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \cdots \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{ccc|ccc} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & \mathbf{0} & \\ \nu & \nu & 1-\nu & & & \\ \cdots & \cdots & \cdots & & & \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & \mathbf{0} & & 0 & \frac{1-2\nu}{2} & 0 \\ & & & 0 & 0 & \frac{(1-2\nu)}{2} \end{array}\right] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \cdots \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}
$$

$$
\boldsymbol{\sigma} = \mathbf{E} \boldsymbol{\epsilon} \tag{1.18}
$$

For uniaxial tension, with the normal stress in the x direction given a constant positive value σ_0, and all other stresses set equal to zero,

$$\sigma_x = \sigma_0 > 0 \qquad \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0 \tag{1.19a}$$

The normal strains are given by Hooke's law as

$$\epsilon_x = \frac{\sigma_0}{E} \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_0}{E} \tag{1.19b}$$

and the shear strains are all zero. Under this loading condition, the material undergoes extension in the axial direction x and contraction in the transverse directions y and z. This shows that the material constants ν and E are both positive:

$$E > 0 \qquad \nu > 0 \tag{1.20}$$

In hydrostatic compression p, the material is subjected to identical compressive stresses in all three coordinate directions:

$$\sigma_x = \sigma_y = \sigma_z = -p \qquad p > 0 \tag{1.21}$$

while all shear stresses are zero. The dilatation under this loading condition is

$$e = -\frac{3p}{3\lambda + 2G} = -\frac{3p(1-2\nu)}{E} \tag{1.22}$$

Since the volume change in hydrostatic compression is negative, this expression for e implies that Poisson's ratio must be less than $\frac{1}{2}$:

$$\nu < \tfrac{1}{2} \tag{1.23}$$

and the following properties of the elastic constants are established:

$$E > 0 \qquad G > 0 \qquad \lambda > 0 \qquad 0 < \nu < \tfrac{1}{2} \tag{1.24}$$

Materials for which $\nu \approx 0$ and $\nu \approx \frac{1}{2}$ are very compressible or very incompressible, respectively. Cork is an example of a very compressible material, whereas rubber is very incompressible.

1.1.3 Equations of Equilibrium

Equilibrium at a point in a solid is characterized by a relationship between internal (volume or body) forces $\overline{p}_{Vx}$, $\overline{p}_{Vy}$, $\overline{p}_{Vz}$, such as those generated by gravity or acceleration, and differential equations involving stress. Prescribed forces are designated with a bar placed over a letter. These *equilibrium* or *static* relations appear as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \overline{p}_{Vx} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \overline{p}_{Vy} = 0 \tag{1.25}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \overline{p}_{Vz} = 0$$

where $\overline{p}_{Vx}, \overline{p}_{Vy}, \overline{p}_{Vz}$ are the body forces per unit volume. In matrix form,

$$\underbrace{\begin{bmatrix} \partial_x & 0 & 0 & \vdots & \partial_y & \partial_z & 0 \\ 0 & \partial_y & 0 & \vdots & \partial_x & 0 & \partial_z \\ 0 & 0 & \partial_z & \vdots & 0 & \partial_x & \partial_y \end{bmatrix}}_{\mathbf{D}^{\mathrm{T}}} \underbrace{\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \cdots \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}}_{\boldsymbol{\sigma}} + \underbrace{\begin{bmatrix} \overline{p}_{Vx} \\ \overline{p}_{Vy} \\ \overline{p}_{Vz} \end{bmatrix}}_{\overline{\mathbf{p}}_V} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{0}} \tag{1.26}$$

where the matrix of differential operators $\mathbf{D}^{\mathrm{T}}$ is the transpose of the $\mathbf{D}$ of Eq. (1.5). These relationships are derived in books dealing with the theory of elasticity and, also, in many basic strength-of-materials textbooks.

1.1.4 Surface Forces and Boundary Conditions

The forces applied to a surface (i.e., the boundary) of a body must be in equilibrium with the stress components on the surface. Let S_p denote the part of the surface of the body on which forces are prescribed, and let displacements be specified on the remaining surface S_u. The surface conditions on S_p are

$$\begin{aligned} p_x &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ p_y &= \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z \\ p_z &= \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z \end{aligned} \tag{1.27}$$

where n_x, n_y, n_z are the components of the unit vector $\mathbf{n}$ normal to the surface and p_x, p_y, p_z are the surface forces per unit area.

In matrix form,

$$\underbrace{\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}}_{\mathbf{p}} = \underbrace{\begin{bmatrix} n_x & 0 & 0 & \vdots & n_y & n_z & 0 \\ 0 & n_y & 0 & \vdots & n_x & 0 & n_z \\ 0 & 0 & n_z & \vdots & 0 & n_x & n_y \end{bmatrix}}_{\mathbf{N}^{\mathrm{T}}} \underbrace{\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \cdots \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}}_{\boldsymbol{\sigma}} \tag{1.28}$$

Note that $\mathbf{N}^T$ is similar in form to $\mathbf{D}^T$ of Eq. (1.26) in that the components of $\mathbf{N}^T$ correspond to the derivatives of $\mathbf{D}^T$. The relations of Eq. (1.27) are referred to as *Cauchy's formula*.

Surface forces (per unit area) $\mathbf{p}$ applied externally are called prescribed surface *tractions* $\overline{\mathbf{p}}$. Equilibrium demands that the resultant stress be equal to the applied surface tractions $\overline{\mathbf{p}}$ on S_p:

$$\mathbf{p} = \overline{\mathbf{p}} \qquad \text{on } S_p \tag{1.29}$$

These are the *static (force, stress*, or *mechanical) boundary conditions.* Continuity requires that on the surface S_u, the displacements $\mathbf{u}$ be equal to the specified displacements $\overline{\mathbf{u}}$:

$$\mathbf{u} = \overline{\mathbf{u}} \qquad \text{on } S_u \tag{1.30}$$

These are the *displacement* (*kinematic*) *boundary conditions.*

Unit Vectors on a Boundary Curve It is helpful to identify several useful relationships between vectors on a boundary curve. Consider a boundary curve lying in the yz plane as shown in Fig. 1.2a. The vector $\mathbf{n}$ is the unit outward normal $\mathbf{n} = n_y\mathbf{j} + n_z\mathbf{k}$ and $\mathbf{t}$ is the unit tangent vector $\mathbf{t} = t_y\mathbf{j} + t_z\mathbf{k}$, where $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the y and z axes. The quantity s, the coordinate along the arc of the boundary, is chosen to increase in the counterclockwise sense. As shown in Fig. 1.2a, the unit tangent vector $\mathbf{t}$ is directed along increasing s. Since $\mathbf{n}$ and $\mathbf{t}$ are unit vectors, $n_y^2 + n_z^2 = 1$ and $t_y^2 + t_z^2 = 1$. The components of $\mathbf{n}$ are its direction cosines, that is, from Fig. 1.2b,

$$n_y = \cos\theta_y \quad \text{and} \quad n_z = \cos\theta_z \tag{1.31}$$

since, for example, $\cos\theta_y = n_y/\sqrt{n_y^2 + n_z^2} = n_y$.

From Fig. 1.2c it can be observed that

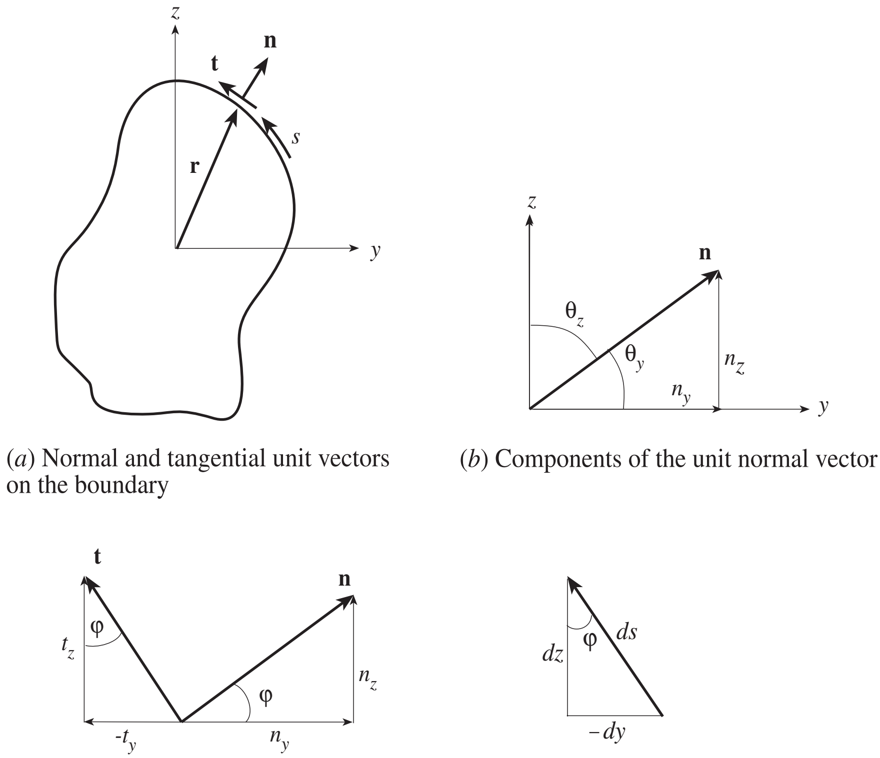
$$\begin{aligned} \cos\varphi &= n_y & \sin\varphi &= n_z \\ \sin\varphi &= -t_y & \cos\varphi &= t_z \end{aligned} \tag{1.32}$$

As a consequence,

$$n_y = t_z \qquad n_z = -t_y \tag{1.33}$$

and the unit outward normal is defined in terms of the components t_y and t_z of the unit tangent as

$$\mathbf{n} = t_z\mathbf{j} - t_y\mathbf{k} = \mathbf{t} \times \mathbf{i} \tag{1.34}$$



(a) Normal and tangential unit vectors on the boundary

(b) Components of the unit normal vector

(c) Unit normal and tangential vectors

(d) Differential components

Figure 1.2 Geometry of the unit normal and tangential vectors.

From Fig. 1.2d it is apparent that

$$\sin\varphi = -\frac{dy}{ds} \quad \text{and} \quad \cos\varphi = \frac{dz}{ds} \tag{1.35}$$

Thus,

$$n_y = t_z = \frac{dz}{ds} \qquad n_z = -t_y = -\frac{dy}{ds} \tag{1.36}$$

The vector **r** to any point on the boundary is

$$\mathbf{r} = y\mathbf{j} + z\mathbf{k}$$

Then

$$d\mathbf{r} = dy\,\mathbf{j} + dz\,\mathbf{k} = \frac{d\mathbf{r}}{ds}\,ds = \left(\frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right) ds = \mathbf{t}\,ds \tag{1.37}$$

1.1.5 Other Forms of the Governing Differential Equations

The general problem of the theory of elasticity is to calculate the stresses, strains, and displacements throughout a solid. The kinematic equations $\boldsymbol{\epsilon} = \mathbf{Du}$ (Eq. 1.4) are written in terms of six strains and three displacements, while the static equations $\mathbf{D}^{\mathrm{T}}\boldsymbol{\sigma} + \overline{\mathbf{p}}_V = \mathbf{0}$ (Eq. 1.26) are expressed as functions of the six stress components. The constitutive equations $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\epsilon}$ (Eq. 1.18) are relations between the stresses and strains. The boundary conditions of Eqs. (1.29) and (1.30) need to be satisfied by the solution for the 15 unknowns.

