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The Physics of the DC Motor

The principles of operation of a direct current (DC) motor are presented based on fundamental concepts from electricity and magnetism contained in any basic physics course. The DC motor is used as a concrete example for reviewing the concepts of magnetic fields, magnetic force, Faraday's law, and induced electromotive forces (emf) that will be used throughout the remainder of the book for the modeling of electric machines. All of the Physics concepts referred to in this chapter are contained in the book *Physics* by Halliday and Resnick [34].

1.1 Magnetic Force

Motors work on the basic principle that magnetic fields produce forces on wires carrying a current. In fact, this experimental phenomenon is what is used to define the magnetic field. If one places a current carrying wire between the poles of a magnet as in Figure 1.1, a force is exerted on the wire. Experimentally, the magnitude of this force is found to be proportional to both the amount of current in the wire and to the length of the wire that is between the poles of the magnet. That is, F_{magnetic} is proportional to ℓi. The direction of the magnetic field $\vec{\mathbf{B}}$ at any point is defined to be the direction that a small compass needle would point at that location. This direction is indicated by arrows in between the north and south poles in Figure 1.1.

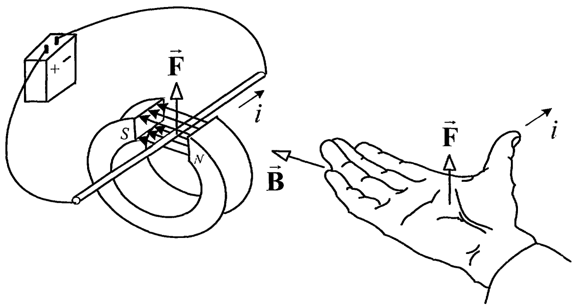

FIGURE 1.1. Magnetic force law. From *PSSC Physics*, 7th edition, by Haber-Schaim, Dodge, Gardner, and Shore, published by Kendall/Hunt, 1991.

With the direction of $\vec{\mathbf{B}}$ perpendicular to the wire, the strength (magnitude) of the *magnetic induction field* $\vec{\mathbf{B}}$ is defined to be

$$B = |\vec{\mathbf{B}}| \triangleq \frac{F_{\text{magnetic}}}{\ell i}$$

where F_{magnetic} is the magnetic force, i is the current, and ℓ is the length of wire perpendicular to the magnetic field carrying the current. That is, B is the proportionality constant so that $F_{\text{magnetic}} = i\ell B$. As illustrated in Figure 1.1, the direction of the force can be determined using the right-hand rule. Specifically, using your right hand, point your fingers in the direction of the magnetic field and point your thumb in the direction of the current. Then the direction of the force is out of your palm.

Further experiments show that if the wire is parallel to the $\vec{\mathbf{B}}$ field rather than perpendicular as in Figure 1.1, then no force is exerted on the wire. If the wire is at some angle θ with respect to $\vec{\mathbf{B}}$ as in Figure 1.2, then the force is proportional to the *component* of $\vec{\mathbf{B}}$ perpendicular to the wire; that is, it is proportional to $B_\perp = B\sin(\theta)$. This is summarized in the *magnetic force law*: Let $\vec{\boldsymbol{\ell}}$ denote a vector whose magnitude is the length ℓ of the wire in the magnetic field and whose direction is defined as the positive direction of current in the bar; then the magnetic force on the bar of length ℓ carrying the current i is given by

$$\vec{\mathbf{F}}_{\text{magnetic}} = i\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}$$

or, in scalar terms, $F_{\text{magnetic}} = i\ell B\sin(\theta) = i\ell B_\perp$. Again, $B_\perp \triangleq B\sin(\theta)$ is the component of $\vec{\mathbf{B}}$ perpendicular to the wire.[1]

