Chapter Two

# RATE OF CHANGE: THE DERIVATIVE

**Chapter 1 introduced the average rate of change of a function on an interval. In this chapter, we investigate the *instantaneous* rate of change of a function at a point. The notion of rate of change at a given instant leads us to the concept of the *derivative*.**

**The derivative can be interpreted geometrically as the slope of a curve and physically as a rate of change. Derivatives can be used to represent everything from fluctuations in interest rates, to the rate at which fish are dying, to the rate of growth of a tumor.**

# 2.1 INSTANTANEOUS RATE OF CHANGE

Chapter 1 introduced the average rate of change of a function over an interval. In this section, we consider the rate of change of a function at a point. We saw in Chapter 1 that when an object is moving along a straight line, the average rate of change of position with respect to time is the average velocity. If position is expressed as $y = f(t)$, where $t$ is time, then

$$\begin{array}{c}\text{Average rate of change in position} \\ \text{between } t = a \text{ and } t = b\end{array} = \frac{\Delta y}{\Delta t} = \frac{f(b) - f(a)}{b - a}.$$

If you drive 200 miles in 4 hours, your average velocity is $200/4 = 50$ miles per hour. Of course, this does not mean that you travel at exactly 50 mph the entire trip. Your velocity at a given instant during the trip is shown on your speedometer, and this is the quantity that we investigate now.

## Instantaneous Velocity

We throw a grapefruit straight upward into the air. Table 2.1 gives its height, $y$, at time $t$. What is the velocity of the grapefruit at exactly $t = 1$? We use average velocities to estimate this quantity.

**Table 2.1** *Height of the grapefruit above the ground*

| $t$ (sec) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $y = s(t)$ (feet) | 6 | 90 | 142 | 162 | 150 | 106 | 30 |

The average velocity on the interval $0 \leq t \leq 1$ is 84 ft/sec and the average velocity on the interval $1 \leq t \leq 2$ is 52 ft/sec. Notice that the average velocity before $t = 1$ is larger than the average velocity after $t = 1$ since the grapefruit is slowing down. We expect the velocity *at* $t = 1$ to be between these two average velocities. How can we find the velocity at *exactly* $t = 1$? We look at what happens near $t = 1$ in more detail. Suppose that we find the average velocities on either side of $t = 1$ over smaller and smaller intervals, as in Figure 2.1. Then, for example,

$$\begin{array}{c}\text{Average velocity} \\ \text{between } t = 1 \text{ and } t = 1.01\end{array} = \frac{\Delta y}{\Delta t} = \frac{s(1.01) - s(1)}{1.01 - 1} = \frac{90.678 - 90}{0.01} = 67.8 \text{ ft/sec.}$$

We expect the instantaneous velocity at $t = 1$ to be between the average velocities on either side of $t = 1$. In Figure 2.1, the values of the average velocity before $t = 1$ and the average velocity after $t = 1$ get closer together as the size of the interval shrinks. For the smallest intervals in Figure 2.1, both velocities are 68.0 ft/sec (to one decimal place), so we say the velocity at $t = 1$ is 68.0 ft/sec (to one decimal place).

| $t$ | 0 | 0.9 | 0.99 | 0.999 | 1 | 1.001 | 1.01 | 1.1 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $y = s(t)$ | 6.000 | 83.040 | 89.318 | 89.932 | 90.000 | 90.068 | 90.678 | 96.640 | 142.000 |

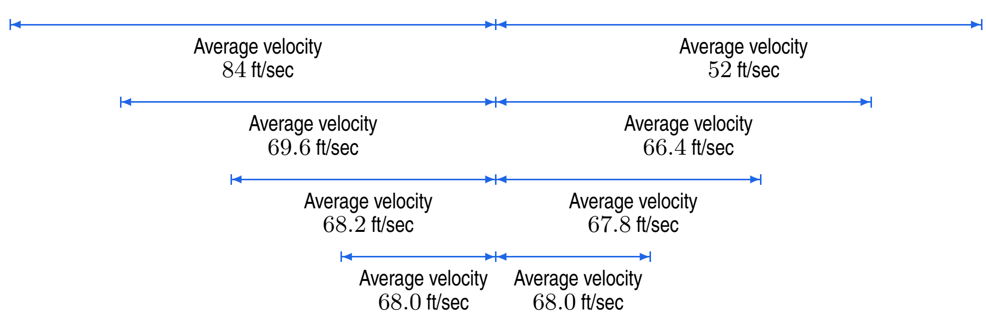



**Figure 2.1:** Average velocities over intervals on either side of $t = 1$ showing successively smaller intervals

Of course, if we showed more decimal places, the average velocities before and after $t = 1$ would no longer agree. To calculate the velocity at $t = 1$ to more decimal places of accuracy, we take smaller and smaller intervals on either side of $t = 1$ until the average velocities agree to the number of decimal places we want. In this way, we can estimate the velocity at $t = 1$ to any accuracy.

## Defining Instantaneous Velocity Using the Idea of a Limit

When we take smaller intervals near $t = 1$, it turns out that the average velocities for the grapefruit are always just above or just below 68 ft/sec. It seems natural, then, to define velocity at the instant $t = 1$ to be 68 ft/sec. This is called the *instantaneous velocity* at this point. Its definition depends on our being convinced that smaller and smaller intervals provide average velocities that come arbitrarily close to 68. This process is referred to as *taking the limit*.

> The **instantaneous velocity** of an object at time $t$ is defined to be the limit of the average velocity of the object over shorter and shorter time intervals containing $t$.

Notice that the instantaneous velocity seems to be exactly 68, but what if it were 68.000001? How can we be sure that we have taken small enough intervals? Showing that the limit is exactly 68 requires more precise knowledge of how the velocities were calculated and of the limiting process; see the Focus on Theory section on page 135.

## Instantaneous Rate of Change

We can define the *instantaneous rate of change* of any function $y = f(t)$ at a point $t = a$. We mimic what we did for velocity and look at the average rate of change over smaller and smaller intervals.

> The **instantaneous rate of change** of $f$ at $a$, also called the **rate of change** of $f$ at $a$, is defined to be the limit of the average rates of change of $f$ over shorter and shorter intervals around $a$.

Since the average rate of change is a difference quotient of the form $\Delta y/\Delta t$, the instantaneous rate of change is a limit of difference quotients. In practice, we often approximate a rate of change by one of these difference quotients.

**Example 1** The quantity (in mg) of a drug in the blood at time $t$ (in minutes) is given by $Q = 25(0.8)^t$. Estimate the rate of change of the quantity at $t = 3$ and interpret your answer.

**Solution** We estimate the rate of change at $t = 3$ by computing the average rate of change over intervals near $t = 3$. We can make our estimate as accurate as we like by choosing our intervals small enough. Let's look at the average rate of change over the interval $3 \le t \le 3.01$:

$$\text{Average rate of change } = \frac{\Delta Q}{\Delta t} = \frac{25(0.8)^{3.01} - 25(0.8)^3}{3.01 - 3.00} = \frac{12.7715 - 12.80}{3.01 - 3.00} = -2.85.$$

A reasonable estimate for the rate of change of the quantity at $t = 3$ is $-2.85$. Since $Q$ is in mg and $t$ in minutes, the units of $\Delta Q/\Delta t$ are mg/minute. Since the rate of change is negative, the quantity of the drug is decreasing. After 3 minutes, the quantity of the drug in the body is decreasing at 2.85 mg/minute.

In Example 1, we estimated the rate of change using an interval to the right of the point ($t = 3$ to $t = 3.01$). We could use an interval to the left of the point, or we could average the rates of change to the left and the right. In this text, we usually use an interval to the right of the point.

## The Derivative at a Point

The instantaneous rate of change of a function $f$ at a point $a$ is so important that it is given its own name, the *derivative of $f$ at $a$*, denoted $f'(a)$ (read "$f$-prime of a"). If we want to emphasize that $f'(a)$ is the rate of change of $f(x)$ as the variable $x$ increases, we call $f'(a)$ the derivative of $f$ *with respect to* $x$ at $x = a$. Notice that the derivative is just a new name for the rate of change of a function.

> The **derivative of $f$ at $a$**, written $f'(a)$, is defined to be the instantaneous rate of change of $f$ at the point $a$.

A definition of the derivative using a formula is given in the Focus on Theory section on page 135.

**Example 2** Estimate $f'(2)$ if $f(x) = x^3$.

**Solution** Since $f'(2)$ is the derivative, or rate of change, of $f(x) = x^3$ at 2, we look at the average rate of change over intervals near 2. Using the interval $2 \le x \le 2.001$, we see that

$$\begin{array}{c}\text{Average rate of change}\\ \text{on } 2 \le x \le 2.001\end{array} = \frac{(2.001)^3 - 2^3}{2.001 - 2} = \frac{8.012 - 8}{0.001} = 12.0.$$

The rate of change of $f(x)$ at $x = 2$ appears to be approximately 12, so we estimate $f'(2) = 12$.

## Visualizing the Derivative: Slope of the Graph and Slope of the Tangent Line

Figure 2.2 shows the average rate of change of a function represented by the slope of the secant line joining points $A$ and $B$. The derivative is found by taking the average rate of change over smaller and smaller intervals. In Figure 2.3, as point $B$ moves toward point $A$, the secant line becomes the tangent line at point $A$. Thus, the derivative is represented by the slope of the tangent line to the graph at the point.

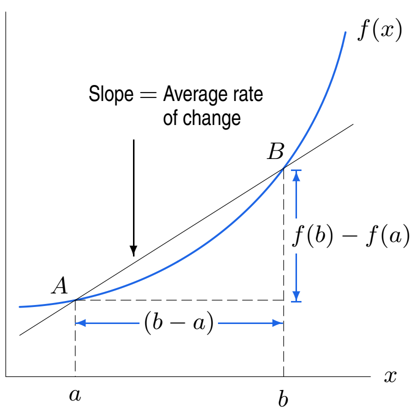



**Figure 2.2:** Visualizing the average rate of change of $f$ between $a$ and $b$

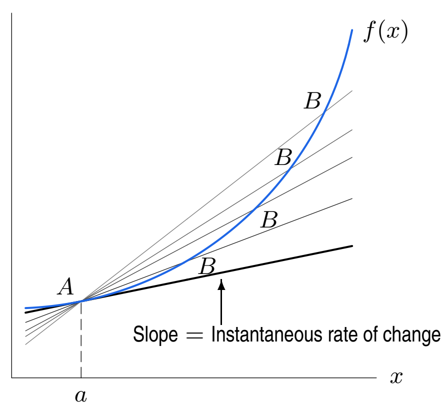



**Figure 2.3:** Visualizing the instantaneous rate of change of $f$ at $a$

Alternatively, take the graph of a function around a point and "zoom in" to get a close-up view. (See Figure 2.4.) The more we zoom in, the more the graph appears to be straight. We call the slope of this line the *slope of the graph* at the point; it also represents the derivative.

> The derivative of a function at the point $A$ is equal to
>
> - The slope of the graph of the function at $A$.
> - The slope of the line tangent to the curve at $A$.

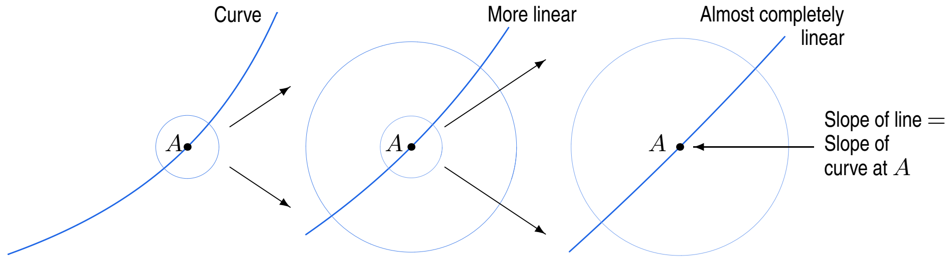



**Figure 2.4:** Finding the slope of a curve at a point by "zooming in"

The slope interpretation is often useful in gaining rough information about the derivative, as the following examples show.

**Example 3** Use a graph of $f(x) = x^2$ to determine whether each of the following quantities is positive, negative, or zero: (a) $f'(1)$ (b) $f'(-1)$ (c) $f'(2)$ (d) $f'(0)$

**Solution** Figure 2.5 shows tangent line segments to the graph of $f(x) = x^2$ at the points $x = 1$, $x = -1$, $x = 2$, and $x = 0$. Since the derivative is the slope of the tangent line at the point, we have:

(a) $f'(1)$ is positive.
(b) $f'(-1)$ is negative.
(c) $f'(2)$ is positive (and larger than $f'(1)$).
(d) $f'(0) = 0$ since the graph has a horizontal tangent at $x = 0$.

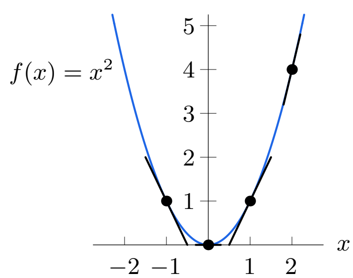



**Figure 2.5:** Tangent lines showing sign of derivative of $f(x) = x^2$

**Example 4** Estimate the derivative of $f(x) = 2^x$ at $x = 0$ graphically and numerically.

**Solution** Graphically: If we draw a tangent line at $x = 0$ to the exponential curve in Figure 2.6, we see that it has a positive slope between 0.5 and 1.

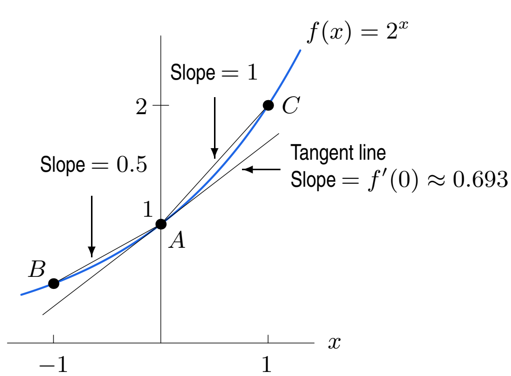



**Figure 2.6:** Graph of $f(x) = 2^x$ showing the derivative at $x = 0$

Numerically: To estimate the derivative at $x = 0$, we compute the average rate of change on an interval around 0.

$$\begin{array}{c}\text{Average rate of change}\\ \text{on } 0 \le x \le 0.0001\end{array} = \frac{2^{0.0001} - 2^0}{0.0001 - 0} = \frac{1.000069317 - 1}{0.0001} = 0.69317.$$

Since using smaller intervals gives approximately the same values, it appears that the derivative is approximately 0.69317; that is, $f'(0) \approx 0.693$.

**Example 5** The graph of a function $y = f(x)$ is shown in Figure 2.7. Indicate whether each of the following quantities is positive or negative, and illustrate your answers graphically.

(a) $f'(1)$ (b) $\dfrac{f(3) - f(1)}{3 - 1}$ (c) $f(4) - f(2)$

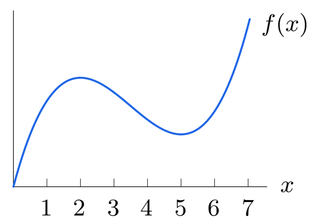



Figure 2.7

**Solution** (a) Since $f'(1)$ is the slope of the graph at $x = 1$, we see in Figure 2.8 that $f'(1)$ is positive.

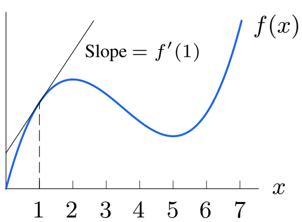



Figure 2.8

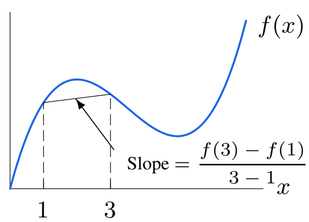



Figure 2.9

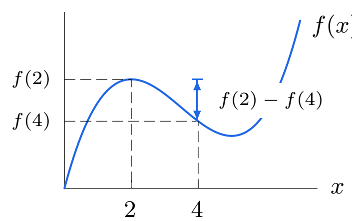



Figure 2.10

(b) The difference quotient $(f(3) - f(1))/(3 - 1)$ is the slope of the secant line between $x = 1$ and $x = 3$. We see from Figure 2.9 that this slope is positive.

(c) Since $f(4)$ is the value of the function at $x = 4$ and $f(2)$ is the value of the function at $x = 2$, the expression $f(4) - f(2)$ is the change in the function between $x = 2$ and $x = 4$. Since $f(4)$ lies below $f(2)$, this change is negative. See Figure 2.10.

## Estimating the Derivative of a Function Given Numerically

If we are given a table of values for a function, we can estimate values of its derivative. To do this, we have to assume that the points in the table are close enough together that the function does not change wildly between them.

**Example 6** The total acreage of farms in the US[1] has decreased since 1980. See Table 2.2.

**Table 2.2** *Total farm land in million acres*

| Year | 1980 | 1985 | 1990 | 1995 | 2000 |
|---|---|---|---|---|---|
| Farm land (million acres) | 1039 | 1012 | 987 | 963 | 945 |

(a) What was the average rate of change in farm land between 1980 and 2000?
(b) Estimate $f'(1995)$ and interpret your answer in terms of farm land.

[1] *Statistical Abstracts of the United States 2004–2005*, Table 796.

**Solution**

(a) Between 1980 and 2000,

$$\text{Average rate of change} = \frac{945-1039}{2000-1980} = \frac{-94}{20} = -4.7 \text{ million acres per year.}$$

Between 1980 and 2000, the amount of farm land was decreasing at an average rate of 4.7 million acres per year.

(b) We use the interval from 1995 to 2000 to estimate the instantaneous rate of change at 1995:

$$f'(1995) = \begin{array}{c}\text{Rate of change}\\ \text{in 1995}\end{array} \approx \frac{945-963}{2000-1995} = \frac{-18}{5} = -3.6 \text{ million acres per year.}$$

In 1995, the amount of farm land was decreasing at a rate of approximately 3.6 million acres per year.