In terms of achieving solutions, it is useful to derive alternative forms of the governing equations. The elasticity problem can be formulated in terms of the displacement functions u_x, u_y, u_z. The stress–strain equations allow the equilibrium equations to be written in terms of the strains. When the strains are replaced in the resulting equations by the expressions given by the strain–displacement relations, the equilibrium equations become a set of partial differential equations for the displacements. Thus, substitute $\boldsymbol{\epsilon} = \mathbf{Du}$ into $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\epsilon}$ to give the stress–displacement relations $\boldsymbol{\sigma} = \mathbf{EDu}$. The conditions of equilibrium become

$$\mathbf{D}^{\mathrm{T}}\boldsymbol{\sigma} + \overline{\mathbf{p}}_V = \mathbf{D}^{\mathrm{T}}\mathbf{EDu} + \overline{\mathbf{p}}_V = 0 \tag{1.38}$$

or, in scalar form,

$$\begin{aligned}
(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 u_x + \overline{p}_{Vx} &= 0 \\
(\lambda + G)\frac{\partial e}{\partial y} + G\nabla^2 u_y + \overline{p}_{Vy} &= 0 \\
(\lambda + G)\frac{\partial e}{\partial z} + G\nabla^2 u_z + \overline{p}_{Vz} &= 0
\end{aligned} \tag{1.39}$$

where ∇^2 is the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.40}$$

The dilatation e is a function of displacements

$$e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \boldsymbol{\nabla} \cdot \mathbf{u} \tag{1.41}$$

where $\mathbf{u}$ is the displacement vector, whose components along the x, y, z axes are u_x, u_y, u_z, and $\boldsymbol{\nabla}$ is the gradient operator. The displacement vector is expressed as $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit base vectors along the coordinates x, y, z, respectively. The gradient operator appears as

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{1.42}$$

To complete the displacement formulation, the surface conditions on S_p must also be written in terms of the displacements. This is done by first writing these surface conditions of Eq. (1.27) in terms of strains using the material laws, and then expressing the strains in terms of the displacements, using the strain–displacement relations. The resulting conditions are

$$\begin{aligned} \lambda e n_x + G\mathbf{n} \cdot \nabla u_x + G\mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial x} &= p_x \\ \lambda e n_y + G\mathbf{n} \cdot \nabla u_y + G\mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial y} &= p_y \\ \lambda e n_z + G\mathbf{n} \cdot \nabla u_z + G\mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial z} &= p_z \end{aligned} \tag{1.43}$$

where $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$. If boundary conditions exist for both S_p and S_u, the boundary value problem is called *mixed*. The equations of equilibrium written in terms of the displacements together with boundary conditions on S_p and S_u constitute the displacement formulation of the elasticity problem. In this formulation, the displacement functions are found first. The strain–displacement relations then give the strains, and the material laws give the stresses.

1.2 BENDING STRESSES IN A BEAM IN PURE BENDING

A beam is said to be in *pure bending* if the force–couple equivalent of the stresses over any cross section is a couple $\mathbf{M}$ in the plane of the section

$$\mathbf{M} = M_y\mathbf{j} + M_z\mathbf{k} \tag{1.44}$$

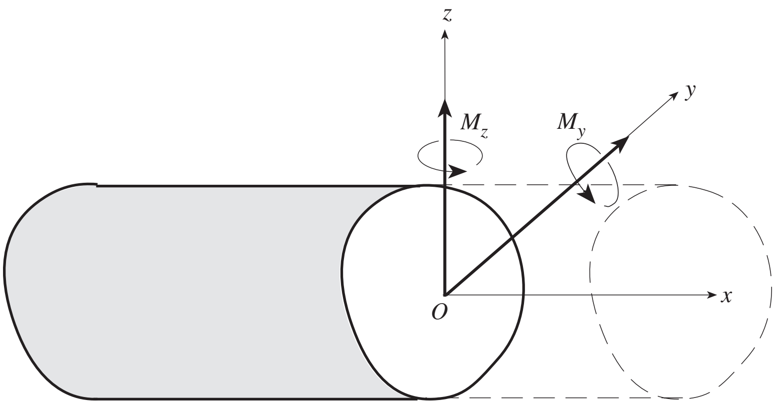


Figure 1.3 Beam in pure bending.

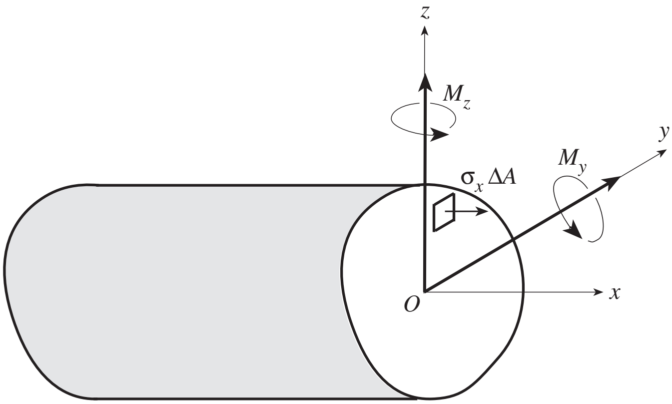


Figure 1.4 Stress resultants on a beam cross section.

where **j**, **k** are the unit vectors parallel to the y, z axes, and the x axis is the beam axis, as shown in Fig. 1.3. In terms of the stress σ_x, the bending moments may be calculated as stress resultants by summing the moments about the origin O of the axes (Fig. 1.4)

$$M_y = \int z\sigma_x \, dA \qquad M_z = -\int y\sigma_x \, dA \tag{1.45}$$

The point about which the moments are taken is arbitrary because the moment of a couple has the same value about any point.

Since in pure bending there is no axial stress resultant,

$$\int \sigma_x \, dA = 0 \tag{1.46}$$

According to the Bernoulli–Euler theory of bending, the cross-sectional planes of the beam remain plane and normal to the beam axis as it deforms. Choose the x axis (i.e. the beam axis) such that it passes through a reference point with coordinates $(x, 0, 0)$. This point is designated by O in Fig. 1.5. The axial displacement u_x of a point on the cross section with coordinates $(x,\ y,\ z)$ can be expressed in terms of the rotations of the cross section about the y and z axes and the axial displacement $u(x)$ of the reference point (Fig. 1.5). Thus

$$u_x(x,\ y,\ z) = u(x) + z\theta_y(x) - y\theta_z(x) \tag{1.47}$$

where θ_y, θ_z are the angles of rotation of the section about the y, z axes. Thus, the displacement u_x at a point on the cross section has been expressed in terms of the beam axis variables u, θ_y, and θ_z. Note that the quantities u, θ_y, and θ_z do not vary over a particular cross section. The terms $z\theta_y$ and $y\theta_z$ vary linearly. Figure 1.6 shows

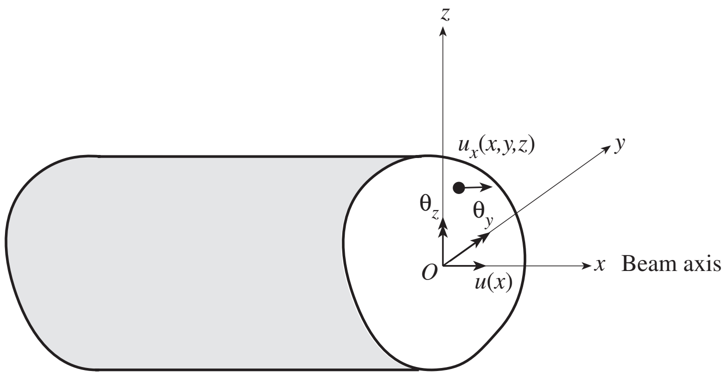


Figure 1.5 Axial displacement. During bending the cross-sectional plane remains plane and normal to the beam axis.

the displacement of a point P of the section with respect to point O as a result of a rotation about the y axis.

The axial strain at the point (x, y, z) is found from the strain–displacement equation (Eq. 1.3)

$$\epsilon_x = \frac{\partial u_x}{\partial x} = \kappa_\epsilon + \kappa_y z - \kappa_z y \tag{1.48}$$

where

$$\kappa_\epsilon = \frac{du}{dx} \qquad \kappa_y = \frac{d\theta_y}{dx} \qquad \kappa_z = \frac{d\theta_z}{dx}$$

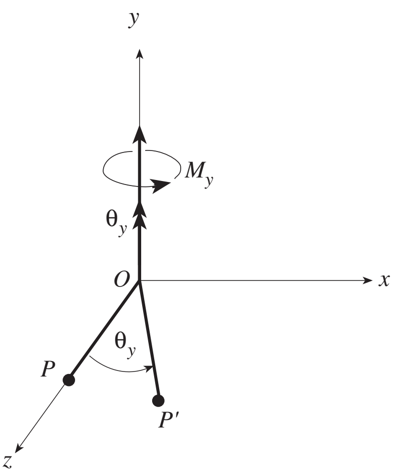


Figure 1.6 Rotation of a cross section about the y axis.

At a given cross section, $\kappa_\epsilon, \kappa_y, \kappa_z$ are constants, so that the normal strain distribution over the section is linear in y and z.

In pure bending, the only nonzero stress is assumed to be the normal stress σ_x, which is given by the material law for linearly elastic isotropic materials as $\sigma_x = E\epsilon_x$, so that

$$\sigma_x = E(\kappa_\epsilon + \kappa_y z - \kappa_z y) \tag{1.49}$$

For a nonhomogeneous beam, the elastic modulus takes on different values over different parts of the section, making E a function of position:

$$E = E(y, z)$$

The stress distribution at a given cross section is then expressed as

$$\sigma_x(y, z) = E(y, z)(\kappa_\epsilon + \kappa_y z - \kappa_z y) \tag{1.50}$$

The stress distribution is statically equivalent to the couple at the section so that the total axial force calculated as a stress resultant is zero and the moments are equal to the bending moments at the section. Thus, from Eqs. (1.45), (1.46), and (1.50),

$$\begin{aligned}
\int \sigma_x\, dA &= \kappa_\epsilon \int E\, dA + \kappa_y \int zE\, dA - \kappa_z \int yE\, dA = 0 \\
\int z\sigma_x\, dA &= \kappa_\epsilon \int zE\, dA + \kappa_y \int z^2E\, dA - \kappa_z \int yzE\, dA = M_y \\
\int y\sigma_x\, dA &= \kappa_\epsilon \int yE\, dA + \kappa_y \int yzE\, dA - \kappa_z \int y^2E\, dA = -M_z
\end{aligned} \tag{1.51}$$

Define geometric properties of the cross section as

$$Q_y = \int z\, dA \qquad Q_z = \int y\, dA \tag{1.52a}$$

$$I_y = \int z^2\, dA \qquad I_z = \int y^2\, dA \qquad I_{yz} = \int yz\, dA \tag{1.52b}$$

where Q_y and Q_z are first moments of the cross-sectional area and I_y, I_z, and I_{yz} are the second moments of a plane area or the area moments of inertia. Place the definitions of Eqs. (1.52) in Eq. (1.51):

$$\begin{aligned}
\kappa_\epsilon A + \kappa_y Q_y - \kappa_z Q_z &= 0 \\
\kappa_\epsilon Q_y + \kappa_y I_y - \kappa_z I_{yz} &= \frac{M_y}{E} \\
\kappa_\epsilon Q_z + \kappa_y I_{yz} - \kappa_z I_z &= -\frac{M_z}{E}
\end{aligned} \tag{1.53}$$

where the elastic modulus E is assumed to be constant for the cross section.