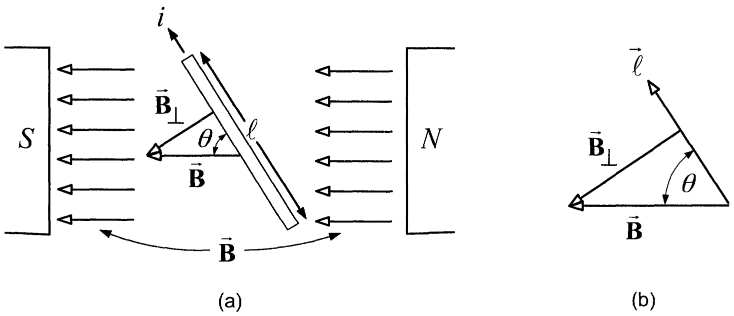

FIGURE 1.2. Only the component $B_\perp$ of the magnetic field which is perpendicular to the wire produces a force on the current.

[1]Motors are designed so that the conductors are perpendicular to the external magnetic field.

Example *A Linear DC Machine* [19]

Consider the simple linear DC machine in Figure 1.3 where a sliding bar rests on a simple circuit consisting of two rails. An external magnetic field is going through the loop of the circuit up out of the page indicated by the $\otimes$ in the plane of the loop. Closing the switch results in a current flowing around the circuit and the external magnetic field produces a force on the bar which is free to move. The force on the bar is now computed.

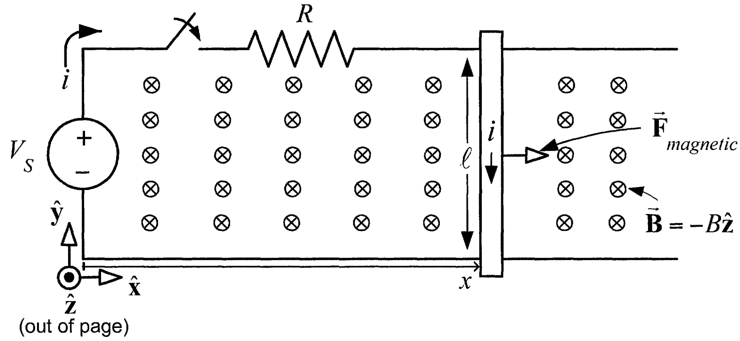

FIGURE 1.3. A linear DC motor.

The magnetic field is constant and points into the page (indicated by $\otimes$) so that written in vector notation, $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$ with $B > 0$. By the right hand rule, the magnetic force on the sliding bar points to the right. Explicitly, with $\vec{\boldsymbol{\ell}} = -\ell\hat{\mathbf{y}}$, the force is given by

$$\begin{aligned}\vec{\mathbf{F}}_{\text{magnetic}} &= i\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}} = i(-\ell\hat{\mathbf{y}}) \times (-B\hat{\mathbf{z}}) \\ &= i\ell B\hat{\mathbf{x}}.\end{aligned}$$

To find the equations of motion for the bar, let f be the coefficient of viscous (sliding) friction of the bar so that the friction force is given by $F_f = -f dx/dt$. Then, with m_ℓ denoting the mass of the bar, Newton's law gives

$$i\ell B - f dx/dt = m_\ell d^2x/dt^2.$$

Just after closing the switch at $t = 0$, but before the bar starts to move, the current is $i(0^+) = V_S(0^+)/R$. However, it turns out that as the bar moves the current does *not* stay at this value, but instead decreases due to electromagnetic induction. This will be explained later.

1.2 Single-Loop Motor

As a first step to modeling a DC motor, a simplistic single-loop motor is considered. It is first shown how torque is produced and then how the current in the single loop can be reversed (commutated) every half turn to keep the torque constant.

1.2.1 Torque Production

Consider the magnetic system in Figure 1.4, where a cylindrical core is cut out of a block of a permanent magnet and replaced with a soft iron core. The term "soft" iron refers to the fact that material is easily magnetized (a permanent magnet is referred to as "hard" iron).