## Problems for Section 2.1

**1.** The distance (in feet) of an object from a point is given by $s(t) = t^2$, where time $t$ is in seconds.

**(a)** What is the average velocity of the object between $t = 3$ and $t = 5$?

**(b)** By using smaller and smaller intervals around 3, estimate the instantaneous velocity at time $t = 3$.

**2.** In a time of $t$ seconds, a particle moves a distance of $s$ meters from its starting point, where $s = 4t^2 + 3$.

**(a)** Find the average velocity between $t = 1$ and $t = 1 + h$ if:

(i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$.

**(b)** Use your answers to part (a) to estimate the instantaneous velocity of the particle at time $t = 1$.

**3.** The size, $S$, of a tumor (in cubic millimeters) is given by $S = 2^t$, where $t$ is the number of months since the tumor was discovered. Give units with your answers.

**(a)** What is the total change in the size of the tumor during the first six months?

**(b)** What is the average rate of change in the size of the tumor during the first six months?

**(c)** Estimate the rate at which the tumor is growing at $t = 6$. (Use smaller and smaller intervals.)

**4.** Match the points labeled on the curve in Figure 2.11 with the given slopes.

| Slope | Point |
|---|---|
| $-3$ | |
| $-1$ | |
| $0$ | |
| $1/2$ | |
| $1$ | |
| $2$ | |

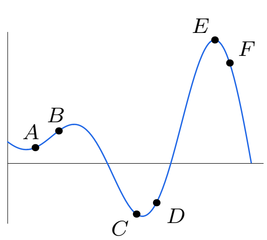



Figure 2.11

**5.** If $t$ is in years since 2000, the population, in thousands, of the McAllen, Texas metropolitan area was given by $P(t) = 570(1.037)^t$. Estimate the rate of growth, in people per year, in 2006.

**6.** Figure 2.12 shows the cost, $y = f(x)$, of manufacturing $x$ kilograms of a chemical.

**(a)** Is the average rate of change of the cost greater between $x = 0$ and $x = 3$, or between $x = 3$ and $x = 5$? Explain your answer graphically.

**(b)** Is the instantaneous rate of change of the cost of producing $x$ kilograms greater at $x = 1$ or at $x = 4$? Explain your answer graphically.

**(c)** What are the units of these rates of change?

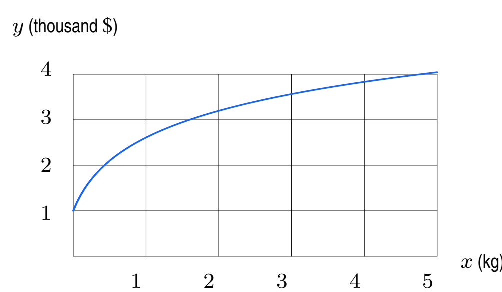



Figure 2.12

**7.** Find the average velocity over the interval $0 \le t \le 0.8$, and estimate the velocity at $t = 0.2$ of a car whose position, $s$, is given by the following table.

| $t$ (sec) | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $s$ (ft) | 0 | 0.5 | 1.8 | 3.8 | 6.5 | 9.6 |

**8.** The following table gives the percent of the US population living in urban areas as a function of year.[2]

| Year | 1800 | 1830 | 1860 | 1890 | 1920 |
|---|---|---|---|---|---|
| Percent | 6.0 | 9.0 | 19.8 | 35.1 | 51.2 |
| Year | 1950 | 1980 | 1990 | 2000 | |
| Percent | 64.0 | 73.7 | 75.2 | 79.0 | |

- **(a)** Find the average rate of change of the percent of the population living in urban areas between 1890 and 1990.
- **(b)** Estimate the rate at which this percent is increasing at the year 1990.
- **(c)** Estimate the rate of change of this function for the year 1830 and explain what it is telling you.
- **(d)** Is this function increasing or decreasing?

**9.** **(a)** The function $f$ is given in Figure 2.13. At which of the labeled points is $f'(x)$ positive? Negative? Zero?
- **(b)** At which labeled point is $f'$ largest? At which labeled point is $f'$ most negative?

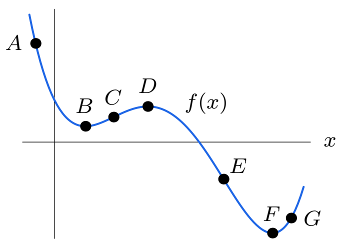



Figure 2.13

**10.** For $-3 \le x \le 7$, use a calculator or computer to graph

$$f(x) = (x^3 - 6x^2 + 8x)(2 - 3^x).$$

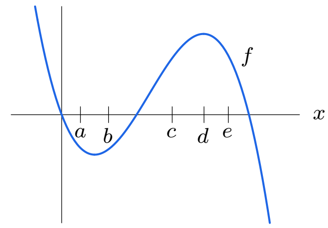
- **(a)** How many zeros does $f$ have in this interval?
- **(b)** Is $f$ increasing or decreasing at $x = 0$? At $x = 2$? At $x = 4$?
- **(c)** On which interval is the average rate of change of $f$ greater: $-1 \le x \le 0$ or $2 \le x \le 3$?
- **(d)** Is the instantaneous rate of change of $f$ greater at $x = 0$ or at $x = 2$?

**11.** Let $f(x) = 5^x$. Use a small interval to estimate $f'(2)$. Now improve your accuracy by estimating $f'(2)$ again, using an even smaller interval.

**12.** **(a)** Let $g(t) = (0.8)^t$. Use a graph to determine whether $g'(2)$ is positive, negative, or zero.
- **(b)** Use a small interval to estimate $g'(2)$.

**13.** **(a)** Use a graph of $f(x) = 2 - x^3$ to decide whether $f'(1)$ is positive or negative. Give reasons.
- **(b)** Use a small interval to estimate $f'(1)$.

**14.** Figure 2.14 shows the graph of $f$. Match the derivatives in the table with the points $a, b, c, d, e$.




| $x$ | $f'(x)$ |
|---|---|
| | 0 |
| | 0.5 |
| | 2 |
| | $-0.5$ |
| | $-2$ |

Figure 2.14

**15.** Estimate $P'(0)$ if $P(t) = 200(1.05)^t$. Explain how you obtained your answer.

**16.** For the function $f(x) = 3^x$, estimate $f'(1)$. From the graph of $f(x)$, would you expect your estimate to be greater than or less than the true value of $f'(1)$?

**17.** Table 2.3 gives $P = f(t)$, the percent of households in the US with cable television $t$ years since 1990.[3]

- **(a)** Does $f'(6)$ appear to be positive or negative? What does this tell you about the percent of households with cable television?
- **(b)** Estimate $f'(2)$. Estimate $f'(10)$. Explain what each is telling you, in terms of cable television.

**Table 2.3**

| $t$ (years since 1990) | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $P$ (% with cable) | 59.0 | 61.5 | 63.4 | 66.7 | 67.4 | 67.8 | 68.9 |

**18.** Figure 2.15 shows $N = f(t)$, the number of farms in the US[4] between 1930 and 2000 as a function of year, $t$.

- **(a)** Is $f'(1950)$ positive or negative? What does this tell you about the number of farms?
- **(b)** Which is more negative: $f'(1960)$ or $f'(1980)$? Explain.

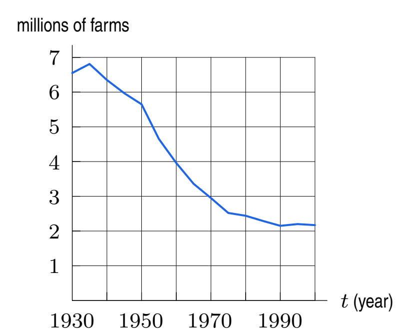



Figure 2.15

[2] *Statistical Abstracts of the US*, 1985, US Department of Commerce, Bureau of the Census, p. 22, and *World Almanac and Book of Facts 2005*, p. 624 (New York).
[3] *The World Almanac and Book of Facts 2005*, p. 310, (New York).
[4] www.nass.usda.gov:81/ipedb/farmnum.htm, accessed April 11, 2005.

**19.** Estimate the instantaneous rate of change of the function $f(x) = x \ln x$ at $x = 1$ and at $x = 2$. What do these values suggest about the concavity of the graph between 1 and 2?

**20.** Use the graph in Figure 2.7 on page 102 to decide if each of the following quantities is positive, negative or approximately zero. Illustrate your answers graphically.

**(a)** The average rate of change of $f(x)$ between $x = 3$ and $x = 7$.
**(b)** The instantaneous rate of change of $f(x)$ at $x = 3$.

**21.** Use Figure 2.16 to fill in the blanks in the following statements about the function $g$ at point $B$.

**(a)** $g(\_\_) = \_\_$ **(b)** $g'(\_\_) = \_\_$

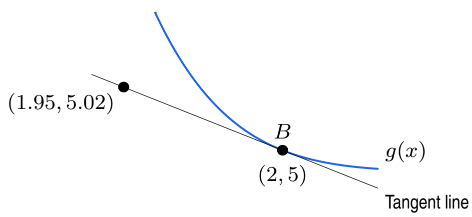



Figure 2.16

**22.** Use Figure 2.17 to fill in the blanks in the following statements about the function $f$ at point $A$.
**(a)** $f(\_\_) = \_\_$ **(b)** $f'(\_\_) = \_\_$

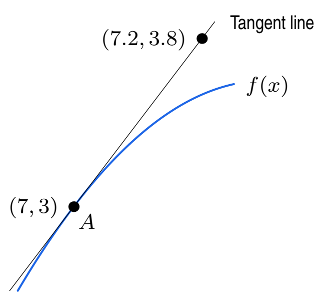



Figure 2.17

**23.** Show how to represent the following on Figure 2.18.

**(a)** $f(4)$ **(b)** $f(4) - f(2)$
**(c)** $\dfrac{f(5) - f(2)}{5 - 2}$ **(d)** $f'(3)$

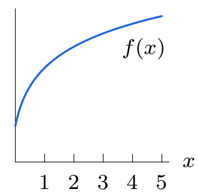



Figure 2.18

**24.** For each of the following pairs of numbers, use Figure 2.18 to decide which is larger. Explain your answer.

**(a)** $f(3)$ or $f(4)$?
**(b)** $f(3) - f(2)$ or $f(2) - f(1)$?
**(c)** $\dfrac{f(2) - f(1)}{2 - 1}$ or $\dfrac{f(3) - f(1)}{3 - 1}$?
**(d)** $f'(1)$ or $f'(4)$?

**25.** **(a)** Graph $f(x) = x^2$ and $g(x) = x^2 + 3$ on the same axes. What can you say about the slopes of the tangent lines to the two graphs at the point $x = 0$? $x = 1$? $x = 2$? $x = a$, where $a$ is any value?
**(b)** Explain why adding a constant to any function will not change the value of the derivative at any point.

**26.** The following table shows the number of hours worked in a week, $f(t)$, hourly earnings, $g(t)$, in dollars, and weekly earnings, $h(t)$, in dollars, of production workers as functions of $t$, the year.[5]

**(a)** Indicate whether each of the following derivatives is positive, negative, or zero: $f'(t)$, $g'(t)$, $h'(t)$. Interpret each answer in terms of hours or earnings.
**(b)** Estimate each of the following derivatives, and interpret your answers:
(i) $f'(1970)$ and $f'(1995)$
(ii) $g'(1970)$ and $g'(1995)$
(iii) $h'(1970)$ and $h'(1995)$

| $t$ | 1970 | 1975 | 1980 | 1985 | 1990 | 1995 | 2000 |
|---|---|---|---|---|---|---|---|
| $f(t)$ | 37.0 | 36.0 | 35.2 | 34.9 | 34.3 | 34.3 | 34.3 |
| $g(t)$ | 3.40 | 4.73 | 6.84 | 8.73 | 10.09 | 11.64 | 14.00 |
| $h(t)$ | 125.80 | 170.28 | 240.77 | 304.68 | 349.29 | 399.53 | 480.41 |

[5] *The World Almanac and Book of Facts 2005,* p. 151 (New York). Production workers includes nonsupervisory workers in mining, manufacturing, construction, transportation, public utilities, wholesale and retail trade, finance, insurance, real estate, and services.

# 2.2 THE DERIVATIVE FUNCTION

In Section 2.1 we looked at the derivative of a function at a point. In general, the derivative takes on different values at different points and is itself a function. Recall that the derivative is the slope of the tangent line to the graph at the point.

## Finding the Derivative of a Function Given Graphically

**Example 1** Estimate the derivative of the function $f(x)$ graphed in Figure 2.19 at $x = -2, -1, 0, 1, 2, 3, 4, 5$.

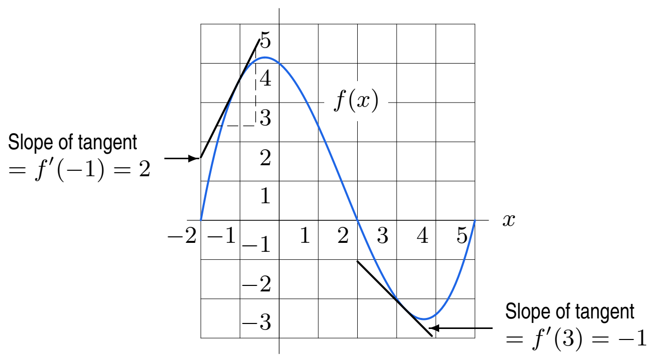



**Figure 2.19:** Estimating the derivative graphically as the slope of a tangent line

**Solution** From the graph, we estimate the derivative at any point by placing a straight edge so that it forms the tangent line at that point, and then using the grid to estimate the slope of the tangent line. For example, the tangent at $x = -1$ is drawn in Figure 2.19, and has a slope of about 2, so $f'(-1) \approx 2$. Notice that the slope at $x = -2$ is positive and fairly large; the slope at $x = -1$ is positive but smaller. At $x = 0$, the slope is negative, by $x = 1$ it has become more negative, and so on. Some estimates of the derivative, to the nearest integer, are listed in Table 2.4. You should check these values yourself. Is the derivative positive where you expect? Negative?

**Table 2.4** *Estimated values of derivative of function in Figure 2.19*

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|---|
| Derivative at $x$ | $6$ | $2$ | $-1$ | $-2$ | $-2$ | $-1$ | $1$ | $4$ |

The important point to notice is that for every $x$-value, there is a corresponding value of the derivative. The derivative, therefore, is a function of $x$.

For a function $f$, we define the **derivative function**, $f'$, by

$$f'(x) = \text{Instantaneous rate of change of } f \text{ at } x.$$

**Example 2** Plot the values of the derivative function calculated in Example 1. Compare the graphs of $f'$ and $f$.

**Solution** Graphs of $f$ and $f'$ are in Figures 2.20 and 2.21, respectively. Notice that $f'$ is positive (its graph is above the $x$-axis) where $f$ is increasing, and $f'$ is negative (its graph is below the $x$-axis) where $f$ is decreasing. The value of $f'(x)$ is 0 where $f$ has a maximum or minimum value (at approximately $x = -0.5$ and $x = 3.7$).

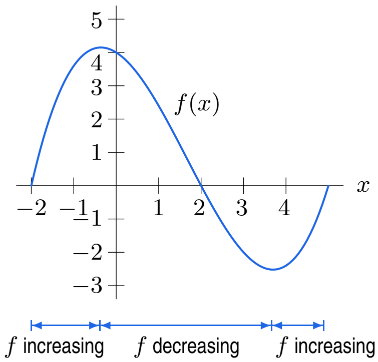



Figure 2.20: The function $f$

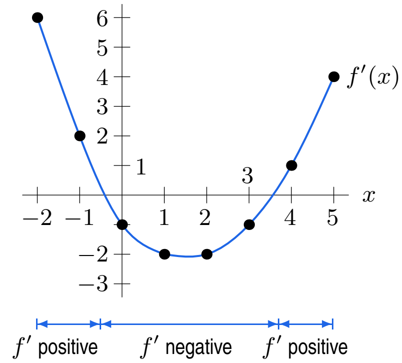



Figure 2.21: Estimates of the derivative, $f'$

**Example 3** The graph of $f$ is in Figure 2.22. Which of the graphs (a)–(c) is a graph of the derivative, $f'$?

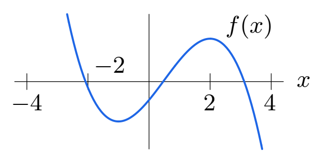
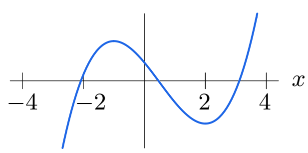
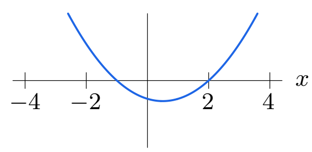



Figure 2.22

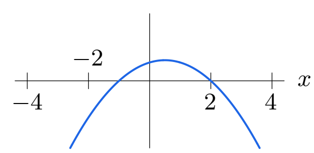
(a)



(b)



(c)



**Solution** Since the graph of $f(x)$ is horizontal at $x = -1$ and $x = 2$, the derivative is zero there. Therefore, the graph of $f'(x)$ has $x$-intercepts at $x = -1$ and $x = 2$.

The function $f$ is decreasing for $x < -1$, increasing for $-1 < x < 2$, and decreasing for $x > 2$. The derivative is positive (its graph is above the $x$-axis) where $f$ is increasing, and the derivative is negative (its graph is below the $x$-axis) where $f$ is decreasing. The correct graph is (c).

## What Does the Derivative Tell Us Graphically?

Where the derivative, $f'$, of a function is positive, the tangent to the graph of $f$ is sloping up; where $f'$ is negative, the tangent is sloping down. If $f' = 0$ everywhere, then the tangent is horizontal everywhere and so $f$ is constant. The sign of the derivative $f'$ tells us whether the function $f$ is increasing or decreasing.