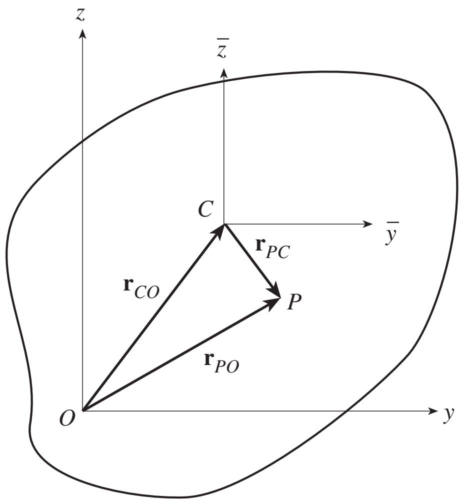


Figure 1.7 Translation of the origin O to the centroid C.

The three simultaneous relations of Eq. (1.53) for the constants κ_ϵ, κ_y, and κ_z become simpler if advantage is taken of the arbitrariness of the choice of origin O. Let a new coordinate system with origin C be defined as shown in Fig. 1.7 by a translation of axes. Figure 1.7 shows that the coordinate transformation equation for any point P of the section is $\mathbf{r}_{PO} = \mathbf{r}_{PC} + \mathbf{r}_{CO}$, or

$$\mathbf{r}_{PC} = \mathbf{r}_{PO} - \mathbf{r}_{CO} \tag{1.54a}$$

The components of this vector equation are

$$\overline{y} = y - y_C \qquad \overline{z} = z - z_C \tag{1.54b}$$

where y and z are the coordinates of P relative to the y, z coordinate axes and y_C and z_C are the coordinates of C relative to the y, z coordinate axes. Choose the origin C such that the first moments of area in the coordinate system $C\overline{y}\,\overline{z}$ are zero:

$$\begin{aligned} Q_{\overline{y}} &= \int \overline{z}\; dA = \int (z - z_C)\; dA = 0 \\ Q_{\overline{z}} &= \int \overline{y}\; dA = \int (y - y_C)\; dA = 0 \end{aligned} \tag{1.55}$$

With the definitions of Eq. (1.52a), these conditions yield the familiar geometric *centroid* of the cross section:

$$y_C = \frac{Q_z}{A} \qquad z_C = \frac{Q_y}{A} \tag{1.56}$$

Transform Eq. (1.53) to the centroidal coordinate system by assuming that Q_y, Q_z, I_y, I_z, and I_{yz} are measured from the centroidal coordinates. Since $Q_{\overline{y}}$ and $Q_{\overline{z}}$ are equal to zero (Eq. 1.55), Eq. (1.53) reduces to

$$\overline{\kappa}_\epsilon A = 0$$

$$\overline{\kappa}_y I_{\overline{y}} - \overline{\kappa}_z I_{\overline{y}\,\overline{z}} = \frac{M_y}{E}$$

$$\overline{\kappa}_y I_{\overline{y}\,\overline{z}} - \overline{\kappa}_z I_{\overline{z}} = -\frac{M_z}{E}$$

where $I_{\overline{y}}$, $I_{\overline{z}}$, and $I_{\overline{yz}}$ are the moments of inertia about the $\overline{y}$, $\overline{z}$ centroidal axes. Solve these expressions for $\overline{\kappa}_\epsilon$, $\overline{\kappa}_y$, and $\overline{\kappa}_z$, and substitute the results into σ_x of Eq. (1.49) expressed in terms of centroidal coordinates [i.e., $\sigma_x = E(\overline{\kappa}_\epsilon + \overline{\kappa}_y \overline{z} - \overline{\kappa}_z \overline{y})$]. This leads to an expression for the normal stress:

$$\sigma_x = -\frac{I_{\overline{y}\,\overline{z}} M_y + I_{\overline{y}} M_z}{I_{\overline{y}} I_{\overline{z}} - I^2_{\overline{y}\,\overline{z}}}\overline{y} + \frac{I_{\overline{z}} M_y + I_{\overline{y}\,\overline{z}} M_z}{I_{\overline{y}} I_{\overline{z}} - I^2_{\overline{y}\,\overline{z}}}\overline{z} \qquad (1.57)$$

The *neutral axis* is defined as the line on the cross section for which the normal stress σ_x is zero. This axis is the line of intersection of the neutral surface, which passes through the centroid of the section, and the cross-sectional plane. By equating Eq. (1.57) to zero, we find that the neutral axis is a straight line defined by

$$-(I_{\overline{y}\,\overline{z}} M_y + I_{\overline{y}} M_z)\overline{y} + (I_{\overline{z}} M_y + I_{\overline{y}\,\overline{z}} M_z)\overline{z} = 0$$

or

$$\overline{y} = \frac{I_{\overline{z}} M_y + I_{\overline{y}\,\overline{z}} M_z}{I_{\overline{y}\,\overline{z}} M_y + I_{\overline{y}} M_z}\overline{z} \qquad (1.58)$$

If $M_z = 0$, Eq. (1.57) reduces to

$$\sigma_x = M_y \frac{I_{\overline{z}}\overline{z} - I_{\overline{y}\,\overline{z}}\overline{y}}{I_{\overline{y}} I_{\overline{z}} - I^2_{\overline{y}\,\overline{z}}} \qquad (1.59)$$

The centroidal coordinates can be located using Eq. (1.56). Sometimes it is convenient to calculate the area moment of inertia first about a judiciously selected coordinate system and then transform them to the centroidal coordinate system. The calculation for $I_{\overline{z}}$, for instance, is

$$\begin{aligned} I_{\overline{z}} &= \int \overline{y}^2\, dA = \int (y - y_C)^2\, dA \\ &= \int (y^2 - 2yy_C + y_C^2)\, dA \\ &= I_z - y_C^2 A \end{aligned}$$

From Eq. (1.55), the integral $\int y \; dA$ in this expression is equal to $\int y_C \; dA$. This is one of *Huygens's* or *Steiner's laws* and is referred to as a *parallel axis theorem*. The complete set of equations is

$$\begin{aligned} I_{\overline{y}} &= I_y - z_C^2 A \\ I_{\overline{z}} &= I_z - y_C^2 A \\ I_{\overline{y}\,\overline{z}} &= I_{yz} - y_C z_C A \end{aligned} \tag{1.60}$$

Here I_y, I_z, I_{yz} and $I_{\overline{y}}, I_{\overline{z}}, I_{\overline{yz}}$ are the moments of inertia about the y, z and $\overline{y}, \overline{z}$ (centroidal) axes, respectively.

Example 1.1 Thin-Walled Cantilevered Beam with an Asymmetrical Cross Section. Find the normal stress distribution for the cantilevered angle shown in Fig. 1.8. The beam is fixed at one end and loaded with a vertical concentrated force $\overline{P}$ at the other end.

SOLUTION. The centroid for this asymmetrical section is found to be located as shown in Fig. 1.9a. Assume that the thickness t is much smaller than the dimension a. Then the moments of inertia can be calculated from Eq. (1.52b) as

$$I_{\overline{y}} = \int \overline{z}^2 \, dA = \int_{-4a/3}^{2a/3-t/2} \int_{-a/6-t/2}^{-a/6+t/2} \overline{z}^2 \, d\overline{y} \, d\overline{z}$$

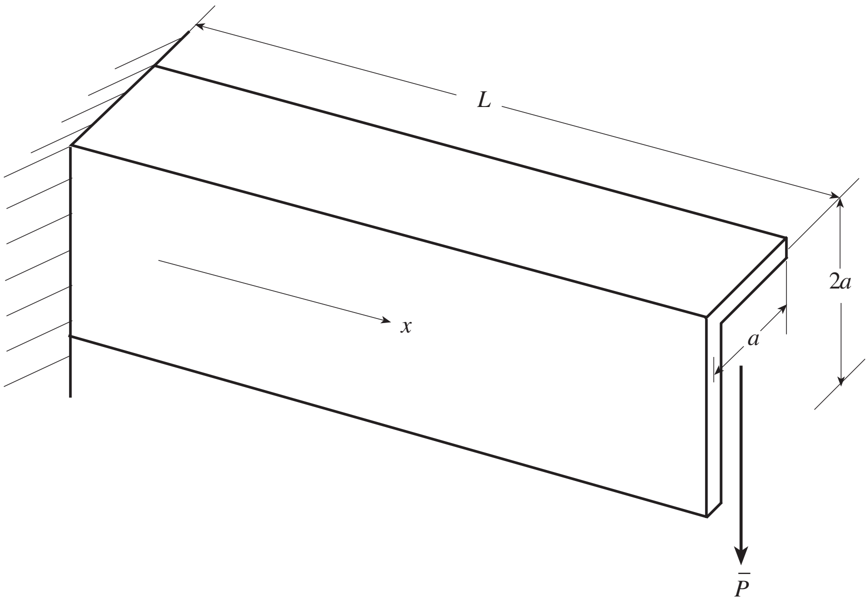


Figure 1.8 Thin-walled cantilevered beam with an asymmetrical cross section.