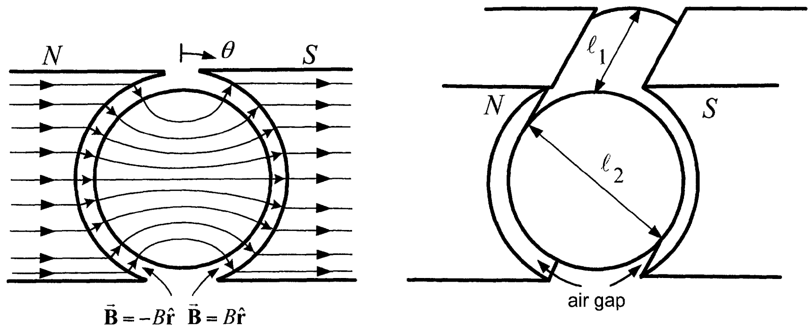

FIGURE 1.4. Soft iron cylindrical core placed inside a hollowed out permanent magnet to produce a radial magnetic field in the air gap.

An important property of soft magnetic materials is that the magnetic field at the surface of such materials tends to be normal (perpendicular) to the surface. Consequently, the cylindrical shape of the surfaces of the soft iron core and the stator permanent magnet has the effect of making the field in the air gap *radially* directed; furthermore, it is reasonably constant (uniform) in magnitude. A mathematical description of the magnetic field in the air gap due to the permanent magnet is simply

$$\vec{\mathbf{B}} = \begin{cases} +B\hat{\mathbf{r}} & \text{for } 0 < \theta < \pi \\ -B\hat{\mathbf{r}} & \text{for } \pi < \theta < 2\pi \end{cases}$$

where $B > 0$ is the magnitude or strength of the magnetic field and θ is an arbitrary location in the air gap.[2]

Figure 1.5 shows a rotor loop wound around the iron core of Figure 1.4. The length of the rotor is ℓ_1 and its diameter is ℓ_2. The torque on this rotor loop is now calculated by considering the magnetic forces on sides a and a' of the loop. On the other two sides of the loop, that is, the front and

[2]Actually it will be shown in a later chapter that the magnetic field must be of the form $\vec{\mathbf{B}} = \pm B(r_0/r)\hat{\mathbf{r}}$ in the air gap, that is, it varies as $1/r$ in the air gap. However, as the air gap is small, the $\vec{\mathbf{B}}$ field is essentially constant across the air gap.

back sides, the magnetic field has negligible strength so that no significant force is produced on these sides. As illustrated in Figure 1.5(b), the rotor angular position is taken to be the angle θ_R from the vertical to side a of the rotor loop.

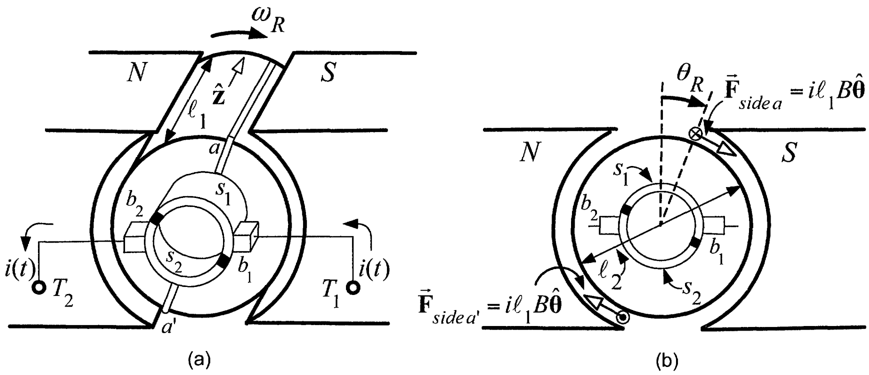

FIGURE 1.5. A single-loop motor. From *Electromagnetic and Electromechanical Machines*, 3rd edition, L. W. Matsch and J. Derald Morgan, 1986. Reprinted by permisson of John Wiley & Sons.