> If $f' > 0$ on an interval, then $f$ is *increasing* over that interval.
> If $f' < 0$ on an interval, then $f$ is *decreasing* over that interval.
> If $f' = 0$ on an interval, then $f$ is *constant* over that interval.

The magnitude of the derivative gives us the magnitude of the rate of change of $f$. If $f'$ is large in magnitude, then the graph of $f$ is steep (up if $f'$ is positive or down if $f'$ is negative); if $f'$ is small in magnitude, the graph of $f$ is gently sloping.

## Estimating the Derivative of a Function Given Numerically

If we are given a table of function values instead of a graph of the function, we can estimate values of the derivative.

**Example 4** Table 2.5 gives values of $c(t)$, the concentration (mg/cc) of a drug in the bloodstream at time $t$ (min). Construct a table of estimated values for $c'(t)$, the rate of change of $c(t)$ with respect to $t$.

**Table 2.5** *Concentration of a drug as a function of time*

| $t$ (min) | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $c(t)$ (mg/cc) | 0.84 | 0.89 | 0.94 | 0.98 | 1.00 | 1.00 | 0.97 | 0.90 | 0.79 | 0.63 | 0.41 |

**Solution** To estimate the derivative of $c$ using the values in the table, we assume that the data points are close enough together that the concentration does not change wildly between them. From the table, we see that the concentration is increasing between $t = 0$ and $t = 0.4$, so we expect a positive derivative there. From $t = 0.5$ to $t = 1.0$, the concentration starts to decrease, and the rate of decrease gets larger and larger, so we would expect the derivative to be negative and of greater and greater magnitude.

We estimate the derivative for each value of $t$ using a difference quotient. For example,

$$c'(0) \approx \frac{c(0.1) - c(0)}{0.1 - 0} = \frac{0.89 - 0.84}{0.1} = 0.5 \text{ (mg/cc) per minute.}$$

Similarly, we get the estimates

$$c'(0.1) \approx \frac{c(0.2) - c(0.1)}{0.2 - 0.1} = \frac{0.94 - 0.89}{0.1} = 0.5$$
$$c'(0.2) \approx \frac{c(0.3) - c(0.2)}{0.3 - 0.2} = \frac{0.98 - 0.94}{0.1} = 0.4$$

and so on. These values are tabulated in Table 2.6. Notice that the derivative has small positive values up until $t = 0.4$, and then it gets more and more negative, as we expected.

**Table 2.6** *Derivative of concentration*

| $t$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $c'(t)$ | 0.5 | 0.5 | 0.4 | 0.2 | 0.0 | −0.3 | −0.7 | −1.1 | −1.6 | −2.2 |

### Improving Numerical Estimates for the Derivative

In the previous example, our estimate for the derivative of $c(t)$ at $t = 0.2$ used the point to the right. We found the average rate of change between $t = 0.2$ and $t = 0.3$. However, we could equally well have gone to the left and used the rate of change between $t = 0.1$ and $t = 0.2$ to approximate the derivative at 0.2. For a more accurate result, we could average these slopes, getting the approximation

$$c'(0.2) \approx \frac{1}{2}\left(\begin{array}{c}\text{Slope to left}\\ \text{of } 0.2\end{array} + \begin{array}{c}\text{Slope to right}\\ \text{of } 0.2\end{array}\right) = \frac{0.5 + 0.4}{2} = 0.45.$$

Each of these methods of approximating the derivative gives a reasonable answer. We will usually estimate the derivative by going to the right.

## Finding the Derivative of a Function Given by a Formula

If we are given a formula for a function $f$, can we come up with a formula for $f'$? Using the definition of the derivative, we often can. Indeed, much of the power of calculus depends on our ability to find formulas for the derivatives of all the familiar functions. This is explained in detail in Chapter 3. In the next example, we see how to guess a formula for the derivative.

**Example 5** Guess a formula for the derivative of $f(x) = x^2$.

**Solution** We use difference quotients to estimate the values of $f'(1)$, $f'(2)$, and $f'(3)$. Then we look for a pattern in these values which we use to guess a formula for $f'(x)$.

Near $x = 1$, we have

$$f'(1) \approx \frac{1.001^2 - 1^2}{0.001} = \frac{1.002 - 1}{0.001} = \frac{0.002}{0.001} = 2.$$

Similarly,

$$f'(2) \approx \frac{2.001^2 - 2^2}{0.001} = \frac{4.004 - 4}{0.001} = \frac{0.004}{0.001} = 4$$

$$f'(3) \approx \frac{3.001^2 - 3^2}{0.001} = \frac{9.006 - 9}{0.001} = \frac{0.006}{0.001} = 6.$$

Knowing the value of $f'$ at specific points cannot tell us the formula for $f'$, but it can be suggestive: knowing $f'(1) \approx 2$, $f'(2) \approx 4$, $f'(3) \approx 6$ suggests that $f'(x) = 2x$. In Chapter 3, we show that this is indeed the case.

## Problems for Section 2.2

**1.** The graph of $f(x)$ is given in Figure 2.23. Draw tangent lines to the graph at $x = -2$, $x = -1$, $x = 0$, and $x = 2$. Estimate $f'(-2)$, $f'(-1)$, $f'(0)$, and $f'(2)$.

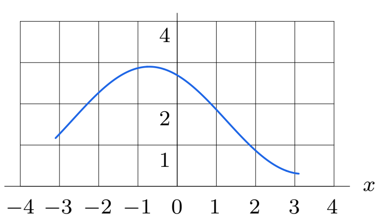
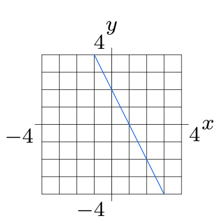


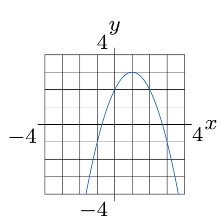

Figure 2.23

For Problems 2–7, graph the derivative of the given functions.

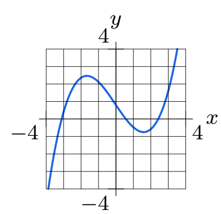
**2.**



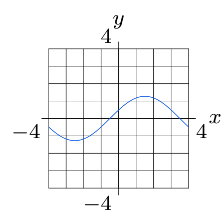

**3.**



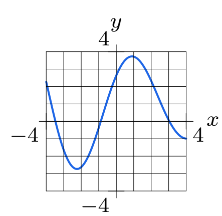

**4.**



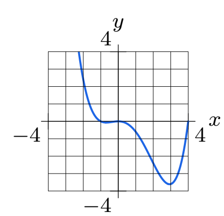

**5.**




**6.**




**7.**




**8.** A city grew in population throughout the 1980s. The population was at its largest in 1990, and then shrank throughout the 1990s. Let $P = f(t)$ represent the population of the city $t$ years since 1980. Sketch graphs of $f(t)$ and $f'(t)$, labeling the units on the axes.

Match the functions in Problems 9–12 with one of the derivatives in Figure 2.24.

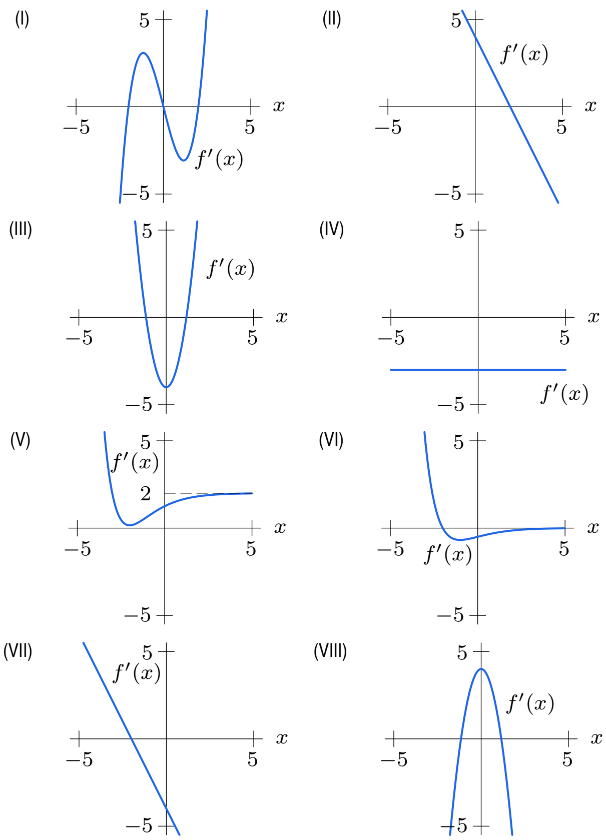
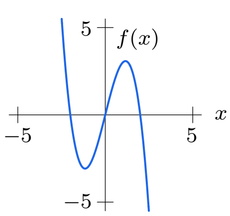
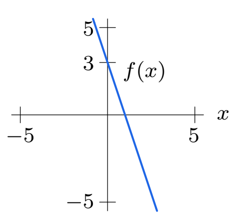



Figure 2.24

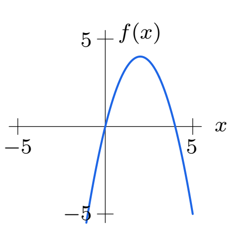
**9.**



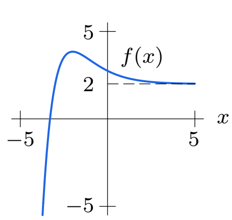

**10.**




**11.**




**12.**




**13.** In the graph of $f$ in Figure 2.25, at which of the labeled $x$-values is

**(a)** $f(x)$ greatest? **(b)** $f(x)$ least?
**(c)** $f'(x)$ greatest? **(d)** $f'(x)$ least?

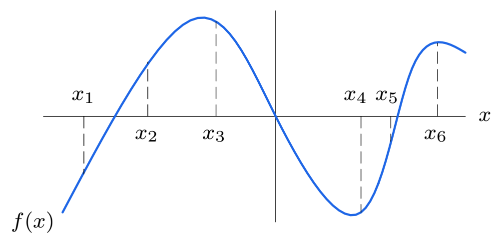



Figure 2.25

**14.** **(a)** Estimate $f'(2)$ using the values of $f$ in the table.
**(b)** For what values of $x$ does $f'(x)$ appear to be positive? Negative?

| $x$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 10 | 18 | 24 | 21 | 20 | 18 | 15 |

**15.** Find approximate values for $f'(x)$ at each of the $x$-values given in the following table.

| $x$ | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| $f(x)$ | 100 | 70 | 55 | 46 | 40 |

**16.** Draw a possible graph of $y = f(x)$ given the following information about its derivative.

- $f'(x) > 0$ for $x < -1$
- $f'(x) < 0$ for $x > -1$
- $f'(x) = 0$ at $x = -1$

**17.** Draw a possible graph of $y = f(x)$ given the following information about its derivative.

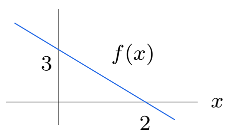
- $f'(x) > 0$ on $1 < x < 3$
- $f'(x) < 0$ for $x < 1$ and $x > 3$
- $f'(x) = 0$ at $x = 1$ and $x = 3$

**18.** Values of $x$ and $g(x)$ are given in the table. For what value of $x$ is $g'(x)$ closest to 3?

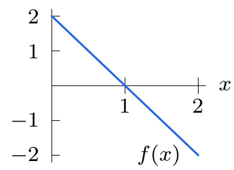
| $x$ | 2.7 | 3.2 | 3.7 | 4.2 | 4.7 | 5.2 | 5.7 | 6.2 |
|---|---|---|---|---|---|---|---|---|
| $g(x)$ | 3.4 | 4.4 | 5.0 | 5.4 | 6.0 | 7.4 | 9.0 | 11.0 |

For Problems 19–26, sketch the graph of $f'(x)$.

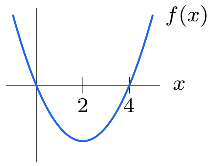
**19.**



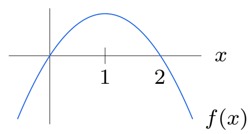

**20.**



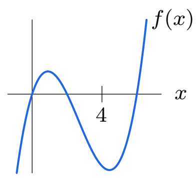

**21.**



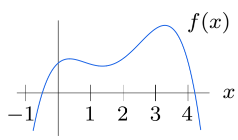

**22.**



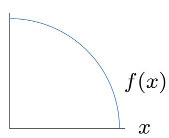

**23.**



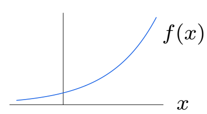

**24.**




**25.**




**26.**




**27.** **(a)** Let $f(x) = \ln x$. Use small intervals to estimate $f'(1)$, $f'(2)$, $f'(3)$, $f'(4)$, and $f'(5)$.
**(b)** Use your answers to part (a) to guess a formula for the derivative of $f(x) = \ln x$.

**28.** Suppose $f(x) = \frac{1}{3}x^3$. Estimate $f'(2)$, $f'(3)$, and $f'(4)$. What do you notice? Can you guess a formula for $f'(x)$?

**29.** Figure 2.26 is the graph of $f'$, the derivative of a function $f$. On what interval(s) is the function $f$

**(a)** Increasing? **(b)** Decreasing?

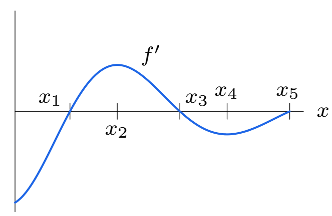



**Figure 2.26**: Graph of $f'$, not $f$

**30.** A child inflates a balloon, admires it for a while and then lets the air out at a constant rate. If $V(t)$ gives the volume of the balloon at time $t$, then Figure 2.27 shows $V'(t)$ as a function of $t$. At what time does the child:

**(a)** Begin to inflate the balloon?
**(b)** Finish inflating the balloon?
**(c)** Begin to let the air out?
**(d)** What would the graph of $V'(t)$ look like if the child had alternated between pinching and releasing the open end of the balloon, instead of letting the air out at a constant rate?

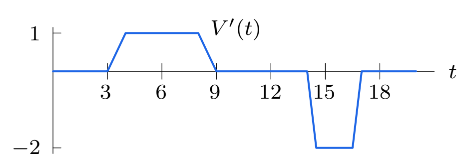



**Figure 2.27**

# 2.3 INTERPRETATIONS OF THE DERIVATIVE

We have seen the derivative interpreted as a slope and as a rate of change. In this section, we see other interpretations. The purpose of these examples is not to make a catalog of interpretations but to illustrate the process of obtaining them. There is another notation for the derivative that is often helpful.

## An Alternative Notation for the Derivative

So far we have used the notation $f'$ to stand for the derivative of the function $f$. An alternative notation for derivatives was introduced by the German mathematician Gottfried Wilhelm Leibniz (1646–1716) when calculus was first being developed. We know that $f'(x)$ is approximated by the average rate of change over a small interval. If $y = f(x)$, then the average rate of change is given by $\Delta y/\Delta x$. For small $\Delta x$, we have

$$f'(x) \approx \frac{\Delta y}{\Delta x}.$$

Leibniz's notation for the derivative, $dy/dx$, is meant to remind us of this. If $y = f(x)$, then we write
$$f'(x) = \frac{dy}{dx}.$$

Leibniz's notation is quite suggestive, especially if we think of the letter $d$ in $dy/dx$ as standing for "small difference in . . . ." The notation $dy/dx$ reminds us that the derivative is a limit of ratios of the form
$$\frac{\text{Difference in } y\text{-values}}{\text{Difference in } x\text{-values}}.$$
The notation $dy/dx$ is useful for determining the units for the derivative: the units for $dy/dx$ are the units for $y$ divided by (or "per") the units for $x$.

The separate entities $dy$ and $dx$ officially have no independent meaning: they are part of one notation. In fact, a good formal way to view the notation $dy/dx$ is to think of $d/dx$ as a single symbol meaning "the derivative with respect to $x$ of . . .". Thus, $dy/dx$ could be viewed as
$$\frac{d}{dx}(y), \quad \text{meaning "the derivative with respect to } x \text{ of } y\text{."}$$

On the other hand, many scientists and mathematicians really do think of $dy$ and $dx$ as separate entities representing "infinitesimally" small differences in $y$ and $x$, even though it is difficult to say exactly how small "infinitesimal" is. It may not be formally correct, but it is very helpful intuitively to think of $dy/dx$ as a very small change in $y$ divided by a very small change in $x$.

For example, recall that if $s = f(t)$ is the position of a moving object at time $t$, then $v = f'(t)$ is the velocity of the object at time $t$. Writing
$$v = \frac{ds}{dt}$$
reminds us that $v$ is a velocity since the notation suggests a distance, $ds$, over a time, $dt$, and we know that distance over time is velocity. Similarly, we recognize
$$\frac{dy}{dx} = f'(x)$$
as the slope of the graph of $y = f(x)$ by remembering that slope is vertical rise, $dy$, over horizontal run, $dx$.

The disadvantage of the Leibniz notation is that it is awkward to specify the $x$-value at which a derivative is evaluated. To specify $f'(2)$, for example, we have to write
$$\left.\frac{dy}{dx}\right|_{x=2}.$$

## Using Units to Interpret the Derivative

Suppose $s = f(t)$ gives the position in meters of a body from a fixed point as a function of time, $t$, in seconds. Then, knowing that
$$\frac{ds}{dt} = f'(2) = 10 \text{ meters/sec}$$
tells us that when $t = 2$ sec, the body is moving at a velocity of 10 meters/sec. If the body continues to move at this velocity for a whole second (from $t = 2$ to $t = 3$), it would move an additional 10 meters. In general:

- The units of the derivative of a function are the units of the dependent variable divided by the units of the independent variable.
- If the derivative of a function is not changing rapidly near a point, then the derivative is approximately equal to the change in the function when the independent variable increases by 1 unit.

We define the derivative of velocity, $dv/dt$, as *acceleration*.

**Example 1** If the velocity of a body at time $t$ seconds is measured in meters/sec, what are the units of the acceleration?

Solution Since acceleration, $dv/dt$, is the derivative of velocity, the units of acceleration are units of velocity divided by units of time, or (meters/sec)/sec, written meters/sec$^2$.