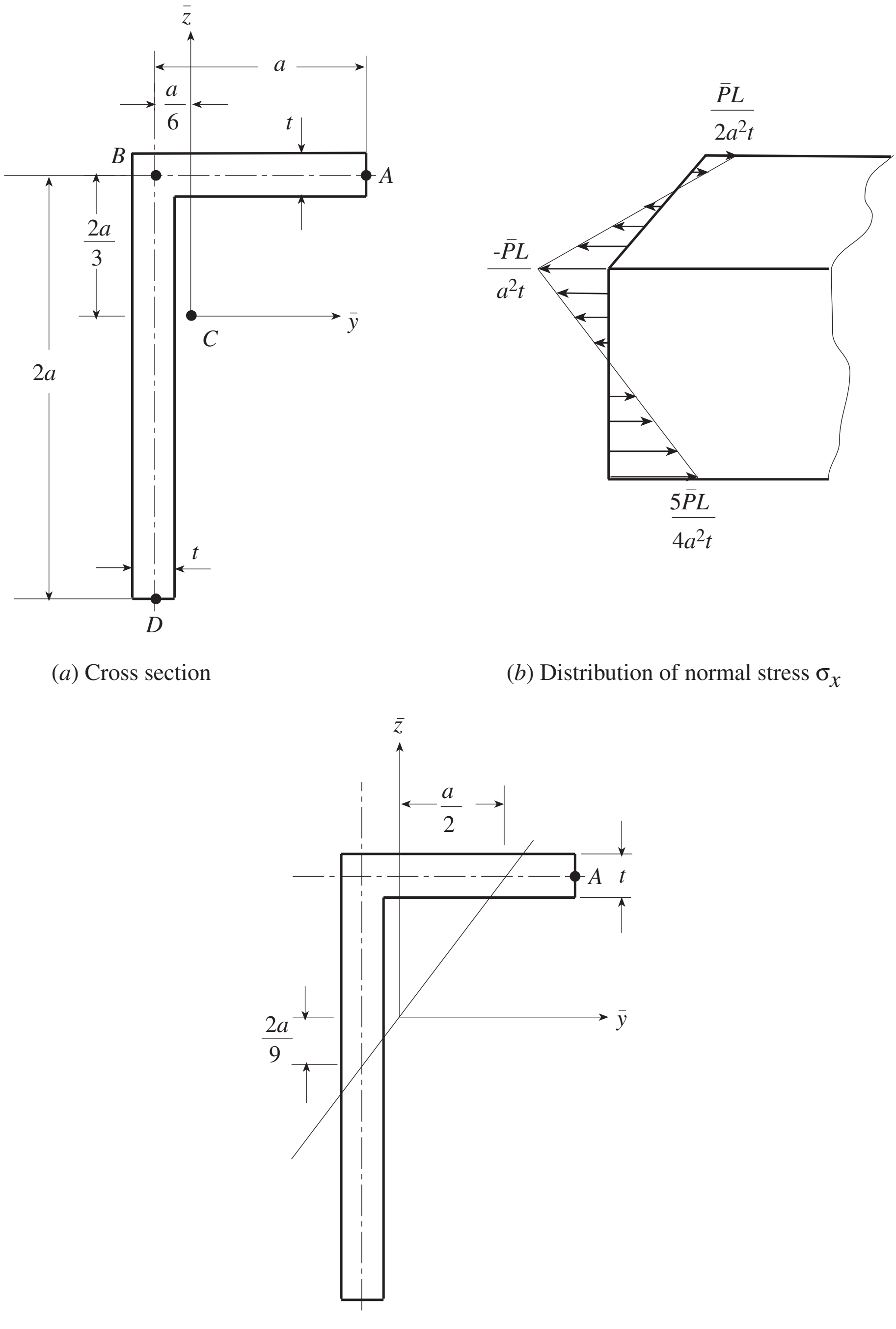


Figure 1.9 Normal stress distribution of the thin-walled beam of Example 1.1.

$$+\int_{2a/3-t/2}^{2a/3+t/2}\int_{-a/6-t/2}^{5a/6}\bar{z}^2\,d\bar{y}\,d\bar{z}$$

$$=\tfrac{4}{3}a^3t+\tfrac{1}{4}at^3\approx\tfrac{4}{3}a^3t \tag{1}$$

$$I_{\bar{z}}=\int\bar{y}^2\,dA=\tfrac{1}{4}a^3t+\tfrac{5}{24}at^3\approx\tfrac{1}{4}a^3t$$

$$I_{\overline{yz}}=\int\overline{yz}\,dA=\tfrac{1}{3}a^3t-\tfrac{5}{48}at^3\approx\tfrac{1}{3}a^3t$$

Numerical values for these and other cross-sectional parameters are given in Table 1.1.

The normal stress σ_x on a cross-sectional face is given by Eq. (1.57). The sign convention for M_y and M_z is detailed in Chapter 2. At a distance L from the free end, $M_y=-\overline{P}L$ and $M_z=0$, so that Eq. (1.57) becomes

$$\sigma_x=-\frac{I_{\bar{y}\bar{z}}M_y+I_{\bar{y}}M_z}{I_{\bar{y}}I_{\bar{z}}-I_{\bar{y}\bar{z}}^2}\bar{y}+\frac{I_{\bar{z}}M_y+I_{\bar{y}\bar{z}}M_z}{I_{\bar{y}}I_{\bar{z}}-I_{\bar{y}\bar{z}}^2}\bar{z}$$

$$=-\frac{(a^3t/3)(-\overline{P}L)}{(4a^3t/3)(a^3t/4)-(a^3t/3)^2}\bar{y}+\frac{(a^3t/4)(-\overline{P}L)}{(4a^3t/3)(a^3t/4)-(a^3t/3)^2}\bar{z} \tag{2}$$

$$=\frac{3\overline{P}L}{2a^3t}\bar{y}-\frac{9\overline{P}L}{8a^3t}\bar{z}$$

TABLE 1.1 Part of the Output File for the Computer Program of the Appendixes for the Angle Section of Examples 1.1 and 1.2 with *a* = 1 and *t* = 0.1[a]

Cross-Sectional Properties		Corresponds to Equation:
Cross-Sectional Area	0.3000	
Y Moment of Area	−0.1999	1.52*a*
Z Moment of Area	0.0499	1.52*a*
Y Centroid	0.1663	1.56
Z Centroid	−0.6663	1.56
Moment of Inertia $I_{\bar{y}}$	0.1336	1.52*b*
Moment of Inertia $I_{\bar{z}}$	0.0252	1.52*b*
Product of Inertia $I_{\bar{y}\bar{z}}$	0.0332	1.52*b*
Principal Bending Angle (deg)	−15.7589	1.82 or 1.95
Principal Moment of Inertia (max)	0.1430	1.88
Principal Moment of Inertia (min)	0.0158	1.88

[a] See Fig. 5.26b for coordinate systems.

If the terms involving t^3 in (1) are not neglected, we would find

$$\sigma_x = \frac{48\overline{P}L}{at}\frac{16a^2 - 5t^2}{512a^4 + 944a^2t^2 + 95t^4}\overline{y} - \frac{48\overline{P}L}{at}\frac{12a^2 + 10t^2}{512a^4 + 944a^2t^2 + 95t^4}\overline{z} \tag{3}$$

Equation (2) is the desired distribution of σ_x on the cross section. The stress at point A of Fig. 1.9a is found by substituting $\overline{z} = 2a/3$ and $\overline{y} = 5a/6$ into (2):

$$(\sigma_x)_A = \frac{\overline{P}L}{2a^2t} \tag{4}$$

At point B, $\overline{y} = -a/6$, $\overline{z} = 2a/3$, and σ_x becomes

$$(\sigma_x)_B = -\frac{\overline{P}L}{a^2t} \tag{5}$$

Finally, at point D, $\overline{y} = -a/6$, $\overline{z} = -4a/3$, and σ_x is found to be

$$(\sigma_x)_D = \frac{5\overline{P}L}{4a^2t} \tag{6}$$

The distribution of the normal stresses is illustrated in Fig. 1.9b.

The neutral axis is defined by Eq. (1.58) as

$$\overline{y} = \frac{I_{\overline{z}}M_y}{I_{\overline{yz}}M_y}\overline{z} = \frac{I_{\overline{z}}}{I_{\overline{yz}}}\overline{z} = \frac{3}{4}\overline{z} \tag{7}$$

This line is plotted in Fig. 1.9c. The angle between the $\overline{y}$ axis and the neutral axis is 53.13°.

If the asymmetrical nature of the cross section is ignored, $I_{\overline{yz}}$ would be zero and the normal stress σ_x of Eq. (1.57) would be

$$\sigma_x = \frac{M_y}{I_{\overline{y}}}\overline{z} \tag{8}$$

The maximum stress occurs at point D with $\overline{z} = -4a/3$, so that (8) becomes

$$(\sigma_x)_D = \frac{\overline{P}L}{a^2t} \tag{9}$$

At points A and B, $\overline{z} = 2a/3$ and (8) becomes

$$(\sigma_x)_A = (\sigma_x)_B = -\frac{\overline{P}L}{2a^2t} \tag{10}$$

Note that these values are not consistent with (4), (5), and (6).

The radii of gyration about the centroidal axes $\overline{y}, \overline{z}$ are defined by

$$\overline{r}_y = \sqrt{\frac{I_{\overline{y}}}{A}} \qquad \overline{r}_z = \sqrt{\frac{I_{\overline{z}}}{A}} \tag{1.61}$$

The elastic section moduli Y_e, Z_e about the centroidal axes $\overline{y}, \overline{z}$ are defined by

$$Y_e = \frac{I_{\overline{y}}}{\overline{z}_{\max}} \qquad Z_e = \frac{I_{\overline{z}}}{\overline{y}_{\max}} \tag{1.62}$$

where $\overline{z}_{\max}$ is the maximum distance between the $\overline{y}$ axis and the material points of the cross section, that is, $\overline{z}_{\max}$ is the distance from the $\overline{y}$ axis to the outermost fiber; $\overline{y}_{\max}$ is the maximum distance between the $\overline{z}$ axis and the material points of the cross section. The polar moment of inertia I_p with respect to the centroid of the section is the sum of the area moments of inertia about the $\overline{y}$ and $\overline{z}$ axes:

$$I_p = I_{\overline{y}} + I_{\overline{z}} \tag{1.63}$$

Modulus-Weighted Properties If the material properties are not homogeneous on the cross section, it is useful to introduce a reference modulus E_r and to define a *modulus-weighted* differential area by

$$d\tilde{A} = \frac{E}{E_r}\, dA \tag{1.64}$$

Then, Eq. (1.51) appears as

$$\begin{aligned} \kappa_\epsilon \tilde{A} + \kappa_y \tilde{Q}_y - \kappa_z \tilde{Q}_z &= 0 \\ \kappa_\epsilon \tilde{Q}_y + \kappa_y \tilde{I}_y - \kappa_z \tilde{I}_{yz} &= \frac{M_y}{E_r} \\ \kappa_\epsilon \tilde{Q}_z + \kappa_y \tilde{I}_{yz} - \kappa_z \tilde{I}_z &= -\frac{M_z}{E_r} \end{aligned} \tag{1.65}$$

In this equation, modulus-weighted section properties of the beam are utilized. The modulus-weighted first moments of area are

$$\tilde{Q}_y = \int z\, d\tilde{A} \qquad \tilde{Q}_z = \int y\, d\tilde{A} \tag{1.66a}$$

The modulus-weighted area moments of inertia are given by

$$\tilde{I}_y = \int z^2 d\tilde{A} \qquad \tilde{I}_z = \int y^2 d\tilde{A} \tag{1.66b}$$

and the modulus-weighted area product of inertia is

$$\tilde{I}_{yz} = \int yz\, d\tilde{A} \tag{1.66c}$$

For a homogeneous beam, the elastic modulus E has the same value at any point of the section, and E_r is chosen equal to E. The modulus-weighted properties then become purely geometric properties of the cross section of Eq. (1.52).

Equation (1.65) is simplified if the relationships are transformed to the centroidal coordinates. For the modulus-weighted case, the components of Eq. (1.54b) are

$$\overline{y} = y - \tilde{y}_C \qquad \overline{z} = z - \tilde{z}_C \tag{1.67}$$

As in the homogeneous case, the origin C is chosen such that the first moments of area in the coordinate system $C\,\overline{yz}$ are zero:

$$\begin{aligned} \tilde{Q}_{\overline{y}} &= \int \overline{z}\, d\tilde{A} = \int (z - \tilde{z}_C)\, d\tilde{A} = 0 \\ \tilde{Q}_{\overline{z}} &= \int \overline{y}\, d\tilde{A} = \int (y - \tilde{y}_C)\, d\tilde{A} = 0 \end{aligned} \tag{1.68}$$

These conditions give

$$\tilde{y}_C = \frac{\tilde{Q}_z}{\tilde{A}} \qquad \tilde{z}_C = \frac{\tilde{Q}_y}{\tilde{A}} \tag{1.69}$$

The point C is the *modulus-weighted centroid* of the cross section. When the material is homogeneous, C becomes the familiar geometric centroid, given by Eq. (1.56).