Figure 1.6 shows the cylindrical coordinate system used in Figure 1.5. Here $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$ denote unit cylindrical coordinate vectors. The unit vector $\hat{\mathbf{z}}$ points along the rotor axis into the paper in Figure 1.5(b), $\hat{\boldsymbol{\theta}}$ is in the direction of increasing θ, and $\hat{\mathbf{r}}$ is in the direction of increasing r.

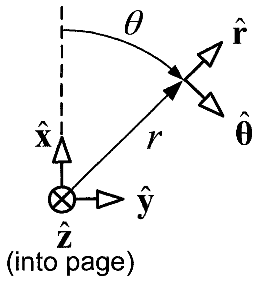

FIGURE 1.6. Cylindrical coordinate system used in Figure 1.5.

Referring back to Figure 1.5, for $i > 0$, the current in side a of the loop is going into the page (denoted by $\otimes$) and then comes out of the page (denoted by $\odot$) on side a'. Thus, on side a, $\vec{\boldsymbol{\ell}} = \ell_1 \hat{\mathbf{z}}$ (as $\vec{\boldsymbol{\ell}}$ points in the direction of positive current flow) and the magnetic force $\vec{\mathbf{F}}_{\text{side } a}$ on side a

is then

$$\begin{aligned}\vec{\mathbf{F}}_{\text{side } a} &= i\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}} \\ &= i(\ell_1\hat{\mathbf{z}}) \times (B\hat{\mathbf{r}}) \\ &= i\ell_1 B\hat{\boldsymbol{\theta}}\end{aligned}$$

which is tangential to the motion as shown in Figure 1.5(b). The resulting torque is

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{\text{side } a} &= (\ell_2/2)\hat{\mathbf{r}} \times \vec{\mathbf{F}}_{\text{side } a} \\ &= (\ell_2/2)i\ell_1 B\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \\ &= (\ell_2/2)i\ell_1 B\hat{\mathbf{z}}.\end{aligned}$$

Similarly, the magnetic force on side a' of the rotor loop is

$$\begin{aligned}\vec{\mathbf{F}}_{\text{side } a'} &= i\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}} \\ &= i(-\ell_1\hat{\mathbf{z}}) \times (-B\hat{\mathbf{r}}) \\ &= i\ell_1 B\hat{\boldsymbol{\theta}}\end{aligned}$$

so that the corresponding torque is then

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{\text{side } a'} &= (\ell_2/2)\hat{\mathbf{r}} \times \vec{\mathbf{F}}_{\text{side } a'} \\ &= (\ell_2/2)i\ell_1 B\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \\ &= (\ell_2/2)i\ell_1 B\hat{\mathbf{z}}.\end{aligned}$$

The total torque on the rotor loop is then

$$\begin{aligned}\vec{\boldsymbol{\tau}}_m &= \vec{\boldsymbol{\tau}}_{\text{side } a} + \vec{\boldsymbol{\tau}}_{\text{side } a'} \\ &= 2(\ell_2/2)i\ell_1 B\hat{\mathbf{z}} \\ &= \ell_1\ell_2 Bi\hat{\mathbf{z}}.\end{aligned}$$

The torque points along the z axis, which is the axis of rotation. In scalar form,

$$\tau_m = K_T i$$

where $K_T \triangleq \ell_1\ell_2 B$. The force is proportional to the strength B of magnetic field $\vec{\mathbf{B}}$ in the air gap due to the permanent magnet.

In order to increase the strength of the magnetic field in the air gap, the permanent magnet can be replaced with a soft iron material with wire wound around the periphery of the magnetic material as shown in Figure 1.7(a). This winding is referred to as the *field winding*, and the current it carries is called the *field current*. In normal operation, the field current is held constant. The strength of the magnetic field in the air gap is then proportional to the field current i_f at lower current levels (i.e., $B = K_f i_f$) and then saturates as the current increases. This may be written as $B =$

$f(i_f)$ where $f(\cdot)$ is a saturation curve satisfying $f(0) = 0, f'(0) = K_f$ as shown in Figure 1.7(b).

