CHAPTER 1

THE KINEMATICS AND DYNAMICS OF AIRCRAFT MOTION

1.1 INTRODUCTION

In this chapter the end point will be the equations of motion of a rigid aircraft moving over the oblate, rotating Earth. The flat-Earth equations, describing motion over a small area of a nonrotating Earth, with constant gravity, will be derived as a special case. To reach this end point we will use the vector analysis of classical mechanics to set up the equations of motion, matrix algebra to describe operations with coordinate systems, and concepts from geodesy, gravitation, and navigation to introduce the effects of the Earth's shape and mass attraction.

The moments and forces acting on the vehicle, other than the mass attraction of the Earth, will be abstract until Chapter 2 is reached. At this stage the equations can be used to describe the motion of any type of aerospace vehicle, including an Earth satellite, provided that suitable force and moment models are available. The term *rigid* means that structural flexibility is not allowed for, and all points in the vehicle are assumed to maintain the same relative position at all times. This assumption is good enough for flight simulation in most cases, and good enough for flight control system design provided that we are not trying to design a system to control structural modes or to alleviate aerodynamic loads on the aircraft structure.

The vector analysis needed for the treatment of the equations of motion often causes difficulties for the student, particularly the concept of the angular velocity vector. Therefore, a review of the relevant topics is provided. In some cases we have gone beyond the traditional approach to flight mechanics. For example, quaternions have been introduced because of their "all-attitude" capability and numerical advantages in simulation and control. They are now widely used in simulation, robotics, guidance and navigation calculations, attitude control, and graphics animation. Topics from geodesy (a branch of mathematics dealing with the shape of the Earth), gravitation (the mass attraction effect of the Earth), and navigation have also been introduced. This is because aircraft can now fly autonomously at very high altitudes and over long distances and there is a need to simulate navigation of such vehicles.

The equations of motion will be organized as a set of simultaneous first-order differential equations, explicitly solved for the derivatives. For n dependent variables, X_i, and m control inputs, U_i, the general form will be:

$$\begin{aligned} \dot{X}_1 &= f_1(X_1, X_2, \ldots, X_n, U_1, \ldots, U_m) \\ &\vdots \\ \dot{X}_n &= f_n(X_1, X_2, \ldots, X_n, U_1, \ldots, U_m), \end{aligned} \tag{1.1-1}$$

where the functions f_i are the nonlinear functions that can arise from modeling real systems. If the variables X_i constitute the smallest set of variables that, together with given inputs U_i, completely describe the behavior of the system, then the X_i are a set of *state variables* for the system. Equations (1.1-1) become a *state-space* description of the system. The functions f_i are required to be single-valued continuous functions. Equations (1.1-1) are often written symbolically as:

$$\dot{X} = f(X, U), \tag{1.1-2}$$

where the *state vector* X is an $(n \times 1)$ column array of the n state variables, the *control vector* U is an $(m \times 1)$ column array of the control variables, and f is an array of nonlinear functions. The nonlinear state equations (1.1-1), or a subset of them, usually have one or more *equilibrium points* in the multidimensional state and control space, where the derivatives vanish. The equations are often approximately linear for small perturbations from equilibrium, and can be written in matrix form as the *linear state equation*:

$$\dot{x} = Ax + Bu \tag{1.1-3}$$

Here, the lowercase notation for the state and control vectors indicates that they are perturbations from equilibrium, although the derivative vector contains the actual values (i.e., perturbations from zero). The "A matrix" is square and the "B matrix" has dimensions determined by the number of states and controls.

The state-space formulation will be described in more detail in Chapters 2 and 3. At this point we will simply note that a major advantage of this formulation is that the nonlinear state equations can be solved numerically. The simplest numerical solution method is *Euler integration*, given by:

$$X = X + f(X, U)\delta t, \tag{1.1-4}$$

where "=" indicates replacement of X in computer memory by the value on the right-hand side of the equation. The *integration time-step*, δt, must be made small enough that, for every δt interval, U can be approximated by a constant, and $\dot{X}\delta t$ provides a good approximation to the increment in the state vector. This numerical integration allows the state vector to be stepped forward in time, in time-increments of δt, to obtain a *time-history* simulation (Problem 1.5-2).

1.2 VECTOR KINEMATICS

Definitions and Notation

Kinematics can be defined as the study of the motion of objects without regard to the mechanisms that cause the motion. The motion of physical objects can be described by means of vectors in three dimensions, and in performing kinematic analysis with vectors we will make use of the following definitions:

Frame of Reference: a rigid body or set of rigidly related points that can be used to establish distances and directions (denoted by F_i, F_e, etc.). In general, a subscript used to indicate a frame will be lowercase, while a subscript used to indicate a point will be uppercase.

Inertial Frame: a frame of reference in which Newton's laws apply. Our best inertial approximation is probably a "helio-astronomic" frame in which the center of mass (cm) of the sun is a fixed point, and fixed directions are established by the normal to the plane of the ecliptic and the projection on that plane of certain stars that appear to be fixed in position.

Vector: a vector is an abstract geometrical object that has both magnitude and direction. It exists independently of any coordinate system. The vectors used here are Euclidean vectors that exist only in three-dimensional space.

Coordinate System: a measurement system for locating points in space, set up within a frame of reference. We may have multiple coordinate systems (with no relative motion) within one frame of reference, and we sometimes loosely refer to them as "frames."

In choosing a notation the following facts must be taken into account. For position vectors, the notation should specify the two points whose relative position the vector describes. Velocity and acceleration vectors are relative to a frame of reference, and the notation should specify the frame of reference as well as the moving point. The derivative of a vector depends on the observer's frame of reference, and this frame must be specified in the notation. A derivative may be taken in a different frame from that in which a vector is defined, so the notation may require two frame designators with one vector. We will use the following notation:

Vectors will be in boldface typefonts.

A right subscript will be used to designate two points for a position vector, and a point and a frame for a velocity or acceleration vector. A "/" in a subscript will mean "with respect to."

A left superscript will specify the frame in which a derivative is taken, and the dot notation will indicate a derivative.

A right superscript on a vector will specify a coordinate system. It will therefore denote an array of the components of that vector in the specified system.

Vector length will be denoted by single bars, for example, $|\mathbf{p}|$.

Examples of the notation are:

$\mathbf{p}_{A/B} \equiv$ position vector of point A with respect to point B
$\mathbf{v}_{A/i} \equiv$ velocity of point A in frame i (F_i)
${}^{b}\dot{\mathbf{v}}_{A/i} \equiv$ derivative of $\mathbf{v}_{A/i}$ taken in F_b
$\mathbf{v}^{c}_{A/i} \equiv (\mathbf{v}_{A/i})^{c} \equiv$ components of $\mathbf{v}_{A/i}$ in coordinate system c
${}^{b}\dot{\mathbf{v}}^{c}_{A/i} \equiv$ components in system c of the derivative in F_b

The components of a vector will be denoted by subscripts that indicate the coordinate system, or by the vector symbol with subscripts x, y, and z. All component arrays will be column arrays unless otherwise indicated by the transpose symbol, a right superscript T. For example,

$$\mathbf{p}^{b}_{A/B} = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \quad \text{or} \quad \mathbf{v}^{b} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^{T}$$

are arrays of components in a coordinate system b.

The Derivative Vector

The derivative of a vector can be defined in the same way as the derivative of a scalar:

$$\frac{d\mathbf{p}_{A/B}}{dt} = \underset{\delta t \to 0}{\text{Lim.}} \left[\frac{\mathbf{p}_{A/B}(t + \delta t) - \mathbf{p}_{A/B}(t)}{\delta t} \right]$$

This is a new vector created by the changes in length and direction of $\mathbf{p}_{A/B}$. Different answers will be obtained for the derivative depending on how the observer's frame is rotating. As another example of the notation above, consider

$${}^{i}\dot{\mathbf{p}}_{A/B} = \text{derivative of the vector } \mathbf{p}_{A/B}\text{, taken in frame } i$$

Note that if $\mathbf{p}_{A/B}$ is a position vector, the derivative is a velocity vector only if it is taken in the frame in which B is a fixed point. Similarly, the derivative of a velocity vector is an acceleration vector only if it is taken in the frame in which the velocity vector is defined.

If the derivative of a general vector $\mathbf{v}$ is taken in frame a, the components of the derivative vector in a coordinate system fixed in frame a are given by the rates of change of the components of $\mathbf{v}$ in that coordinate system. For example, if

$$\mathbf{v}^c = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T,$$

where system c is fixed in frame a, then

$$^a\dot{\mathbf{v}}^c = \begin{bmatrix} \dot{v}_x & \dot{v}_y & \dot{v}_z \end{bmatrix}^T$$

The vector derivative deserves special attention, and is discussed further in connection with angular velocity.

Vector Properties

Vectors are independent of any coordinate system, but some vector operations yield *pseudo-vectors* that are not independent of a "handedness" convention. For example, the result of the *vector cross-product* operation is a vector whose direction depends on whether a right-handed or left-handed convention is being used. We will always use the right-hand rule in connection with vector direction. Similarly, we will always use Cartesian coordinate systems that are right-handed. Figure 1.2-1 shows a vector **p** and a reference coordinate system (fixed in some frame) used to describe the direction of **p**. The axes of the coordinate system are aligned with the unit vectors **i**, **j**, **k**, which (in that order) form a right-handed set (i.e., $\mathbf{i} \times \mathbf{j} = \mathbf{k}$). The direction of **p**, relative to this coordinate system, is described by the three *direction angles* α, β, γ. The *direction cosines* of **p**—$\cos\alpha$, $\cos\beta$, and $\cos\gamma$—give the projections of **p** on the coordinate axes, and two applications of the theorem of Pythagoras yield

$$|\mathbf{p}|^2 \cos^2\alpha + |\mathbf{p}|^2 \cos^2\beta + |\mathbf{p}|^2 \cos^2\gamma = |\mathbf{p}|^2$$

Therefore, the direction cosines satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{1.2-1}$$

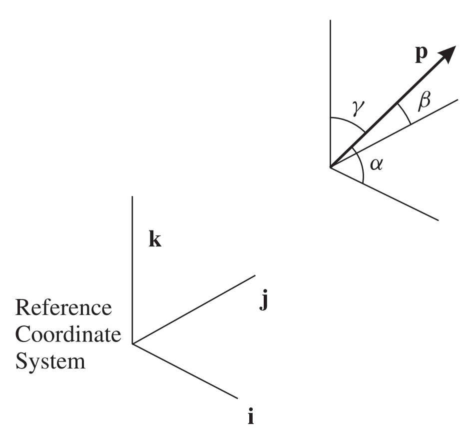

Figure 1.2-1 The direction angles of a vector.

Addition and subtraction of vectors can be defined independently of coordinate systems by means of geometrical constructions (the "parallelogram law"). The *dot product* of two vectors is a scalar defined by

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta, \tag{1.2-2}$$

where θ is the included angle between the vectors (it may be necessary to translate the vectors so that they intersect). The dot product is commutative and distributive; thus,

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

and

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$$

The principal uses of the dot product are to find the projection of a vector, to establish orthogonality, and to find length. For example, if (1.2-2) is divided by $|\mathbf{v}|$, we have the projection of $\mathbf{u}$ on $\mathbf{v}$,

$$(\mathbf{u} \cdot \mathbf{v})/|\mathbf{v}| = |\mathbf{u}| \cos \theta$$

If $\cos \theta = 0$, $\mathbf{u} \cdot \mathbf{v} = 0$, and the vectors are said to be *orthogonal*. If a vector is dotted with itself, then $\cos \theta = 1$ and we obtain the square of its length.

Orthogonal unit vectors satisfy the dot product relationships

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$
$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

Using these relationships, the dot product of two vectors can be expressed in terms of components,

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z, \tag{1.2-3}$$

where the vector are taken in any orthogonal Cartesian coordinate system.

The *cross-product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is a vector $\mathbf{w}$ that is normal to the plane of $\mathbf{u}$ and $\mathbf{v}$ and is in a direction such that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ (in that order) form a right-handed system (again, it may be necessary to translate the vectors so that they intersect). The length of $\mathbf{w}$ is defined to be $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

It has the following properties:

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \qquad \text{(anticommutative)}$$
$$a(\mathbf{u} \times \mathbf{v}) = (a\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (a\mathbf{v}) \qquad \text{(associative)}$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \qquad \text{(distributive)}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \qquad \text{(scalar triple product)}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{w} \cdot \mathbf{u}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) \qquad \text{(vector triple product)} \qquad (1.2\text{-}4)$$

As an aid for remembering the form of the triple products, note the cyclic permutation of the vectors involved. Alternatively, the vector triple product can be remembered phonetically using "ABC = BAC − CAB."

The cross-products of the unit vectors describing a right-handed orthogonal coordinate system satisfy the equations

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Also remember that $\mathbf{j} \times \mathbf{i} = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}$, and so on. From these properties we can derive a formula for the cross-product of two vectors; a convenient way of remembering the formula is to write it so that it resembles the expansion of a determinant.

The mnemonic is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \mathbf{i} \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}, \qquad (1.2\text{-}5)$$

where subscripts x, y, z indicate components in a coordinate system whose axes are aligned respectively with the unit vectors **i**, **j**, **k**.

An example of the use of the cross-product is to find the moment **M** of a force **F**, acting at a point whose position vector is **r**; the vector moment about the origin of **r** is given by

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

Other examples are given in the following subsections.

Rotation of a Vector

It is intuitively obvious that a vector can be made to point in an arbitrary direction by means of a single rotation around an appropriate axis. Here we follow Goldstein (Goldstein, 1980) to derive a formula for vector rotation.

Consider Figure 1.2-2, in which a free vector **u** has been rotated to form a new vector **v** by defining a rotation axis along a unit vector **n** and performing a left-handed rotation through μ around **n**. NV and NU have been constructed to find the projections of **v** and **u** on the rotation axis and hence identify μ. A vector expression for **v** is

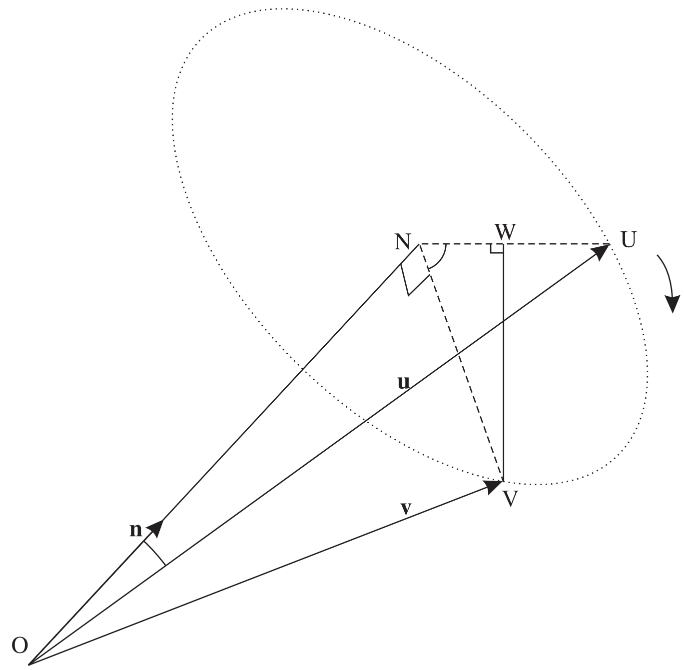

Figure 1.2-2 Rotation of a vector.

$$\begin{aligned}\mathbf{v} &= \overrightarrow{ON} \qquad + \overrightarrow{NW} \qquad\qquad\qquad\qquad + \overrightarrow{WV} \\ &= (\mathbf{u}\cdot\mathbf{n})\,\mathbf{n} + \frac{(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n})}{|\mathbf{u}-(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}|}\,\mathrm{NV}\cos\mu + \frac{(\mathbf{u}\times\mathbf{n})}{|\mathbf{u}|\sin\phi}\,\mathrm{NV}\sin\mu\end{aligned}$$

Now,

$$\mathrm{NV} = \mathrm{NU} = |\mathbf{u}-(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}| = |\mathbf{u}|\sin\phi$$

Therefore,

$$\mathbf{v} = \mathbf{n}(\mathbf{n}\cdot\mathbf{u}) + \cos\mu\;(\mathbf{u}-\mathbf{n}(\mathbf{n}\cdot\mathbf{u})) - \sin\mu\;(\mathbf{n}\times\mathbf{u})$$

or,

$$\mathbf{v} = (1-\cos\mu)\,\mathbf{n}\,(\mathbf{n}\cdot\mathbf{u}) + \cos\mu\,\mathbf{u} - \sin\mu\;(\mathbf{n}\times\mathbf{u}) \qquad (1.2\text{-}6)$$

Equation (1.2-6) is sometimes called the *rotation formula*; it shows that, after choosing $\mathbf{n}$ and μ, we can operate on $\mathbf{u}$ with dot and cross-product operations to get the desired rotation.

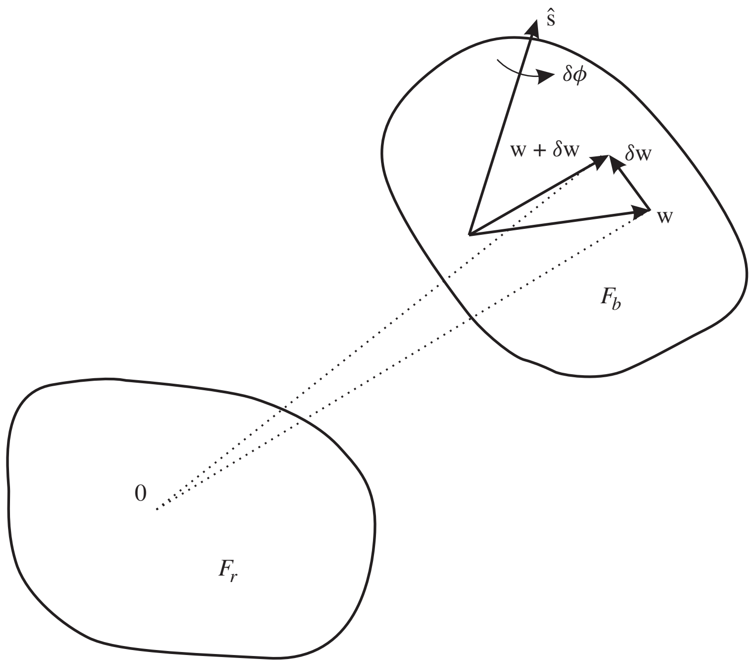

Figure 1.2-3 A vector derivative in a rotating frame.