Transform Eq. (1.65) for the constants $\kappa_\epsilon, \kappa_y, \kappa_z$ to the centroidal coordinate system. Introduce Eq. (1.68). Then

$$\begin{aligned} \overline{\kappa}_\epsilon \tilde{A} &= 0 \\ \overline{\kappa}_y \tilde{I}_{\overline{y}} - \overline{\kappa}_z \tilde{I}_{\overline{y}\,\overline{z}} &= \frac{M_y}{E_r} \\ \overline{\kappa}_y \tilde{I}_{\overline{y}\,\overline{z}} - \overline{\kappa}_z \tilde{I}_{\overline{z}} &= -\frac{M_z}{E_r} \end{aligned} \tag{1.70}$$

Solve these equations for $\overline{\kappa}_\epsilon$, $\overline{\kappa}_y$, and $\overline{\kappa}_z$, and substitute the results into σ_x of Eq. (1.50), expressed in terms of $\overline{\kappa}_\epsilon$, $\overline{\kappa}_y$, and $\overline{\kappa}_z$. This leads to the normal stress

$$\sigma_x = \frac{E}{E_r}\left(-\frac{\tilde{I}_{\overline{y}\,\overline{z}} M_y + \tilde{I}_{\overline{y}} M_z}{\tilde{I}_{\overline{y}} \tilde{I}_{\overline{z}} - \tilde{I}^2_{\overline{y}\,\overline{z}}}\, \overline{y} + \frac{\tilde{I}_{\overline{z}} M_y + \tilde{I}_{\overline{y}\,\overline{z}} M_z}{\tilde{I}_{\overline{y}} \tilde{I}_{\overline{z}} - \tilde{I}^2_{\overline{yz}}}\, \overline{z} \right) \tag{1.71}$$

The parallel axis theorem transformation equations for the modulus-weighted properties are

$$
\begin{aligned}
\tilde{I}_{\overline{y}} &= \tilde{I}_y - \tilde{z}_C^2 \tilde{A} \\
\tilde{I}_{\overline{z}} &= \tilde{I}_z - \tilde{y}_C^2 \tilde{A} \\
\tilde{I}_{\overline{y}\,\overline{z}} &= \tilde{I}_{yz} - \tilde{y}_C \tilde{z}_C \tilde{A}
\end{aligned}
\tag{1.72}
$$

The radii of gyration about the centroidal axes $\overline{y}$, $\overline{z}$ are defined as

$$
\overline{r}_y = \sqrt{\frac{\tilde{I}_{\overline{y}}}{\tilde{A}}} \qquad \overline{r}_z = \sqrt{\frac{\tilde{I}_{\overline{z}}}{\tilde{A}}} \tag{1.73}
$$

and the elastic section moduli about the centroidal axes are

$$
Y_e = \frac{\tilde{I}_{\overline{y}}}{\overline{z}_{\max}} \qquad Z_e = \frac{\tilde{I}_{\overline{z}}}{\overline{y}_{\max}} \tag{1.74}
$$

Finally, the polar moment of inertia with respect to the centroid is

$$
\tilde{I}_p = \tilde{I}_{\overline{y}} + \tilde{I}_{\overline{z}} \tag{1.75}
$$

1.3 PRINCIPAL BENDING AXES

Figure 1.10 shows centroidal axes $\overline{y}$, $\overline{z}$ and a rotated set of centroidal axes y', z'. The unit vectors $\mathbf{j}$, $\mathbf{k}$ are directed along the $\overline{y}$, $\overline{z}$ axes and the unit vectors $\mathbf{j}'$, $\mathbf{k}'$ along the

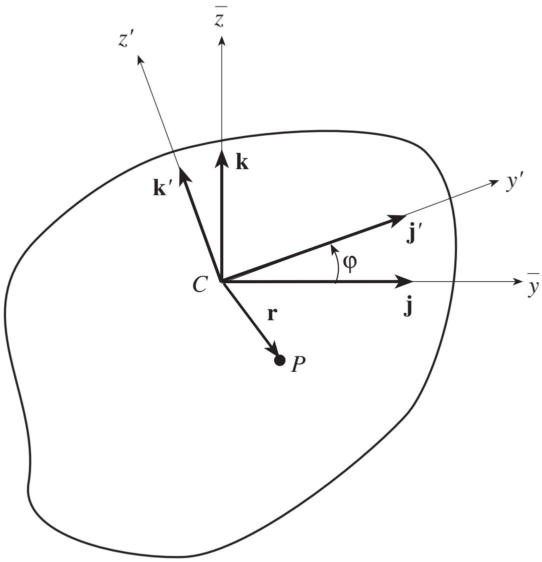


Figure 1.10 Rotated centroidal coordinate system.

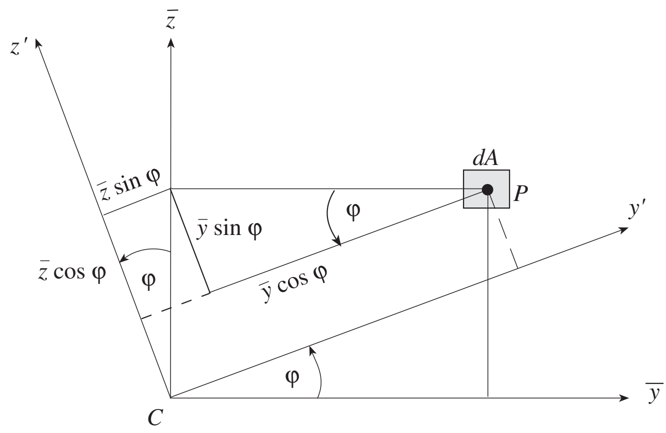


Figure 1.11 Rotation of the centroidal coordinate system.

y', z' axes. The position vector $\mathbf{r}$ of a point P on the cross section may be expressed as

$$\mathbf{r} = \overline{y}\mathbf{j} + \overline{z}\mathbf{k} = y'\mathbf{j}' + z'\mathbf{k}' \tag{1.76}$$

where $\overline{y}$ and $\overline{z}$ are the coordinates of P from the $\overline{y}, \overline{z}$ axes. Similarly, y' and z' are the coordinates of P from the y', z' axes. The y', z' coordinates can be obtained in terms of the $\overline{y}, \overline{z}$ coordinates (Fig. 1.11):

$$\begin{aligned} y' &= \overline{y}\mathbf{j}\cdot\mathbf{j}' + \overline{z}\mathbf{k}\cdot\mathbf{j}' = \overline{y}\cos\varphi + \overline{z}\sin\varphi \\ z' &= \overline{y}\mathbf{j}\cdot\mathbf{k}' + \overline{z}\mathbf{k}\cdot\mathbf{k}' = -\overline{y}\sin\varphi + \overline{z}\cos\varphi \end{aligned} \tag{1.77}$$

Suppose that the differential area dA is located at point P. The second moments of the area (i.e., the area moments of inertia) in the rotated coordinate system are

$$\begin{aligned} I_{y'} &= \int z'^2\, dA = I_{\overline{z}}\sin^2\varphi + I_{\overline{y}}\cos^2\varphi - 2I_{\overline{yz}}\sin\varphi\cos\varphi \\ I_{z'} &= \int y'^2\, dA = I_{\overline{z}}\cos^2\varphi + I_{\overline{y}}\sin^2\varphi + 2I_{\overline{yz}}\sin\varphi\cos\varphi \\ I_{y'z'} &= \int y'z'\, dA = I_{\overline{yz}}(\cos^2\varphi - \sin^2\varphi) + (I_{\overline{y}} - I_{\overline{z}})\sin\varphi\cos\varphi \end{aligned} \tag{1.78}$$

where the relations of Eq. (1.77) have been utilized. The use of the familiar trigonometric identities $2\cos^2\varphi = 1+\cos 2\varphi$, $2\sin^2\varphi = 1-\cos 2\varphi$, $2\sin\varphi\cos\varphi = \sin 2\varphi$, leads to an alternative form:

$$I_{y'} = \frac{I_{\overline{y}} + I_{\overline{z}}}{2} + \frac{I_{\overline{y}} - I_{\overline{z}}}{2}\cos 2\varphi - I_{\overline{yz}}\sin 2\varphi \tag{1.79a}$$

$$I_{z'} = \frac{I_{\overline{y}} + I_{\overline{z}}}{2} - \frac{I_{\overline{y}} - I_{\overline{z}}}{2} \cos 2\varphi + I_{\overline{yz}} \sin 2\varphi \tag{1.79b}$$

$$I_{y'z'} = \frac{I_{\overline{y}} - I_{\overline{z}}}{2} \sin 2\varphi + I_{\overline{yz}} \cos 2\varphi \tag{1.79c}$$

Equations (1.78) and (1.79) provide the area moments of inertia $I_{y'}$, $I_{z'}$, and $I_{y'x'}$ about coordinate axes y', x' at rotation angle φ. These three area moments of inertia as functions of φ are shown in Fig. 1.12. Note that these moments of inertia are bounded. The upper bound for $I_{y'}$ and $I_{z'}$ is $I_{\max} = I_1$ and the lower bound is $I_{\min} = I_2$. Also, for the product of inertia $I_{y'z'}$,

$$-\tfrac{1}{2}(I_1 - I_2) \le I_{y'z'} \le +\tfrac{1}{2}(I_1 - I_2) \tag{1.80}$$

The extreme values of the moments of inertia I_1 and I_2 are called *principal moments of inertia* and the corresponding angles define the *principal directions*. In the case shown in Fig. 1.12, both I_1 and I_2 are positive. As observed in Fig. 1.12 by the vertical dashed lines, the product of inertia is zero at the principal directions, which are 90° apart. That is, the two principal directions are perpendicular to each other.