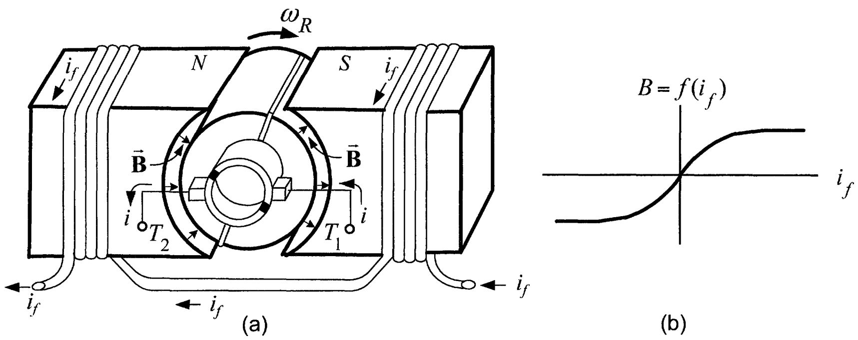

FIGURE 1.7. (a) DC motor with a field winding. (b) Radial magnetic field strength in the air gap produced by the field current.

1.2.2 Commutation of the Single-Loop Motor

The above derivation for the torque $\tau_m = K_T i$ assumes that the current in the side of the rotor loop[3] under the south pole face is into the page and the current in the side of the loop under the north pole face is out of the page as in Figure 1.8(a). In order to make this assumption valid, the direction of the current in the loop must be changed each time the rotor loop passes through the vertical.

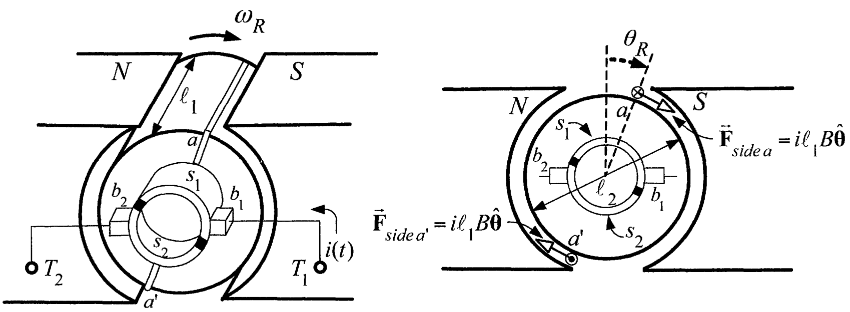

FIGURE 1.8. (a) $0 < \theta_R < \pi$. From *Electromagnetic and Electromechanical Machines,* 3rd edition, L. W. Matsch and J. Derald Morgan, 1986. Reprinted by permisson of John Wiley & Sons.

[3]The rotor loop is also referred to as the *armature* winding and the current in it as the *armature* current.

The process of changing the direction of the current is referred to as *commutation* and is done at $\theta_R = 0$ and $\theta_R = \pi$ through the use of the slip rings s_1, s_2 and brushes b_1, b_2 drawn in Figure 1.8. The slip rings are rigidly attached to the loop and thus rotate with it. The brushes are fixed in space with the slip rings making a sliding electrical contact with the brushes as the loop rotates.