The following examples illustrate how useful units can be in suggesting interpretations of the derivative.

**Example 2** The cost $C$ (in dollars) of building a house $A$ square feet in area is given by the function $C = f(A)$. What is the practical interpretation of the function $f'(A)$?

Solution In the Leibniz notation,

$$f'(A) = \frac{dC}{dA}.$$

This is a cost divided by an area, so it is measured in dollars per square foot. You can think of $dC$ as the extra cost of building an extra $dA$ square feet of house. Thus, $dC/dA$ is the additional cost per square foot. So if you are planning to build a house roughly $A$ square feet in area, $f'(A)$ is the cost per square foot of the *extra* area involved in building a slightly larger house, and is called the *marginal cost*. The marginal cost is not necessarily the same thing as the average cost per square foot for the entire house, since once you are already set up to build a large house, the cost of adding a few square feet could be comparatively small.

**Example 3** The cost of extracting $T$ tons of ore from a copper mine is $C = f(T)$ dollars. What does it mean to say that $f'(2000) = 100$?

Solution In the Leibniz notation,

$$f'(2000) = \left.\frac{dC}{dT}\right|_{T=2000}.$$

Since $C$ is measured in dollars and $T$ is measured in tons, $dC/dT$ must be measured in dollars per ton. So the statement

$$\left.\frac{dC}{dT}\right|_{T=2000} = 100$$

says that when 2000 tons of ore have already been extracted from the mine, the cost of extracting the next ton is approximately \$100. Another way of saying this is that it costs about \$100 to extract the 2001$^{\text{st}}$ ton. Note that this may well be different from the cost of extracting the tenth ton, which is likely to be more accessible.

**Example 4** If $q = f(p)$ gives the number of pounds of sugar produced when the price per pound is $p$ dollars, then what are the units and the meaning of

$$\left.\frac{dq}{dp}\right|_{p=3} = 50?$$

Solution The units of $dq/dp$ are the units of $q$ over the units of $p$, or pounds/dollar. The statement

$$\left.\frac{dq}{dp}\right|_{p=3} = f'(3) = 50 \text{ pounds/dollar}$$

tells us that the rate of change of $q$ with respect to $p$ is 50 when $p = 3$. This means that when the price is \$3, the quantity produced is increasing at 50 pounds for each dollar increase in price. This is an instantaneous rate of change, meaning that if the rate were to remain 50 pounds/dollar and if the price were to increase by a whole dollar, the quantity produced would increase by $50$ pounds. In fact, the rate probably does not remain constant, so the quantity produced would probably not increase by exactly $50$ pounds.

**Example 5** The length of time, $L$, (in hours) that a drug stays in a person's system is a function of the quantity administered, $q$, in mg, so $L = f(q)$.

(a) Interpret the statement $f(10) = 6$. Give units for the numbers 10 and 6.
(b) Write the derivative of the function $L = f(q)$ in Leibniz notation. If $f'(10) = 0.5$, what are the units of the $0.5$?
(c) Interpret the statement $f'(10) = 0.5$ in terms of dose and duration.

Solution

(a) We know that $f(q) = L$. In the statement $f(10) = 6$, we have $q = 10$ and $L = 6$, so the units are 10 mg and 6 hours. The statement $f(10) = 6$ tells us that a dose of 10 mg lasts 6 hours.
(b) Since $L = f(q)$, we see that $L$ depends on $q$. The derivative of this function is $dL/dq$. Since $L$ is in hours and $q$ is in mg, the units of the derivative are hours per mg. In the statement $f'(10) = 0.5$, the $0.5$ is the derivative and the units are hours per mg.
(c) The statement $f'(10) = 0.5$ tells us that, at a dose of 10 mg, the rate of change of duration is $0.5$ hour per mg. In other words, if we increase the dose by 1 mg, the drug stays in the body approximately 30 minutes longer.

**Example 6** You are told that water is flowing through a pipe at a rate of 10 cubic feet per second. Interpret this rate as the derivative of some function.

Solution

You might think at first that the statement has something to do with the velocity of the water, but in fact a flow rate of 10 cubic feet per second could be achieved either with very slowly moving water through a large pipe, or with very rapidly moving water through a narrow pipe. If we look at the units—cubic feet per second—we realize that we are being given the rate of change of a quantity measured in cubic feet. But a cubic foot is a measure of volume, so we are being told the rate of change of a volume. If you imagine all the water that is flowing through ending up in a tank somewhere and let $V(t)$ be the volume of the tank at time $t$, then we are being told that the rate of change of $V(t)$ is 10, or that

$$V'(t) = \frac{dV}{dt} = 10.$$

## Using the Derivative to Estimate Values of a Function

Since the derivative tells us how fast the value of a function is changing, we can use the derivative at a point to estimate values of the function at nearby points.

**Example 7** The number of new subscriptions to a newspaper, $y$, in a month is a function of the amount, $x$, in dollars spent on advertising in that month, so $y = f(x)$.

(a) Interpret the statements $f(250) = 180$ and $f'(250) = 2$.
(b) Use the statements given in part (a) to estimate $f(251)$ and $f(260)$. Which estimate is more reliable?

Solution

(a) The statement $f(250) = 180$ tells us that $y = 180$ when $x = 250$. This means that if \$250 a month is spent on advertising, there are 180 new subscriptions a month. Since the derivative is $dy/dx$, the statement $f'(250) = 2$ tells us that

$$\frac{dy}{dx} = 2 \quad \text{when } x = 250.$$

This means that if the amount spent on advertising is \$250 and increases by \$1, the number of new subscriptions will go up by about 2.

(b) The statement $f(250) = 180$ says that when \$250 is spent on advertising, there are 180 new subscriptions. The statement $f'(250) = 2$ means that the number of new subscriptions increases at a rate of 2 subscriptions per additional dollar spent on advertising. If one more dollar is spent on advertising (so $x = 251$), we expect 2 more subscriptions in addition to the 180, so

$$f(251) \approx 180 + 2 = 182.$$

Similarly, if 10 dollars more were spent on advertising (so $x = 260$), we expect about $10(2) = 20$ new subscriptions, so

$$f(260) \approx 180 + 10(2) = 200.$$

Note that to estimate $f(260)$, we have to assume that the rate of 2 new subscriptions for each additional dollar continues all the way from $x = 250$ to $x = 260$. This means that the estimate of $f(251)$ is more reliable.

In Example 7, representing the change in $y$ by $\Delta y$ and the change in $x$ by $\Delta x$, we used the following result:

**Local Linear Approximation**

$$\Delta y \approx f'(x)\Delta x \qquad \text{for } \Delta x \text{ near } 0.$$

**Example 8** Climbing health care costs have been a source of concern for some time. Use the data[6] in Table 2.7 to estimate average (per consumer unit) expenditures in 2005 and 2020.

**Table 2.7** *Average yearly health care costs (per consumer unit) for various years since 1990*

| Year | 1990 | 1995 | 1998 | 2000 | 2002 |
|---|---|---|---|---|---|
| Per capita expenditure (\$) | 1480 | 1732 | 1903 | 2066 | 2350 |

Solution Health care costs increased throughout the period shown. Between 2000 and 2002, they increased $(2350 - 2066)/2 = \$142$ per year. To make estimates beyond 2002 we assume that costs continue to climb at the same rate. Therefore, we estimate

$$\begin{aligned} \text{Costs in 2005} &= \text{Costs in 2002} + \text{Change in costs} \\ &\approx \$2350 + \$142 \cdot 3 = \$2776. \end{aligned}$$

Since 2020 is 18 years beyond 2002,

$$\text{Costs in 2020} \approx \$2350 + \$142 \cdot 18 = \$4906.$$

The estimate for 2005 in the preceding example is much more likely to be close to the true value than the estimate for 2020. The further we extrapolate from the given data, the less accurate we are likely to be. It is unlikely that the rate of change of health care costs will stay at \$142/year until 2020.

Graphically, what we have done is to extend the line joining the points for 2000 and 2002 to make projections for the future. See Figure 2.28. You might be concerned that we used only the last two pieces of data to make the estimates. Isn't there valuable information to be gained from the rest of the data? Yes, indeed—though there's no fixed way of taking this information into account. You might look at the rate of change for the years before 2000 and take an average, or you might use linear or exponential regression.

[6] *Statistical Abstracts of the United States 2004–2005*, Table 125.

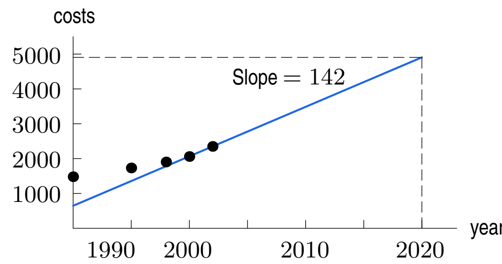



**Figure 2.28:** Graph of health care costs

## Problems for Section 2.3

**1.** The average weight $W$ of an oak tree in kilograms that is $x$ meters tall is given by the function $W = f(x)$. What are the units of measurement of $f'(x)$?

**2.** Figure 2.29 shows world solar energy output, in megawatts, as a function of years since 1990.[7] Estimate $f'(6)$. Give units and interpret your answer.

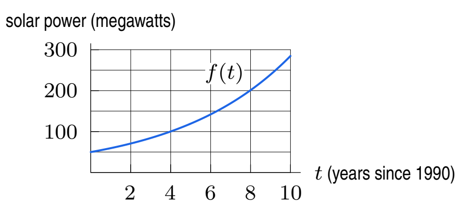



**Figure 2.29**

**3.** The cost, $C = f(w)$, in dollars of buying a chemical is a function of the weight bought, $w$, in pounds.

**(a)** In the statement $f(12) = 5$, what are the units of the 12? What are the units of the 5? Explain what this is saying about the cost of buying the chemical.
**(b)** Do you expect the derivative $f'$ to be positive or negative? Why?
**(c)** In the statement $f'(12) = 0.4$, what are the units of the 12? What are the units of the 0.4? Explain what this is saying about the cost of buying the chemical.

**4.** The cost, $C$ (in dollars) to produce $g$ gallons of ice cream can be expressed as $C = f(g)$. Using units, explain the meaning of the following statements in terms of ice cream.

**(a)** $f(200) = 350$ **(b)** $f'(200) = 1.4$

**5.** The time for a chemical reaction, $T$ (in minutes), is a function of the amount of catalyst present, $a$ (in milliliters), so $T = f(a)$.

**(a)** If $f(5) = 18$, what are the units of 5? What are the units of 18? What does this statement tell us about the reaction?
**(b)** If $f'(5) = -3$, what are the units of 5? What are the units of $-3$? What does this statement tell us?

**6.** The percent, $P$, of US households with a personal computer is a function of the number of years, $t$, since 1982 (when the percent was essentially zero), so $P = f(t)$. Interpret the statements $f(20) = 57$ and $f'(20) = 3$.

**7.** A yam has just been taken out of the oven and is cooling off before being eaten. The temperature, $T$, of the yam (measured in degrees Fahrenheit) is a function of how long it has been out of the oven, $t$ (measured in minutes). Thus, we have $T = f(t)$.

**(a)** Is $f'(t)$ positive or negative? Why?
**(b)** What are the units for $f'(t)$?

**8.** The quantity sold, $q$, of a certain product is a function of the price, $p$, so $q = f(p)$. Interpret each of the following statements in terms of demand for the product:

**(a)** $f(15) = 200$ **(b)** $f'(15) = -25$.

**9.** The weight, $W$, in lbs, of a child is a function of its age, $a$, in years, so $W = f(a)$.

**(a)** Do you expect $f'(a)$ to be positive or negative? Why?
**(b)** What does $f(8) = 45$ tell you? Give units for the numbers 8 and 45.
**(c)** What are the units of $f'(a)$? Explain what $f'(a)$ tells you in terms of age and weight.
**(d)** What does $f'(8) = 4$ tell you about age and weight?
**(e)** As $a$ increases, do you expect $f'(a)$ to increase or decrease? Explain.

**10.** The thickness, $P$, in mm, of pelican eggshells depends on the concentration, $c$, of PCBs in the eggshell, measured in ppm (parts per million); that is, $P = f(c)$.

**(a)** The derivative $f'(c)$ is negative. What does this tell you?
**(b)** Give units and interpret $f(200) = 0.28$ and $f'(200) = -0.0005$ in terms of PCBs and eggs.

[7] The Worldwatch Institute, *Vital Signs* 2001, p. 47, (New York: W.W. Norton, 2001).

Problems 11–14 concern $g(t)$ in Figure 2.30, which gives the weight of a human fetus as a function of its age.

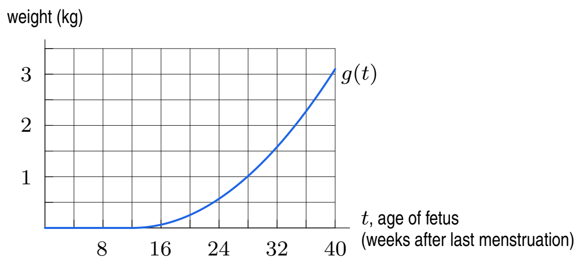



Figure 2.30

**11.** **(a)** What are the units of $g'(24)$?
**(b)** What is the biological meaning of $g'(24) = 0.096$?

**12.** **(a)** Which is greater, $g'(20)$ or $g'(36)$?
**(b)** What does your answer say about fetal growth?

**13.** Is the instantaneous weight growth rate greater or less than the average rate of change of weight over the 40 week period

**(a)** At week 20? **(b)** At week 36?

**14.** Estimate **(a)** $g'(20)$ **(b)** $g'(36)$
**(c)** The average rate of change of weight for the entire 40 week gestation.

**15.** The wind speed $W$ in meters per second at a distance $x$ kilometers from the center of a hurricane is given by the function $W = h(x)$. What does the fact that $h'(15) > 0$ tell you about the hurricane?

**16.** You drop a rock from a high tower. After it falls $x$ meters its speed $S$ in meters per second is $S = h(x)$. What is the meaning of $h'(20) = 0.5$ ?

**17.** If $t$ is the number of years since 2003, the population, $P$, of China, in billions, can be approximated by the function

$$P = f(t) = 1.291(1.006)^t.$$

Estimate $f(6)$ and $f'(6)$, giving units. What do these two numbers tell you about the population of China?

**18.** Figure 2.31 shows the length, $L$, in cm, of a sturgeon (a type of fish) as a function of the time, $t$, in years.[8] Estimate $f'(10)$. Give units and interpret your answer.

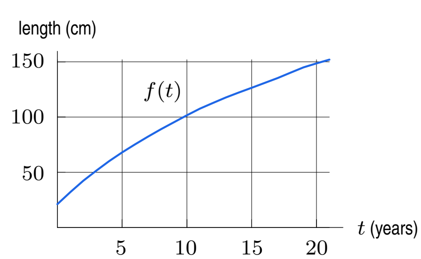



Figure 2.31

**19.** After investing \$1000 at an annual interest rate of 7% compounded continuously for $t$ years, your balance is \$$B$, where $B = f(t)$. What are the units of $dB/dt$? What is the financial interpretation of $dB/dt$?

**20.** For some painkillers, the size of the dose, $D$, given depends on the weight of the patient, $W$. Thus, $D = f(W)$, where $D$ is in milligrams and $W$ is in pounds.

**(a)** Interpret the statements $f(140) = 120$ and $f'(140) = 3$ in terms of this painkiller.
**(b)** Use the information in the statements in part (a) to estimate $f(145)$.

**21.** For a function $f(x)$, we know that $f(20) = 68$ and $f'(20) = -3$. Estimate $f(21)$, $f(19)$ and $f(25)$.

**22.** Suppose that $f(x)$ is a function with $f(20) = 345$ and $f'(20) = 6$. Estimate $f(22)$.

**23.** The quantity, $Q$ mg, of nicotine in the body $t$ minutes after a cigarette is smoked is given by $Q = f(t)$.

**(a)** Interpret the statements $f(20) = 0.36$ and $f'(20) = -0.002$ in terms of nicotine. What are the units of the numbers 20, 0.36, and $-0.002$?
**(b)** Use the information given in part (a) to estimate $f(21)$ and $f(30)$. Justify your answers.

**24.** Table 2.8 shows world gold production,[9] $G = f(t)$, as a function of year, $t$.

**(a)** Does $f'(t)$ appear to be positive or negative? What does this mean in terms of gold production?
**(b)** In which time interval does $f'(t)$ appear to be greatest?
**(c)** Estimate $f'(2002)$. Give units and interpret your answer in terms of gold production.
**(d)** Use the estimated value of $f'(2002)$ to estimate $f(2003)$ and $f(2010)$, and interpret your answers.

**Table 2.8** *World gold production*

| $t$ (year) | 1990 | 1993 | 1996 | 1999 | 2002 |
|---|---|---|---|---|---|
| $G$ (mn troy ounces) | 70.2 | 73.3 | 73.6 | 82.6 | 82.9 |

[8] Data from von Bertalanffy, L., *General System Theory*, p. 177, (New York: Braziller, 1968).
[9] *The World Almanac and Book of Facts 2005*, p. 135, (New York).

**25.** Figure 2.32 shows how the contraction velocity, $v(x)$, of a muscle changes as the load on it changes.

**(a)** Find the slope of the line tangent to the graph of contraction velocity at a load of 2 kg. Give units.
**(b)** Using your answer to part (a), estimate the change in the contraction velocity if the load is increased from 2 kg by adding 50 grams.
**(c)** Express your answer to part (a) as a derivative of $v(x)$.

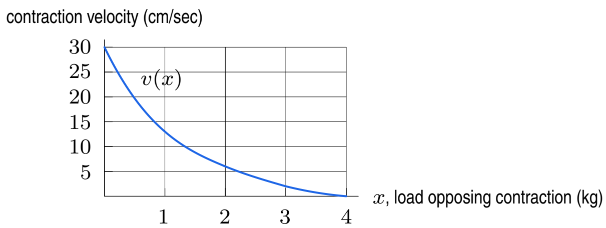



Figure 2.32

**26.** Suppose $C(r)$ is the total cost of paying off a car loan borrowed at an annual interest rate of $r\%$. What are the units of $C'(r)$? What is the practical meaning of $C'(r)$? What is its sign?