Vector Derivatives and the Angular Velocity Vector

Figure 1.2-3 shows a vector **w** that is fixed in a frame F_b, and F_b is rotating with respect to a reference frame F_r. The derivative of **w** taken in F_r is nonzero if **w** is changing direction and/or changing length when observed from F_r, and is independent of translational motion between the frames. The change in direction with respect to F_r can be found by using the rotation theorem. In the figure, let $\hat{\mathbf{s}}$ be a unit vector parallel to the instantaneous axis of rotation at time t. To an observer in F_r, **w** becomes a new vector $\mathbf{w} + \boldsymbol{\delta}\mathbf{w}$ at time $t + \delta t$, due to the small rotation $\delta\phi$. The rotation formula can be used to find $\boldsymbol{\delta}\mathbf{w}$. Then, by taking the limit of $\boldsymbol{\delta}\mathbf{w}/\delta t$ as δt becomes infinitesimal, the derivative of **w** in F_r can be found. The rotation formula, with small-angle approximations and positive $\delta\phi$ right-handed around $\hat{\mathbf{s}}$, gives

$$\frac{\boldsymbol{\delta}\mathbf{w}}{\delta t} \approx \left(\hat{\mathbf{s}}\frac{\delta\phi}{\delta t}\right) \times \mathbf{w}$$

Taking the limit as $\delta t \to 0$,

$${}^{r}\dot{\mathbf{w}} = (\hat{\mathbf{s}}\,\dot{\phi}) \times \mathbf{w}$$

The quantity in parentheses has the properties of a vector, with direction along the axis of rotation and magnitude equal to the angular rotation rate. It is defined to be the instantaneous *angular velocity vector*, $\boldsymbol{\omega}_{b/r}$, of F_b with respect to F_r. A right-handed rotation around $\hat{\mathbf{s}}$ corresponds to a positive angular velocity vector. If $\mathbf{w}$ is also changing in length in F_b, we must add this effect to the right-hand side of the above equation, so that

$$ {}^r\dot{\mathbf{w}} = {}^b\dot{\mathbf{w}} + \boldsymbol{\omega}_{b/r} \times \mathbf{w} \tag{1.2-7} $$

Equation (1.2-7) is sometimes called the *equation of Coriolis* (Blakelock, 1965) and will be an essential tool in developing equations of motion from Newton's laws. It is much more general than is indicated above, and applies to any physical quantity that has a vector representation. The derivatives need not even be taken with respect to time. Angular velocity can be defined as the vector that relates the derivatives of any arbitrary vector in two different frames, according to (1.2-7). In our context we have a physical interpretation of this vector as a right-handed angular rate around a directed axis with, in general, both rate and direction changing with time. An alternative derivation of the angular velocity vector can be found in many texts (McGill and King, 1995; Kane, 1983).

Some formal properties of the angular velocity vector are:

(i) It is a unique vector that relates the derivatives of a vector taken in two different frames.
(ii) It satisfies the relative motion condition $\boldsymbol{\omega}_{b/a} = -\boldsymbol{\omega}_{a/b}$.
(iii) It is additive over multiple frames, for example, $\boldsymbol{\omega}_{c/a} = \boldsymbol{\omega}_{c/b} + \boldsymbol{\omega}_{b/a}$ (this is not true of angular acceleration).
(iv) Its derivative is the same in either frame, ${}^a\dot{\boldsymbol{\omega}}_{b/a} = {}^b\dot{\boldsymbol{\omega}}_{b/a}$. This is made evident by using (1.2-7) to find the derivative of $\boldsymbol{\omega}$.

A common problem is the determination of an angular velocity vector after the frames have been defined in a practical application. This can be achieved by finding one or more intermediate frames in which an axis of rotation and an angular rate are physically evident. Then the additive property can be invoked to combine the intermediate angular velocities. An example of this is given later, with the "rotating-Earth" equations of motion of an aerospace vehicle.

Velocity and Acceleration in Moving Frames

Figure 1.2-4 shows a point P moving with respect to two frames F_a and F_b, with fixed points O and Q, respectively. Suppose that we wish to relate the velocities in the two frames and also the accelerations. First, we must relate the position vectors shown in the figure, and then take derivatives in F_a to introduce velocity:

$$ \mathbf{r}_{P/O} = \mathbf{r}_{Q/O} + \mathbf{r}_{P/Q} \tag{1.2-8} $$

$$ {}^a\dot{\mathbf{r}}_{P/O} = {}^a\dot{\mathbf{r}}_{Q/O} + {}^a\dot{\mathbf{r}}_{P/Q} \tag{1.2-9} $$

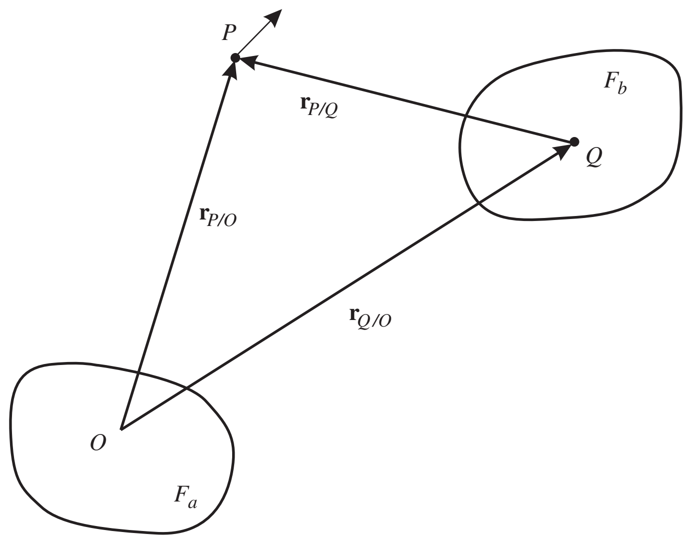

Figure 1.2-4 Velocity and acceleration in moving frames.

Starting from the left-hand side of Equation (1.2-9), the first two terms are velocities in F_a but the last term involves the position of P relative to a fixed point in F_b, with the derivative taken in F_a. Let $\mathbf{v}$ with an appropriate subscript represent a velocity vector. Then, by applying the equation of Coriolis, Equation (1.2-9) gives

$$\mathbf{v}_{P/a} = \mathbf{v}_{Q/a} + \mathbf{v}_{P/b} + \boldsymbol{\omega}_{b/a} \times \mathbf{r}_{P/Q} \tag{1.2-10}$$

As an application of Equation (1.2-10), let F_a be an inertial reference frame and F_b a body moving with respect to the reference frame. Assume that a navigator on the moving body determines, from an onboard inertial navigation system, his velocity in the inertial reference frame ($\mathbf{v}_{Q/a}$) and his inertial angular velocity vector ($\boldsymbol{\omega}_{b/a}$). Also, using a radar set, he measures the velocity of P in $F_b(\mathbf{v}_{P/b})$ and the position of P with respect to $Q(\mathbf{r}_{P/Q})$. He can then use Equation (1.2-10) to calculate the velocity of the object in the inertial reference frame and, knowing the equation of motion in the inertial frame, predict its trajectory. The word *measure* should always evoke the thought "coordinate system?" and Equation (1.2-10) cannot be evaluated without choosing coordinate systems for this example. In Section 1.3 it will become clear how the coordinate systems calculations can be performed.

We next find the acceleration of P by taking derivatives of (1.2-10) in F_a. Starting from the left, the first two terms are velocities in F_a and these become accelerations in F_a. The third term is a velocity in F_b and must be differentiated by the equation of Coriolis. The last term involving a cross-product can be differentiated by the "product rule," and the derivative of angular velocity is an angular acceleration vector, denoted by $\boldsymbol{\alpha}$. Therefore, denoting translational acceleration vectors by $\mathbf{a}$, (1.2-10) yields,

$$\mathbf{a}_{P/a} = \mathbf{a}_{Q/a} + (\mathbf{a}_{P/b} + \boldsymbol{\omega}_{b/a} \times \mathbf{v}_{P/b}) + \boldsymbol{\alpha}_{b/a} \times \mathbf{r}_{P/Q} + \boldsymbol{\omega}_{b/a} \times (\mathbf{v}_{P/b} + \boldsymbol{\omega}_{b/a} \times \mathbf{r}_{P/Q})$$

Regrouping terms, we get,

$$\underbrace{\mathbf{a}_{P/a}}_{\substack{\text{total}\\ \text{accl.}}} = \underbrace{\mathbf{a}_{P/b}}_{\substack{\text{relative}\\ \text{accl.}}} + \underbrace{\boldsymbol{a}_{Q/a} + \boldsymbol{\alpha}_{b/a} \times \mathbf{r}_{P/Q} + \underbrace{\boldsymbol{\omega}_{b/a} \times (\boldsymbol{\omega}_{b/a} \times \mathbf{r}_{P/Q})}_{\substack{\text{Centripetal}\\ \text{accl.}}}}_{\text{Transport accln. of } P \text{ in } F_a} + \underbrace{2\boldsymbol{\omega}_{b/a} \times \mathbf{v}_{P/b}}_{\substack{\text{Coriolis}\\ \text{accl.}}} \qquad (1.2\text{-}11)$$

The term labeled "transport acceleration" is the acceleration in F_a of a fixed point in F_b that is instantaneously coincident with P. This is evident because the two remaining right-hand-side terms vanish when P is fixed in F_b. Note that (1.2-10) can be written as

$$\mathbf{v}_{P/a} = \mathbf{v}_{P/b} + (\mathbf{v}_{Q/a} + \boldsymbol{\omega}_{b/a} \times \mathbf{r}_{P/Q}),$$

where the term in parentheses is the velocity in F_a of a fixed point in F_b that is instantaneously coincident with P. Therefore, the acceleration equation does not have the same form as this velocity equation because of the "Coriolis acceleration" term.

Example 1.2-1: Coriolis Acceleration in an Earth-Fixed Frame. As an example of the application of (1.2-11), let F_b be fixed in the Earth, and let F_a also translate with the Earth but be nonrotating (i.e., chosen to be an approximation to an inertial frame). Let P be a point moving over the surface of the Earth, and let the points Q and O coincide, at the Earth's cm, so that the acceleration $\mathbf{a}_{Q/a}$ vanishes and $\mathbf{r}_{P/Q}$ is a geocentric position vector. The Earth's angular velocity is quite closely constant and so the derivative of $\boldsymbol{\omega}_{b/a}$ vanishes. This leaves only the relative acceleration, centripetal acceleration, and Coriolis acceleration terms. Solving for the relative acceleration gives:

$$\mathbf{a}_{P/b} = \mathbf{a}_{P/a} - \boldsymbol{\omega}_{b/a} \times (\boldsymbol{\omega}_{b/a} \times \mathbf{r}_{P/Q}) - 2\boldsymbol{\omega}_{b/a} \times \mathbf{v}_{P/b} \qquad (1.2\text{-}12)$$

For a particle of mass m at P, the relative acceleration corresponds to an "apparent force" on the particle and produces the trajectory observed by a stationary observer on the Earth. The true acceleration ($\mathbf{a}_{P/a}$) corresponds to "true" forces (e.g., mass attraction, drag), therefore,

$$\text{apparent force} = \text{true force} - m\boldsymbol{\omega}_{b/a} \times (\boldsymbol{\omega}_{b/a} \times \mathbf{r}_{P/Q}) + 2m\mathbf{v}_{P/b} \times \boldsymbol{\omega}_{b/a}$$

The second term on the right is the "centrifugal" force, directed normal to the angular velocity vector. The third term is usually referred to as the Coriolis force and will cause a ballistic trajectory over the Earth to curve to the left or right. A stationary observer on the Earth might realize that the Earth is not an inertial frame by seeing this curvature, which is really just the kinematic effect of the Earth's rotation.

An often quoted example of the Coriolis force is the circulation of winds around a low-pressure area (a cyclone) on the Earth. The true force is radially inward along the pressure gradient. In the Northern Hemisphere, for example, the Earth's angular velocity vector points outward from the Earth's surface and, whichever way the velocity vector $\mathbf{v}_{P/b}$ is directed, the Coriolis force is directed to the right of $\mathbf{v}_{P/b}$. Therefore, in the Northern Hemisphere the winds spiral inward in a counterclockwise direction around a cyclone.

Example 1.2-2: Accelerometer Measurements. This example will illustrate the principle of an accelerometer and the contribution of angular motion to the linear acceleration at a point away from the cm of a rigid body. Figure 1.2-5 shows a very simple accelerometer mounted on a rigid body, and aligned so as to measure z-axis components in the body-fixed coordinate system shown. The accelerometer consists of a "proof mass," m, a suspension spring, a viscous damper for the motion of the mass, and a means of measuring its displacement. The proof mass is constrained to move in one dimension only, in this case, in the body-z direction. Point P is the deflected position of the cm of the proof mass, $R(x_R, 0, 0)$ is the rest-position, and d is the deflection. Applying Equation (1.2-11) to find the acceleration of P in the inertial reference frame F_i yields

$$\mathbf{a}_{P/i} = \mathbf{a}_{P/b} + \mathbf{a}_{CM/i} + \boldsymbol{\alpha}_{b/i} \times \mathbf{r}_{P/Q} + \boldsymbol{\omega}_{b/i} \times (\boldsymbol{\omega}_{b/i} \times \mathbf{r}_{P/Q}) + 2\boldsymbol{\omega}_{b/i} \times \mathbf{v}_{P/b}$$

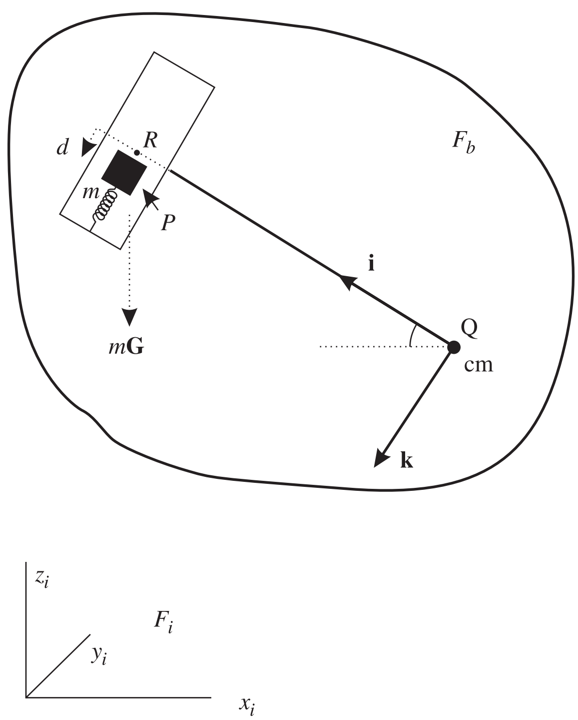

Figure 1.2-5 An accelerometer on a rigid body.

Now write this equation in terms of orthogonal unit vectors, **i**, **j**, **k**, fixed in F_b, with

$$\mathbf{a}_{P/i} = ap_x\mathbf{i} + ap_y\mathbf{j} + ap_z\mathbf{k} \quad \mathbf{a}_{P/b} = \ddot{d}\mathbf{k} \quad \mathbf{r}_{P/Q} = x_R\mathbf{i} + d\mathbf{k} \quad \mathbf{v}_{P/b} = \dot{d}\mathbf{k}$$

$$\mathbf{a}_{CM/i} \equiv a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \quad \boldsymbol{\alpha}_{b/i} \equiv \alpha_x\mathbf{i} + \alpha_y\mathbf{j} + \alpha_z\mathbf{k} \quad \boldsymbol{\omega}_{b/i} \equiv \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

and consider only the components along **k**:

$$ap_z = \ddot{d} + a_z - \alpha_y x_R - d\left(\omega_x^2 + \omega_y^2\right) + x_R\omega_x\omega_z$$

The z-component of force required to produce this acceleration is given by $m(ap_z)$, and is provided by the mass attraction force ($m\mathbf{G}$) toward the Earth's cm (Section 1.4), spring force, and viscous-damping force, that is,

$$m(ap_z) = mG_z - k_s d - b\dot{d},$$

where k_s is the accelerometer spring constant and b is the accelerometer viscous damping constant. Equating these two force expressions and rearranging terms gives

$$\ddot{d} + \frac{b}{m}\dot{d} + \frac{k_s}{m}d - d\left(\omega_x^2 + \omega_y^2\right) = G_z - (a_z - \alpha_y x_R + x_R\omega_x\omega_z)$$

Note that the last term on the right is the transport acceleration of point R in F_i.

The reading of the accelerometer is represented by d. The derivatives of d come into play when the accelerometer has to respond to a changing acceleration; here we will focus on the steady-state behavior with a constant acceleration input and neglect the derivatives. The variable position of the proof mass will cause a measurement error through the term $d(\omega_x^2 + \omega_y^2)$; this is eliminated in high-sensitivity accelerometers by using a "force rebalancing" technique, which measures the force required to maintain d and its derivatives very close to zero.

Acceleration in the z direction corresponds to negative d, so the steady-state accelerometer reading is

$$\text{reading} \propto \left(a_z' - G_z\right), \tag{1.2-13a}$$

where

$$a_z' = (a_z - \alpha_y x_R + x_R\omega_x\,\omega_z) \tag{1.2-13b}$$

Equation (1.2-13a) shows that, in general, this type of accelerometer responds to the pertinent component of $(\mathbf{a} - \mathbf{G})$ at its location. If the mass of the rigid body plus accelerometer is M, and the applied "contact force" (i.e., not counting the gravitational field force) is **F**, the accelerometer responds to $(\mathbf{F} + M\mathbf{G})/M - \mathbf{G}$, or simply $\mathbf{F}/M$. This quantity is a *specific force*, denoted by **f**, and so an accelerometer measures a component of the specific contact force given by

$$\mathbf{f} = \mathbf{a} - \mathbf{G} \tag{1.2-14}$$

in whatever acceleration units are chosen.