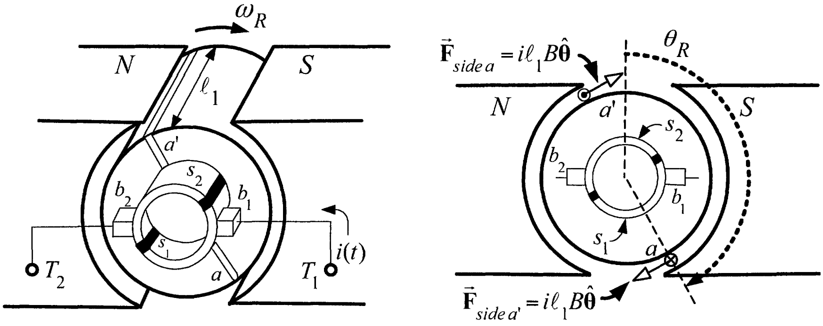

Figure 1.8(b) Rotor loop just prior to commutation where $0 < \theta_R < \pi$.

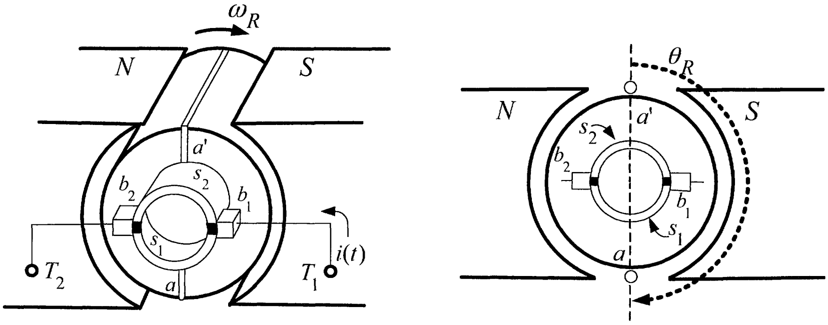

Figure 1.8(c) The ends of the rotor loop are shorted when $\theta_R = \pi$.

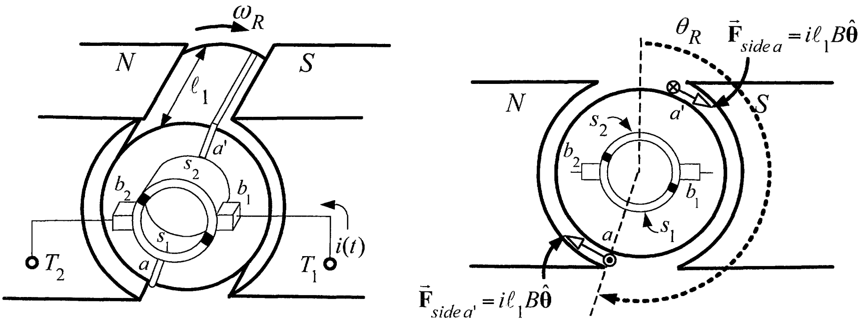

Figure 1.8(d) Rotor loop just after commutation where $\pi < \theta_R < 2\pi$.

To see how the commutation of the current is accomplished using the brushes and slip rings, consider the sequence of Figures 1.8(a)−(d). As shown in Figure 1.8(a), the current goes through brush b_1 into the slip ring s_1. From there, it travels down (into the page $\otimes$) side a of the loop, comes back up side a' (out of the page $\odot$) into the slip ring s_2, and, finally, comes out the brush b_2. Note that side a of the loop is under the south pole face while side a' is under the north pole face. Figure 1.8(b) shows the rotor loop just before commutation where the same comments as in Figure 1.8(a) apply.

Figure 1.8(c) shows that when $\theta_R = \pi$, the slip rings at the ends of the loop are shorted together by the brushes forcing the current in the loop to drop to zero. Subsequently, as shown in Figure 1.8(d), with $\pi < \theta_R < 2\pi$, the current is now going through brush b_1 into slip ring s_2. From there, the current travels down (into the page $\otimes$) side a' of the loop and comes back up (out of the page $\odot$) side a. In other words, the current has *reversed* its direction in the loop from that in Figures 1.8(a) and 1.8(b). This is precisely what is desired, as side a is now under the north pole face and side a' is under the south pole face. As a result of the brushes and slip rings, the current direction in the loop is reversed every half-turn.

1.3 Faraday's Law

Figure 1.9 shows a magnet moving upwards into a wire loop producing a changing magnetic flux in the loop.