To find the angle φ at which the moment of inertia $I_{y'}$ assumes its maximum value, set $\partial I_{y'}/\partial \varphi$ equal to zero. From Eq. (1.79a) this gives

$$(I_{\overline{y}} - I_{\overline{z}})(-\sin 2\varphi) - 2I_{\overline{yz}} \cos 2\varphi = 0 \tag{1.81}$$

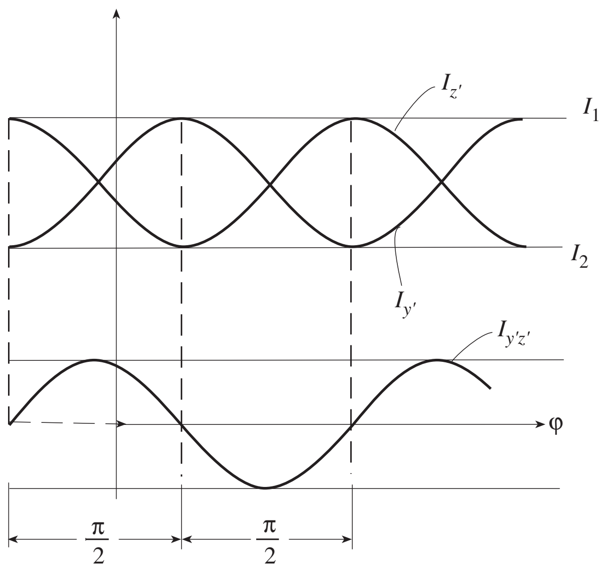


Figure 1.12 Three moments of inertia as a function of the rotation angle φ.

or

$$\tan 2\varphi = \frac{2I_{\overline{yz}}}{I_{\overline{z}} - I_{\overline{y}}} \tag{1.82}$$

This angle φ identifies the so-called *centroidal principal bending axes*. Note that φ of Eq. (1.82) also corresponds to the rotation for which the product of inertia $I_{y'z'}$ is zero. This result, which also was observed in Fig. 1.12, is verified by substituting Eq. (1.81) into Eq. (1.79c). Equation (1.81) determines two values of 2φ that are 180° apart, that is, two values of φ that are 90° apart. At these values, the moments of inertia $I_{y'}$ and $I_{z'}$ assume their maximum or minimum possible values, that is, the principal moments of inertia I_1 and I_2. The magnitudes of I_1 and I_2 can be obtained by substituting φ of Eq. (1.82) into Eq. (1.79a and b). These same values will be obtained below by a different technique. The corresponding directions defined by $\pm\mathbf{j}'$ and $\pm\mathbf{k}'$ are the principal directions. As a particular case, if a cross section is symmetric about an axis, this axis of symmetry is a principal axis.

Consider another approach to finding the magnitudes of the principal moments of inertia I_1 and I_2. It is possible to derive some relationships that are invariant with respect to the rotating coordinate system. It follows from Eq. (1.78) or (1.79) that

$$\begin{aligned} I_{y'} + I_{z'} &= I_{\overline{y}} + I_{\overline{z}} \\ I_{y'} I_{z'} - I_{y'z'}^2 &= I_{\overline{y}} I_{\overline{z}} - I_{\overline{yz}}^2 \end{aligned} \tag{1.83}$$

As noted above, the product of inertia $I_{y'z'}$ is zero at the principal directions and $I_{y'}$ and $I_{z'}$ become I_1 and I_2. Then

$$\begin{aligned} I_{y'} + I_{z'} &= I_{\overline{y}} + I_{\overline{z}} = I_1 + I_2 \\ I_{y'} I_{z'} - I_{y'z'}^2 &= I_{\overline{y}} I_{\overline{z}} - I_{\overline{yz}}^2 = I_1 I_2 \end{aligned} \tag{1.84}$$

The principal moments of inertia I_1 and I_2 can be considered to be the roots of the equation

$$(I - I_1)(I - I_2) = 0 \tag{1.85}$$

Expand Eq. (1.85):

$$I^2 - (I_1 + I_2)I + I_1 I_2 = 0 \tag{1.86}$$

and introduce Eq. (1.84):

$$I^2 - (I_{\overline{y}} + I_{\overline{z}})I + I_{\overline{y}} I_{\overline{z}} - I_{\overline{yz}}^2 = 0 \tag{1.87}$$

The two roots of this equation are the principal moments of inertia

$$
\begin{aligned}
I_1 = I_{\max} &= \frac{I_{\overline{y}} + I_{\overline{z}}}{2} + \Delta \\
I_2 = I_{\min} &= \frac{I_{\overline{y}} + I_{\overline{z}}}{2} - \Delta
\end{aligned}
\tag{1.88}
$$

where

$$
\Delta = \sqrt{\left(\frac{I_{\overline{y}} - I_{\overline{z}}}{2}\right)^2 + I_{\overline{yz}}^2}
$$

Numerical values for some of these parameters are given in Table 1.1 for an angle section.

It is useful to place the transformation relations of Eq. (1.78) or (1.79) in a particular matrix form. Equation (1.87) can be expressed as

$$
(I - I_{\overline{y}})(I - I_{\overline{z}}) - I_{\overline{yz}}^2 = 0 \tag{1.89}
$$

or

$$
\begin{vmatrix} I - I_{\overline{y}} & I_{\overline{yz}} \\ I_{\overline{yz}} & I - I_{\overline{z}} \end{vmatrix} = 0 \tag{1.90}
$$

This determinant is the characteristic equation for the symmetric 2×2 matrix $\mathbf{A}$:

$$
\mathbf{A} = \begin{bmatrix} I_{\overline{y}} & -I_{\overline{yz}} \\ -I_{\overline{yz}} & I_{\overline{z}} \end{bmatrix} \tag{1.91}
$$

With the negative signs on $I_{\overline{yz}}$, $\mathbf{A}$ transforms according to the rotation conventions implied by Fig. 1.10 with φ measured counterclockwise positive from the $\overline{y}$ axis. Define a rotation matrix

$$
\mathbf{R} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \tag{1.92}
$$

It may be verified that the transformation

$$
\mathbf{A}' = \begin{bmatrix} I_{y'} & -I_{y'z'} \\ -I_{y'z'} & I_{z'} \end{bmatrix} = \mathbf{RAR}^{-1} = \mathbf{RAR}^{\mathrm{T}} \tag{1.93}
$$

is identical to the rotation transformation equations of Eq. (1.78) or (1.79) derived from Fig. 1.10. If the off-diagonal elements of $\mathbf{A}$ are taken to be $+I_{yz}$, the relationship between $\mathbf{A}$ and $\mathbf{A}'$ no longer matches these equations.

An alternative approach is to base the determination of the principal axes on the diagonalization of matrix $\mathbf{A}$ of Eq. (1.91). When the product of inertia $I_{\overline{yz}}$ is zero, the $\overline{y}$, $\overline{z}$ axes are already the principal axes and no further computation is necessary.

In the special case when $I_{\overline{y}} = I_{\overline{z}}$, any axis is a principal axis. If $I_{\overline{yz}}$ is not zero, the two vectors

$$\mathbf{v}_1 = I_{\overline{yz}}\mathbf{j} + (I_{\overline{y}} - I_1)\mathbf{k}$$
$$\mathbf{v}_2 = (I_{\overline{z}} - I_2)\mathbf{j} + I_{\overline{yz}}\mathbf{k} \tag{1.94}$$

are two orthogonal eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues I_1 and I_2. Some characteristics of eigenvectors are discussed in Chapter 8. The angle φ between the $\overline{y}$ axis and the axis belonging to the larger principal moment of inertia can be computed as the angle between $\pm\mathbf{v}_1$ and $\mathbf{j}$:

$$\varphi = \tan^{-1} \frac{I_{\overline{y}} - I_1}{I_{\overline{yz}}} \tag{1.95}$$

Since the angle between the smaller principal value and the $\overline{y}$ axis is $\varphi + 90°$, the specification of φ is enough to determine both principal axes.

The results of this section apply also to nonhomogeneous beams. It is only necessary to replace all geometric section properties with modulus-weighted ones. If a nonhomogeneous section has an axis of geometric as well as elastic symmetry, it may be concluded that this axis is a principal axis.

Normal Stresses from the Principal Bending Axes If y', z' are the centroidal principal bending axes, Eq. (1.71) simplifies to

$$\sigma_x = \frac{E}{E_r}\left(-\frac{M_z y'}{\tilde{I}_{z'}} + \frac{M_y z'}{\tilde{I}_{y'}}\right) \tag{1.96}$$

For homogeneous materials, Eq. (1.96) reduces to

$$\sigma_x = -\frac{M_z y'}{I_{z'}} + \frac{M_y z'}{I_{y'}} \tag{1.97}$$

In general, the bending moment components are initially calculated in any convenient coordinate system, and when using Eq. (1.96) or (1.97), it is necessary to compute the bending moment components along the principal bending axes.

Example 1.2 Thin-Walled Cantilevered Beam with an Asymmetrical Cross Section. Return to the cantilevered angle of Fig. 1.8 and find the normal stresses using Eq. (1.97), which is based on the principal bending axes.

SOLUTION. From Eq. (1) of Example 1.1 and Eq. (1.88),

$$I_{\overline{y}} = \tfrac{4}{3}a^3t \qquad I_{\overline{z}} = \tfrac{1}{4}a^3t \qquad I_{\overline{yz}} = \tfrac{1}{3}a^3t \tag{1}$$

$$\Delta = \sqrt{\left(\frac{I_{\overline{y}} - I_{\overline{z}}}{2}\right)^2 + I_{\overline{yz}}^2} = \frac{\sqrt{233}}{24}a^3t \tag{2}$$

$$I_1 = \frac{I_{\overline{y}} + I_{\overline{z}}}{2} + \Delta = \frac{a^3 t}{24}(19 + \sqrt{233})$$
$$I_2 = \frac{I_{\overline{y}} + I_{\overline{z}}}{2} - \Delta = \frac{a^3 t}{24}(19 - \sqrt{233}) \tag{3}$$

The centroidal principal bending axes are located by the angle φ, where (Eq. 1.82)

$$\tan 2\varphi = \frac{2I_{\overline{yz}}}{I_{\overline{z}} - I_{\overline{y}}} = -\frac{8}{13} \tag{4}$$

This relationship leads to the two angles $\varphi = -15.8^\circ$ and $\varphi = 74.2^\circ$, one of which corresponds to I_1 and the other to I_2. Further manipulations are necessary to determine which angle corresponds to I_1 and which to I_2. For example, place $\varphi = -15.8^\circ$ into $I_{y'}$ of Eq. (1.78) and find $I_{y'} = (a^3t/24)(19 + \sqrt{233})$, which is equal to I_1.

The problem of the uncertainty of which value of φ corresponds to I_1 is avoided if Eq. (1.95) is used. In this case,

$$\varphi = \tan^{-1} \frac{I_{\overline{y}} - I_1}{I_{\overline{yz}}} = \tan^{-1}\left[\tfrac{1}{8}\left(13 - \sqrt{233}\right)\right] \tag{5}$$

so that $\varphi = -15.8^\circ$ and 164.2°, both of which identify I_1 (Fig. 1.13).