**27.** A climber on Mount Everest is 6000 meters from the start of his trail and at elevation 8000 meters above sea level. At $x$ meters from the start, the elevation of the trail is $h(x)$ meters above sea level. If $h'(x) = 0.5$ for $x$ near 6000, what is the approximate elevation another 3 meters along the trail?

**28.** Let $f(v)$ be the gas consumption (in liters/km) of a car going at velocity $v$ (in km/hr). In other words, $f(v)$ tells you how many liters of gas the car uses to go one kilometer at velocity $v$. Explain what the following statements tell you about gas consumption:

$$f(80) = 0.05 \quad \text{and} \quad f'(80) = 0.0005.$$

**29.** Figure 2.33 shows how the pumping rate of a person's heart changes after bleeding.

**(a)** Find the slope of the line tangent to the graph at time 2 hours. Give units.
**(b)** Using your answer to part (a), estimate how much the pumping rate increases during the minute beginning at time 2 hours.
**(c)** Express your answer to part (a) as a derivative of $g(t)$.

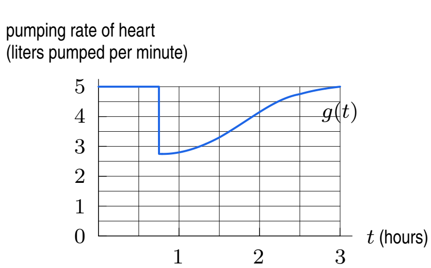



Figure 2.33

**30.** To study traffic flow, a city installs a device which records $C(t)$, the total number of cars that have passed by $t$ hours after 4:00 am. The graph of $C(t)$ is in Figure 2.34.

**(a)** When is the traffic flow the greatest?
**(b)** Estimate $C'(2)$.
**(c)** What does $C'(2)$ mean in practical terms?

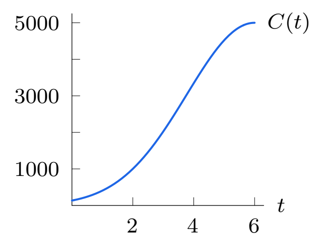



Figure 2.34

Problems 31–35 refer to Figure 2.35, which shows the depletion of food stores in the human body during starvation.

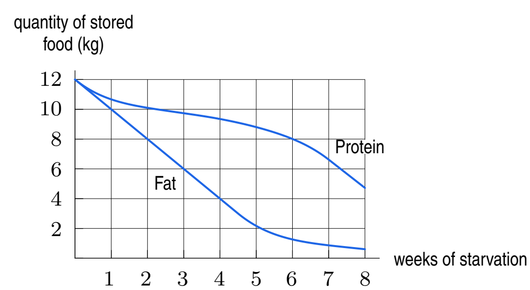



Figure 2.35

**31.** Which is being consumed at a greater rate, fat or protein, during the

**(a)** Third week? **(b)** Seventh week?

**32.** The fat storage graph is linear for the first four weeks. What does this tell you about the use of stored fat?

**33.** Estimate the rate of fat consumption after

**(a)** 3 weeks **(b)** 6 weeks **(c)** 8 weeks

**34.** What seems to happen during the sixth week? Why do you think this happens?

**35.** Figure 2.36 shows the derivatives of the protein and fat storage functions. Which graph is which?

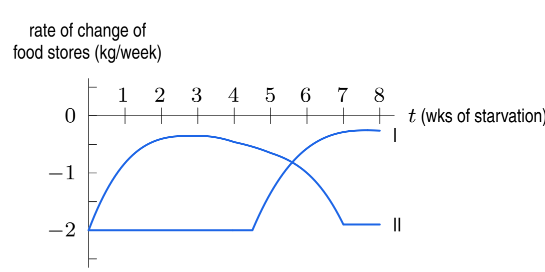



Figure 2.36

**36.** The table[10] shows $f(t)$, total sales of music compact discs (CDs), in millions, and $g(t)$, total sales of music cassettes, in millions, as a function of year $t$.

**(a)** Estimate $f'(2002)$ and $g'(2002)$. Give units with your answers and interpret each answer in terms of sales of CDs or cassettes.
**(b)** Use $f'(2002)$ to estimate $f(2003)$ and $f(2010)$. Interpret your answers in terms of sales of CDs.

| Year, $t$ | 1994 | 1996 | 1998 | 2000 | 2002 |
|---|---|---|---|---|---|
| CD sales, $f(t)$ | 662.1 | 778.9 | 847.0 | 942.5 | 803.3 |
| Cassette sales, $g(t)$ | 345.4 | 225.3 | 158.5 | 76.0 | 31.1 |

**37.** When you breathe, a muscle (called the diaphragm) reduces the pressure around your lungs and they expand to fill with air. The table shows the volume of a lung as a function of the reduction in pressure from the diaphragm. Pulmonologists (lung doctors) define the *compliance* of the lung as the derivative of this function.[11]

**(a)** What are the units of compliance?
**(b)** Estimate the maximum compliance of the lung.
**(c)** Explain why the compliance gets small when the lung is nearly full (around 1 liter).

| Pressure reduction (cm of water) | Volume (liters) |
|---|---|
| 0 | 0.20 |
| 5 | 0.29 |
| 10 | 0.49 |
| 15 | 0.70 |
| 20 | 0.86 |
| 25 | 0.95 |
| 30 | 1.00 |

**38.** A person with a certain liver disease first exhibits larger and larger concentrations of certain enzymes (called SGOT and SGPT) in the blood. As the disease progresses, the concentration of these enzymes drops, first to the predisease level and eventually to zero (when almost all of the liver cells have died). Monitoring the levels of these enzymes allows doctors to track the progress of a patient with this disease. If $C = f(t)$ is the concentration of the enzymes in the blood as a function of time,

**(a)** Sketch a possible graph of $C = f(t)$.
**(b)** Mark on the graph the intervals where $f' > 0$ and where $f' < 0$.
**(c)** What does $f'(t)$ represent, in practical terms?

# 2.4 THE SECOND DERIVATIVE

Since the derivative is itself a function, we can calculate its derivative. For a function $f$, the derivative of its derivative is called the *second derivative*, and written $f''$. If $y = f(x)$, the second derivative can also be written as $\dfrac{d^2y}{dx^2}$, which means $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$, the derivative of $\dfrac{dy}{dx}$.

## What Does the Second Derivative Tell Us?

Recall that the derivative of a function tells us whether the function is increasing or decreasing:

If $f' > 0$ on an interval, then $f$ is increasing over that interval.
If $f' < 0$ on an interval, then $f$ is decreasing over that interval.

Since $f''$ is the derivative of $f'$, we have

If $f'' > 0$ on an interval, then $f'$ is increasing over that interval.
If $f'' < 0$ on an interval, then $f'$ is decreasing over that interval.

So the question becomes: What does it mean for $f'$ to be increasing or decreasing? The case in which $f'$ is increasing is shown in Figure 2.37, where the graph of $f$ is bending upward, or is *concave up*. In the case when $f'$ is decreasing, shown in Figure 2.38, the graph is bending downward, or is *concave down*.

$f'' > 0$ on an interval means $f'$ is increasing, so the graph of $f$ is concave up there.
$f'' < 0$ on an interval means $f'$ is decreasing, so the graph of $f$ is concave down there.

[10] *The World Almanac and Book of Facts 2005*, p. 309 (New York).
[11] Adapted from John B. West, *Respiratory Physiology* 4th Ed. (New York: Williams and Wilkins, 1990).

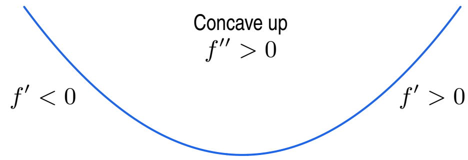



**Figure 2.37**: Meaning of $f''$: The slope increases from negative to positive as you move from left to right, so $f''$ is positive and $f$ is concave up

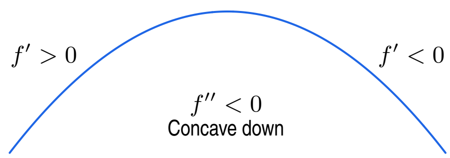



**Figure 2.38**: Meaning of $f''$: The slope decreases from positive to negative as you move from left to right, so $f''$ is negative and $f$ is concave down

**Example 1** For the functions whose graphs are given in Figure 2.39, decide where their second derivatives are positive and where they are negative.

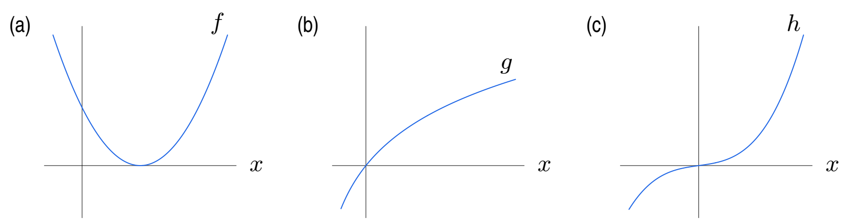



**Figure 2.39**: What signs do the second derivatives have?

Solution From the graphs it appears that

(a) $f'' > 0$ everywhere, because the graph of $f$ is concave up everywhere.
(b) $g'' < 0$ everywhere, because the graph is concave down everywhere.
(c) $h'' > 0$ for $x > 0$, because the graph of $h$ is concave up there; $h'' < 0$ for $x < 0$, because the graph of $h$ is concave down there.

## Interpretation of the Second Derivative as a Rate of Change

If we think of the derivative as a rate of change, then the second derivative is a rate of change of a rate of change. If the second derivative is positive, the rate of change is increasing; if the second derivative is negative, the rate of change is decreasing.

The second derivative is often a matter of practical concern. In 1985 a newspaper headline reported the Secretary of Defense as saying that Congress and the Senate had cut the defense budget. As his opponents pointed out, however, Congress had merely cut the rate at which the defense budget was increasing.[12] In other words, the derivative of the defense budget was still positive (the budget was increasing), but the second derivative was negative (the budget's rate of increase had slowed).

[12] In the *Boston Globe*, March 13, 1985, Representative William Gray (D–Pa.) was reported as saying: "It's confusing to the American people to imply that Congress threatens national security with reductions when you're really talking about a reduction in the increase."

**Example 2** A population, $P$, growing in a confined environment often follows a *logistic* growth curve, like the graph shown in Figure 2.40. Describe how the rate at which the population is increasing changes over time. What is the sign of the second derivative $d^2P/dt^2$? What is the practical interpretation of $t^*$ and $L$?

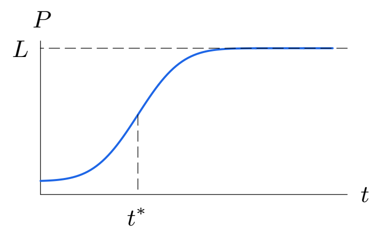



**Figure 2.40:** Logistic growth curve

**Solution** Initially, the population is increasing, and at an increasing rate. So, initially $dP/dt$ is increasing and $d^2P/dt^2 > 0$. At $t^*$, the rate at which the population is increasing is a maximum; the population is growing fastest then. Beyond $t^*$, the rate at which the population is growing is decreasing, so $d^2P/dt^2 < 0$. At $t^*$, the graph changes from concave up to concave down and $d^2P/dt^2 = 0$.

The quantity $L$ represents the limiting value of the population that is approached as $t$ tends to infinity; $L$ is called the *carrying capacity* of the environment and represents the maximum population that the environment can support.

**Example 3** Table 2.9 shows the number of abortions per year, $A$, reported in the US[13] in the year $t$.

**Table 2.9** *Abortions reported in the US (1972–2000)*

| Year, $t$ | 1972 | 1975 | 1980 | 1985 | 1990 | 1995 | 2000 |
|---|---|---|---|---|---|---|---|
| Thousands of abortions reported, $A$ | 587 | 1034 | 1554 | 1589 | 1609 | 1359 | 1313 |

(a) Calculate the average rate of change for the time intervals shown between 1972 and 2000.
(b) What can you say about the sign of $d^2A/dt^2$ during the period 1972–1995?

**Solution** (a) For each time interval we can calculate the average rate of change of the number of abortions per year over this interval. For example, between 1972 and 1975

$$\begin{array}{c}\text{Average rate}\\ \text{of change}\end{array} = \frac{\Delta A}{\Delta t} = \frac{1034 - 587}{1975 - 1972} = \frac{447}{3} \approx 149.$$

Thus, between 1972 and 1975, there were approximately 149,000 more abortions reported each year. Values of $\Delta A/\Delta t$ are listed in Table 2.10:

**Table 2.10** *Rate of change of number of abortions reported*

| Time | 1972–1975 | 1975–1980 | 1980–1985 | 1985–1990 | 1990–1995 | 1995–2000 |
|---|---|---|---|---|---|---|
| Average rate of change, $\Delta A/\Delta t$ (1000s/year) | 149 | 104 | 7 | 4 | −50 | −9.2 |

[13] *Statistical Abstracts of the United States 2004–2005*, Table 89.

(b) We assume the data lies on a smooth curve. Since the values of $\Delta A/\Delta t$ are decreasing dramatically for 1975–1995, we can be pretty certain that $dA/dt$ also decreases, so $d^2A/dt^2$ is negative for this period. For 1972–1975, the sign of $d^2A/dt^2$ is less clear; abortion data from 1968 would help. Figure 2.41 confirms this; the graph appears to be concave down for 1975–1995. The fact that $dA/dt$ is positive during the period 1972–1980 tells us that the number of abortions reported increased from 1972 to 1980. The fact that $dA/dt$ is negative during the period 1990–2000 tells us that the number of abortions reported decreased from 1990 to 2000. The fact that $d^2A/dt^2$ is negative for 1975–1995 tells us that the rate of increase slowed over this period.

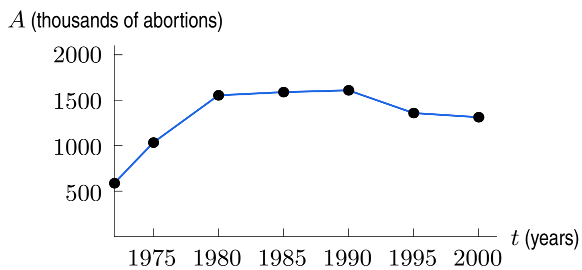



**Figure 2.41:** How the number of reported abortions in the US is changing with time

## Problems for Section 2.4

**1.** For the function graphed in Figure 2.42, are the following nonzero quantities positive or negative?

**(a)** $f(2)$ **(b)** $f'(2)$ **(c)** $f''(2)$

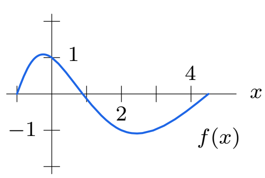
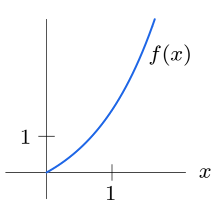


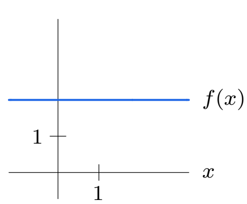

**Figure 2.42**

For Problems 2–7, give the signs of the first and second derivatives for each of the following functions. Each derivative is either positive everywhere, zero everywhere, or negative everywhere.

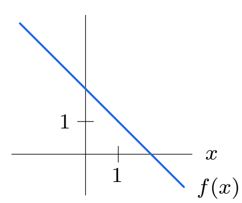
**2.**



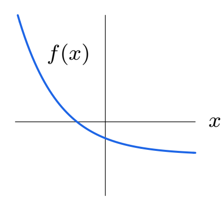

**3.**



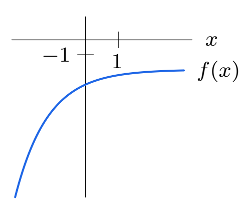

**4.**



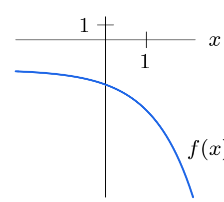

**5.**




**6.**




**7.**




In Problems 8–9, use the values given for each function.

**(a)** Does the derivative of the function appear to be positive or negative over the given interval? Explain.

**(b)** Does the second derivative of the function appear to be positive or negative over the given interval? Explain.

**8.**

| $t$ | 100 | 110 | 120 | 130 | 140 |
|---|---|---|---|---|---|
| $w(t)$ | 10.7 | 6.3 | 4.2 | 3.5 | 3.3 |

**9.**

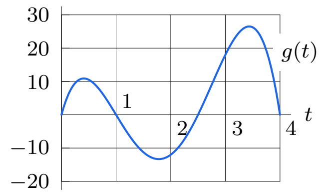
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $s(t)$ | 12 | 14 | 17 | 20 | 31 | 55 |

In Problems 10–11, use the graph given for each function.

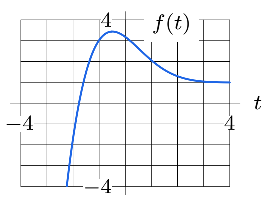
**(a)** Estimate the intervals on which the derivative is positive and the intervals on which the derivative is negative.

**(b)** Estimate the intervals on which the second derivative is positive and the intervals on which the second derivative is negative.

**10.**




**11.**



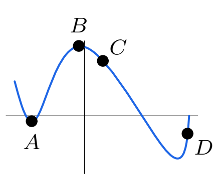

**12.** **(a)** Graph a function whose first and second derivatives are everywhere positive.

**(b)** Graph a function whose second derivative is everywhere negative but whose first derivative is everywhere positive.

**(c)** Graph a function whose second derivative is everywhere positive but whose first derivative is everywhere negative.