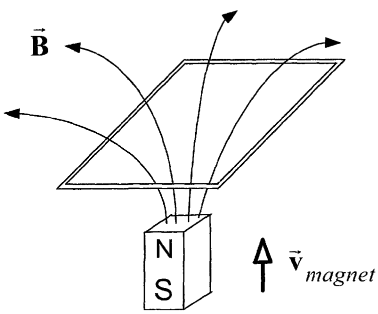

FIGURE 1.9. A magnet moving upwards produces a changing flux in the loop which in turn results in an induced emf and current in the loop.

Recall that a changing flux within a loop produces an induced *electro-*

motive force (emf) ξ in the loop according to Faraday's law.[4] That is,

$$\xi = -d\phi/dt$$

where

$$\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}$$

is the flux in the loop and S is any surface with the loop as its boundary. Faraday's law is now reviewed in some detail.

1.3.1 The Surface Element Vector $d\vec{\mathbf{S}}$

The surface element $d\vec{\mathbf{S}}$ is a vector whose magnitude is a differential (small) element of area dS and whose direction is normal (perpendicular) to the surface element. As there are two possibilities for the normal to the surface, one must choose the normal in a consistent manner. In particular, depending on the particular normal chosen, a convention is used to characterize the positive and negative directions of travel around the surface boundary. To describe this, consider Figure 1.10(a) which shows a small surface element with the normal direction taken to be up in the positive z direction. In this case, with $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, $dS = dxdy$, the surface element vector is defined by

$$d\vec{\mathbf{S}} \triangleq dxdy\hat{\mathbf{z}}.$$

The corresponding direction of travel around the surface boundary is indicated by the curved arrow in the figure.

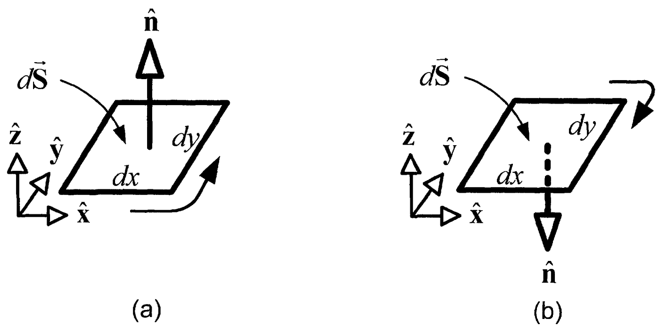

FIGURE 1.10. (a) Positive direction of travel around a surface element with the normal up. (b) Positive direction of travel around a surface element with the normal down.

[4] ξ is the Greek letter "xi" and is pronounced "ksi".

In Figure 1.10(b) a surface element with the normal direction taken to be down in the negative z direction is shown. In this case $\hat{\mathbf{n}} = -\hat{\mathbf{z}}, dS = dxdy$ so that the surface element vector is defined as

$$d\vec{\mathbf{S}} = -dxdy\hat{\mathbf{z}}.$$

The direction of positive travel around the surface element is indicated by the curved arrow in Figure 1.10(b) and is opposite to that of Figure 1.10(a).

As illustrated in Figure 1.10, the vector differential surface element $d\vec{\mathbf{S}}$ is defined to be a vector whose magnitude is the area of the differential surface element and whose direction is normal to the surface. One may choose either normal, and the corresponding direction of positive travel around the surface is then determined.

Two surface elements may be connected together as in Figure 1.11 and travel around the total surface is defined as shown. Note that along the common boundary of the two joined surface elements, the directions of travel "cancel" out each other, resulting in a net travel path around both surface elements. The normals for the surface elements must both be up or both be down; that is, the normal must be continuous as one goes from one surface element to the next.