As an example of Equation (1.2-14) consider a stationary accelerometer on the surface of the Earth, with its sensitive axis aligned with a plumb-bob measurement of the vertical. Neglecting the inertial acceleration of the Earth's cm, the term **a** will be the small centripetal acceleration due to the Earth's rotation. The **G** term depends on distance from Earth's cm, but is close to 9.8 m/s^2 in magnitude and directed toward the Earth's cm (see Section 1.4). The measurement **f** is the specific force due to the upward reaction of the Earth on the accelerometer, and will be exactly equal to the negative of the weight $m\mathbf{g}$ divided by the mass m (i.e., it will be $-\mathbf{g}$), where **g** is the local *gravity* vector. Alternatively, if the accelerometer is in free fall above the Earth, $\mathbf{a} = \mathbf{G}$, $\mathbf{f} = 0$, and the accelerometer reading is zero.

The accelerometer reading can be made dimensionless by dividing by $|\mathbf{g}|$, and accelerometers are commonly calibrated to read 1.0 g-units when stationary on the Earth's surface and having their sensitive axis parallel to the plumb-bob vertical. Therefore, the accelerometer measurement of specific force, at any location, can be obtained by multiplying the scale reading by the gravity value used for calibration, in whatever units are desired. It is evident that **G** must be known accurately to get an accurate value of acceleration from an accelerometer measurement of specific force (Section 1.4).

Quaternions and Vectors

Here we will show that the vector rotation formula can be expressed much more compactly in terms of quaternions. W. R. Hamilton (1805–1865) introduced the quaternion form:

$$x_0 + x_1 i + x_2 j + x_3 k$$

with,

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = k,\, jk = i,\, ki = j = -ik$$

in an attempt to generalize complex numbers in a plane to three dimensions. Quaternions obey the normal laws of algebra, except that multiplication is not commutative. Multiplication is defined by the associative law, for example, if,

$$r = (p_0 + p_1 i + p_2 j + p_3 k) \times (q_0 + q_1 i + q_2 j + q_3 k)$$

then,

$$r = p_0 q_0 + p_0 q_1 i + p_0 q_2 j + p_0 q_3 k + p_1 q_0 i + p_1 q_1 i^2 + \ \ldots$$

By using the rules for i, j, k products, and collecting terms, the answer can be written in various forms, for example,

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Alternatively, by interpreting i, j, k as unit vectors, the quaternion can be treated as $(q_0 + \mathbf{q})$, where $\mathbf{q}$ is the quaternion vector part, with components q_1, q_2, q_3, along $\mathbf{i}, \mathbf{j}, \mathbf{k}$. We will write the quaternion as an array, formed from q_0 and the vector components, thus

$$p = \begin{bmatrix} p_0 \\ \mathbf{p}^r \end{bmatrix} \qquad q = \begin{bmatrix} q_0 \\ \mathbf{q}^r \end{bmatrix}, \tag{1.2-15}$$

where components of the vector are taken in a reference system r, to be chosen when the quaternion is applied. The above multiplication can be written as

$$p * q = \begin{bmatrix} p_0 q_0 - \mathbf{p} \cdot \mathbf{q} \\ (p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q})^r \end{bmatrix}, \tag{1.2-16}$$

where "$*$" indicates quaternion multiplication. We will use (1.2-15) and (1.2-16) as the definitions of quaternions and quaternion multiplication. Quaternion properties can now be derived using ordinary vector operations.

Quaternion Properties

(i) Quaternion Noncommutativity Consider the following identity:

$$p * q - q * p = \begin{bmatrix} 0 \\ (\mathbf{p} \times \mathbf{q} - \mathbf{q} \times \mathbf{p})^r \end{bmatrix} = \begin{bmatrix} 0 \\ 2(\mathbf{p} \times \mathbf{q})^r \end{bmatrix}$$

It is apparent that, in general,

$$p * q \neq q * p$$

(ii) The Quaternion Norm The norm of a quaternion is defined to be the sum of the squares of its elements:

$$\text{norm}(q) = \sum_{i=0}^{i=3} q_i^2$$

(iii) Norm of a Product Using the definition of the norm, and vector operations, it is straightforward to show (Problem 1.2-9) that the norm of a product is equal to the product of the individual norms:

$$\text{norm}(p * q) = \text{norm}(p) \times \text{norm}(q)$$

(iv) Associative Property over Multiplication The associative property:

$$(p * q) * r = p * (q * r)$$

is proven in a straightforward manner.

(v) The Quaternion Inverse Consider the following product,

$$\begin{bmatrix} q_0 \\ \mathbf{q}^r \end{bmatrix} * \begin{bmatrix} q_0 \\ -\mathbf{q}^r \end{bmatrix} = \begin{bmatrix} q_0^2 + \mathbf{q} \cdot \mathbf{q} \\ (q_0 \mathbf{q} - q_0 \mathbf{q} - \mathbf{q} \times \mathbf{q})^r \end{bmatrix} = \begin{bmatrix} \sum q_i^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We see that multiplying a quaternion by another quaternion, which differs only by a change in sign of the vector part, produces a quaternion with a scalar part only. A quaternion of the latter form will have very simple properties in multiplication (i.e., multiplication by a constant) and, when divided by the quaternion norm, will serve as the "identity quaternion." Therefore, the inverse of a quaternion is defined by

$$q^{-1} = \begin{bmatrix} q_0 \\ \mathbf{q}^r \end{bmatrix}^{-1} = \frac{1}{\mathrm{norm}(q)} \begin{bmatrix} q_0 \\ -\mathbf{q}^r \end{bmatrix} \tag{1.2-17}$$

However, we will work entirely with unit-norm quaternions, thus simplifying many expressions.

(vi) Inverse of a Product The inverse of a quaternion product is given by the product of the individual inverses in the reverse order. This can be seen as follows:

$$\begin{aligned} (p * q)^{-1} &= \frac{1}{\mathrm{norm}(p * q)} \begin{bmatrix} p_0 q_0 - \mathbf{p} \cdot \mathbf{q} \\ -(p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q})^r \end{bmatrix} \\ &= \frac{1}{\mathrm{norm}(q)} \begin{bmatrix} q_0 \\ -\mathbf{q}^r \end{bmatrix} * \begin{bmatrix} p_0 \\ -\mathbf{p}^r \end{bmatrix} \frac{1}{\mathrm{norm}(p)} \end{aligned}$$

Therefore,

$$(p * q)^{-1} = q^{-1} * p^{-1}$$

Vector Rotation by Quaternions

A quaternion can be used to rotate a Euclidean vector in the same manner as the rotation formula, and the quaternion rotation is much simpler in form. The vector part of the quaternion is used to define the rotation axis, and the scalar part to define the angle of rotation. The rotation axis is specified by its direction cosines in the reference coordinate system, and it is convenient to impose a unity norm constraint on the quaternion. Therefore, if the direction angles of the axis are α, β, and γ, and a measure of the rotation angle is δ, the rotation quaternion is written as

$$q = \begin{bmatrix} \cos\delta \\ \cos\alpha \;\; \sin\delta \\ \cos\beta \;\; \sin\delta \\ \cos\gamma \;\; \sin\delta \end{bmatrix} = \begin{bmatrix} \cos\delta \\ \sin\delta \;\; \mathbf{n}^r \end{bmatrix}, \tag{1.2-18}$$

where $\mathbf{n}$ is a unit vector along the rotation axis,

$$\mathbf{n}^r = \begin{bmatrix} \cos\alpha & \cos\beta & \cos\gamma \end{bmatrix}^T$$

and,

$$\text{norm}(q) = \cos^2\delta + \sin^2\delta\,(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 1$$

This formulation also guarantees that there is a unique quaternion for every value of δ in the range ± 180 degrees, thus encompassing all possible rotations.

Now consider the form of the transformation, which must involve multiplication. For compatibility of multiplication between vectors and quaternions, a Euclidean vector is written as a quaternion with a scalar part of zero, thus

$$u = \begin{bmatrix} 0 \\ \mathbf{u}^r \end{bmatrix}$$

The result of the rotation must also be a quaternion with a scalar part of zero, the transformation must be reversible by means of the quaternion inverse, and Euclidean length must be preserved. The transformation $v = q * u$ obviously does not satisfy the first of these requirements. Therefore, we consider the transformations:

$$v = q * u * q^{-1} \quad \text{or} \quad v = q^{-1} * u * q,$$

which are reversible by performing the inverse operations on v. The second of these transformations leads to the convention most commonly used:

$$v = q^{-1} * u * q = \begin{bmatrix} q_0(\mathbf{q}\cdot\mathbf{u}) - (q_0\mathbf{u} - \mathbf{q}\times\mathbf{u})\cdot\mathbf{q} \\ ((\mathbf{q}\cdot\mathbf{u})\mathbf{q} + q_0(q_0\mathbf{u} - \mathbf{q}\times\mathbf{u}) + (q_0\mathbf{u} - \mathbf{q}\times\mathbf{u})\times\mathbf{q})^r \end{bmatrix},$$

which reduces to

$$v = q^{-1} * u * q = \begin{bmatrix} 0 \\ (2\mathbf{q}(\mathbf{q}\cdot\mathbf{u}) + \left(q_0^2 - \mathbf{q}\cdot\mathbf{q}\right)\mathbf{u} - 2q_0(\mathbf{q}\times\mathbf{u}))^r \end{bmatrix} \tag{1.2-19}$$

Therefore, this transformation meets the requirement of zero scalar part. Also, because of the properties of quaternion norms, the Euclidean length is preserved. For a match with the rotation formula, we require:

rotation formula	*quaternion rotation*
$(1 - \cos\mu)\, \mathbf{n}(\mathbf{n} \cdot \mathbf{u})$	$2\ \sin^2\delta\ \mathbf{n}(\mathbf{n} \cdot \mathbf{u})$
$\cos\mu\ \mathbf{u}$	$(\cos^2\delta - \sin^2\delta)\ \mathbf{u}$
$-\sin\mu\ (\mathbf{n} \times \mathbf{u})$	$-2\ \cos\delta\ \sin\delta\ (\mathbf{n} \times \mathbf{u})$

The corresponding terms agree if $\delta = \mu/2$ and half-angle trigonometric identities are applied. Therefore, the quaternion

$$q = \begin{bmatrix} \cos(\mu/2) \\ \sin(\mu/2)\mathbf{n}^r \end{bmatrix} \tag{1.2-20a}$$

and transformation

$$q^{-1} * u * q \tag{1.2-20b}$$

give a left-handed rotation of a vector $\mathbf{u}$ through an angle μ, around $\mathbf{n}$, when μ is positive.

1.3 MATRIX ANALYSIS OF KINEMATICS

Properties of Linear Transformations

Before studying matrix representation of kinematic relationships, we will review some pertinent matrix theory. Consider the matrix equation

$$v = Au, \tag{1.3-1}$$

where v and u are $(n \times 1)$ matrices (e.g., vector component arrays) and A is an $(n \times n)$ constant matrix, not necessarily nonsingular. Each element of v is a linear combination of the elements of u, and so this equation is a *linear transformation* of the matrix u. Next, suppose that in an analysis we change to a new set of variables through a reversible linear transformation. If L is the matrix of this transformation, then L^{-1} must exist (i.e., L is nonsingular) for the transformation to be reversible, and the new variables corresponding to u and v are

$$u_1 = Lu, \qquad v_1 = Lv$$

Therefore, the relationship between the new variables must be

$$v_1 = LAu = LAL^{-1}u_1 \tag{1.3-2a}$$

The transformation LAL^{-1} is a *similarity transformation* of the original coefficient matrix A. A special case of this transformation occurs when the inverse of the matrix L is given by its transpose (i.e., L is an orthogonal matrix) and the similarity transformation becomes a *congruence transformation*, LAL^T.

As an important example of a linear transformation, consider the linear state equation (1.1-3) with a nonsingular change of variables $z = Lx$. The state equation in terms of the z-variables is

$$\dot{z} = (LAL^{-1})z + (LB)u \tag{1.3-2b}$$

and L can be chosen so that the state equations have a much simpler form, as shown below.

Eigenvalues and Eigenvectors

A square-matrix linear transformation has the property that vectors exist whose components are only scaled by the transformation. If **v** is such an "invariant" vector, its components must satisfy the equation

$$Av = \lambda v, \qquad v(n \times 1), \tag{1.3-3}$$

where A is the transformation matrix and λ is a (scalar) constant of proportionality. A rearrangement of (1.3-3) gives the set of homogeneous linear equations

$$(A - \lambda I)v = 0, \tag{1.3-4}$$

which has a non-null solution for v if, and only if, the determinant of the coefficient matrix is zero (Strang, 1980); that is,

$$|A - \lambda I| = 0 \tag{1.3-5}$$

This determinant is an n-th order polynomial in λ, called the *characteristic polynomial* of A, so there may be up to n distinct solutions for λ. Each solution, λ_i, is known as an *eigenvalue* or *characteristic value* of the matrix A. The associated invariant vector defined by (1.3-3) is known as a *right eigenvector* of A (the left eigenvectors of A are the right eigenvectors of its transpose A^T).

In the mathematical model of a physical system, a reversible change of internal variables does not usually change the behavior of the system if observed at the same outputs. An example of this is the invariance of the eigenvalues of a linear system, described by Equation (1.1-3), under the similarity transformation (1.3-2*b*). After the similarity transformation, the eigenvalues are given by

$$\left|(\lambda I - LAL^{-1})\right| = 0,$$

which can be rewritten as

$$\left|(\lambda L L^{-1} - L A L^{-1})\right| = 0$$

The determinant of a product of square matrices is equal to the product of the individual determinants; therefore,

$$|L| \times |(\lambda I - A)| \times \left|L^{-1}\right| = 0 \tag{1.3-6}$$

This equation is satisfied by the eigenvalues of the matrix A, so the eigenvalues are unchanged by the transformation.

Now consider a special similarity transformation that will reduce the linear equations to a canonical (standard) form. First, consider the case when all of the n eigenvalues of the coefficient matrix A are distinct. Then the n eigenvectors $\mathbf{v}_i$ can be shown to form a linearly independent set; therefore, their components can be used to form the columns of a nonsingular transformation matrix. This matrix is called the *modal matrix,* M, and

$$M = [v_1 \; v_2 \cdots v_n]$$

According to the eigenvector/eigenvalue defining equation (1.3-3), if M is a modal matrix, we find that

$$AM = MJ \quad \text{and} \quad J = \text{diag}(\lambda_1 \cdots \lambda_n),$$

or

$$M^{-1}AM = J \tag{1.3-7a}$$

When some of the eigenvalues of A are repeated (i.e., multiple), it may not be possible to find a set of n linearly independent eigenvectors. Also, in the case of repeated eigenvalues, the result of the similarity transformation (1.3-7a) is in general a *Jordan-form matrix* (Wilkinson and Golub, 1976). In this case the matrix J may have some unit entries on the superdiagonal. These entries are associated with blocks of repeated eigenvalues on the main diagonal.

As an example, the linear state equation (1.3-2b), with $L^{-1} = M$, becomes

$$\dot{z} = Jz + M^{-1}Bu \tag{1.3-7b}$$

This corresponds to a set of state equations with minimal coupling between them. For example, if the eigenvalue λ_i is of multiplicity 2, and the associated Jordan block has a superdiagonal 1, we can write the corresponding equations as

$$\begin{aligned} \dot{z}_i &= \lambda_i z_i + z_{i+1} + b_i' u \\ \dot{z}_{i+1} &= \lambda_i z_i + b_{i+1}' u \end{aligned} \tag{1.3-7c}$$

The variables z_i are called the *modal coordinates.* When the eigenvalues are all distinct, the modal coordinates yield a set of uncoupled first-order differential equations.

Their homogeneous solutions are the exponential functions $e^{\lambda_i t}$, and these are the *natural modes* of (behavior of) the dynamic system (see Section 3.2). A disadvantage of the modal coordinates is that the state variables usually lose their original physical significance.

The Scalar Product

If $\mathbf{u}^a$ and $\mathbf{v}^a$ are column arrays of the same dimension, their scalar product is $(\mathbf{u}^a)^T\mathbf{v}^a$, for example,

$$(\mathbf{u}^a)^T\mathbf{v}^a = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = u_x v_x + u_y v_y + u_z v_z \tag{1.3-8a}$$

and this result is identical to (1.2-3) obtained from the vector dot product. The scalar product allows us to find the norm of a column matrix:

$$|\mathbf{v}^a| = \left((\mathbf{v}^a)^T\mathbf{v}^a\right)^{1/2} \tag{1.3-8b}$$

In Euclidean space this is the length of a vector.

The Cross-Product Matrix

Suppose that the cross-product $\boldsymbol{\omega} \times \mathbf{v}$ is to be evaluated in system a, where $\boldsymbol{\omega}$ and $\mathbf{v}$ have components given by

$$\boldsymbol{\omega}^a = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad \mathbf{v}^a = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then it is easy to show (Problem 1.3-1), using the determinant formula for the cross-product, that

$$(\boldsymbol{\omega} \times \mathbf{v})^a = \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv \Omega^a \, \mathbf{v}^a \tag{1.3-9}$$

The same idea can be applied to the vector triple product. For example,

$$(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{v}))^a = \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix}^2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv (\Omega^a)^2 \, \mathbf{v}^a \tag{1.3-10}$$

The symbol Ω will be used throughout to denote the *cross-product matrix* corresponding to the operation $(\boldsymbol{\omega} \times \,)$ when $\boldsymbol{\omega}$ is an angular velocity vector. For other vectors, a tilde symbol over the vector will be used to denote the cross-product matrix. A cross-product matrix is skew-symmetric, that is,

$$\Omega^T = -\Omega \equiv -\tilde{\boldsymbol{\omega}} \tag{1.3-11}$$

and therefore the square of the cross-product matrix is symmetric. Note that in the general case the matrix operations must be written in the same order as the vector operations, but may be performed in any order (the associative property for matrix multiplication).