**(d)** Graph a function whose first and second derivatives are everywhere negative.

**13.** At exactly two of the labeled points in Figure 2.43, the derivative $f'$ is 0; the second derivative $f''$ is not zero at any of the labeled points. On a copy of the table, give the signs of $f$, $f'$, $f''$ at each marked point.




| Point | $f$ | $f'$ | $f''$ |
|---|---|---|---|
| $A$ | | | |
| $B$ | | | |
| $C$ | | | |
| $D$ | | | |

**Figure 2.43**

**14.** The length $L$ of the day in minutes (sunrise to sunset) $x$ kilometers north of the equator on June 21 is given by $L = f(x)$. What are the units of

**(a)** $f'(3000)$? **(b)** $f''(3000)$?

**15.** For three minutes the temperature of a feverish person has had positive first derivative and negative second derivative. Which of the following is correct?

**(a)** The temperature rose in the last minute more than it rose in the minute before.

**(b)** The temperature rose in the last minute, but less than it rose in the minute before.

**(c)** The temperature fell in the last minute but less than it fell in the minute before.

**(d)** The temperature rose two minutes ago but fell in the last minute.

**16.** Yesterday's temperature at $t$ hours past midnight was $f(t)$ °C. At noon the temperature was 20°C. The first derivative, $f'(t)$, decreased all morning, reaching a low of 2°C/hour at noon, then increased for the rest of the day. Which one of the following must be correct?

**(a)** The temperature fell in the morning and rose in the afternoon.

**(b)** At 1 pm the temperature was 18°C.

**(c)** At 1 pm the temperature was 22°C.

**(d)** The temperature was lower at noon than at any other time.

**(e)** The temperature rose all day.

**17.** Values of $f(t)$ are given in the following table.

**(a)** Does this function appear to have a positive or negative first derivative? Second derivative? Explain.

**(b)** Estimate $f'(2)$ and $f'(8)$.

| $t$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| $f(t)$ | 150 | 145 | 137 | 122 | 98 | 56 |

**18.** Sketch the graph of a function $f$ such that $f(2) = 5$, $f'(2) = 1/2$, and $f''(2) > 0$.

**19.** Sketch a graph of a continuous function $f$ with the following properties:

- $f'(x) > 0$ for all $x$
- $f''(x) < 0$ for $x < 2$ and $f''(x) > 0$ for $x > 2$.

**20.** At which of the marked $x$-values in Figure 2.44 can the following statements be true?

**(a)** $f(x) < 0$

**(b)** $f'(x) < 0$

**(c)** $f(x)$ is decreasing

**(d)** $f'(x)$ is decreasing

**(e)** Slope of $f(x)$ is positive

**(f)** Slope of $f(x)$ is increasing

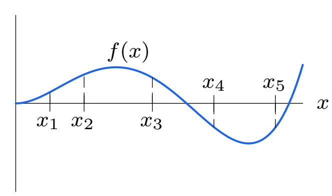



**Figure 2.44**

**21.** A function $f$ has $f(5) = 20$, $f'(5) = 2$, and $f''(x) < 0$, for $x \geq 5$. Which of the following are possible values for $f(7)$ and which are impossible?

**(a)** 26 **(b)** 24 **(c)** 22

**22.** The table gives the number of passenger cars, $C = f(t)$, in millions,[14] in the US in the year $t$.

**(a)** Do $f'(t)$ and $f''(t)$ appear to be positive or negative during the period 1940–1980?
**(b)** Estimate $f'(1975)$. Using units, interpret your answer in terms of passenger cars.

| $t$ | 1940 | 1950 | 1960 | 1970 | 1980 | 1990 | 2000 |
|---|---|---|---|---|---|---|---|
| $C$ | 27.5 | 40.3 | 61.7 | 89.2 | 121.6 | 133.7 | 133.6 |

**23.** "Winning the war on poverty" has been described cynically as slowing the rate at which people are slipping below the poverty line. Assuming that this is happening:

**(a)** Graph the total number of people in poverty against time.
**(b)** If $N$ is the number of people below the poverty line at time $t$, what are the signs of $dN/dt$ and $d^2N/dt^2$? Explain.

**24.** Let $P(t)$ represent the price of a share of stock of a corporation at time $t$. What does each of the following statements tell us about the signs of the first and second derivatives of $P(t)$?

**(a)** "The price of the stock is rising faster and faster."
**(b)** "The price of the stock is close to bottoming out."

**25.** In economics, *total utility* refers to the total satisfaction from consuming some commodity. According to the economist Samuelson:[15]

> As you consume more of the same good, the total (psychological) utility increases. However, . . . with successive new units of the good, your total utility will grow at a slower and slower rate because of a fundamental tendency for your psychological ability to appreciate more of the good to become less keen.

**(a)** Sketch the total utility as a function of the number of units consumed.
**(b)** In terms of derivatives, what is Samuelson saying?

**26.** An industry is being charged by the Environmental Protection Agency (EPA) with dumping unacceptable levels of toxic pollutants in a lake. Over a period of several months, an engineering firm makes daily measurements of the rate at which pollutants are being discharged into the lake. The engineers produce a graph similar to either Figure 2.45(a) or Figure 2.45(b). For each case, give an idea of what argument the EPA might make in court against the industry and of the industry's defense.

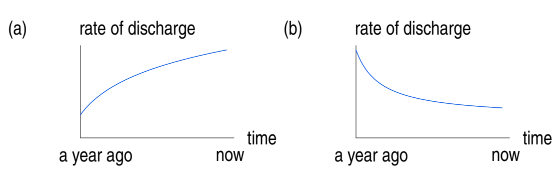



Figure 2.45

**27.** Sketch the graph of the height of a particle against time if velocity is positive and acceleration is negative.

**28.** Figure 2.46 gives the position, $f(t)$, of a particle at time $t$. At which of the marked values of $t$ can the following statements be true?

**(a)** The position is positive
**(b)** The velocity is positive
**(c)** The acceleration is positive
**(d)** The position is decreasing
**(e)** The velocity is decreasing

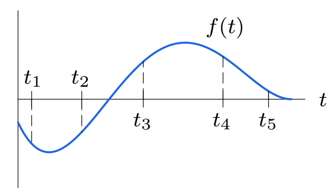



Figure 2.46

**29.** Each of the graphs in Figure 2.47 shows the position of a particle moving along the $x$-axis as a function of time, $0 \le t \le 5$. The vertical scales of the graphs are the same. During this time interval, which particle has

**(a)** Constant velocity?
**(b)** The greatest initial velocity?
**(c)** The greatest average velocity?
**(d)** Zero average velocity?
**(e)** Zero acceleration?
**(f)** Positive acceleration throughout?

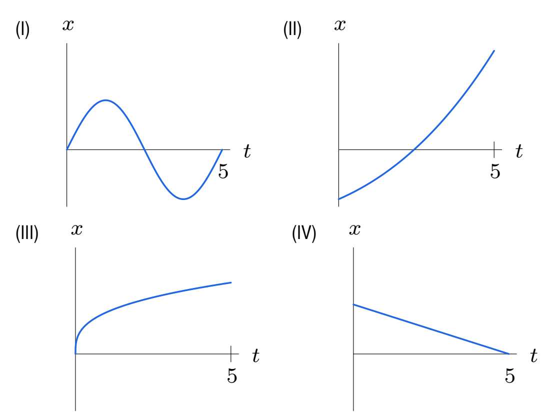



Figure 2.47

[14] *The World Almanac and Book of Facts 2005*, p. 237 (New York).
[15] From Paul A. Samuelson, *Economics*, 11th edition (New York: McGraw-Hill, 1981).

# 2.5 MARGINAL COST AND REVENUE

Management decisions within a particular firm or industry usually depend on the costs and revenues involved. In this section we look at the cost and revenue functions.

## Graphs of Cost and Revenue Functions

The graph of a cost function may be linear, as in Figure 2.48, or it may have the shape shown in Figure 2.49. The intercept on the $C$-axis represents the fixed costs, which are incurred even if nothing is produced. (This includes, for instance, the cost of the machinery needed to begin production.) In Figure 2.49, the cost function increases quickly at first and then more slowly because producing larger quantities of a good is usually more efficient than producing smaller quantities—this is called *economy of scale*. At still higher production levels, the cost function increases faster again as resources become scarce; sharp increases may occur when new factories have to be built. Thus, the graph of a cost function, $C$, may start out concave down and become concave up later on.

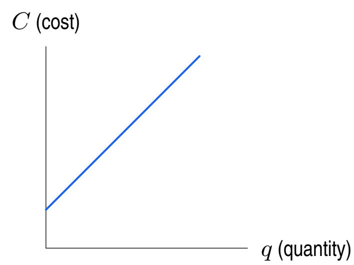



**Figure 2.48:** A linear cost function

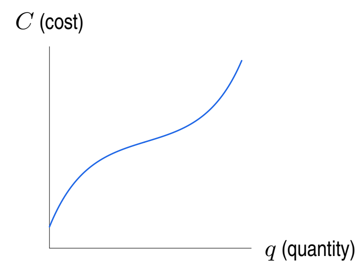



**Figure 2.49:** A nonlinear cost function

The revenue function is $R = pq$, where $p$ is price and $q$ is quantity. If the price, $p$, is a constant, the graph of $R$ against $q$ is a straight line through the origin with slope equal to the price. (See Figure 2.50). In practice, for large values of $q$, the market may become glutted, causing the price to drop and giving $R$ the shape in Figure 2.51.

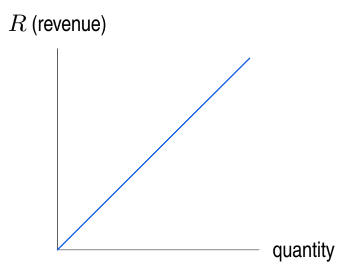



**Figure 2.50:** Revenue: Constant price

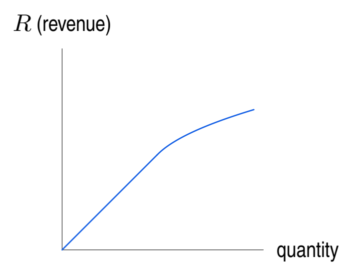



**Figure 2.51:** Revenue: Decreasing price

**Example 1** If cost, $C$, and revenue, $R$, are given by the graph in Figure 2.52, for what production quantities does the firm make a profit?

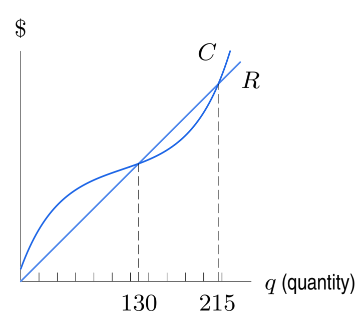



**Figure 2.52:** Costs and revenues for Example 1

Solution The firm makes a profit whenever revenues are greater than costs, that is, when $R > C$. The graph of $R$ is above the graph of $C$ approximately when $130 < q < 215$. Production between 130 units and 215 units will generate a profit.

## Marginal Analysis

Many economic decisions are based on an analysis of the costs and revenues "at the margin." Let's look at this idea through an example.

Suppose you are running an airline and you are trying to decide whether to offer an additional flight. How should you decide? We'll assume that the decision is to be made purely on financial grounds: if the flight will make money for the company, it should be added. Obviously you need to consider the costs and revenues involved. Since the choice is between adding this flight and leaving things the way they are, the crucial question is whether the *additional costs* incurred are greater or smaller than the *additional revenues* generated by the flight. These additional costs and revenues are called *marginal costs* and *marginal revenues*.

Suppose $C(q)$ is the function giving the cost of running $q$ flights. If the airline had originally planned to run 100 flights, its costs would be $C(100)$. With the additional flight, its costs would be $C(101)$. Therefore,

$$\text{Marginal cost} = C(101) - C(100).$$

Now

$$C(101) - C(100) = \frac{C(101) - C(100)}{101 - 100},$$

and this quantity is the average rate of change of cost between 100 and 101 flights. In Figure 2.53 the average rate of change is the slope of the secant line. If the graph of the cost function is not curving too fast near the point, the slope of the secant line is close to the slope of the tangent line there. Therefore, the average rate of change is close to the instantaneous rate of change. Since these rates of change are not very different, many economists choose to define marginal cost, $MC$, as the instantaneous rate of change of cost with respect to quantity:

$$\boxed{\text{Marginal cost} = MC = C'(q).}$$

Marginal cost is represented by the slope of the cost curve.

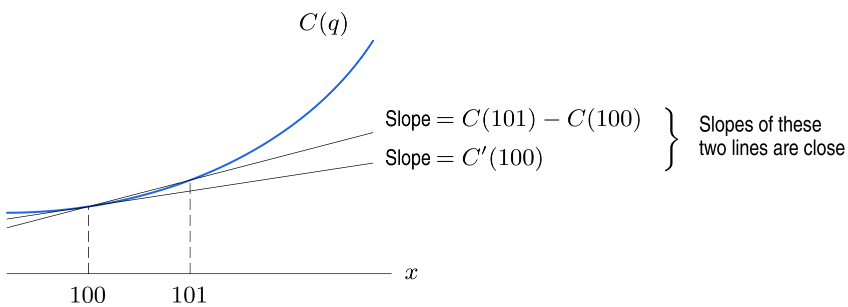



**Figure 2.53:** Marginal cost: Slope of one of these lines

Similarly, if the revenue generated by $q$ flights is $R(q)$, then the additional revenue generated by increasing the number of flights from 100 to 101 is

$$\text{Marginal revenue} = R(101) - R(100).$$

Now $R(101) - R(100)$ is the average rate of change of revenue between 100 and 101 flights. As before, the average rate of change is approximately equal to the instantaneous rate of change, so economists often define

$$\boxed{\text{Marginal revenue} = MR = R'(q).}$$

Marginal revenue is represented by the slope of the revenue curve.

**Example 2** If $C(q)$ and $R(q)$ for the airline are given in Figure 2.54, should the company add the $101^{\text{st}}$ flight?

**Solution** The marginal revenue is the slope of the revenue curve at $q = 100$. The marginal cost is the slope of the graph of $C$ at $q = 100$. Figure 2.54 suggests that the slope at the point $A$ is smaller than the slope at $B$, so $MC < MR$ for $q = 100$. This means that the airline will make more in extra revenue than it will spend in extra costs if it runs another flight, so it should go ahead and run the $101^{\text{st}}$ flight.

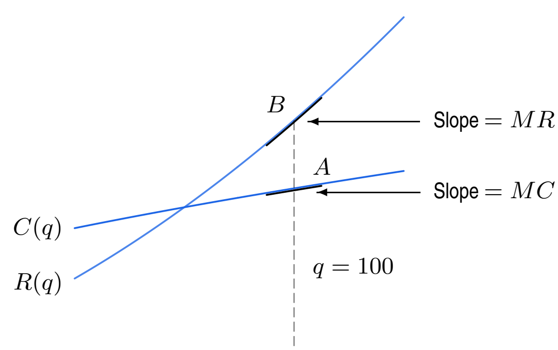



**Figure 2.54:** Cost and revenue for Example 2

**Example 3** The graph of a cost function is given in Figure 2.55. Does it cost more to produce the $500^{\text{th}}$ item or the $2000^{\text{th}}$? Does it cost more to produce the $3000^{\text{th}}$ item or the $4000^{\text{th}}$? At approximately what production level is marginal cost smallest? What is the total cost at this production level?

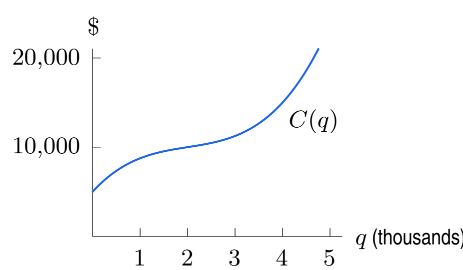



**Figure 2.55:** Estimating marginal cost: Where is marginal cost smallest?

**Solution** The cost to produce an additional item is the marginal cost, which is represented by the slope of the cost curve. Since the slope of the cost function in Figure 2.55 is greater at $q = 0.5$ (when the quantity produced is 0.5 thousand, or 500) than at $q = 2$, it costs more to produce the $500^{\text{th}}$ item than the $2000^{\text{th}}$ item. Since the slope is greater at $q = 4$ than $q = 3$, it costs more to produce the $4000^{\text{th}}$ item than the $3000^{\text{th}}$ item.

The slope of the cost function is close to zero at $q = 2$, and is positive everywhere else, so the slope is smallest at $q = 2$. The marginal cost is smallest at a production level of 2000 units. Since $C(2) \approx 10{,}000$, the total cost to produce 2000 units is about \$10,000.

**Example 4** If the revenue and cost functions, $R$ and $C$, are given by the graphs in Figure 2.56, sketch graphs of the marginal revenue and marginal cost functions, $MR$ and $MC$.


Figure 2.56: Total revenue and total cost for Example 4

Solution The revenue graph is a line through the origin, with equation

$$R = pq$$

where $p$ represents the constant price, so the slope is $p$ and

$$MR = R'(q) = p.$$

The total cost is increasing, so the marginal cost is always positive. For small $q$ values, the graph of the cost function is concave down, so the marginal cost is decreasing. For larger $q$, say $q > 100$, the graph of the cost function is concave up and the marginal cost is increasing. Thus, the marginal cost has a minimum at about $q = 100$. (See Figure 2.57.)