The cross-sectional normal stress σ_x is given by Eq. (1.97). At an axial distance L from the free end, the bending moment components along the principal bending axes are

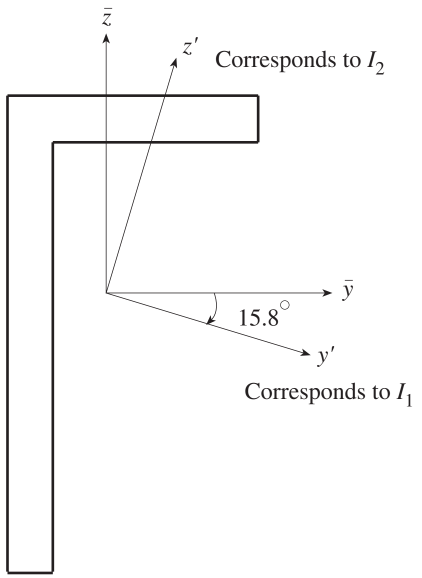


Figure 1.13 Principal bending axes of an asymmetrical cross section.

$$M_{y'} = -\overline{P}L\cos(-15.8°)$$
$$M_{z'} = -[-\overline{P}L\sin(-15.8°)] = \overline{P}L\sin(-15.8°) \tag{6}$$

The sign convention for these moments is discussed in Chapter 2. Equation (1.97) becomes

$$\sigma_x = -\frac{M_z y'}{I_{z'}} + \frac{M_y z'}{I_{y'}} = -\frac{\overline{P}L\sin(-15.8°)y'}{(a^3t/24)(19-\sqrt{233})} + \frac{-\overline{P}L\cos(-15.8°)z'}{(a^3t/24)(19+\sqrt{233})}$$
$$= \frac{1.75\overline{P}L}{a^3t}y' - \frac{0.674\overline{P}L}{a^3t}z' \tag{7}$$

At point A of the cross section shown in Fig. 1.9a, $\overline{y} = 5a/6$ and $\overline{z} = 2a/3$, and from Eq. (1.77),

$$y' = \frac{5a}{6}\cos(-15.8°) + \frac{2a}{3}\sin(-15.8°) = 0.620a$$
$$z' = -\frac{5a}{6}\sin(-15.8°) + \frac{2a}{3}\cos(-15.8°) = 0.869a \tag{8}$$

Substitution of these coordinates into (7) gives $(\sigma_x)_A = \overline{P}L/2a^2t$. At point B, $\overline{y} = -a/6$, $\overline{z} = 2a/3$, and Eq. (1.77) gives $y' = -0.342a$ and $z' = 0.596a$. From (7), $(\sigma_x)_B = -\overline{P}L/a^2t$. At point D, $\overline{y} = -a/6$, $\overline{z} = -4a/3$, $y' = 0.203a$, $z' = -1.328a$, and $(\sigma_x)_D = 5\overline{P}L/4a^2t$. These are the values calculated in Example 1.1.

1.4 AXIAL LOADS

An axial load N_x applied in the x direction at point P of the beam cross section shown in Fig. 1.7 causes additional normal stress. In this case, it is necessary to replace the force at P with its force–couple equivalent at the centroid C. The moment of the equivalent couple is

$$\mathbf{r}_{PC} \times N_x\mathbf{i} = (z_P\mathbf{k} + y_P\mathbf{j}) \times N_x\mathbf{i} = z_P N_x\mathbf{j} - y_P N_x\mathbf{k} \tag{1.98}$$

where y_P, z_P are the coordinates of point P in the coordinate system $C\overline{yz}$. The additional bending moments due to the axial force are added to the pure bending moments M_y^0, M_z^0 at the section

$$M_y = M_y^0 + z_P N_x \qquad M_z = M_z^0 - y_P N_x \tag{1.99}$$

With the inclusion of the axial force, Eqs. (1.57) and (1.71) for normal stress become

$$\sigma_x = \frac{N_x}{A} - \frac{I_{\overline{y}\overline{z}}M_y + I_{\overline{y}}M_z}{I_{\overline{y}}I_{\overline{z}} - I_{\overline{y}\overline{z}}^2}\overline{y} + \frac{I_{\overline{z}}M_y + I_{\overline{y}\overline{z}}M_z}{I_{\overline{y}}I_{\overline{z}} - I_{\overline{y}\overline{z}}^2}\overline{z} \tag{1.100}$$

and

$$\sigma_x = \frac{E}{E_r}\left(\frac{N_x}{\tilde{A}} - \frac{\tilde{I}_{\overline{y}\overline{z}}M_y + \tilde{I}_{\overline{y}}M_z}{\tilde{I}_{\overline{y}}\tilde{I}_{\overline{z}} - \tilde{I}^2_{\overline{y}\overline{z}}}\overline{y} + \frac{\tilde{I}_{\overline{z}}M_y + \tilde{I}_{\overline{y}\overline{z}}M_z}{\tilde{I}_{\overline{y}}\tilde{I}_{\overline{z}} - \tilde{I}^2_{\overline{y}\overline{z}}}\overline{z}\right) \tag{1.101}$$

1.5 ELASTICITY SOLUTION FOR PURE BENDING

A beam for which the moments M_y and M_z are constant along the length is said to be in the state of *pure bending*. The elasticity solution for the displacements u_x, u_y, and u_z of a homogeneous beam in pure bending is obtained by assuming a strain field and attempting to satisfy the equations of elasticity. The axes are chosen as shown in Fig. 1.3. The origin O is at the centroid C of the cross section, so that the (x, y, z) and $(\overline{x}, \overline{y}, \overline{z})$ axes coincide. The beam material is assumed to be homogeneous and body forces are assumed to be absent. It follows from the displacement of Eq. (1.47) that a reasonable form of the strains is

$$\begin{aligned}
\epsilon_x &= \kappa_\epsilon + \kappa_y z - \kappa_z y \\
\epsilon_y &= -\nu\epsilon_x \\
\epsilon_z &= -\nu\epsilon_x \\
\gamma_{xy} &= 0 \\
\gamma_{yz} &= 0 \\
\gamma_{zx} &= 0
\end{aligned} \tag{1.102}$$

As shown in Eq. (1.48), the strain ϵ_x is obtained from $\partial u/\partial x$. This strain field identically satisfies the conditions of compatibility of Eq. (1.9). Substitution of the strains of Eq. (1.102) into the Hooke's law formulas of Eq. (1.18) shows that the only nonzero stress is the axial stress:

$$\sigma_x = E\epsilon_x \tag{1.103}$$

The total axial force N_x acting on the cross section is

$$N_x = \int \sigma_x \, dA = E\int (\kappa_\epsilon + \kappa_y z - \kappa_z y)\, dA = E\kappa_\epsilon A \tag{1.104}$$

In this calculation, the factors multiplying κ_y and κ_z, that is, the integrals $E\int z\, dA$ and $E\int y\, dA$, are proportional to the y, z coordinates of the centroid, which are both zero because the centroid C is at the origin O of the coordinates. For pure bending the axial force N_x will be zero (Eq. 1.46). It follows from Eq. (1.104) that the constant κ_ϵ is zero. Then the y, z components of the bending moment as expressed by Eq. (1.45) are

$$
\begin{aligned}
M_y &= \int z\sigma_x\, dA = E(\kappa_y I_y - \kappa_z I_{yz}) \\
M_z &= -\int y\sigma_x\, dA = E(-\kappa_y I_{yz} + \kappa_z I_z)
\end{aligned}
\tag{1.105}
$$

where the moments of inertia are given by Eq. (1.52b). Equation (1.105) can be solved for the constants κ_y and κ_z, giving

$$
\kappa_y = \frac{I_z M_y + I_{yz} M_z}{E(I_y I_z - I_{yz}^2)} \qquad \kappa_z = \frac{I_{yz} M_y + I_y M_z}{E(I_y I_z - I_{yz}^2)}
\tag{1.106}
$$

It follows from $\sigma_x = E\epsilon_x = E(\kappa_y z - \kappa_z y)$ that the axial stress is again given by Eq (1.57), with $\overline{y} = y$ and $\overline{z} = z$.

The displacements can be obtained from the strain–displacement relations of Eq. (1.3). With the strains given by Eq. (1.102),

$$
\begin{aligned}
\epsilon_x &= \frac{\partial u_x}{\partial x} = \kappa_y z - \kappa_z y & \epsilon_y &= \frac{\partial u_y}{\partial y} = -\nu(\kappa_y z - \kappa_z y) \\
\epsilon_z &= \frac{\partial u_z}{\partial z} = -\nu(\kappa_y z - \kappa_z y) & \gamma_{xy} &= \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = 0 \\
\gamma_{xz} &= \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} = 0 & \gamma_{yz} &= \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = 0
\end{aligned}
\tag{1.107}
$$

The displacements will be determined from these six equations by direct integration. From the first equation, the axial displacement may be expressed as

$$
u_x = \kappa_y xz - \kappa_z xy + u_{x0}(y,\, z)
\tag{1.108}
$$

where u_{x0} is an unknown function of y and z. The derivatives of u_y and u_z with respect to x are given in terms of u_{x0} by $\gamma_{xy} = 0$ and $\gamma_{xz} = 0$:

$$
\begin{aligned}
\frac{\partial u_y}{\partial x} &= -\frac{\partial u_x}{\partial y} = \kappa_z x - \frac{\partial u_{x0}}{\partial y} \\
\frac{\partial u_z}{\partial x} &= -\frac{\partial u_x}{\partial z} = -\kappa_y x - \frac{\partial u_{x0}}{\partial z}
\end{aligned}
\tag{1.109}
$$

from which the displacements u_y and u_z are obtained in the form

$$
\begin{aligned}
u_y &= \kappa_z \frac{x^2}{2} - x\frac{\partial u_{x0}}{\partial y} + u_{y0}(y, z) \\
u_z &= -\kappa_y \frac{x^2}{2} - x\frac{\partial u_{x0}}{\partial z} + u_{z0}(y, z)
\end{aligned}
\tag{1.110}
$$

where u_{y0} and u_{z0} are unknown functions of y and z, respectively.

The second and third strain–displacement relations of Eq. (1.107) become

$$\begin{aligned}
\epsilon_y &= \frac{\partial u_y}{\partial y} = -x\frac{\partial^2 u_{x0}}{\partial y^2} + \frac{\partial u_{y0}}{\partial y} = -\nu(\kappa_y z - \kappa_z y) \\
\text{or} \quad & -x\frac{\partial^2 u_{x0}}{\partial y^2} + \frac{\partial u_{y0}}{\partial y} + \nu(\kappa_y z - \kappa_z y) = 0 \\
\epsilon_z &= \frac{\partial u_z}{\partial z} = -x\frac{\partial^2 u_{x0}}{\partial z^2} + \frac{\partial u_{z0}}{\partial z} = -\nu(\kappa_y z - \kappa_z y) \\
\text{or} \quad & -x\frac{\partial^2 u_{x0}}{\partial z^2} + \frac{\partial u_{z0}}{\partial z} + \nu(\kappa_y z - \kappa_z y) = 0
\end{aligned} \tag{1.111}$$

Note the functional form of these equations, with the coordinate x occurring only once as a factor multiplying a second partial derivative of u_{x0}. Since these equations must hold for all values of x,

$$\frac{\partial^2 u_{x0}}{\partial y^2} = 0 \qquad \frac{\partial^2 u_{x0}}{\partial z^2} = 0 \tag{1.112}$$

Consequently, u_{y0} and u_{z0} can be obtained by integration of Eq. (1.111):

$$\begin{aligned}
u_{y0} &= -\nu\left(\kappa_y yz - \kappa_z \frac{y^2}{2}\right) + u_{y1}(z) \\
u_{z0} &= -\nu\left(\kappa_y \frac{z^2}{2} - \kappa_z yz\right) + u_{z1}(y)
\end{aligned} \tag{1.113}$$

The final strain–displacement relation of Eq. (1.107),

$$\gamma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = 0 \tag{1.114}$$

becomes

$$-2x\frac{\partial^2 u_{x0}}{\partial y\,\partial z} + \frac{\partial u_{z0}}{\partial y} + \frac{\partial u_{y0}}{\partial z} = -2x\frac{\partial^2 u_{x0}}{\partial y\,\partial z} + \frac{du_{z1}}{dy} + \nu\kappa_z z + \frac{du_{y1}}{dz} - \nu\kappa_y y = 0 \tag{1.115}$$

The functional form of this equality shows that the factor multiplying x is zero

$$\frac{\partial^2 u_{x0}}{\partial y\,\partial z} = 0 \tag{1.116}$$

Hence

$$\frac{du_{z1}}{dy} - \nu\kappa_y y + \frac{du_{y1}}{dz} + \nu\kappa_z z = 0 \tag{1.117}$$

By a separation of variables,

$$\frac{du_{z1}}{dy} - \nu\kappa_y y = C_0 \qquad \frac{du_{y1}}{dz} + \nu\kappa_z z = -C_0 \tag{1.118}$$

where C_0 is a constant.