Coordinate Rotation

When the rotation formula (1.2-6) is resolved in a coordinate system a, the result is

$$\mathbf{v}^a = \left[(1 - \cos\mu)\, \mathbf{n}^a(\mathbf{n}^a)^T + \cos(\mu)I - \sin(\mu)\, \tilde{\mathbf{n}}^a\right] \mathbf{u}^a, \tag{1.3-12}$$

where $\mathbf{n}^a(\mathbf{n}^a)^T$ is a square matrix, I is the identity matrix, and $\tilde{\mathbf{n}}^a$ is a cross-product matrix. This formula was developed as an "active" vector operation in that a vector was being rotated to a new position by means of a left-handed rotation about the specified unit vector. In component form, the new array can be interpreted as the components of a new vector in the same coordinate system, or as the components of the original vector in a new coordinate system, obtained by a right-handed coordinate rotation around the specified axis. This can be visualized in Figure 1.3-1, which shows a b coordinate system obtained by a right-handed rotation around the z-axis. If the vector is next given a left-handed rotation through μ, then (x_b, y_b) will become the components in the original system. Taking the coordinate-system rotation viewpoint, and combining the matrices in (1.3-12) into a single coefficient matrix, this linear transformation can be written as

$$\mathbf{u}^b = C_{b/a}\, \mathbf{u}^a \tag{1.3-13}$$

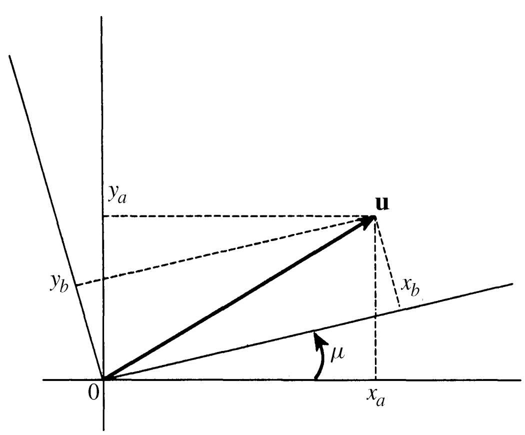

Figure 1.3-1 A plane rotation of coordinates.

Here $C_{b/a}$ is a matrix that transforms the components of the vector **u** from system a to system b, and is called a *direction cosine matrix*, or simply a *rotation matrix*.

We will look briefly at some of the properties of the rotation matrix, and then at how it may be determined in applications. A coordinate rotation must leave the length of a vector unchanged. The change of length under the rotation above is

$$\left(\mathbf{u}^b\right)^T \mathbf{u}^b = \left(C_{b/a}\mathbf{u}^a\right)^T C_{b/a}\mathbf{u}^a = \left(\mathbf{u}^a\right)^T C_{b/a}^T C_{b/a}\,\mathbf{u}^a$$

and the length is preserved if

$$C_{b/a}^T\, C_{b/a} = I = C_{b/a}\, C_{b/a}^T \tag{1.3-14}$$

This is the definition of an orthogonal matrix, and it makes the inverse matrix particularly easy to determine ($C^{-1} = C^T$). It also implies that the columns (and also the rows) of the rotation matrix form an orthonormal set:

$$C = [c_1\ c_2\ c_3] \qquad c_i^T c_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Also, since

$$c_1 = C \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

columns of the rotation matrix give us the components, in the new system, of a unit vector in the old system.

If a vector is expressed in a new coordinate system by a sequence of rotations as

$$\mathbf{u}^d = C_{d/c}\, C_{c/b}\, C_{b/a}\, \mathbf{u}^a, \tag{1.3-15}$$

then the inverse operation is given by

$$\mathbf{u}^a = \left(C_{d/c}C_{c/b}C_{b/a}\right)^{-1} \mathbf{u}^d = C_{b/a}^{-1}C_{c/b}^{-1}C_{d/c}^{-1}\mathbf{u}^d = C_{b/a}^T C_{c/b}^T C_{d/c}^T \mathbf{u}^d$$

or

$$\mathbf{u}^a = \left(C_{d/c}C_{c/b}C_{b/a}\right)^T \mathbf{u}^d = \left(C_{d/a}\right)^T\ \mathbf{u}^d \tag{1.3-16}$$

A better understanding of coordinate rotations can be obtained by examining the eigenvalues of the (3×3) rotation matrix. Goldstein (1980) shows that any nontrivial rotation matrix has one, and only one, eigenvalue equal to $+1$, and that this corresponds to a theorem proved by Leonhard Euler (1707–1783) for a rigid body. The other two eigenvalues are a complex conjugate pair with unit magnitude, and can be written as $(\cos\phi \pm j\sin\phi)$. Therefore, using a similarity transformation, and writing separate equations for the real and imaginary parts, it is possible to transform any rotation matrix C to the form of a plane rotation matrix P, for example,

$$P = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.3-17}$$

The matrix (1.3-17) corresponds to a single rotation through an angle ϕ about the z-axis. It shows that the orientation of one coordinate frame with respect to another is uniquely determined by a single rotation about a unique axis (the Euler axis), and this is the essence of Euler's theorem. This principle is used as the basis of the quaternion representation of rotation.

Summary of Rotation Matrix Properties

(i) Successive rotations can be described by the product of the individual rotation matrices; cf. (1.3-15).
(ii) Rotation matrices are not commutative, for example, $C_{c/b}C_{b/a} \neq C_{b/a}C_{c/b}$.
(iii) Rotation matrices are orthogonal matrices, for example, (1.3-14).
(iv) The determinant of a rotation matrix is unity.
(v) A nontrivial rotation matrix has one, and only one, eigenvalue equal to unity.

Euler Rotations

The direction cosine matrix is so-called because its elements can be determined from dot products that involve the direction cosines between corresponding axes of the new and old coordinate systems. Here we will determine the rotation matrix in a way that is better suited to visualizing aircraft orientation.

The orientation of one Cartesian coordinate system with respect to another can always be described by three successive rotations, and the angles of rotation are called the *Euler angles* (or Eulerian angles). These angles are specified in various ways in different fields of science, and the reader should be aware that there are small differences in many of the formulae in the literature as a result of this. In the aerospace field the rotations are performed, in a specified order, about each of the three Cartesian axes in succession. That is, they are performed in each of the three coordinate planes, and are therefore called *plane rotations*.

Figure 1.3-1 shows a plane rotation, in which coordinate system b has been rotated relative to system a. The systems are right-handed, with the z-axis coming out of the page, and the rotation is a right-handed rotation about the z-axis, through the angle μ. Assume that the components of the vector **u** are known in system a, and that we need to know its components in b. Equation (1.3-12) readily gives the rotation matrix, or simple trigonometry can be applied to the figure (Problem 1.3-2); the result is

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} \cos\mu & \sin\mu & 0 \\ -\sin\mu & \cos\mu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \tag{1.3-18}$$

Henceforth the plane rotation matrix will be written immediately by inspection. The unit and zero elements correspond to the coordinate that does not change, and the remaining elements are always cosines on the main diagonal and sines off the diagonal (so that zero rotation produces the identity matix). The negative sine element always occurs on the row above the one containing the unit element when the second system is reached by a right-handed rotation (note that the third row is considered as being above the first row). Note that changing the sign of the rotation angle yields the matrix transpose.

Three-dimensional coordinate rotations can now be built up as a sequence of plane rotations. The fact that the individual rotations are not commutative can be checked by performing sequences of rotations with any convenient solid object. Therefore, although the order of the sequence can be chosen arbitrarily, the same order must be maintained ever after. For example, standard aircraft practice is to describe the aircraft orientation by the z, y, x (also called 3, 2, 1) right-handed rotation sequence that is required to get from a reference system on the surface of the Earth into alignment with an aircraft body-fixed coordinate system. Therefore, starting from the reference system, the sequence of rotations is:

1. Right-handed rotation about the z-axis (positive ψ)
2. Right-handed rotation about the new y-axis (positive θ)
3. Right-handed rotation about the new x-axis (positive ϕ)

The reference system, on the Earth, normally has its z-axis pointing down and the aircraft axes are normally aligned forward, starboard, and down. Starting with the aircraft axes aligned with the corresponding reference axes, we see that this sequence corresponds first to a right-handed rotation around the aircraft z-axis, which is a positive "yaw." This is followed by a right-handed rotation around the aircraft y-axis, which is a positive "pitch," and a right-handed rotation around the aircraft x-axis, which is a positive "roll." Therefore, the rotations are often described as a yaw-pitch-roll sequence, starting from the reference system.

The plane rotation matrices can be written down immediately with the help of the rules established above. Thus, abbreviating cosine and sine to c and s, and using r and b to denote reference and body systems, we get

$$\mathbf{u}^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\phi & \mathrm{s}\phi \\ 0 & -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta & 0 & -\mathrm{s}\theta \\ 0 & 1 & 0 \\ \mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} \mathrm{c}\psi & \mathrm{s}\psi & 0 \\ -\mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{u}^r \tag{1.3-19}$$

Let $C_{b/r}$ denote the complete transformation from the reference system to the body system. Then, multiplying out these transformations, we get:

$$\mathrm{C}_{b/r} = \begin{bmatrix} \mathrm{c}\theta\ \mathrm{c}\psi & \mathrm{c}\theta\ \mathrm{s}\psi & -\mathrm{s}\theta \\ (-\mathrm{c}\phi\ \mathrm{s}\psi + \mathrm{s}\phi\ \mathrm{s}\theta\ \mathrm{c}\psi) & (\mathrm{c}\phi\ \mathrm{c}\psi + \mathrm{s}\phi\ \mathrm{s}\theta\ \mathrm{s}\psi) & \mathrm{s}\phi\ \mathrm{c}\theta \\ (\mathrm{s}\phi\ \mathrm{s}\psi + \mathrm{c}\phi\ \mathrm{s}\theta\ \mathrm{c}\psi) & (-\mathrm{s}\phi\ \mathrm{c}\psi + \mathrm{c}\phi\ \mathrm{s}\theta\ \mathrm{s}\psi) & \mathrm{c}\phi\ \mathrm{c}\theta \end{bmatrix} \tag{1.3-20}$$

This matrix represents a standard transformation, and will be used throughout the text.

The Euler angles are not unique for a given orientation. For example, imagine an aircraft performing a vertical loop with the pilot's head inside the loop. This could be represented by a pitch angle that is continuous in the range $-\pi < \theta \leq \pi$, and zero roll and yaw angles. Alternatively, we can restrict the pitch-attitude angle to $\pm\pi/2$ and, when the pitch attitude reaches $\pi/2$, we can allow the roll and yaw angles to change abruptly by π radians (inverted and heading in the opposite direction). The pitch attitude will then begin to decrease, passing through zero when the plane is at the top of the loop and reaching $-\pi/2$ when it is nose down, at which point the roll and yaw angles change back to zero. This is a more mathematically convenient choice, and so pitch is normally restricted to $\pm\pi/2$. The Euler angles are then unique, apart from the case when the pitch is exactly $\pm\pi/2$ and the roll and yaw are undefined during their abrupt transition.

Matrix Kinematic Relationships for Rotation

Given a set of time-varying Euler angles describing a rotating frame, it is not difficult to determine the components of the angular velocity vector. For example, let the orientation of a coordinate system in frame F_b, relative to a system in reference frame F_r, be described by the aircraft standard yaw (ψ), pitch (θ), roll (ϕ) sequence of Euler rotations. Also, let the Euler angles have derivatives $\dot{\psi}, \dot{\theta}, \dot{\phi}$. Starting from F_r, using two intermediate frames whose relative angular velocities are given by the Euler angle rates, and the additive property of angular velocity, we obtain

$$\boldsymbol{\omega}_{b/r}^{b} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + C_\phi \left(\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + C_\theta \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \right),$$

where C_ϕ and C_θ are the right-handed plane rotations through the particular Euler angles, as given in Equation (1.3-19). After multiplying out the matrices, the final result is

$$\boldsymbol{\omega}_{b/r}^{b} \equiv \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\mathrm{s}\theta \\ 0 & \mathrm{c}\phi & \mathrm{s}\phi\,\mathrm{c}\theta \\ 0 & -\mathrm{s}\phi & \mathrm{c}\phi\,\mathrm{c}\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}, \tag{1.3-21}$$

where P, Q, R are standard symbols for, respectively, the roll, pitch, and yaw rate components of the aircraft angular-velocity vector. The inverse transformation is

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{t}\theta\,\mathrm{s}\phi & \mathrm{t}\theta\,\mathrm{c}\phi \\ 0 & \mathrm{c}\phi & -\mathrm{s}\phi \\ 0 & \mathrm{s}\phi/\mathrm{c}\theta & \mathrm{c}\phi/\mathrm{c}\theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \tag{1.3-22a}$$

We will use the following matrix notation for these equations:

$$\dot{\Phi} = H(\Phi)\boldsymbol{\omega}_{b/r}^{b} \tag{1.3-22b}$$

Equations (1.3-21) and (1.3-22) will be referred to as the *Euler kinematical equations*. Note that the Euler-angle derivatives are each in a different coordinate system, and so the array of derivatives does not represent the components of a vector. Therefore, the equations do not represent coordinate transformations, and the coefficient matrices are not orthogonal matrices. Note also that Equations (1.3-22) have a singularity when $\theta = \pm\pi/2$. In addition, if these equations are used in a simulation, the Euler-angle rates may integrate up to values outside the Euler-angle range. Therefore, logic to deal with this problem must be included in the computer code. Despite these disadvantages the Euler kinematical equations are commonly used in aircraft simulation.

An alternative set of kinematic equations can be derived as follows. The reference system to body-fixed coordinate system transformation was

$$\mathbf{u}^b = C_{b/r}\,\mathbf{u}^r$$

Performing the operations of matrix multiplication in terms of the columns of C shows us that a vector $\mathbf{c}_i$, whose (time-varying) components in F_b are given by the i-th column of $C_{b/r}$, represents a (fixed) unit vector in F_r. Now, applying the equation of Coriolis to the derivative of this vector in the two frames, we have

$$0 = {}^{r}\dot{\mathbf{c}}_i = {}^{b}\dot{\mathbf{c}}_i + \boldsymbol{\omega}_{b/r} \times \mathbf{c}_i \qquad i = 1, 2, 3$$

Resolving in F_b,

$$0 = {}^{b}\dot{\mathbf{c}}_i^{b} + \Omega_{b/r}^{b}\mathbf{c}_i^{b} \qquad i = 1, 2, 3$$

The term ${}^{b}\dot{\mathbf{c}}_i^{b}$ is the derivative of the i-th column of $C_{b/r}$. If we combine the three equations into one matrix equation, the result is

$$\dot{C}_{b/r} = -\Omega_{b/r}^{b}\,C_{b/r} \tag{1.3-23}$$

These equations are known as *Poisson's kinematical equations* or, in inertial navigation, as the *strapdown equation*. Whereas Equations (1.3-22) deal with the Euler angles, this equation deals directly with the elements of the rotation matrix. The components P, Q, R, of the angular velocity vector are, of course, contained in Ω. Compared to the Euler kinematical equations, the strapdown equation has the advantage of being singularity-free and the disadvantage of a large amount of redundancy (nine scalar equations).

When the strapdown equation is used in a simulation, the Euler angles are not directly available and must be calculated from the direction cosine matrix as follows. Let the elements of the rotation matrix (1.3-20) be denoted by c_{ij}. Then for this definition of Euler angles and rotation order, we see that

$$\begin{aligned} \theta &= -\sin^{-1}(c_{13}) \\ \phi &= \text{atan2}(c_{23}, c_{33}) \\ \psi &= \text{atan2}(c_{12}, c_{11}), \end{aligned} \tag{1.3-24}$$

where atan2( ) is the four-quadrant inverse tangent function, available in most programming languages. These equations automatically put the Euler angles into the ranges discussed earlier.

Derivative of an Array

It is interesting to consider formulae for the derivative of an array, and look for a parallel to the equation of Coriolis. Starting from a time-varying coordinate transformation of the components of a general vector,

$$\mathbf{u}^b = C_{b/a}\,\mathbf{u}^a$$

with coordinate systems a and b fixed in F_a and F_b, differentiate the arrays on both sides of the equation. Differentiating the $\mathbf{u}^b$ array is equivalent to taking the derivative in F_b with components taken in system b, therefore,

$${}^b\dot{\mathbf{u}}^b = C_{b/a}\,{}^a\dot{\mathbf{u}}^a + \dot{C}_{b/a}\,\mathbf{u}^a$$

or,

$${}^b\dot{\mathbf{u}}^b = {}^a\dot{\mathbf{u}}^b + \dot{C}_{b/a}\,\mathbf{u}^a$$

Now use the Poisson equations to replace $\dot{C}_{b/a}$ (note that we used the equation of Coriolis to derive the Poisson equations, but they could have been derived in other ways),

$${}^b\dot{\mathbf{u}}^b = {}^a\dot{\mathbf{u}}^b - \Omega^b_{b/a} C_{b/a}\mathbf{u}^a$$

or,

$${}^b\dot{\mathbf{u}}^b = {}^a\dot{\mathbf{u}}^b + \Omega^b_{a/b}\mathbf{u}^b \tag{1.3-25}$$

Equation (1.3-25) is Equation (1.2-7) (the equation of Coriolis) resolved in coordinate system b.

Quaternion Coordinate Rotation

Referring to the quaternion rotation formulae (1.2-20) and the discussion of Equation (1.3-13), we again take the viewpoint that positive μ is a right-handed coordinate rotation rather than a left-handed rotation of a vector. We will define the quaternion that performs the coordinate rotation from system a to system b to be $q_{b/a}$, therefore,

$$q_{b/a} \equiv \begin{bmatrix} \cos(\mu/2) \\ \sin(\mu/2)\mathbf{n}^r \end{bmatrix} \tag{1.3-26a}$$

and the coordinate transformation is

$$\mathbf{u}^b = q_{b/a}^{-1} * \mathbf{u}^a * q_{b/a} \tag{1.3-26b}$$

Equation (1.3-26*b*) can take the place of the direction cosine matrix transformation (1.3-13), and the coordinate transformation is thus achieved by a single rotation around an axis aligned with the quaternion vector $\mathbf{n}\sin(\mu/2)$. Euler's theorem shows that the same coordinate rotation can be achieved by a plane rotation around the unique axis corresponding to an eigenvector of the rotation matrix. Therefore, the vector $\mathbf{n}$ must be parallel to this eigenvector, and so

$$\mathbf{n}^b = C_{b/a}\,\mathbf{n}^a = \mathbf{n}^a,$$

which shows that the quaternion vector part has the same components in system a or system b. In (1.3-26*a*) the reference coordinate system r may be either a or b. We will postpone, for the moment, the problem of finding the rotation quaternion without finding the direction cosine matrix and its eigenstructure, and instead examine the properties of the quaternion transformation.