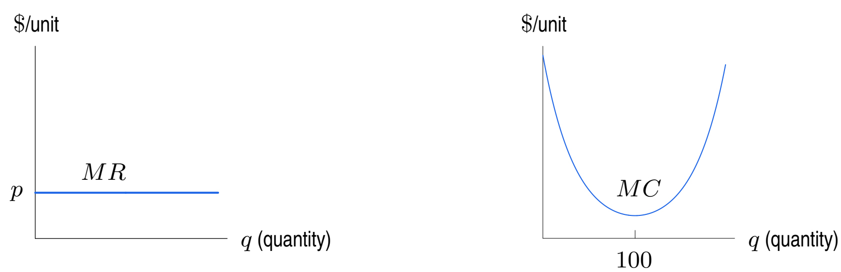



Figure 2.57: Marginal revenue and costs for Example 4

## Problems for Section 2.5

1. In Figure 2.58, estimate the marginal cost when the level of production is 10,000 units and interpret it.

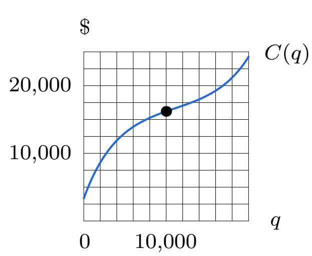



Figure 2.58

2. In Figure 2.59, estimate the marginal revenue when the level of production is 600 units and interpret it.

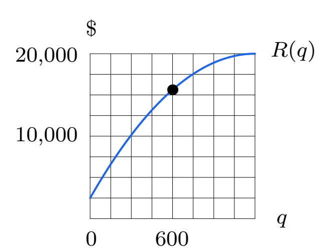



Figure 2.59

**3.** The function $C(q)$ gives the cost in dollars to produce $q$ barrels of olive oil.

**(a)** What are the units of marginal cost?
**(b)** What is the practical meaning of the statement $MC = 3$ for $q = 100$?

**4.** It costs \$4800 to produce 1295 items and it costs \$4830 to produce 1305 items. What is the approximate marginal cost at a production level of 1300 items?

**5.** In Figure 2.60, is marginal cost greater at $q = 5$ or at $q = 30$? At $q = 20$ or at $q = 40$? Explain.

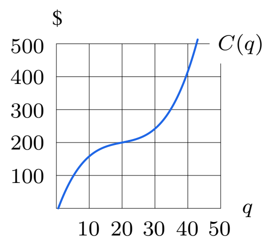



**Figure 2.60**

**6.** Figure 2.61 shows part of the graph of cost and revenue for a car manufacturer. Which is greater, marginal cost or marginal revenue, at

**(a)** $q_1$? **(b)** $q_2$?

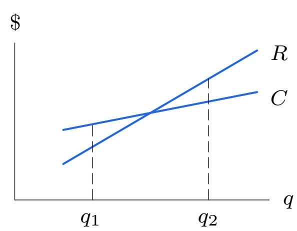



**Figure 2.61**

**7.** Let $C(q)$ represent the total cost of producing $q$ items. Suppose $C(15) = 2300$ and $C'(15) = 108$. Estimate the total cost of producing: **(a)** 16 items **(b)** 14 items.

**8.** To produce 1000 items, the total cost is \$5000 and the marginal cost is \$25 per item. Estimate the costs of producing 1001 items, 999 items, and 1100 items.

**9.** Let $C(q)$ represent the cost and $R(q)$ represent the revenue, in dollars, of producing $q$ items.

**(a)** If $C(50) = 4300$ and $C'(50) = 24$, estimate $C(52)$.
**(b)** If $C'(50) = 24$ and $R'(50) = 35$, approximately how much profit is earned by the $51^{\text{st}}$ item?
**(c)** If $C'(100) = 38$ and $R'(100) = 35$, should the company produce the $101^{\text{st}}$ item? Why or why not?

**10.** Cost and revenue functions for a charter bus company are shown in Figure 2.62. Should the company add a $50^{\text{th}}$ bus? How about a $90^{\text{th}}$? Explain your answers using marginal revenue and marginal cost.

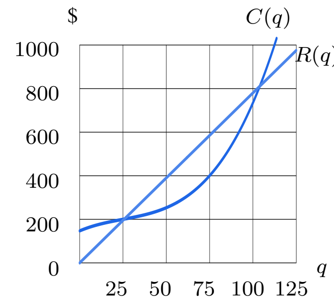



**Figure 2.62**

**11.** For $q$ units of a product, a manufacturer's cost is $C(q)$ dollars and revenue is $R(q)$ dollars, with $C(500) = 7200$, $R(500) = 9400$, $MC(500) = 15$, and $MR(500) = 20$.

**(a)** What is the profit or loss at $q = 500$?
**(b)** If production is increased from 500 to 501 units, by approximately how much does profit change?

**12.** A company's cost of producing $q$ liters of a chemical is $C(q)$ dollars; this quantity can be sold for $R(q)$ dollars. Suppose $C(2000) = 5930$ and $R(2000) = 7780$.

**(a)** What is the profit at a production level of 2000?
**(b)** If $MC(2000) = 2.1$ and $MR(2000) = 2.5$, what is the approximate change in profit if $q$ is increased from 2000 to 2001? Should the company increase or decrease production from $q = 2000$?
**(c)** If $MC(2000) = 4.77$ and $MR(2000) = 4.32$, should the company increase or decrease production from $q = 2000$?

**13.** An industrial production process costs $C(q)$ million dollars to produce $q$ million units; these units then sell for $R(q)$ million dollars. If $C(2.1) = 5.1$, $R(2.1) = 6.9$, $MC(2.1) = 0.6$, and $MR(2.1) = 0.7$, calculate

**(a)** The profit earned by producing 2.1 million units
**(b)** The change in revenue if production increases from 2.1 to 2.14 million units.
**(c)** The change in revenue if production decreases from 2.1 to 2.05 million units.
**(d)** The change in profit in parts (b) and (c).

**14.** The cost of recycling $q$ tons of paper is given in the following table. Estimate the marginal cost at $q = 2000$. Give units and interpret your answer in terms of cost. At approximately what production level does marginal cost appear smallest?

| $q$ (tons) | 1000 | 1500 | 2000 | 2500 | 3000 | 3500 |
|---|---|---|---|---|---|---|
| $C(q)$ (dollars) | 2500 | 3200 | 3640 | 3825 | 3900 | 4400 |

**15.** Let $C(q)$ be the total cost of producing a quantity $q$ of a certain product. See Figure 2.63.

**(a)** What is the meaning of $C(0)$?

**(b)** Describe in words how the marginal cost changes as the quantity produced increases.
**(c)** Explain the concavity of the graph (in terms of economics).
**(d)** Explain the economic significance (in terms of marginal cost) of the point at which the concavity changes.
**(e)** Do you expect the graph of $C(q)$ to look like this for all types of products?

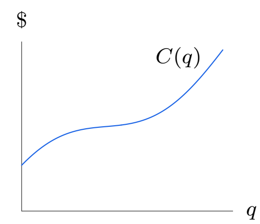



Figure 2.63

## CHAPTER SUMMARY

- **Rate of change**
  Average, instantaneous
- **Estimating derivatives**
  Estimate derivatives from a graph, table of values, or formula.
- **Interpretation of derivatives**
  Rate of change, slope, using units, instantaneous velocity.
- **Marginality**
  Marginal cost and marginal revenue
- **Second derivative**
  Concavity
- **Derivatives and graphs**
  Understand relation between sign of $f'$ and whether $f$ is increasing or decreasing. Sketch graph of $f'$ from graph of $f$. Marginal analysis.

## REVIEW PROBLEMS FOR CHAPTER TWO

**1.** For the function shown in Figure 2.64, at what labeled points is the slope of the graph positive? Negative? At which labeled point does the graph have the greatest (i.e., most positive) slope? The least slope (i.e., negative and with the largest magnitude)?

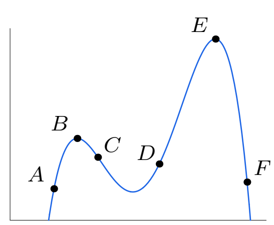



Figure 2.64

**2.** The function in Figure 2.65 has $f(4) = 25$ and $f'(4) = 1.5$. Find the coordinates of the points $A$, $B$, $C$.

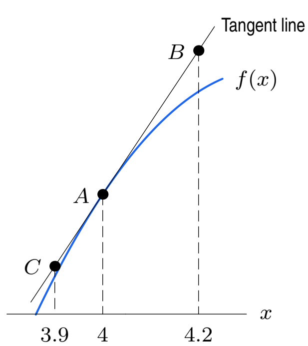



Figure 2.65

**3.** Estimate $f'(2)$ for $f(x) = 3^x$. Explain your reasoning.

**4.** In a time of $t$ seconds, a particle moves a distance of $s$ meters from its starting point, where $s = 3t^2$.

**(a)** Find the average velocity between $t = 1$ and $t = 1 + h$ if:

(i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$.

**(b)** Use your answers to part (a) to estimate the instantaneous velocity of the particle at time $t = 1$.

**5.** The population of the world reached 1 billion in 1804, 2 billion in 1927, 3 billion in 1960, 4 billion in 1974, 5 billion in 1987 and 6 billion in 1999. Find the average rate of change of the population of the world, in people per minute, during each of these intervals. [That is, from 1804 to 1927, 1927 to 1960, etc.]

**6.** In a time of $t$ seconds, a particle moves a distance of $s$ meters from its starting point, where $s = \sin(2t)$.

**(a)** Find the average velocity between $t = 1$ and $t = 1 + h$ if:

(i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$.

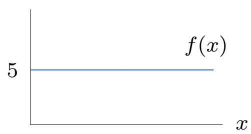
**(b)** Use your answers to part (a) to estimate the instantaneous velocity of the particle at time $t = 1$.

**7.** Given the numerical values shown, find approximate values for the derivative of $f(x)$ at each of the $x$-values given. Where is the rate of change of $f(x)$ positive? Where is it negative? Where does the rate of change of $f(x)$ seem to be greatest?

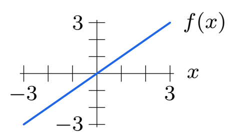
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 18 | 13 | 10 | 9 | 9 | 11 | 15 | 21 | 30 |

Sketch the graphs of the derivatives of the functions shown in Problems 8–13. Be sure your sketches are consistent with the important features of the graphs of the original functions.

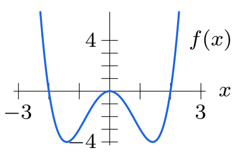
**8.**



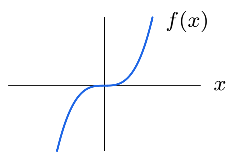

**9.**



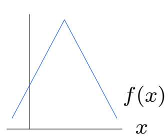

**10.**



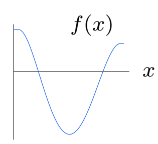

**11.**




**12.**




**13.**




**14.** A vehicle moving along a straight road has distance $f(t)$ from its starting point at time $t$. Which of the graphs in Figure 2.66 could be $f'(t)$ for the following scenarios? (Assume the scales on the vertical axes are all the same.)

**(a)** A bus on a popular route, with no traffic
**(b)** A car with no traffic and all green lights
**(c)** A car in heavy traffic conditions

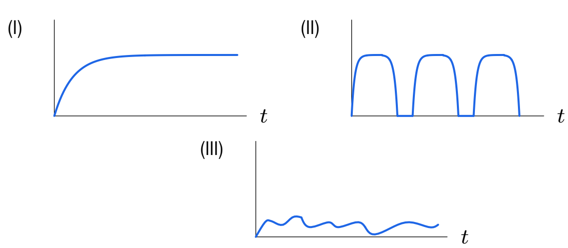



Figure 2.66

**15.** The temperature, $H$, in degrees Celsius, of a cup of coffee placed on the kitchen counter is given by $H = f(t)$, where $t$ is in minutes since the coffee was put on the counter.

**(a)** Is $f'(t)$ positive or negative? Give a reason for your answer.
**(b)** What are the units of $f'(20)$? What is its practical meaning in terms of the temperature of the coffee?

**16.** The temperature, $T$, in degrees Fahrenheit, of a cold yam placed in a hot oven is given by $T = f(t)$, where $t$ is the time in minutes since the yam was put in the oven.

**(a)** What is the sign of $f'(t)$? Why?
**(b)** What are the units of $f'(20)$? What is the practical meaning of the statement $f'(20) = 2$?

**17.** Suppose that $f(x)$ is a function with $f(100) = 35$ and $f'(100) = 3$. Estimate $f(102)$.

**18.** Suppose that $f(t)$ is a function with $f(25) = 3.6$ and $f'(25) = -0.2$. Estimate $f(26)$ and $f(30)$.

**19.** A mutual fund is currently valued at \$80 per share and its value per share is increasing at a rate of \$0.50 a day. Let $V = f(t)$ be the value of the share $t$ days from now.

**(a)** Express the information given about the mutual fund in term of $f$ and $f'$.
**(b)** Assuming that the rate of growth stays constant, estimate and interpret $f(10)$.

**20.** The average weight, $W$, in pounds, of an adult is a function, $W = f(c)$, of the average number of Calories per day, $c$, consumed.

**(a)** Interpret the statements $f(1800) = 155$ and $f'(2000) = 0$ in terms of diet and weight.
**(b)** What are the units of $f'(c) = dW/dc$?

**21.** Investing \$1000 at an annual interest rate of $r\%$, compounded continuously, for 10 years gives you a balance of \$$B$, where $B = g(r)$. Give a financial interpretation of the statements:

**(a)** $g(5) \approx 1649$.
**(b)** $g'(5) \approx 165$. What are the units of $g'(5)$?

**22.** Suppose $P(t)$ is the monthly payment, in dollars, on a mortgage which will take $t$ years to pay off. What are the units of $P'(t)$? What is the practical meaning of $P'(t)$? What is its sign?

**23.** Let $f(x)$ be the elevation in feet of the Mississippi river $x$ miles from its source. What are the units of $f'(x)$? What can you say about the sign of $f'(x)$?

**24.** An economist is interested in how the price of a certain item affects its sales. At a price of \$$p$, a quantity, $q$, of the item is sold. If $q = f(p)$, explain the meaning of each of the following statements:

**(a)** $f(150) = 2000$ **(b)** $f'(150) = -25$

**25.** A company's revenue from car sales, $C$ (in thousands of dollars), is a function of advertising expenditure, $a$, in thousands of dollars, so $C = f(a)$.

**(a)** What does the company hope is true about the sign of $f'$?
**(b)** What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?
**(c)** Suppose the company plans to spend about $\$100{,}000$ on advertising. If $f'(100) = 2$, should the company spend more or less than $100,000 on advertising? What if $f'(100) = 0.5$?

**26.** At one of the labeled points on the graph in Figure 2.67 both $dy/dx$ and $d^2y/dx^2$ are positive. Which is it?

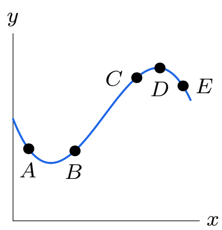



Figure 2.67

**27.** Sketch the graph of a function whose first and second derivatives are everywhere positive.

**28.** Sketch the graph of a function whose first derivative is everywhere negative and whose second derivative is positive for some $x$-values and negative for other $x$-values.

**29.** IBM-Peru uses second derivatives to assess the relative success of various advertising campaigns. They assume that all campaigns produce some increase in sales. If a graph of sales against time shows a positive second derivative during a new advertising campaign, what does this suggest to IBM management? Why? What does a negative second derivative suggest?

**30.** A high school principal is concerned about the drop in the percentage of students who graduate from her school, shown in the following table.

| Year entered school, $t$ | 1992 | 1995 | 1998 | 2001 | 2004 |
|---|---|---|---|---|---|
| Percent graduating, $P$ | 62.4 | 54.1 | 48.0 | 43.5 | 41.8 |

**(a)** Calculate the average rate of change of $P$ for each of the three-year intervals between 1992 and 2004.
**(b)** Does $d^2P/dt^2$ appear to be positive or negative between 1992 and 2004?
**(c)** Explain why the values of $P$ and $dP/dt$ are troublesome to the principal.
**(d)** Explain why the sign of $d^2P/dt^2$ and the magnitude of $dP/dt$ in the year 2001 may give the principal some cause for optimism.

**31.** Students were asked to evaluate $f'(4)$ from the following table which shows values of the function $f$:

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $f(x)$ | 4.2 | 4.1 | 4.2 | 4.5 | 5.0 | 5.7 |

- Student A estimated the derivative as $f'(4) \approx \dfrac{f(5) - f(4)}{5 - 4} = 0.5$.
- Student B estimated the derivative as $f'(4) \approx \dfrac{f(4) - f(3)}{4 - 3} = 0.3$.
- Student C suggested that they should split the difference and estimate the average of these two results, that is, $f'(4) \approx \frac{1}{2}(0.5 + 0.3) = 0.4$.

**(a)** Sketch the graph of $f$, and indicate how the three estimates are represented on the graph.
**(b)** Explain which answer is likely to be best.

**32.** Given all of the following information about a function $f$, sketch its graph.

- $f(x) = 0$ at $x = -5, x = 0$, and $x = 5$
- $f(x) \to \infty$ as $x \to -\infty$
- $f(x) \to -3$ as $x \to \infty$
- $f'(x) = 0$ at $x = -3, x = 2.5,$ and $x = 7$

**33.** Figure 2.68 shows the rate at which energy, $f(v)$, is consumed by a bird flying at speed $v$ meters/sec.

**(a)** What rate of energy consumption is needed by the bird to keep aloft, without moving forward?
**(b)** What does the shape of the graph tell you about how birds fly?
**(c)** Sketch $f'(v)$.

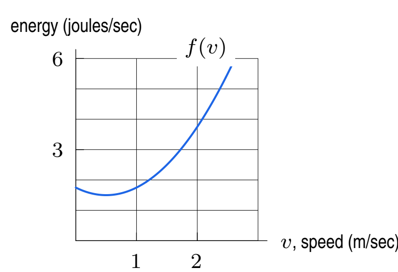



Figure 2.68

## PROJECTS FOR CHAPTER TWO

**1. Estimating the Temperature of a Yam**

Suppose you put a yam in a hot oven, maintained at a constant temperature of $200°$C. As the yam picks up heat from the oven, its temperature rises.[16]

**(a)** Draw a possible graph of the temperature $T$ of the yam against time $t$ (minutes) since it is put into the oven. Explain any interesting features of the graph, and in particular explain its concavity.