The relations

$$\frac{\partial^2 u_{x0}}{\partial y^2} = 0 \qquad \frac{\partial^2 u_{x0}}{\partial z^2} = 0 \qquad \frac{\partial^2 u_{x0}}{\partial y\,\partial z} = 0 \tag{1.119}$$

of Eqs. (1.112) and (1.116) show that the function u_{x0} has the form

$$u_{x0} = C_1 y + C_2 z + C_3 \tag{1.120}$$

in which the C_k are constants. It follows from Eq. (1.108) that the axial displacement appears as

$$u_x = \kappa_y xz - \kappa_z xy + C_1 y + C_2 z + C_3 \tag{1.121}$$

To find u_{y1}, u_{z1} of Eq. (1.113), integrate Eq. (1.118):

$$\begin{aligned} u_{y1} &= -C_0 z - \nu\kappa_z \frac{z^2}{2} + C_4 \\ u_{z1} &= C_0 y + \nu\kappa_y \frac{y^2}{2} + C_5 \end{aligned} \tag{1.122}$$

so that, from Eq. (1.113),

$$\begin{aligned} u_{y0} &= -\nu\left(\kappa_y yz - \kappa_z \frac{y^2 - z^2}{2}\right) - C_0 z + C_4 \\ u_{z0} &= -\nu\left(\kappa_y \frac{z^2 - y^2}{2} - \kappa_z yz\right) + C_0 y + C_5 \end{aligned} \tag{1.123}$$

The displacements u_y and u_z of Eq. (1.110) may therefore be expressed as

$$\begin{aligned} u_y &= \kappa_z \frac{x^2}{2} - \nu\left(\kappa_y yz - \kappa_z \frac{y^2 - z^2}{2}\right) - C_0 z - C_1 x + C_4 \\ u_z &= -\kappa_y \frac{x^2}{2} - \nu\left(\kappa_y \frac{z^2 - y^2}{2} - \kappa_z yz\right) + C_0 y - C_2 x + C_5 \end{aligned} \tag{1.124}$$

The constants of integration in the expressions derived for the displacements depend on the support conditions. For example, suppose that the centroid at the origin of the coordinates ($x = 0,\ y = 0,\ z = 0$) at the left end ($x = 0$) of the horizontal beam is fixed such that no translational or rotational motion is possible. Then $u_x = 0,\ u_y = 0,\ u_z = 0$ at $x = 0,\ y = 0,\ z = 0$. Also, at $x = 0,\ y = 0,\ z = 0$, there is no rotation in the z direction ($\partial u_z/\partial x = 0$), no rotation in the y direction ($\partial u_y/\partial x = 0$), and no rotation about the x axis ($\partial u_y/\partial z = 0$).

The enforcement of these boundary conditions amounts to restraining the beam at $x = 0,\ y = 0,\ z = 0$ against rigid-body translation and rotation. From Eqs. (1.121) and (1.124) these boundary conditions require that

$$C_0 = 0, \quad C_1 = 0, \quad C_2 = 0, \quad C_3 = 0, \quad C_4 = 0, \quad C_5 = 0 \tag{1.125}$$

The displacements can now be written as (Eqs. 1.121, 1.124, and 1.125)

$$u_x = (\kappa_y z - \kappa_z y)x \tag{1.126a}$$

$$u_y = \kappa_z \frac{x^2}{2} - \nu \left(\kappa_y yz - \kappa_z \frac{y^2 - z^2}{2} \right) \tag{1.126b}$$

$$u_z = -\kappa_y \frac{x^2}{2} - \nu \left(\kappa_y \frac{z^2 - y^2}{2} - \kappa_z yz \right) \tag{1.126c}$$

Consider a special case of a beam with a cross section symmetric about the z axis and for which $M_z = 0$. Then, from Eq. (1.106), $\kappa_z = 0$ and $\kappa_y = M_y/EI_y$.For this case, the displacements of Eq. (1.126) become

$$u_x = \kappa_y z x \tag{1.127a}$$

$$u_y = -\nu \kappa_y yz \tag{1.127b}$$

$$u_z = -\frac{\kappa_y}{2}[x^2 + \nu(z^2 - y^2)] \tag{1.127c}$$

The deflection of the centroidal beam axis is given by Eq. (1.127c) with y and z equal to zero; that is,

$$u_z(x, 0, 0) = w = -\kappa_y \frac{x^2}{2} = -\frac{M_y}{EI_y} \frac{x^2}{2} \tag{1.128}$$

This is the same deflection given by engineering beam theory (Chapter 2) for a cantilevered beam loaded with a concentrated moment at the free end. Some interesting characteristics of beams in bending can be studied by considering the displacements away from the central axis.

To find the axial displacement at a particular cross section, say at $x = a$, consider $u_x(x, y, z)$ of Eq. (1.127a). Thus

$$u_x(a,\ y,\ z) = \kappa_y z a \tag{1.129}$$

We see that cross-sectional planes remain planar. This is not surprising since the assumed strain ϵ_x corresponds to a linear variation in the displacements in the y and z directions (Eq. 1.47).

Note from Eq. (1.127a) that beam fibers in the $z = 0$ plane do not displace in the x direction [i.e., $u_x(x, y, 0) = 0$]. Consequently, this plane is referred to as the *neutral plane*. The x axis before deformation is designated as the *neutral axis*.

To illustrate the distortion of the cross-section profile, consider the rectangular section of Fig. 1.14. From Eq. (1.127b), the horizontal displacements u_y of the vertical sides are

$$u_y\left(x, \pm\frac{b}{2}, z\right) = \pm\frac{b}{2}(-\nu\kappa_y z) \tag{1.130}$$

Thus, the vertical sides rotate. The vertical displacements of the top and bottom ($z = \pm h/2$) are (Eq. 1.127c)

$$u_z\left(x, y, \pm\frac{h}{2}\right) = -\frac{\kappa_y}{2}\left[x^2 + \nu\left(\frac{h^2}{4} - y^2\right)\right] = \frac{M}{2EI_y}\left[x^2 + \nu\left(\frac{h^2}{4} - y^2\right)\right] \tag{1.131}$$

where, as seen in Fig. 1.15a, $\kappa_y = M_y/EI_y = -M/EI_y$. This shows that the top and bottom are deformed into parabolic shapes. Assume that $b \gg h$. Note that if the curvature of the longitudinal axis of the beam is concave upward (Fig. 1.15a), the curvature of the top and bottom surfaces are concave downward (Fig. 1.15b). This is referred to as *anticlastic curvature*. For the thin-walled beam of Fig. 1.15, the anticlastic curvature can be significant. In contrast, if the depth and width are of comparable size (Fig. 1.14), the effect is small.

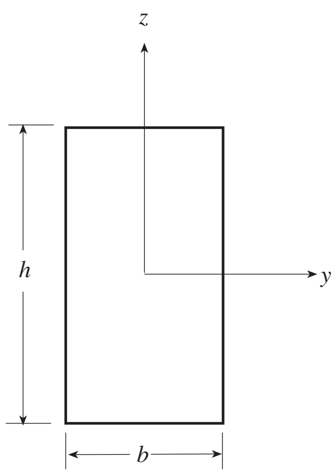


Figure 1.14 Beam cross section.

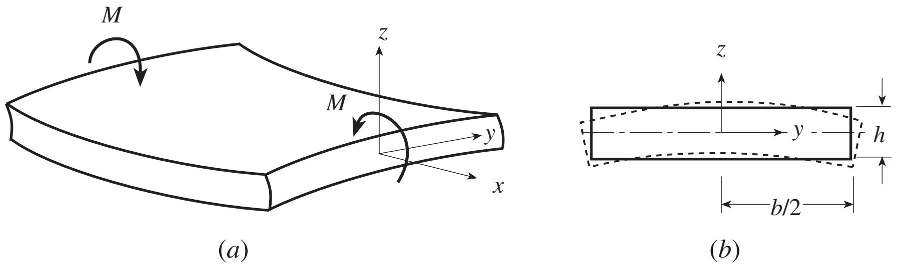


Figure 1.15 Anticlastic curvature.

There is a simple physical interpretation of this behavior in bending. For pure bending as shown in Fig. 1.15a, the upper fibers are in compression and the lower fibers in tension. Strain ϵ_x along the x direction is accompanied by strain $-\nu\epsilon_x$ in the y direction, where ν is Poisson's ratio. It follows that as the upper fibers are compressed in the x direction, they become somewhat longer in the y direction. Conversely, as the lower fibers are extended in the x direction, they shorten in the y direction.

Engineering Beam Theory In contrast to the pure bending assumptions of this section, engineering beam theory, which is presented in Chapter 2, is applied to beams under general lateral loading conditions. The bounding surface of the beam is often not free of stress; body forces are not necessarily zero; the shear force at each section is nonzero; and the bending moment is not constant along the beam. Engineering beam theory neglects the normal stresses σ_y and σ_z, which are much smaller than the axial stress. Also neglected is the influence of Poisson's ratio, so that longitudinal fibers deform independently. For engineering beam theory, the normal stresses and strains are calculated as in the case of pure bending, although the bending moment is no longer constant along the beam axis.

REFERENCES

Atanackovic, T. M., and Guran, A. (2000). *Theory of Elasticity for Scientists and Engineers*, Birkhäuser, Boston.

Boresi, A. P., and Chong, K. P. (1987). *Elasticity in Engineering Mechanics*, Elsevier, Amsterdam, The Netherlands.

Gere, J. M. (2001). *Mechanics of Materials*, 5th ed., PWS, Boston.

Gould, P. L. (1994). *Introduction to Linear Elasticity*, 2nd ed., Springer-Verlag, New York.

Love, A. E. (1944). *A Treatise on the Mathematical Theory of Elasticity*, 4th ed., Dover, New York.

Muskhelishvili, N. I. (1953). *Some Basic Problems of the Mathematical Theory of Elasticity*, Walters-Noordhoff, Groningen, The Netherlands.

Oden, J. T., and Ripperger, E. A. (1981) *Mechanics of Elastic Structures*, 2nd ed., McGraw-Hill, New York.

Pilkey, W. D., and Wunderlich, W. (1994). *Mechanics of Structures; Variational and Computational Methods*, CRC Press, Boca Raton, Fla.

Rivello, R. M. (1969). *Theory and Analysis of Flight Structures*, McGraw-Hill, New York.

Sokolnikoff, I. S. (1956). *Mathematical Theory of Elasticity*, McGraw-Hill, New York.

Uğural, A. C., and Fenster, S. K. (1981) *Advanced Strength and Applied Elasticity*, Elsevier, Amsterdam, The Netherlands.