Performing the inverse transformation to (1.3-26*b*) shows that

$$(q_{b/a})^{-1} = q_{a/b} \tag{1.3-27}$$

Also, for multiple transformations,

$$\mathbf{u}^c = q_{c/b}^{-1} * q_{b/a}^{-1} * \mathbf{u}^a * q_{b/a} * q_{c/b}, \tag{1.3-28}$$

which, because of the associative property, means that we can also perform this transformation with the single quaternion given by

$$q_{c/a}^{-1} = q_{c/b}^{-1} * q_{b/a}^{-1}$$

or,

$$q_{c/a} = q_{b/a} * q_{c/b} \tag{1.3-29}$$

The quaternion coordinate transformation (1.3-26*b*) actually involves more arithmetic operations than premultiplication of $\mathbf{u}^a$ by the direction cosine matrix. However, when the coordinate transformation is evolving with time, the time-update of the quaternion involves differential equations (following shortly) that are numerically preferable to the Euler kinematical equations and more efficient than the Poisson kinematical equations. In addition, the quaternion formulation avoids the singularity of the Euler equations.

In simulation and control, we often choose to keep track of orientation with a quaternion and construct the direction cosine matrix from the quaternion as needed.

It is easy to construct the quaternion for a simple plane rotation, but for a compound rotation (e.g., yaw, pitch, and roll combined) the quaternion rotation axis is not evident. Therefore, we initialize the quaternion from Euler angles or the direction cosine matrix. We now derive the relationships between the quaternion and the Euler angles and direction cosine matrix.

Direction Cosine Matrix from Quaternion

If we write the quaternion rotation formula (1.2-19) in terms of array operations, using the vector part of the quaternion, we get

$$\mathbf{u}^b = \left[2\mathbf{q}^a(\mathbf{q}^a)^T + \left(q_0^2 - (\mathbf{q}^a)^T\mathbf{q}^a\right)I - 2q_0\tilde{\mathbf{q}}^a\right]\,\mathbf{u}^a \qquad (1.3\text{-}30)$$

The cross-product matrix $\tilde{\mathbf{q}}^a$ is given by

$$\tilde{\mathbf{q}}^a = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \qquad (1.3\text{-}31)$$

Now, evaluating the complete transformation matrix in (1.3-30), we find that

$$C_{b/a} = \begin{bmatrix} \left(q_0^2 + q_1^2 - q_2^2 - q_3^2\right) & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & \left(q_0^2 - q_1^2 + q_2^2 - q_3^2\right) & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & \left(q_0^2 - q_1^2 - q_2^2 + q_3^2\right) \end{bmatrix} \qquad (1.3\text{-}32)$$

This expression for the rotation matrix, in terms of quaternion parameters, corresponds to Equations (1.3-26) and the single right-handed rotation around **n**, through the angle μ. Equation (1.3-32) is independent of any choice of Euler angles. We now show how a quaternion may be determined for any given sequence of Euler rotations.

Quaternion from Euler Angles

For the yaw, pitch, roll sequence described by (1.3-19) the quaternion formulation is

$$\mathbf{v}^b = q_{roll}^{-1}q_{pitch}^{-1}q_{yaw}^{-1}\mathbf{v}^r q_{yaw}q_{pitch}q_{roll}$$

If we think of this equation as three successive transformations, with pairs of quaternions, the rotation axes for the quaternions are immediately evident:

$$q_{yaw} = \begin{bmatrix} \cos(\psi/2) \\ 0 \\ 0 \\ \sin(\psi/2) \end{bmatrix} \quad q_{pitch} = \begin{bmatrix} \cos(\theta/2) \\ 0 \\ \sin(\theta/2) \\ 0 \end{bmatrix} \quad q_{roll} = \begin{bmatrix} \cos(\phi/2) \\ \sin(\phi/2) \\ 0 \\ 0 \end{bmatrix}$$

These transformations can be multiplied out, using quaternion multiplication, with only a minor amount of pain. The result is

$$\begin{aligned}
q_0 &= \pm(\cos\phi/2\ \cos\theta/2\ \cos\psi/2 + \sin\phi/2\ \sin\theta/2\ \sin\psi/2) \\
q_1 &= \pm(\sin\phi/2\ \cos\theta/2\ \cos\psi/2 - \cos\phi/2\ \sin\theta/2\ \sin\psi/2) \\
q_2 &= \pm(\cos\phi/2\ \sin\theta/2\ \cos\psi/2 + \sin\phi/2\ \cos\theta/2\ \sin\psi/2) \\
q_3 &= \pm(\cos\phi/2\ \cos\theta/2\ \sin\psi/2 - \sin\phi/2\ \sin\theta/2\ \cos\psi/2)
\end{aligned} \tag{1.3-33}$$

and these are the elements of $q_{b/r}$. A plus or minus sign has been added to these equations because neither (1.3-26*b*) nor (1.3-32) is affected by the choice of sign. The same choice of sign must be used in all of Equations (1.3-33).

Quaternion from Direction Cosine Matrix

The quaternion parameters can also be calculated from the elements $\{c_{ij}\}$ of the general direction cosine matrix. If terms on the main diagonal of (1.3-32) are combined, the following relationships are obtained:

$$\begin{aligned}
4q_0^2 &= 1 + c_{11} + c_{22} + c_{33} \\
4q_1^2 &= 1 + c_{11} - c_{22} - c_{33} \\
4q_2^2 &= 1 - c_{11} + c_{22} - c_{33} \\
4q_3^2 &= 1 - c_{11} - c_{22} + c_{33}
\end{aligned} \tag{1.3-34a}$$

These relationships give the magnitudes of the quaternion elements but not the signs. The off-diagonal terms in (1.3-32) yield the additional relationships

$$\begin{aligned}
4q_0q_1 &= c_{23} - c_{32} \qquad & 4q_1q_2 &= c_{12} + c_{21} \\
4q_0q_2 &= c_{31} - c_{13} \qquad & 4q_2q_3 &= c_{23} + c_{32} \\
4q_0q_3 &= c_{12} - c_{21} \qquad & 4q_1q_3 &= c_{13} + c_{31}
\end{aligned} \tag{1.3-34b}$$

From the first set of equations, (1.3-34*a*), the quaternion element with the largest magnitude (at least one of the four must be nonzero) can be selected. The sign associated with the square root can be chosen arbitrarily, and then this variable can be used as a divisor with (1.3-34*b*) to find the remaining quaternion elements. An interesting quirk of this algorithm is that the quaternion may change sign if the algorithm is restarted with a new set of initial conditions. This will have no effect on the rotation matrix given in (1.3-32). Algorithms like this are discussed in Shoemake (1985) and Shepperd (1978).

The Quaternion Kinematical Equations

When two frames are in relative angular motion, and we wish to keep track of the relative orientation by means of a quaternion, a method is required for continuously updating the quaternion. This takes the form of a differential equation for the quaternion, with the coefficients determined from the relative angular rates. The equation is analagous to the Euler and Poisson kinematical equations.

Let the orientation of a rotating frame F_b, relative to a reference frame F_r, be given, at time t, by the quaternion $q_{b/r}(t)$. Also, as in Figure 1.2-3, let the instantaneous angular velocity of F_b be in the direction of a unit vector $\hat{\mathbf{s}}$, with magnitude ω. Then, in a small time interval δt, the quaternion $\delta q_{b/r}$, which describes the incremental coordinate rotation around $\hat{\mathbf{s}}$, can be found by using small angle approximations in (1.3-26a):

$$\delta q_{b/r}(\delta t) \approx \begin{bmatrix} 1 \\ \hat{\mathbf{s}}^b \ \omega \delta t/2 \end{bmatrix}$$

At time $t + \delta t$ the rotation is given by the quaternion $q_{b/r}(t + \delta t)$, where

$$q_{b/r}(t + \delta t) = q_{b/r}(t) * \delta q_{b/r}(\delta t)$$

(Note that the order of the multiplication matches (1.3-29)). By definition the derivative of $q_{b/r}(t)$ is, temporarily omitting the subscripts,

$$\frac{dq}{dt} = \operatorname*{Lim}_{\delta t \to 0} \frac{q(t) * \left[\delta q - I_q\right]}{\delta t},$$

where I_q is the identity quaternion. Substituting for δq gives

$$\frac{dq}{dt} = \tfrac{1}{2}\, q(t) * \begin{bmatrix} 0 \\ \hat{\mathbf{s}}^b \omega \end{bmatrix} = \tfrac{1}{2}\, q(t) * \boldsymbol{\omega}^b$$

This result can be written formally as:

$$\dot{q}_{b/r} = \tfrac{1}{2}\, q_{b/r} * \boldsymbol{\omega}^b_{b/r} \tag{1.3-35}$$

Replacing the quaternion multiplication by matrix multiplication, Equation (1.3-35) can be put into the form

$$\dot{q} = \tfrac{1}{2} \begin{bmatrix} 0 & -\boldsymbol{\omega}^{\mathrm{T}} \\ \boldsymbol{\omega} & -\Omega \end{bmatrix} \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix}$$

Writing this out in full, using the body-system components of $\boldsymbol{\omega}_{b/r}$, gives

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -P & -Q & -R \\ P & 0 & R & -Q \\ Q & -R & 0 & P \\ R & Q & -P & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{1.3-36}$$

This equation is widely used in simulation of rigid-body motion, and in discrete form it is used in digital attitude control systems (e.g., for satellites) and for inertial navigation digital processing.

1.4 GEODESY, EARTH'S GRAVITATION, TERRESTRIAL NAVIGATION

Geodesy is a branch of mathematics that deals with the shape and area of the Earth. Some ideas and facts from geodesy are needed to simulate the motion of an aerospace vehicle around the Earth. In addition, some knowledge of the Earth's gravitation is required. Useful references are *Encyclopaedia Britannica* (1987), Heiskanen and Moritz (1967), Kuebler and Sommers (1981), NIMA (1997), and Vanicek and Krakiwsky (1982).

The Shape and Gravitation of the Earth, WGS-84

Simulation of high-speed flight over large areas of the Earth's surface, with accurate equations of motion and precise calculation of position, requires an accurate model of the Earth's shape, rotation, and gravity. The shape of the Earth can be well modeled by an ellipsoid of revolution (i.e., a spheroid). The polar radius of the Earth is approximately 21 km less than the equatorial radius, so the generating ellipse must be rotated about its minor axis, to produce an oblate spheroidal model. Organizations from many countries participate in making accurate measurements of the parameters of such models. In the United States the current model is the Department of Defense World Geodetic System 1984, or WGS-84, and the agency responsible for supporting this model is the National Imagery and Mapping Agency (NIMA) (NIMA, 1997). The Global Positioning System (GPS) relies on WGS-84 for the ephemerides of its satellites.

The equipotential surface of the Earth's gravity field that coincides with the undisturbed mean sea level, extended continuously underneath the continents, is called the *geoid*. Earth's irregular mass distribution causes the geoid to be an undulating surface, and this is illustrated in Figure 1.4-1. Note that the *local vertical* is defined by the direction in which a plumb-bob hangs and is accurately normal to the geoid. The angle that it makes with the spheroid normal is called the *deflection of the vertical*, and is usually less than 10 arc-sec (the largest deflections over the entire Earth are about 1 arc-min).

The WGS-84 spheroid has its center at the Earth's cm, and was originally (1976–1979 data) a least-squares best fit to the geoid. More recent estimates have slightly changed the "best fit" parameters, but the current WGS-84 spheroid now uses the

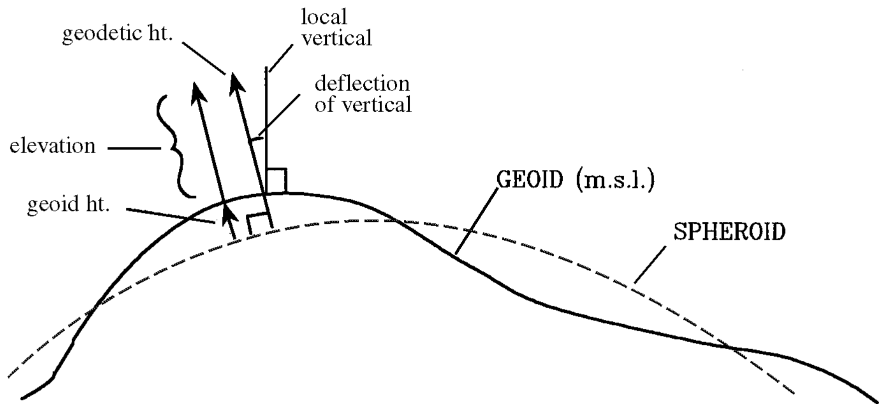

Figure 1.4-1 The geoid and definitions of height.

original parameters as its defining values. Based on a 1° by 1° (latitude, longitude) worldwide grid, the rms deviation of the geoid from the spheroid is only about 30 m! Figure 1.4-2 shows the oblate spheroidal model of the Earth, with the oblateness greatly exaggerated. In the figure, a and b are, respectively, the semimajor and semiminor axes of the generating ellipse. Two other parameters of the ellipse (not

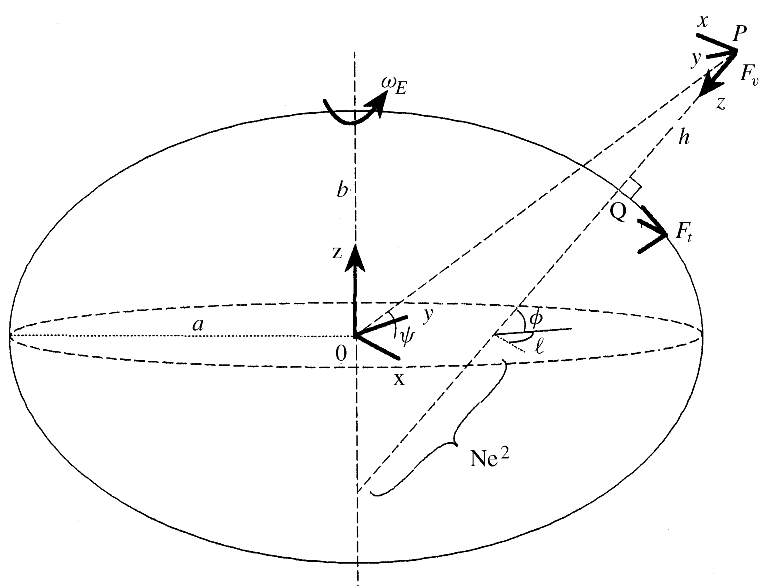

Figure 1.4-2 The oblate spheroidal model of the Earth.

shown) are its flattening, f, and its eccentricity, e. The WGS-84 defined and derived values are:

$$a \equiv 6{,}378{,}137.0 \text{ m} \qquad \text{(defined)} \qquad (1.4\text{-}1a)$$

$$f = \frac{a-b}{a} \equiv 1/298.257223563 \qquad \text{(defined)} \qquad (1.4\text{-}1b)$$

$$b = 6{,}356{,}752 \text{ m} \qquad \text{(derived)} \qquad (1.4\text{-}1c)$$

$$e = \frac{(a^2-b^2)^{1/2}}{a} \approx .08181919 \qquad \text{(derived)} \qquad (1.4\text{-}1d)$$

Two additional parameters are used to define the complete WGS-84 reference frame; these are a fixed Earth rotation rate, ω_E, and the Earth's gravitational constant (GM) with the mass of the atmosphere included. In WGS-84 they are defined to be:

$$\omega_E \equiv 7.2921150 \times 10^{-5} \text{ rad/s} \qquad (1.4\text{-}1e)$$

$$GM \equiv 3986004.418 \times 10^{8} \text{ m}^3/\text{s}^2 \qquad (1.4\text{-}1f)$$

The ω_E value is the sidereal rate of rotation, that is, the "inertial" rate relative to the "fixed" stars (Kaplan, 1981).

Frames and Coordinate Systems

Table 1.4-1 shows the frames and coordinate systems that will be used with Figure 1.4-2. Note that a "geographic" coordinate system has its axes aligned east, north, and up (ENU), or north, east, and down (NED), where "up" or "down" means along the spheroid normal at the system location. It is also called a local navigational system, and the symbol n is used to denote the components in this system. The stability and wind axes systems will be defined in Chapter 2.

Geocentric Coordinates

Geocentric coordinates are referenced to the common origin of the ECI and ECEF systems. Thus, in Figure 1.4-2, the dashed line OP represents the *geocentric radius* of P, and the angle ψ (measured positive north) is the *geocentric latitude* of P. Longitude is measured in the equatorial plane, from one axis of the ECI or ECEF system, to the projection of P on the equatorial plane. We will assume that the ECEF x-axis points to the zero-longitude meridian, and the *terrestrial longitude,* ℓ, (positive east) is shown in the figure. *Celestial longitude,* λ, is measured from the ECI x-axis, which is aligned with some celestial reference direction such as a line from the Sun's cm to the Earth's position in orbit at vernal equinox. In a given time interval, an increment in celestial longitude is equal to the increment in terrestrial longitude plus the increment in Earth's rotation angle. This can be written as

$$\lambda - \lambda_0 = \ell - \ell_0 + \omega_E t, \qquad (1.4\text{-}2)$$

TABLE 1.4-1 Frames and Coordinate Systems Used with Figure 1.4-2

Frame of Reference	Coordinate Systems
F_i, an "inertial" frame, nonrotating but translating with Earth's cm	ECI (Earth-centered inertial), origin at Earth's cm, axes in the equatorial plane and along the spin axis
F_e, a frame defined by the "rigid" Earth	ECEF (Earth-centered, Earth-fixed), axes in the equatorial plane and along the spin axis Tangent-plane system, a geographic system with its origin on the Earth's surface
F_v, a frame translating with the vehicle cm, in which north, east, and down, represent fixed directions	Vehicle-carried system, a geographic system with its origin at the vehicle cm
F_b, a "body" frame defined by the "rigid" vehicle	Vehicle body-fixed system, origin at vehicle cm, axes aligned with vehicle reference directions Vehicle stability-axes system Vehicle wind-axes system

where λ_0 and ℓ_0 are the values at $t = 0$. Absolute celestial longitude is often unimportant, and $\lambda_0 \equiv 0$ can be used.