**(b)** Suppose that, at $t = 30$, the temperature $T$ of the yam is $120°$ and increasing at the (instantaneous) rate of $2°$/min. Using this information, plus what you know about the shape of the $T$ graph, estimate the temperature at time $t = 40$.

**(c)** Suppose in addition you are told that at $t = 60$, the temperature of the yam is $165°$. Can you improve your estimate of the temperature at $t = 40$?

**(d)** Assuming all the data given so far, estimate the time at which the temperature of the yam is $150°$.

**2. Temperature and Illumination**

Alone in your dim, unheated room, you light a single candle rather than curse the darkness. Depressed with the situation, you walk directly away from the candle, sighing. The temperature (in degrees Fahrenheit) and illumination (in $\%$ of one candle power) decrease as your distance (in feet) from the candle increases. In fact, you have tables showing this information.

| Distance (feet) | Temperature ($°$F) |
|---|---|
| 0 | 55 |
| 1 | 54.5 |
| 2 | 53.5 |
| 3 | 52 |
| 4 | 50 |
| 5 | 47 |
| 6 | 43.5 |

| Distance (feet) | Illumination ($\%$) |
|---|---|
| 0 | 100 |
| 1 | 85 |
| 2 | 75 |
| 3 | 67 |
| 4 | 60 |
| 5 | 56 |
| 6 | 53 |

You are cold when the temperature is below $40°$. You are in the dark when the illumination is at most $50\%$ of one candle power.

**(a)** Two graphs are shown in Figures 2.69 and 2.70. One is temperature as a function of distance and one is illumination as a function of distance. Which is which? Explain.

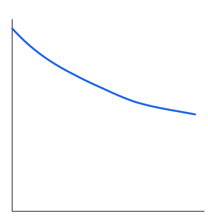


Figure 2.69

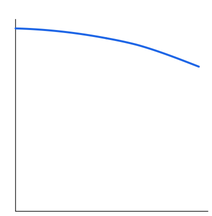


Figure 2.70

**(b)** What is the average rate at which the temperature is changing when the illumination drops from $75\%$ to $56\%$?

**(c)** You can still read your watch when the illumination is about $65\%$. Can you still read your watch at 3.5 feet? Explain.

[16]From Peter D. Taylor, *Calculus: The Analysis of Functions* (Toronto: Wall & Emerson, Inc., 1992).

**(d)** Suppose you know that at $6$ feet the instantaneous rate of change of the temperature is $-4.5$°F/ft and the instantaneous rate of change of illumination is $-3\%$ candle power/ft. Estimate the temperature and the illumination at 7 feet.

**(e)** Are you in the dark before you are cold, or vice-versa?

# FOCUS ON THEORY

## LIMITS, CONTINUITY, AND THE DEFINITION OF THE DERIVATIVE

The velocity at a single instant in time is surprisingly difficult to define precisely. Consider the statement "At the instant it crossed the finish line, the horse was traveling at 42 mph." How can such a claim be substantiated? A photograph taken at that instant will show the horse motionless—it is no help at all. There is some paradox in trying to quantify the property of motion at a particular instant in time, since by focusing on a single instant we stop the motion!

A similar difficulty arises whenever we attempt to measure the rate of change of anything—for example, oil leaking out of a damaged tanker. The statement "One hour after the ship's hull ruptured, oil was leaking at a rate of 200 barrels per second" seems not to make sense. We could argue that at any given instant *no* oil is leaking.

Problems of motion were of central concern to Zeno and other philosophers as early as the fifth century BC. The approach that we took, made famous by Newton's calculus, is to stop looking for a simple notion of speed at an instant, and instead to look at speed over small intervals containing the instant. This method sidesteps the philosophical problems mentioned earlier but brings new ones of its own.

### Definition of the Derivative Using Average Rates

On page 100 of Section 2.1, we defined the derivative as the instantaneous rate of change of a function. We can estimate a derivative by computing average rates of change over smaller and smaller intervals. We use this idea to give a symbolic definition of the derivative. Letting $h$ represent the size of the interval, we have

$$\begin{array}{c}\text{Average rate of change}\\ \text{between } x \text{ and } x+h\end{array} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}.$$

To find the derivative, or instantaneous rate of change at the point $x$, we use smaller and smaller intervals. To find the derivative exactly, we take the limit as $h$, the size of the interval, shrinks to zero, so we say

$$\text{Derivative} = \text{Limit, as } h \text{ approaches zero, of } \frac{f(x+h)-f(x)}{h}.$$

Finally, instead of writing the phrase "limit, as $h$ approaches 0," we use the notation $\lim\limits_{h\to 0}$. This leads to the following symbolic definition:

For any function $f$, we define the **derivative function**, $f'$, by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

provided the limit exists. The function $f$ is said to be **differentiable** at any point $x$ at which the derivative function is defined.

Notice that we have replaced the original difficulty of computing velocity at a point by an argument that the average rates of change approach a number as the time intervals shrink in size. In a sense, we have traded one hard question for another, since we don't yet have any idea how to be certain what number the average velocities are approaching.

## The Idea of a Limit

We used a limit to define the derivative. Now we look a bit more at the idea of the limit of a function at the point $c$. Provided the limit exists:

> We write $\lim\limits_{x\to c} f(x)$ to represent the number approached by $f(x)$ as $x$ approaches $c$.

**Example 1** Investigate $\lim\limits_{x\to 2} x^2$.

**Solution** Notice that we can make $x^2$ as close to 4 as we like by taking $x$ sufficiently close to 2. (Look at the values of $1.9^2$, $1.99^2$, $1.999^2$, and $2.1^2$, $2.01^2$, $2.001^2$ in Table 2.11; they seem to be approaching 4.) We write

$$\lim_{x\to 2} x^2 = 4,$$

which is read "the limit, as $x$ approaches 2, of $x^2$ is 4." Notice that the limit does not ask what happens *at* $x = 2$, so it is not sufficient to substitute 2 to find the answer. The limit describes behavior of a function *near* a point, not *at* the point.

**Table 2.11** *Values of $x^2$ near $x = 2$*

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $x^2$ | 3.61 | 3.96 | 3.996 | 4.004 | 4.04 | 4.41 |

**Example 2** Use a graph to estimate $\lim\limits_{x\to 0} \dfrac{2^x - 1}{x}$.

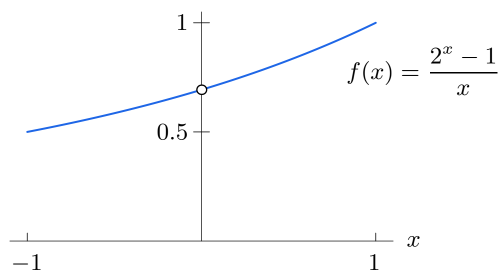



**Figure 2.71:** Find the limit as $x \to 0$ of $\dfrac{2^x - 1}{x}$

**Solution** Notice that the expression $\dfrac{2^x - 1}{x}$ is undefined at $x = 0$. To find out what happens to this expression as $x$ approaches 0, look at a graph of $f(x) = \dfrac{2^x - 1}{x}$. Figure 2.71 shows that as $x$ approaches 0 from either side, the value of $\dfrac{2^x - 1}{x}$ appears to approach 0.7. If we zoom in on the graph near $x = 0$, we can estimate the limit with greater accuracy, giving

$$\lim_{x\to 0} \frac{2^x - 1}{x} \approx 0.693.$$

**Example 3** Estimate $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$ numerically.

**Solution** The limit is the value approached by this expression as $h$ approaches $0$. The values in Table 2.12 seem to be approaching $6$ as $h \to 0$. So it is a reasonable guess that

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = 6.$$

However, we cannot be sure that the limit is *exactly* $6$ by looking at the table. To calculate the limit exactly requires algebra.

**Table 2.12** *Values of* $\left((3+h)^2-9\right)/h$

| $h$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $\left((3+h)^2-9\right)/h$ | $5.9$ | $5.99$ | $5.999$ | $6.001$ | $6.01$ | $6.1$ |

**Example 4** Use algebra to find $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$.

**Solution** Expanding the numerator gives

$$\frac{(3+h)^2-9}{h} = \frac{9+6h+h^2-9}{h} = \frac{6h+h^2}{h}.$$

Since taking the limit as $h \to 0$ means looking at values of $h$ near, but not equal, to $0$, we can cancel a common factor of $h$, giving

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0}(6+h).$$

As $h$ approaches $0$, the values of $(6+h)$ approach $6$, so

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0}(6+h) = 6.$$

## Continuity

Roughly speaking, a function is said to be *continuous* on an interval if its graph has no breaks, jumps, or holes in that interval. A continuous function has a graph that can be drawn without lifting the pencil from the paper.

*Example:* The function $f(x) = 3x^2 - x^2 + 2x + 1$ is continuous on any interval. (See Figure 2.72.)

*Example:* The function $f(x) = 1/x$ is not defined at $x = 0$. It is continuous on any interval not containing the origin. (See Figure 2.73.)

*Example:* Suppose $p(x)$ is the price of mailing a first-class letter weighing $x$ ounces. It costs 34¢ for one ounce or less, 57¢ between the first and second ounces, and so on. So the graph (in Figure 2.74) is a series of steps. This function is not continuous on intervals such as $(0, 2)$ because the graph jumps at $x = 1$.

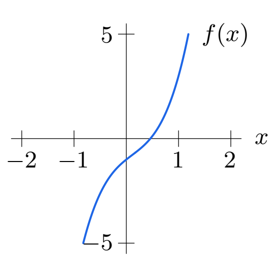



**Figure 2.72:** The graph of $f(x) = 3x^3 - x^2 + 2x - 1$

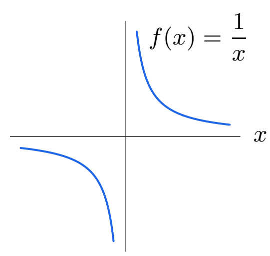



**Figure 2.73:** Graph of $f(x) = 1/x$: Not defined at 0

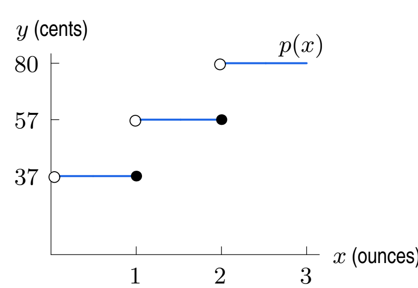



**Figure 2.74:** Cost of mailing a letter

### What Does Continuity Mean Numerically?

Continuity is important in practical work because it means that small errors in the independent variable lead to small errors in the value of the function.

*Example:* Suppose that $f(x) = x^2$ and that we want to compute $f(\pi)$. Knowing $f$ is continuous tells us that taking $x = 3.14$ should give a good approximation to $f(\pi)$, and that we can get a better approximation to $f(\pi)$ by using more decimals of $\pi$.

*Example:* If $p(x)$ is the cost of mailing a letter weighing $x$ ounces, then $p(0.99) = p(1) = 34$¢, whereas $p(1.01) = 57$¢, because as soon as we get over 1 ounce, the price jumps up to 57¢. So a small difference in the weight of a letter can lead to a significant difference in its mailing cost. Hence $p$ is not continuous at $x = 1$.

### Definition of Continuity

We now define continuity using limits. The idea of continuity rules out breaks, jumps, or holes by demanding that the behavior of a function *near* a point be consistent with its behavior *at* the point:

> The function $f$ is **continuous** at $x = c$ if $f$ is defined at $x = c$ and
>
> $$\lim_{x \to c} f(x) = f(c).$$
>
> The function is **continuous on an interval** $(a, b)$ if it is continuous at every point in the interval.

### Which Functions are Continuous?

Requiring a function to be continuous on an interval is not asking very much, as any function whose graph is an unbroken curve over the interval is continuous. For example, exponential functions, polynomials, and sine and cosine are continuous on every interval. Functions created by adding, multiplying, or composing continuous functions are also continuous.

## Using the Definition to Calculate Derivatives

By estimating the derivative of the function $f(x) = x^2$ at several points, we guessed in Example 5 of Section 2.2 that the derivative of $x^2$ is $f'(x) = 2x$. In order to show that this formula is correct, we have to use the symbolic definition of the derivative given on page 135.

In evaluating the expression

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

we simplify the difference quotient first, and then take the limit as $h$ approaches zero.

---

**Example 5** Show that the derivative of $f(x) = x^2$ is $f'(x) = 2x$.

Solution Using the definition of the derivative with $f(x) = x^2$, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} \frac{h(2x+h)}{h}. \end{aligned}$$

To take the limit, look at what happens when $h$ is close to 0, but do not let $h = 0$. Since $h \neq 0$, we cancel the common factor of $h$, giving

$$f'(x) = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h) = 2x,$$

because as $h$ gets close to zero, $2x + h$ gets close to $2x$. So

$$f'(x) = \frac{d}{dx}(x^2) = 2x.$$

**Example 6** Show that if $f(x) = 3x - 2$, then $f'(x) = 3$.

**Solution** Since the slope of the linear function $f(x) = 3x - 2$ is 3 and the derivative is the slope, we see that $f'(x) = 3$. We can also use the definition to get this result:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(3(x+h)-2) - (3x-2)}{h} \\ &= \lim_{h\to 0} \frac{3x + 3h - 2 - 3x + 2}{h} = \lim_{h\to 0} \frac{3h}{h}. \end{aligned}$$

To find the limit, look at what happens when $h$ is close to, but not equal to, 0. Simplifying, we get

$$f'(x) = \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3.$$

## Problems on Limits and the Definition of the Derivative

**1.** On Figure 2.75, mark lengths that represent the quantities in parts (a) – (e). (Pick any $h$, with $h > 0$.)

**(a)** $a + h$ **(b)** $h$ **(c)** $f(a)$
**(d)** $f(a+h)$ **(e)** $f(a+h) - f(a)$

**(f)** Using your answers to parts (a)–(e), show how the quantity $\dfrac{f(a+h) - f(a)}{h}$ can be represented as the slope of a line on the graph.

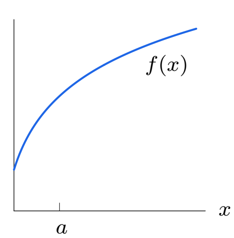



Figure 2.75

**2.** On Figure 2.76, mark lengths that represent the quantities in parts (a)–(e). (Pick any $h$, with $h > 0$.)

**(a)** $a + h$ **(b)** $h$ **(c)** $f(a)$
**(d)** $f(a+h)$ **(e)** $f(a+h) - f(a)$

**(f)** Using your answers to parts (a)–(e), represent the quantity $\dfrac{f(a+h) - f(a)}{h}$ as the slope of a line on the graph.

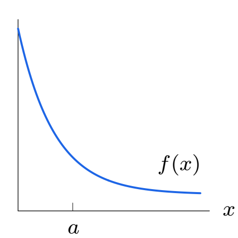



Figure 2.76

Use a graph to estimate the limits in Problems 3–4.

**3.** $\lim_{x \to 0} \frac{\sin x}{x}$ (with $x$ in radians)

**4.** $\lim_{x \to 0} \frac{5^x - 1}{x}$

Estimate the limits in Problems 5–8 by substituting smaller and smaller values of $h$. For trigonometric functions, use radians. Give answers to one decimal place.

**5.** $\lim_{h \to 0} \frac{(3+h)^3 - 27}{h}$

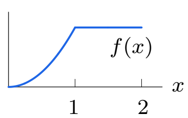
**6.** $\lim_{h \to 0} \frac{7^h - 1}{h}$

**7.** $\lim_{h \to 0} \frac{e^{1+h} - e}{h}$

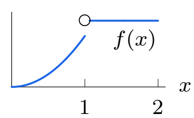
**8.** $\lim_{h \to 0} \frac{\cos h - 1}{h}$

In Problems 9–12, does the function $f(x)$ appear to be continuous on the interval $0 \le x \le 2$? If not, what about on the interval $0 \le x \le 0.5$?

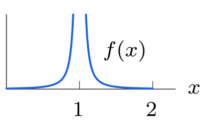
**9.**



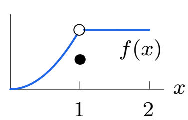

**10.**




**11.**




**12.**




Are the functions in Problems 13–18 continuous on the given intervals?

**13.** $f(x) = x + 2$ on $-3 \le x \le 3$

**14.** $f(x) = 2^x$ on $0 \le x \le 10$

**15.** $f(x) = x^2 + 2$ on $0 \le x \le 5$

**16.** $f(x) = \frac{1}{x-1}$ on $2 \le x \le 3$

**17.** $f(x) = \frac{1}{x-1}$ on $0 \le x \le 2$

**18.** $f(x) = \frac{1}{x^2+1}$ on $0 \le x \le 2$

Which of the functions described in Problems 19–23 are continuous?

**19.** The number of people in a village as a function of time.

**20.** The weight of a baby as a function of time during the second month of the baby's life.

**21.** The number of pairs of pants as a function of the number of yards of cloth from which they are made. Each pair requires 3 yards.

**22.** The distance traveled by a car in stop-and-go traffic as a function of time.

**23.** You start in North Carolina and go westward on Interstate 40 toward California. Consider the function giving the local time of day as a function of your distance from your starting point.

Use the definition of the derivative to show how the formulas in Problems 24–33 are obtained.

**24.** If $f(x) = 5x$, then $f'(x) = 5$.

**25.** If $f(x) = 3x - 2$, then $f'(x) = 3$.

**26.** If $f(x) = x^2 + 4$, then $f'(x) = 2x$.

**27.** If $f(x) = 3x^2$, then $f'(x) = 6x$.

**28.** If $f(x) = -2x^3$, then $f'(x) = -6x^2$.

**29.** If $f(x) = x - x^2$, then $f'(x) = 1 - 2x$.

**30.** If $f(x) = 1 - x^3$, then $f'(x) = -3x^2$.

**31.** If $f(x) = 5x^2 + 1$, then $f'(x) = 10x$.

**32.** If $f(x) = 2x^2 + x$, then $f'(x) = 4x + 1$.

**33.** If $f(x) = 1/x$, then $f'(x) = -1/x^2$.