Geodetic Coordinates

Referring to Figure 1.4-2, *geodetic position* over the surface of the Earth, as used for maps and navigation, is determined by using a normal to the spheroid. *Geodetic latitude,* ϕ, is the angle that the normal makes with the geodetic equatorial plane, and is positive in the Northern Hemisphere. *Geodetic height,* h, is the height above the spheroid, along the normal, as shown in Figure 1.4-1. It can be determined from a database of tabulated geoid height versus latitude and longitude, plus the elevation above mean sea level (msl). The elevation above msl is in turn obtained from a barometric altimeter, or from the land elevation (in a *hypsographic database*) plus the altitude above land (e.g., radar altimeter).

Navigation Calculations

Two important parameters of the spheroid are required for navigation calculations, namely, the radii of curvature. The *meridian radius of curvature,* M, is the radius of curvature in a meridian plane, that is, the radius of curvature of the generating ellipse. Calculations of the radius of curvature for an ellipse can be found in calculus texts and, in terms of geodetic latitude, it is easy to show that M is given by

$$M = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{3/2}} \tag{1.4-3}$$

A radius of curvature, integrated with respect to angle, gives arc length. In this case the integral cannot be found in closed form, and it is much easier to compute distance over the Earth approximately using spherical triangles. The usefulness of the radius of curvature lies in calculating components of velocity. Thus, at geodetic height, h, the geographic-system north component of velocity over the Earth is related to latitude rate by

$$V_N = (M + h)\dot{\phi} \tag{1.4-4}$$

The *prime vertical radius of curvature,* N, is the radius of curvature in a plane containing the spheroid normal and a normal to the meridian plane. It is equal to the distance along the normal, from the spheroid surface to the semiminor axis, and is given by

$$N = \frac{a}{(1 - e^2 \sin^2 \phi)^{1/2}} \tag{1.4-5}$$

Again, N is useful for calculating velocity components. If we take the component of N parallel to the equatorial plane, we obtain the radius of a constant-latitude circle. Therefore, the geographic-system east component of velocity over the Earth is related to longitude rate by

$$V_E = (N + h)\ \cos(\phi)\ \dot{\ell} \tag{1.4-6}$$

Cartesian position coordinates (ECI or ECEF) can be readily calculated from the prime vertical radius of curvature. The projection of N on the x-y plane gives the x and y components. The z-component can be found by dividing N into its parts (Problem 1.4-3) above and below the x-y plane:

$$N = \underset{(\text{below } x\text{-}y)}{Ne^2} + \underset{(\text{above } x\text{-}y)}{N(1 - e^2)} \tag{1.4-7}$$

Therefore, ECEF position can be calculated from geodetic coordinates by

$$\mathbf{p}^e = \begin{bmatrix} (N + h)\ \cos(\phi)\ \cos(\ell) \\ (N + h)\ \cos(\phi)\ \sin(\ell) \\ \left[N(1 - e^2) + h\right]\ \sin(\phi) \end{bmatrix}, \tag{1.4-8}$$

where superscript e indicates ECEF coordinates. Position in ECI coordinates is of the same form as (1.4-8), but with celestial longitude λ replacing terrestrial longitude ℓ.

The reverse of the above transformation is to find the geodetic coordinates from Cartesian coordinates. An exact formula exists but requires the solution of a quartic equation in $\tan(\phi)$ (Vanicek and Krakiwsky, 1982). Therefore, an iterative algorithm is often used. Referring to Figure 1.4-2, we see that

$$\sin\phi = \frac{z}{N(1 - e^2) + h} \tag{1.4-9}$$

Using the large triangle whose hypotenuse is $(N + h)$, and sides $\sqrt{(x^2 + y^2)}$, $[z + Ne^2 \sin(\phi)]$, we can write

$$\tan\phi = \frac{\left[z + Ne^2 \sin\phi\right]}{\sqrt{(x^2 + y^2)}} \tag{1.4-10}$$

If (1.4-9) is substituted for $\sin(\phi)$ in (1.4-10) and simplified, we obtain

$$\tan\phi = \frac{z}{\sqrt{(x^2 + y^2)}[1 - Ne^2/(N + h)]}$$

Because N is a function of ϕ, this formula is implicit in ϕ, but it can be used in the following iterative algorithm for the geodetic coordinates:

$$\begin{aligned}
\ell &= \text{atan2}(y, x) \\
h &= 0, \quad N = a \\
\phi &= \tan^{-1}\left[\frac{z}{(x^2 + y^2)^{1/2}\,[1 - Ne^2/(N + h)]}\right] \\
N &= \frac{a}{(1 - e^2 \sin^2\phi)^{1/2}} \\
(N + h) &= \frac{\sqrt{(x^2 + y^2)}}{\cos\phi} \\
h &= (N + h) - N
\end{aligned} \tag{1.4-11}$$

Latitudes of ±90° must be dealt with as a special case, but elsewhere the iterations converge very rapidly.

Earth-Related Coordinate Transformations

The aircraft equations of motion will require the coordinate rotation matrices between the three systems defined above. The rotation between ECEF and ECI is a plane rotation around the z-axis. Equation (1.4-2) gives the rotation angle as

$$\mu = \lambda_0 - \ell_0 + \omega_E t$$

Therefore, the rotation from ECI to ECEF can be written as

$$C_{e/i} = \begin{bmatrix} \cos\mu & \sin\mu & 0 \\ -\sin\mu & \cos\mu & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.4-12}$$

where subscripts i and e will be used, respectively, to indicate ECI and ECEF systems.

When going from the ECEF to a geographic system, the convention is to perform the longitude rotation first. Now imagine the ECEF system moved to the equator at the correct longitude; a left-handed rotation through 90 degrees, around the y-axis, is needed to get the x-axis pointing north and the z-axis down. It is then only necessary to move to the correct latitude and fall into alignment with the NED system by means of an additional left-handed rotation around the y-axis, through the latitude angle. Therefore, the transformation is

$$C_{n/e} = \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c\ell & s\ell & 0 \\ -s\ell & c\ell & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$C_{n/e} = \begin{bmatrix} -s\phi\, c\ell & -s\phi\, s\ell & c\phi \\ -s\ell & c\ell & 0 \\ -c\phi\, c\ell & -c\phi\, s\ell & -s\phi \end{bmatrix}, \tag{1.4-13}$$

where the n subscript indicates a geographic (local navigational) system.

Gravitation

The term *gravitation* denotes a mass attraction effect, as distinct from *gravity,* meaning the combination of mass attraction and centrifugal force experienced by a body constrained to move with the Earth's surface. Our most accurate equations of motion will contain the centripetal acceleration as a separate term.

The WGS-84 datum includes an amazingly detailed model of the Earth's gravitation. This model is in the form of a (scalar) potential function V, such that a component of specific mass-attraction force along each of three axes can be found from the gradients of the potential function. The current potential function, for use with WGS-84, is Earth Gravitational Model 1996 (EGM96). This has 130,676 coefficients and is intended for very precise satellite and missile calculations. The largest coefficient is two orders of magnitude bigger than the next coefficient and, if we retain only the largest coefficient, the result is still a very accurate model. Neglecting these coefficients removes the dependence on terrestrial longitude, leaving the following potential function:

$$V = \frac{GM}{r}\left[1 - 0.5J_2(a/r)^2(3\sin^2\psi - 1)\right] \tag{1.4-14}$$

in which r is radial distance from the Earth's cm, and ψ is the geocentric latitude. The Earth's gravitational constant GM is the product of the Earth's mass and the universal gravitational constant of the inverse square law. Its EGM96 value, with the mass of the atmosphere included, was given in Equation (1.4-1f). The constant J_2 is given by

$$J_2 = -\sqrt{5}\bar{C}_{2,0} = 1.0826267 \times 10^{-3}, \tag{1.4-15}$$

where $\bar{C}_{2,0}$ is the actual EGM96 coefficient.

The gradients of the potential function are easily evaluated in geocentric coordinates (the same as NED coordinates but with the z-axis pointing to the cm). When this is done and the results are transformed into the ECEF system, we obtain the following gravitation model:

$$\mathbf{G}^e = \frac{-GM}{r^2}\begin{bmatrix} \left[1 + 1.5J_2(a/r)^2(1 - 5\sin^2\psi)\right] p_x/r \\ \left[1 + 1.5J_2(a/r)^2(1 - 5\sin^2\psi)\right] p_y/r \\ \left[1 + 1.5J_2(a/r)^2(3 - 5\sin^2\psi)\right] p_z/r \end{bmatrix}, \tag{1.4-16}$$

where r is equal to the length of the geocentric position vector $\mathbf{p}_e$, whose ECEF components are p_x, p_y, p_z. This model is accurate to about 30–35 $\times$ 10^{-3} cm/s^2 rms, but local deviations can be quite large. Note that the x and y components are identical because there is no longitude dependence. The geocentric latitude is given by

$$\sin\psi = p_z/|\mathbf{p}_e|$$

The model can also be converted to geodetic coordinates. A useful relationship is

$$\tan\psi = (1 - n)\tan\phi, \qquad \text{where} \quad n = Ne^2/(N + h) \tag{1.4-17}$$

The weight of an object on Earth is determined by the gravitational attraction ($m\mathbf{G}$) minus the force needed to produce the centripetal acceleration at the Earth's surface ($m\boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}_e)$). Dividing the weight of the object by its mass gives the *gravity* vector $\mathbf{g}$. Therefore, the vector equation for $\mathbf{g}$ is

$$\mathbf{g} = \mathbf{G} - \boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}_e) \tag{1.4-18}$$

As noted earlier, at the Earth's surface $\mathbf{g}$ is accurately normal to the geoid and defines the local vertical. When Equation (1.4-16) is substituted for $\mathbf{G}$ in (1.4-18), and the equation is resolved in the NED system, we find that $\mathbf{g}$ is almost entirely along the down axis with a variable north component of only a few micro-g's. This is a modeling error, since deflection of the vertical is not explicitly included in the model. The down component of $\mathbf{g}$ given by the model, at the Earth's surface, varies sinusoidally from 9.780 m/s^2 at the equator to 9.806 m/s^2 at 45° geodetic latitude, and 9.832 m/s^2 at the poles. When a constant value of gravity is to be used (e.g., in a simulation), the value at 45° latitude is taken as the standard value of gravity (actually defined to be 9.80665 m/s^2). Our simplified "flat-Earth" equations of motion will use a $\mathbf{g}$ vector that has only a down component, and is measured at the Earth's surface.

1.5 RIGID-BODY DYNAMICS

Angular Motion

By using the vehicle cm as a reference point, the rotational dynamics of the aircraft can be separated from the translational dynamics (Wells, 1967). Here, we develop the equations for the rotational dynamics, which will be the same for both the flat-Earth and oblate-rotating-Earth equations of motion. The following definitions will be needed:

F_i = an inertial reference frame
F_b = a body-fixed frame in the rigid vehicle
$\mathbf{v}_{CM/i}$ = velocity of vehicle cm in F_i
$\boldsymbol{\omega}_{b/i}$ = angular velocity of F_b with respect to F_i
$\mathbf{M}_{A,T}$ = sum of aerodynamic and thrust moments at the cm

The moment is generated by aerodynamic effects, by any reaction-control thrusters, and by any components of the engine thrust not acting through the cm (e.g., due to thrust-vectoring control).

Let the angular momentum vector of a rigid body, in the inertial frame and taken about the cm, be denoted by $\mathbf{h}$. It is shown in textbooks on classical mechanics (Goldstein, 1980) that the derivative of $\mathbf{h}$, taken in the inertial frame, is equal to the vector torque or moment $\mathbf{M}_{A,T}$, applied about the cm. Therefore, analogously to Newton's law for translational momentum, we write

$$\mathbf{M}_{A,T} = {}^{i}\dot{\mathbf{h}} \tag{1.5-1}$$

In order to determine the angular momentum vector, consider an element of mass δm with position vector $\mathbf{r}$ relative to the cm. Its inertial velocity is given by

$$\mathbf{v} = \mathbf{v}_{CM/i} + \boldsymbol{\omega}_{b/i} \times \mathbf{r}$$

The angular momentum of this particle, about the cm, is the moment of the translational momentum about the cm, or

$$\delta\mathbf{h} = \mathbf{r} \times \mathbf{v}\delta m = \mathbf{r} \times \mathbf{v}_{CM/i}\delta m + \mathbf{r} \times (\boldsymbol{\omega}_{b/i} \times \mathbf{r})\,\delta m$$

When this equation is integrated over all the elements of mass, the first term will disappear. This is because $\mathbf{v}_{CM/i}$ is constant for the purposes of the integration and can be taken outside the integral, and the integral of $\mathbf{r}dm$ is zero by the definition of the cm. If the vector triple product formula is applied to the remaining term, and the integration is over all of the elements of mass, the result is:

$$\mathbf{h} = \boldsymbol{\omega}_{b/i} \int (\mathbf{r} \cdot \mathbf{r})\, dm - \int \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}_{b/i})\, dm$$

We must now choose a coordinate system in which to perform the integration. The easiest choice is a body-fixed system in which **r** has constant components. Therefore, let

$$\boldsymbol{\omega}_{b/i}^b = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}, \quad \text{and} \quad \mathbf{r}^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

so that

$$d\mathbf{h}^b = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \left(x^2 + y^2 + z^2\right)\, dm - \begin{bmatrix} x \\ y \\ z \end{bmatrix} (Px + Qy + Rz)\, dm$$

and the integral over the whole vehicle can be written as

$$\mathbf{h}^b = \begin{bmatrix} P \int \left(y^2 + z^2\right) dm - Q \int xy\, dm - R \int xz\, dm \\ Q \int \left(x^2 + z^2\right) dm - R \int yz\, dm - P \int yx\, dm \\ R \int \left(x^2 + y^2\right) dm - P \int zx\, dm - Q \int zy\, dm \end{bmatrix}$$

The various integrals in the angular momentum components are defined to be the moments and cross-products of inertia, for example,

$$\text{moment of inertia about } x\text{-axis} = J_{xx} = \int \left(y^2 + z^2\right) dm$$

$$\text{cross-product of inertia } J_{xy} \equiv J_{yx} = \int xy\, dm$$

On substituting these definitions into the angular momentum, we obtain expressions for the components of $\mathbf{h}^b$ that are bilinear in P, Q, R, and the inertias. This allows $\mathbf{h}^b$ to be written as the matrix product:

$$\mathbf{h}^b = \begin{bmatrix} J_{xx} & -J_{xy} & -J_{xz} \\ -J_{xy} & J_{yy} & -J_{yz} \\ -J_{xz} & -J_{yz} & J_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \equiv J^b \boldsymbol{\omega}_{b/i}^b \tag{1.5-2}$$

The matrix J will be referred to as the *inertia matrix* of the rigid body. It can be calculated or experimentally determined, and is a constant matrix when calculated in body-fixed coordinates for a body with a fixed distribution of mass. It was necessary to choose a coordinate system to obtain this matrix and, consequently, it is not possible to obtain a vector equation of motion that is completely coordinate-free. In more advanced treatments this paradox is avoided by the use of tensors. Note also that J is a real symmetric matrix, and therefore has special properties that are discussed below.

With the angular momentum expressed in terms of the inertia matrix and angular velocity vector of the complete rigid body, Equation (1.5-1) can be evaluated. Since the inertia matrix is known, and constant in the body frame, it will be convenient to replace the derivative in (1.5-1) by a derivative taken in the body frame:

$$\mathbf{M}_{A,T} = {}^i\dot{\mathbf{h}} = {}^b\dot{\mathbf{h}} + \boldsymbol{\omega}_{b/i} \times \mathbf{h}$$

Now, differentiating (1.5-2) in F_b, with J constant, and taking body-fixed components, we obtain,

$$\mathbf{M}^b_{A,T} = J^b\, {}^b\dot{\boldsymbol{\omega}}^b_{b/i} + \Omega^b_{b/i} J^b \boldsymbol{\omega}^b_{b/i},$$

where $\Omega^b_{b/i}$ is a cross-product matrix for $\boldsymbol{\omega}^b_{b/i}$. A rearrangement of this equation gives the *state equation for angular velocity*:

$${}^b\dot{\boldsymbol{\omega}}^b_{b/i} = \left(J^b\right)^{-1} \left[\mathbf{M}^b_{A,T} - \Omega^b_{b/i} J^b \boldsymbol{\omega}^b_{b/i}\right] \tag{1.5-3}$$

This state equation is widely used in simulation and analysis of rigid-body motion, from satellites to ships. It can be solved numerically for $\boldsymbol{\omega}^b_{b/i}$ given the inertia matrix and the torque vector, and its features will now be described.

The assumption that the inertia matrix is constant is not always completely true. For example, with aircraft the inertias may change slowly as fuel is transferred and burned. Also, the inertias may change abruptly if an aircraft is engaged in dropping stores. These effects can usually be adequately accounted for in a simulation by simply changing the inertias in (1.5-3) without accounting for their rates of change. As far as aircraft control system design is concerned, point designs are done for particular flight conditions, and interpolation between point designs can be used when the aircraft mass properties change. This is more likely to be done to deal with movement of the vehicle cm and the resultant effect on static stability.

The inverse of the inertia matrix occurs in (1.5-3), and because of symmetry this has a relatively simple form:

$$J^{-1} = \frac{1}{\Delta} \begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & k_4 & k_5 \\ k_3 & k_5 & k_6 \end{bmatrix}, \tag{1.5-4}$$

where

$$\begin{aligned} k_1 &= \left(J_{yy}J_{zz} - J_{yz}^2\right)/\Delta & k_2 &= \left(J_{yz}J_{zx} + J_{xy}J_{zz}\right)/\Delta \\ k_3 &= \left(J_{xy}J_{yz} + J_{zx}J_{yy}\right)/\Delta & k_4 &= \left(J_{zz}J_{xx} - J_{zx}^2\right)/\Delta \\ k_5 &= \left(J_{xy}J_{zx} + J_{yz}J_{xx}\right)/\Delta & k_6 &= \left(J_{xx}J_{yy} - J_{xy}^2\right)/\Delta \end{aligned}$$

and

$$\Delta = J_{xx}J_{yy}J_{zz} - 2J_{xy}J_{yz}J_{zx} - J_{xx}J_{yz}^2 - J_{yy}J_{zx}^2 - J_{zz}J_{xy}^2$$

A real symmetric matrix has real eigenvalues and, furthermore, a repeated eigenvalue of order p still has associated with it p linearly independent eigenvectors. Therefore, a similarity transformation can be found that reduces the matrix to a real diagonal form. In the case of the inertia matrix this means that we can find a set of coordinate axes in which the inertia matrix is diagonal. These axes are called the *principal axes*.

In principal axes the inverse of the inertia matrix is also diagonal and the angular velocity state equation takes its simplest form, known as *Euler's equations of motion*. If the torque vector has body-axes components given by

$$\mathbf{M}_{A,T}^{b} = \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}, \tag{1.5-5}$$

then Euler's equations are

$$\begin{aligned} \dot{P} &= \frac{(J_y - J_z)\; QR}{J_x} + \frac{\ell}{J_x} \\ \dot{Q} &= \frac{(J_z - J_x)\; RP}{J_y} + \frac{m}{J_y} \\ \dot{R} &= \frac{(J_x - J_y)\; PQ}{J_z} + \frac{n}{J_z} \end{aligned} \tag{1.5-6}$$

The equations involve cyclic permutation of the rate and inertia components; they are inherently coupled because angular rates about any two axes produce an angular acceleration about the third. This *inertia coupling* has important consequences for aircraft maneuvering rapidly at high angles of attack; we examine its effects in Chapter 4. The stability properties of the Euler equations are interesting and will be studied in Problem (1.5-1).

The angular velocity state equation is again simplified when applied to aircraft since for most aircraft the x-z plane is a plane of symmetry. Under this condition, for every product $y_i z_j$ or $y_i x_j$ in an inertia computation there is a product that is identical in magnitude but opposite in sign. Therefore, only the J_{xz} cross-product of inertia is nonzero. A notable exception is an oblique-wing aircraft (Travassos et al., 1980), which does not have a plane of symmetry. Under the plane-of-symmetry assumption the inertia matrix and its inverse reduce to

$$J = \begin{bmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{bmatrix}, \quad J^{-1} = \frac{1}{\Gamma} \begin{bmatrix} J_z & 0 & J_{xz} \\ 0 & \Gamma/J_y & 0 \\ J_{xz} & 0 & J_x \end{bmatrix}, \tag{1.5-7}$$

where $\Gamma = (J_x J_z - J_{xz}^2)$ and the double-subscript notation on the moments of inertia has been dropped.

If the angular velocity state equation (1.5-3) is expanded using the torque vector (1.5-5) and the simple inertia matrix given by (1.5-7), the result is:

$$\begin{aligned}
\Gamma\, \dot{P} &= J_{xz}\left[J_x - J_y + J_z\right] PQ - \left[J_z(J_z - J_y) + J_{xz}^2\right] QR + J_z\ell + J_{xz}n \\
J_y\dot{Q} &= (J_z - J_x)PR - J_{xz}(P^2 - R^2) + m \\
\Gamma\, \dot{R} &= \left[(J_x - J_y)J_x + J_{xz}^2\right] PQ - J_{xz}\left[J_x - J_y + J_z\right] QR + J_{xz}\ell + J_x n
\end{aligned} \tag{1.5-8}$$

In the analysis of angular motion we have so far neglected the angular momentum of any spinning rotors. Technically this violates the rigid-body assumption, but the resulting equations are valid. Note that, strictly, we require axial symmetry of the spinning rotors, otherwise the position of the vehicle cm will vary. This is not a restrictive requirement because it is also a requirement for dynamically balancing the rotors. The effects of the additional angular momentum may be quite significant. For example, a number of World War I aircraft had a single "rotary" engine that had a fixed crankshaft and rotating cylinders. The gyroscopic effects caused by the large angular momentum of the engine gave these aircraft tricky handling characteristics. In the case of a small jet with a single turbofan engine on the longitudinal axis, the effects are smaller. To represent the effect, a constant vector can be added to the angular momentum vector in (1.5-2). Thus,

$$\mathbf{h}^b = J^b \boldsymbol{\omega}_{b/i}^b + \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \tag{1.5-9a}$$

If this analysis is carried through, the effect is to add the following terms, respectively, to the right-hand sides of the three equations (1.5-8):

$$\begin{aligned}
&J_z(Rh_y - Qh_z) + J_{xz}(Qh_x - Ph_y) \\
&- Rh_x + Ph_z \\
&J_{xz}(Rh_y - Qh_z) + J_x(Qh_x - Ph_y)
\end{aligned} \tag{1.5-9b}$$

To complete the set of equations for angular motion, a kinematic equation is required that describes the rigid-body orientation. The changing orientation is a result of the nonzero (in general) angular rates that satisfy the state equation (1.5-3). The kinematics may be described by:

(a) Euler's kinematical equations (1.3-21/22)
(b) Poisson's kinematical equations (1.3-23)
(c) the quaternion kinematical equations (1.3-35/36)

For example, the quaternion state equation

$$\dot{q}_{b/i} = \tfrac{1}{2} q_{b/i} * \boldsymbol{\omega}^{b}_{b/i} \tag{1.5-10}$$

can be solved simultaneously with the angular velocity state equation, for a total of seven state variables. The direction cosine matrix $C_{b/i}$ and the Euler angles will be required in a complete simulation, and these can be derived from the quaternion using (1.3-32) and (1.3-24). The quaternion can be initialized from the initial Euler angles using (1.3-33).

This completes the discussion of the angular motion dynamics. We now turn our attention to the translational motion of the cm.

Translational Motion of the Center of Mass

The first step will be to find the inertial acceleration of the vehicle cm so that Newton's second law may be applied. The frames and coordinate systems related to Figure 1.4-2 are required. These include the ECI system fixed in the inertial frame F_i, the ECEF system fixed in F_e, the NED geographic system in F_v, and the vehicle body-fixed system in frame F_b. In addition to the definitions in the angular motion subsection, we must define the following vectors:

$\mathbf{p}_{CM/O}$ = vehicle cm position relative to ECI origin
$\mathbf{v}_{CM/i} = {}^{i}\dot{\mathbf{p}}_{CM/O}$ = velocity of the cm in F_i
$\mathbf{v}_{CM/e} = {}^{e}\dot{\mathbf{p}}_{CM/O}$ = velocity of the cm in F_e
$\boldsymbol{\omega}_{x/y}$ = angular velocity of frame x with respect to frame y
$\mathbf{F}_{A,T}$ = vector sum of aerodynamic and thrust forces at cm

Note that because $\mathbf{p}_{CM/O}$ is a position vector in both F_i and F_e, $\mathbf{v}_{CM/i}$ and $\mathbf{v}_{CM/e}$ are both velocity vectors.

By using the equation of Coriolis to relate the derivatives of $\mathbf{p}_{CM/O}$ in F_i and F_e, we find that

$$\mathbf{v}_{CM/i} = {}^{i}\dot{\mathbf{p}}_{CM/O} = \mathbf{v}_{CM/e} + \boldsymbol{\omega}_{e/i} \times \mathbf{p}_{CM/O} \tag{1.5-11}$$

Newton's second law applied to the motion of the cm, and neglecting the rate of change of mass of the vehicle, gives

$$(1/\mathrm{m})\, \mathbf{F}_{A,T} + \mathbf{G} = {}^{i}\dot{\mathbf{v}}_{CM/i}$$

where m is the mass of the vehicle.

Now differentiate (1.5-11) and substitute for the right-hand side of this equation. The Earth's angular velocity vector is constant for the differentiation and, introducing the derivative of $\mathbf{v}_{CM/e}$ taken in the body frame, gives

$$(1/\mathrm{m})\, \mathbf{F}_{A,T} + \mathbf{G} = {}^{b}\dot{\mathbf{v}}_{CM/e} + \boldsymbol{\omega}_{b/i} \times \mathbf{v}_{CM/e} + \boldsymbol{\omega}_{e/i} \times {}^{i}\dot{\mathbf{p}}_{CM/O}$$

Substitute (1.5-11) for the inertial position derivative:

$$(1/\mathrm{m})\,\mathbf{F}_{A,T} + \mathbf{G} = {}^b\dot{\mathbf{v}}_{CM/e} + (\boldsymbol{\omega}_{b/i} + \boldsymbol{\omega}_{e/i}) \times \mathbf{v}_{CM/e} + \boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}_{CM/O})$$

or

$${}^b\dot{\mathbf{v}}_{CM/e} = (1/\mathrm{m})\,\mathbf{F}_{A,T} + \mathbf{g} - (\boldsymbol{\omega}_{b/i} + \boldsymbol{\omega}_{e/i}) \times \mathbf{v}_{CM/e}, \qquad (1.5\text{-}12)$$

where

$$\mathbf{g} = \mathbf{G} - \boldsymbol{\omega}_{e/i} \times (\boldsymbol{\omega}_{e/i} \times \mathbf{p}_{CM/O})$$

The kinematic equation (1.5-11) and the dynamic equation (1.5-12) will provide two matrix state equations. When (1.5-11) is resolved in the ECI system, it relates the derivatives of the ECI components of the inertial position vector to themselves and the velocity in F_e. The velocity in F_e must be obtained from the simultaneous solution of (1.5-12). Similarly, Equation (1.5-12) must be resolved in the vehicle body-fixed coordinate system so that it relates derivatives of body-fixed components to themselves and the aerodynamic and thrust forces. The body-fixed components are needed to determine the aerodynamic and thrust forces. Equation (1.5-12) also requires the inertial position vector and the angular velocity vector, and is therefore coupled to (1.5-11) and the angular velocity equation (1.5-3). Because (1.5-11) and (1.5-12) are resolved in different coordinate systems, the direction cosine matrix $C_{b/i}$, obtained from the quaternion (1.5-10), is needed to transform components between these equations. This means that the translational motion equations must be solved simultaneously with the angular motion equations. The complete set of equations of motion, in component form, will be assembled after we have examined the importance of the Earth-rotation terms in (1.5-12).

Significance of the Earth-Rotation Terms

The third term on the right-hand side of (1.5-12) is where these equations will differ from the flat-Earth equations that we will derive. Using the additive property of angular velocities, Equation (1.5-12) can also be written as

$${}^b\dot{\mathbf{v}}_{CM/e} = (1/\mathrm{m})\,\mathbf{F}_{A,T} + \mathbf{g} - (\boldsymbol{\omega}_{b/v} + \boldsymbol{\omega}_{v/e} + 2\boldsymbol{\omega}_{e/i}) \times \mathbf{v}_{CM/e} \qquad (1.5\text{-}13)$$

The angular velocity $\boldsymbol{\omega}_{b/v}$ is null when the vehicle is not maneuvering; the angular velocity $\boldsymbol{\omega}_{v/e}$ can be obtained from geodetic latitude and longitude rates, which can in turn be determined from the NED components of $\mathbf{v}_{CM/e}$ using Equations (1.4-4) and (1.4-6). The term $2\boldsymbol{\omega}_{e/i} \times \mathbf{v}_{CM/e}$ is the Coriolis acceleration, introduced in Equation (1.2-11).

We will use $\boldsymbol{\omega}_{v/e}$ to illustrate how an angular velocity vector can be determined in general. The additive property of angular velocity can be used in conjunction with intermediate frames whose angular velocities are more easily determined. Imagine an intermediate vehicle-carried frame, F_{vi}, at the same longitude as the vehicle in question, but at zero latitude. Summation of angular velocities gives

$$\boldsymbol{\omega}_{v/e} = \boldsymbol{\omega}_{v/vi} + \boldsymbol{\omega}_{vi/e}$$

The geographic components of these angular velocities are easily found from the latitude and longitude rates. The transformation from the geographic system in F_{vi} to that in F_v is a left-handed rotation around the east axis through the latitude angle ϕ. Therefore we obtain

$$\boldsymbol{\omega}^n_{v/e} = \begin{bmatrix} 0 \\ -\dot{\phi} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathrm{c}\phi & 0 & \mathrm{s}\phi \\ 0 & 1 & 0 \\ -\mathrm{s}\phi & 0 & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} \dot{\ell} \\ 0 \\ 0 \end{bmatrix}$$

The latitude and longitude rates are given by (1.4-4) and (1.4-6) and are often adequately approximated by letting $(M + h) = (N + h) = R$, with $R = 21 \times 10^6$ ft. Then

$$\boldsymbol{\omega}^n_{v/e} = 1/R \begin{bmatrix} V_E \\ -V_N \\ -V_E \tan\phi \end{bmatrix} \tag{1.5-14a}$$

In ECEF or ECI coordinates the angular velocity $\boldsymbol{\omega}_{e/i}$ has only a z component, equal to the Earth's spin rate, ω_E. The NED components can be found by using the coordinate rotation (1.4-13):

$$\boldsymbol{\omega}^n_{e/i} = \begin{bmatrix} \omega_E \cos\phi \\ \phi \\ -\omega_E \sin\phi \end{bmatrix} \tag{1.5-14b}$$

Now, adding (1.5-14a) to twice (1.5-14b), we can form a cross-product matrix to premultiply the NED components of $\mathbf{v}_{CM/e}$, and hence evaluate the cross-product terms in the equation of motion (1.5-13) when the vehicle is not maneuvering. Letting the NED components of $\mathbf{v}_{CM/e}$ be $[V_N\, V_E\, V_D]^T$, the result is very cumbersome, so we will take the special case where the vehicle is flying due east at constant altitude; then $V_N = V_D = 0$. The result is

$$\left(\Omega^n_{v/e} + 2\Omega^n_{e/i}\right) \mathbf{v}^n_{CM/e} = \begin{bmatrix} V_E^2 \tan(\phi)/R + 2V_E\omega_E \sin\phi \\ 0 \\ V_E^2/R + 2V_E\omega_E \cos\phi \end{bmatrix} \tag{1.5-15}$$

The centripetal term V_E^2/R is equal to the Coriolis term $2V_E\omega_E$ at about 3000 ft/s. At zero latitude, the down component in (1.5-15) is about 0.9 ft/s^2, which is to be compared with $g_D = 32.2$ ft/s^2. In the "flat-Earth" equations of motion, the Coriolis terms are omitted and the equations have significant errors for velocities over the Earth with magnitude greater than about 2000 ft/s.

The Oblate Rotating-Earth Equations of Motion

We now return to the task of assembling a set of state equations. Position derivatives are obtained from the kinematic equation (1.5-11), resolved in ECI coordinates.

Derivatives of translational velocity components are found from the Newton's law equation (1.5-12) in body-fixed coordinates, and derivatives of angular velocity components are found from the angular velocity equation (1.5-3) as it stands. A time-varying coordinate rotation from ECI to body-fixed coordinates will be needed, and so we will apply the quaternion differential equations (1.5-10), and construct the direction cosine matrix from the quaternion using Equation (1.3-32). This leads to the following set of state equations:

$$C_{b/i} = fn(q_{b/i}) \qquad \left[\text{auxiliary eqn., see 1.3-32}\right] \tag{1.5-16a}$$

$$\dot{q}_{b/i} = \tfrac{1}{2} q_{b/i} * \boldsymbol{\omega}^b_{b/i} \tag{1.5-16b}$$

$$^{i}\dot{\mathbf{p}}^i_{CM/O} = C_{i/b}\mathbf{v}^b_{CM/e} + \Omega^i_{e/i}\mathbf{p}^i_{CM/O} \tag{1.5-16c}$$

$$^{b}\dot{\mathbf{v}}^b_{CM/e} = (1/\mathrm{m})\mathbf{F}^b_{A,T} - \left(\Omega^b_{b/i} + \Omega^b_{e/i}\right)\mathbf{v}^b_{CM/e} + C_{b/i}\mathbf{g}^i \tag{1.5-16d}$$

$$^{b}\dot{\boldsymbol{\omega}}^b_{b/i} = (J^b)^{-1}\left[\mathbf{M}^b_{A,T} - \Omega^b_{b/i} J^b \boldsymbol{\omega}^b_{b/i}\right] \tag{1.5-16e}$$

The state vector for this set of simultaneous differential equations is given by

$$X^T = \left[\left(\mathbf{p}^i_{CM/O}\right)^T (q_{b/i})^T \left(\mathbf{v}^b_{CM/e}\right)^T \left(\boldsymbol{\omega}^b_{b/i}\right)^T\right]$$

The auxiliary equation for the direction cosine matrix must first be calculated from the state vector, before the state equations can be evaluated. Given the mass properties of the vehicle (m and J^b), and the forces and moments, all of the terms on the right-hand side of Equations (1.5–16) should be determined by the state vector.

On the right-hand side of (1.5-16*c*) $\Omega^i_{e/i}$ is the cross-product matrix for the ECI components of Earth's angular velocity vector $[0, 0, \omega_E]^T$. In Equation (1.5-16*d*) a new cross-product matrix for the body-fixed components of this angular velocity is needed. This is given by a similarity transformation:

$$\Omega^b_{e/i} = C_{b/i}\, \Omega^i_{e/i}\, C_{i/b} \tag{1.5-17}$$

since each cross-product matrix is a linear transformation of components in one coordinate system. A simpler calculation is to form a cross-product matrix for

$$\left(\boldsymbol{\omega}^b_{b/i} + C_{b/i}\boldsymbol{\omega}^i_{e/i}\right)$$

and postmultiply it by $\mathbf{v}^b_{CM/e}$.

For convenience the gravity vector has been left in terms of ECI components. Because our gravitation model has no longitude dependence, these can be used instead of ECEF components. If the gravity term must be computed accurately at high altitude, or over a wide range of latitude, we must use

$$\mathbf{g}^b = C_{b/i}\left(\mathbf{G}^i - (\Omega^i_{e/i})^2 \mathbf{p}^i_{CM/O}\right) \tag{1.5-18}$$

with the gravitation model (1.4-16).

In Equations (1.5-16*d*) and (1.5-16*e*) models of the aerodynamic and thrust forces and moments are needed, as derived in Chapter 2. This is where the control inputs enter the model, as throttle settings, aerodynamic control surface deflections, and so on. These forces and moments also depend on the velocity of the vehicle relative to the surrounding air. Therefore, we define a relative velocity vector, $\mathbf{v}_{rel}$, by

$$\mathbf{v}_{rel} = \mathbf{v}_{CM/e} - \mathbf{v}_{W/e}, \tag{1.5-19a}$$

where $\mathbf{v}_{W/e}$ is the wind velocity taken in F_e. Since the wind is normally specified in NED components, and body-fixed components are required for aerodynamic calculations, we will calculate $\mathbf{v}_{rel}^b$ from

$$\mathbf{v}_{rel}^b = \mathbf{v}_{CM/e}^b - C_{b/n}\,\mathbf{v}_{W/e}^n \tag{1.5-19b}$$

Equation (1.5-19*b*) requires $C_{b/n}$, from which we can also find yaw, pitch, and roll Euler angles that describe the attitude of the vehicle relative to the vehicle-carried geographic system. These are usually needed for control purposes, and are calculated as follows.

The algorithm (1.4-11) can be used to find geodetic altitude and latitude, and celestial longitude is found from the inertial position vector. Geodetic altitude can be used to determine the atmospheric properties required for the aerodynamic calculations. The direction cosine matrix $C_{b/n}$ can be computed using

$$C_{b/n} = C_{b/i} C_{i/n}, \tag{1.5-20}$$

where $C_{i/n}$ is found from (1.4-13) with λ replacing ℓ. Equations (1.3-24) can be used to obtain the attitude Euler angles from $C_{b/n}$:

$$\begin{aligned} \phi &= \operatorname{atan2}(c_{23}, c_{33}) \\ \theta &= -\sin^{-1}(c_{13}) \\ \psi &= \operatorname{atan2}(c_{12}, c_{11}) \end{aligned} \tag{1.5-21}$$

Finally, the terrestrial longitude can be calculated from celestial longitude using Equation (1.4-2). Hence position over the Earth is specified.

Equations (1.5-16) and the auxiliary equations (1.5-17) through (1.5-21) constitute the equations needed to simulate the motion of a vehicle around the oblate rotating Earth. They should be used when an accurate simulation is required for a vehicle flying faster than about 2000 ft/s over the Earth, or when accurate long-distance navigation is being simulated. They apply to any type of rigid aerospace vehicle; differences between vehicles begin to appear when the various forces and torques, acting on the vehicle, are modeled. For example simulation of a satellite might require models for gravity-gradient torque, radiation pressure, and residual atmospheric drag. In the next subsection we will derive the so-called flat-Earth equations of motion as a subset of these equations.

The Flat-Earth Equations of Motion

For low-speed flight simulation of aircraft flying over a small region of the Earth, and with no requirement for precise simulation of position, it is usual to neglect the centripetal and Coriolis terms in Equation (1.5-13), as described earlier. Neglecting the centripetal term is equivalent to pretending that the Earth is flat ($R \to \infty$ in (1.5-14*a*)), and neglecting the Coriolis term is equivalent to assuming that the Earth is an inertial frame. The vehicle-carried frame F_v now has zero angular velocity relative to F_e, and so $\boldsymbol{\omega}_{b/v} \equiv \boldsymbol{\omega}_{b/e}$, and the geographic coordinate system in F_v remains aligned with a tangent-plane system in the vicinity of the vehicle. Vehicle attitude can be described, relative to the tangent plane, by yaw, pitch, and roll angles and a direction-cosine matrix $C_{b/n}$ (vehicle body with respect to the local navigational system). Position can conveniently be measured from the origin of the tangent-plane system, T. Therefore, Equations (1.5-16) become:

$$C_{b/n} = fn(\Phi) \qquad \text{(from 1.3-20)} \tag{1.5-22a}$$

$${}^{e}\dot{\mathbf{p}}^{n}_{CM/T} = C_{n/b}\mathbf{v}^{b}_{CM/e} \tag{1.5-22b}$$

$$\dot{\Phi} = H(\Phi)\boldsymbol{\omega}^{b}_{b/e} \qquad \text{(from 1.3-22)} \tag{1.5-22c}$$

$${}^{b}\dot{\mathbf{v}}^{b}_{CM/e} = (1/\mathrm{m})\,\mathbf{F}^{b}_{A,T} + C_{b/n}\mathbf{g}^{n} - \Omega^{b}_{b/e}\mathbf{v}^{b}_{CM/e} \tag{1.5-22d}$$

$${}^{b}\dot{\boldsymbol{\omega}}^{b}_{b/e} = \left(J^{b}\right)^{-1}\left[\mathbf{M}^{b}_{A,T} - \Omega^{b}_{b/e}J^{b}\boldsymbol{\omega}^{b}_{b/e}\right], \tag{1.5-22e}$$

where

$\mathbf{p}_{CM/T}$ = vehicle cm position relative to tangent-system origin
$\mathbf{v}_{CM/e} = {}^{e}\dot{\mathbf{p}}_{CM/T}$ = cm velocity vector in F_e
$\boldsymbol{\omega}_{b/e}$ = angular velocity of F_b with respect to F_e
Φ = Euler angles of body-fixed system relative to NED system

The equation for the velocity vector relative to the surrounding air becomes

$$\mathbf{v}^{b}_{rel} = \mathbf{v}^{b}_{CM/e} - C_{b/n}\,\mathbf{v}^{n}_{W/e} \tag{1.5-23}$$

Gravity appears in the velocity state equation, and in tangent system components this is

$$\mathbf{g}^{t} = [0 \quad 0 \quad g_D] \tag{1.5-24}$$

with g_D equal to the standard gravity (9.80665 $\mathrm{m/s^2}$), or the local value. The state vector can be seen to be

$$X^{T} = \left[\left(\mathbf{p}^{n}_{CM/T}\right)^{T} \;\; \Phi^{T} \;\; \left(\mathbf{v}^{b}_{CM/e}\right)^{T} \;\; \left(\boldsymbol{\omega}^{b}_{b/e}\right)^{T}\right] \tag{1.5-25}$$

The vector form of the relative velocity equation is (1.5-19*a*). If this equation is differentiated in the body-fixed frame, and used to eliminate $\mathbf{v}_{CM/e}$ and its derivative from the vector equation for the translational acceleration, we obtain

$$^{b}\dot{\mathbf{v}}_{rel} = (1/\mathrm{m})\,\mathbf{F}_{A,T} + \mathbf{g} - \boldsymbol{\omega}_{b/e} \times \mathbf{v}_{rel} - {^{e}\dot{\mathbf{v}}_{W/e}}, \qquad (1.5\text{-}26a)$$

where a term $\boldsymbol{\omega}_{b/e} \times \mathbf{v}_{W/e}$ has been canceled from each side of the equation. The last term on the right can be used as a way of introducing gust inputs into the model, or can be set to zero for steady winds. Taking the latter course, and introducing components in the body-fixed system, gives

$$^{b}\dot{\mathbf{v}}^{b}_{rel} = (1/\mathrm{m})\,\mathbf{F}^{b}_{A,T} + C_{b/n}\mathbf{g}^{n} - \Omega^{b}_{b/e}\mathbf{v}^{b}_{rel} \qquad (1.5\text{-}26b)$$

This equation is an alternative to (1.5-22*d*), and Equation (1.5-22*b*) must then be modified to use the sum of the relative and wind velocities.

The dynamic behavior of the rigid vehicle is determined by the force and moment equations and the attitude kinematic equation. There is only a weak dependence on altitude, and hence only a weak coupling to Equation (1.5-22*b*). Furthermore, it is shown in Chapter 2 that the aerodynamic forces and moments depend on the velocity relative to the air mass, with only a weak dependence on altitude. Therefore, the dynamic behavior is essentially determined by $\mathbf{v}_{rel}$, or its negative, the *relative wind*, and is independent of the steady wind velocity. In Chapter 2 we will use (1.5-26) to make a model that is suitable for studying the dynamic behavior.

Equations (1.5-22b–d) are twelve coupled, nonlinear, first-order differential equations. Chapter 2 shows how they can be "solved" analytically. Chapter 3 shows how they can be solved simultaneously by numerical integration for the purposes of flight simulation. Coupling exists because angular acceleration integrates to angular velocity, which determines the Euler angle rates, which in turn determine the direction cosine matrix. The direction cosine matrix is involved in the state equations for position and velocity, and position (the altitude component) and velocity determine aerodynamic effects which determine angular acceleration. Coupling is also present through the translational velocity. These interrelationships will become more apparent in Chapter 2.

1.6 SUMMARY

This chapter provides sufficient background material to enable the reader to deal with many of the dynamical problems that occur in the modern aerospace industry. Classical mechanics is the key to the analysis and solution of many of these problems, and we have reviewed and used many of the vector operations from classical mechanics. Coordinate rotations are everywhere in the analysis and in the software used for computer control of many systems. Therefore, we have attempted to provide an easy approach to setting up the rotations. The use of quaternions for coordinate rotation is very popular in satellite and missile problems, and in computer graphics. We have provided background material in this area because the quaternion avoids any numerical singularity problems with "all-attitude" simulation of aircraft.

The gravity model presented here is more detailed than usual for aircraft simulations and gives the reader an introduction to the more detailed modeling that would be needed to simulate accurate navigation, or simulate spacecraft and launch and reentry vehicles. The rotating-Earth equations of motion may be applied to vehicles intended to reach hypersonic speeds and perhaps go into orbit, or to slowly moving aircraft near the surface of the Earth. The design of these vehicles requires large computer simulations involving the equations of motion, and controlling them may require programming onboard computers with algorithms that employ many of the concepts described here.

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PROBLEMS

Section 1.2

1.2-1 Prove the scalar triple product formula.

1.2-2 Prove the vector triple product formula.

1.2-3 If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are bound vectors (i.e., they have a common origin), show that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ represents the signed volume of the parallepiped that has $\mathbf{u}, \mathbf{v}, \mathbf{w}$ as adjacent edges.

1.2-4 Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0$.

1.2-5 If two particles moving with constant velocity are described by the position vectors

$$\mathbf{p} = \mathbf{p}_0 + \mathbf{v}t, \quad \text{and} \quad \mathbf{s} = \mathbf{s}_0 + \mathbf{w}t$$

(a) show that the shortest distance between their trajectories is given by:

$$d = |(\mathbf{s}_0 - \mathbf{p}_0) \cdot (\mathbf{w} \times \mathbf{v})|/(|\mathbf{w} \times \mathbf{v}|)$$

(b) find the shortest distance between the particles themselves.

1.2-6 If the vectors $\mathbf{u}, \mathbf{v}$ in the rotation formula (1.2-6) are known, what can be determined mathematically about the unit vector $\mathbf{n}$?

1.2-7 (a) Write a vector equation for the specific force at the cm of a moving vehicle, in terms of the gravitational, centripetal, and Coriolis acceleration vectors, and the derivative in F_e of vehicle velocity $\mathbf{v}_{CM/e}$. (b) Rewrite the equation in terms of the derivative taken in $\boldsymbol{F_v}, \boldsymbol{\omega}_{v/i}$ and $\boldsymbol{\omega}_{e/i}$ (see Table 1.4-1). (c) Write the matrix equation for the NED components of the velocity in F_v. (d) Explain how this equation could be used to perform inertial navigation using Equations (1.4-3–6) and measurements from three accelerometers on a servo-driven, NED-aligned platform.

1.2-8 Compare the Coriolis deflections of a mass reaching the ground for the following two cases:

(a) thrown vertically upward with initial velocity u

(b) dropped, with zero initial velocity, from the maximum height reached in (a).

1.2-9 Show that, for a quaternion product, the norm of the product is equal to the product of the individual norms.

1.2-10 Compare the operation count $(+, -, \times, \div)$ of the vector rotation formula (1.2-6) with that of the quaternion formula (1.2-20b).

1.2-11 If a coordinate system b is rotating at a constant rate with respect to a system a, and only the components of the angular velocity vector in system b are given, find an expression for the quaternion that transforms coordinates from b to a.

Section 1.3

1.3-1 Derive the cross-product matrix used in Equation (1.3-9).

1.3-2 Derive the plane rotation matrix given in Equation (1.3-18)
(a) by using Equation (1.3-12);
(b) by trigonometry from Figure 1.3-1.

1.3-3 A "compound" rotation can be represented by a sequence of plane rotations, but the plane rotations do not commute. Start with an airplane heading north in level flight and draw two sequences of pictures to illustrate the difference between a yaw, pitch, roll sequence, and a roll, yaw, pitch sequence. Let the rotations (Euler angles) be yaw $\psi = -90°$, pitch $\theta = -45°$, and roll $\phi = 45°$. State the final orientation.

1.3-4 Find the rotation matrix corresponding to (1.3-20) if the reference system has its z-axis pointing up, not down.

1.3-5 Show that the rotation matrix between two coordinate systems can be calculated from a knowledge of the position vectors of two different objects if the position vectors are known in each system. Specify the rotation matrix in terms of the solution of a matrix equation. Show how this technique could be used to determine vehicle attitude by taking telescope bearings on two known stars, given a star catalog.

1.3-6 Find the eigenvalues of the rotation matrix (1.3-17).

Section 1.4

1.4-1 The ECI position coordinates of a celestial object are (x, y, z). Determine the ENU (east, north, up) position coordinates of the object with respect to a tracking station on the surface of the Earth at celestial longitude λ, geodetic latitude ϕ, and sea-level altitude. Assume a spherical Earth (of radius R), align the ECI system with its z-axis pointing up toward the North Pole, and assume that the ENU frame is obtained by rotating first through the longitude angle and then through the latitude angle. Assume also that longitude is measured east from the ECI x-axis. Show that the Earth's radius appears in only one of the required coordinates.

1.4-2 Starting from a calculus-textbook definition of radius of curvature, and the equation of an ellipse, derive the formula (1.4-3) for the meridian radius of curvature.

1.4-3 Use the fact that the prime vertical radius of curvature is equal to the distance along the normal, from the spheroid surface to the semiminor axis, together with the equation of the generating ellipse, and an expression for the gradient of a normal, to derive the formula (1.4-5) for N. Also confirm Equation (1.4-7) for the two parts of N.

1.4-4 Program the iterative calculation of geodetic coordinates, (1.4-11) and use some test cases to demonstrate that it converges very quickly to many decimal digits.

1.4-5 Derive the formula (1.4-16), for **G**, starting from the potential function, V, in Equation (1.4-14). Use a geocentric coordinate system as mentioned in the text.

1.4-6 Derive the formula (1.4-17) for geocentric latitude in terms of geodetic latitude by using the geometry of the generating ellipse.

1.4-7 Starting from (1.4-16), write and test a program to evaluate $|\mathbf{g}|$ and $|\mathbf{G}|$ as functions of geodetic latitude and altitude. Plot them both on the same axes, against latitude ($0 \rightarrow 90°$). Do this for $h = 0$ and 30,000 m.

1.4-8 Derive the conditions for a body to remain in a geostationary orbit of the Earth. Use the gravity model and geodetic data to determine the geostationary altitude. What are the constraints on the latitude and inclination of the orbit?

Section 1.5

1.5-1 Derive a set of linear state equations from Equations (1.5-6) by considering perturbations from a steady-state condition with angular rates P_e, Q_e, and R_e. Find expressions for the eigenvalues of the coefficient matrix when only one angular rate is nonzero, and show that there is an unstable eigenvalue if the moment of inertia about this axis is either the largest or the smallest of the three inertias. Deduce any practical consequences of this result.

1.5-2 Use Euler's equations of motion (1.5-6) and the Poisson kinematical equations (1.3-23) to simulate the angular motion of a brick tossed in the air and spinning. Write a MATLAB program using Euler integration (1.1-4) to integrate these equations over a 300 s interval, using an integration step of 10 ms. Let the brick have dimensions $8 \times 5 \times 2$ units, corresponding to x, y, z axes at the center of mass. The moments ℓ, m, n are all zero, and the initial conditions are:

(a) $\phi = \theta = \psi = 0, \quad P = 0.1, \quad Q = 0, \quad R = 0.001$ rad/s
(b) $\phi = \theta = \psi = 0, \quad P = 0.001, \quad Q = 0, \quad R = 0.1$ rad/s
(c) $\phi = \theta = \psi = 0, \quad P = 0.0, \quad Q = 0.1, \quad R = 0.001$ rad/s

Plot the three angular rates (deg/s) on one graph, and the three Euler angles (in deg) on another. Which motion is stable and why?

1.5-3 Repeat 1.5-2, but use the Euler kinematical equations (1.3-22a) to represent attitude. Add logic to the program to restrict the Euler angles to the ranges described in Section 1.3.

1.5-4 An aircraft is to be mounted on a platform with a torsional suspension so that its moment of inertia, I_{zz}, can be determined. Treat the wings as one piece, equal to one-third of the aircraft weight, and placed on the fuselage one-third back from the nose.

(a) Find the distance of the aircraft cm from the nose, as a fraction of the fuselage length.

(b) The aircraft weight is 80,000 lbs, the wing planform is a rectangle 40 ft by 16 ft, and the planview of the fuselage is a rectangle 50 ft by 12 ft.

Assuming uniform density, calculate the aircraft moment of inertia (in slug-ft^2).

(c) Calculate the period of oscillation (in s) of the platform if the torsional spring constant is 10,000 lb-ft/rad.

1.5-5 The propeller and crankshaft of a single-engine aircraft have a combined moment of inertia of 45 slug-ft^2 about the axis of rotation, and are rotating at 1500 rpm clockwise when viewed from in front. The moments of inertia of the aircraft are roll: 3000 slug-ft^2, pitch: 6700 slug-ft^2, yaw: 9000 slug-ft^2. If the aircraft rolls at 100 deg/s, while pitching at 20 deg/s, determine the angular acceleration in yaw. All inertias and angular rates are body-axes components.