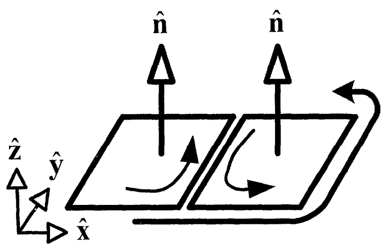

FIGURE 1.11. Positive direction of travel around two joined surface elements.

1.3.2 *Interpreting the Sign of* ξ

The interpretation of positive and negative values of the induced electromotive force ξ is now explained. Faraday's law says that the induced emf (voltage) in a loop is given by

$$\xi = -d\phi/dt$$

where

$$\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}.$$

If $\xi > 0$, the induced emf will force current in the positive direction of travel around the surface while if $\xi < 0$, the induced emf will force current in the opposite direction. As illustrated in problems 1 and 2, this sign convention for Faraday's law is just a precise mathematical way of describing Lenz's law: *"In all cases of electromagnetic induction, an induced voltage will cause a current to flow in a closed circuit in such a direction that the magnetic field which is caused by that current will oppose the change that produced the current"* (pages 873–877 of Ref. [34]).

Faraday's law is now illustrated by some examples. Specifically, it is used to compute the induced emf in the linear DC machine, the induced emf in the single-loop machine and the self-induced voltage in the single-loop machine.

1.3.3 Back Emf in a Linear DC Machine

Figure 1.12 shows the linear DC machine where the back emf it generates is now computed. The magnetic field is constant and points into the page, that is, $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$, where $B > 0$. The magnetic force on the bar is $\vec{\mathbf{F}}_{\text{magnetic}} = i\ell B\hat{\mathbf{x}}$. To compute the induced voltage in the loop of the circuit, let $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ be the normal to the surface so that $d\vec{\mathbf{S}} = dxdy\hat{\mathbf{z}}$, where $dS = dxdy$.

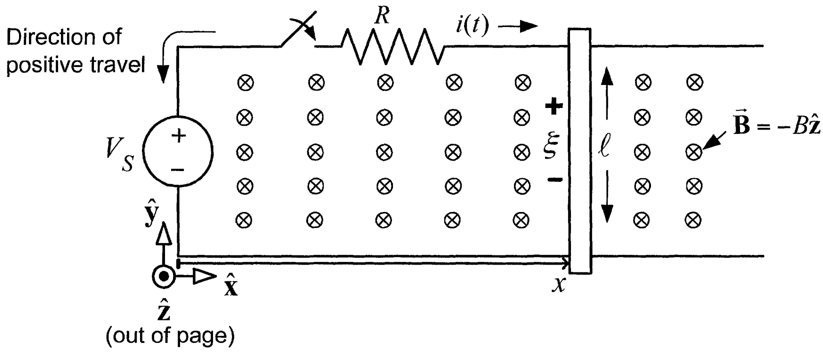

FIGURE 1.12. With $d\vec{\mathbf{S}} = dxdy\hat{\mathbf{z}}$, the direction of positive travel around the flux surface is in the counterclockwise direction.

Then

$$\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_0^\ell \int_0^x (-B\hat{\mathbf{z}}) \cdot (dxdy\hat{\mathbf{z}}) = \int_0^\ell \int_0^x -Bdxdy = -B\ell x.$$

The induced (back) emf is therefore given by

$$\xi = -d\phi/dt = -d(-B\ell x)/dt = B\ell v.$$

In the flux computation, the normal for the surface was taken to be in $+\hat{\mathbf{z}}$ direction. By putting together the differential flux surfaces $d\vec{\mathbf{S}}$ in a fashion similar to Figure 1.11, the positive direction of travel around the surface is counterclockwise around the loop as indicated in Figure 1.12. Here the sign conventions for source voltage V_S and the back emf ξ are opposite so that, as the back emf $\xi = B\ell v > 0$, it is *opposing* the applied source voltage V_S.

Remark $\phi = -B\ell x$ is the flux in the circuit due to the *external* magnetic field $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$. There is also a flux $\psi = Li$ due to the current i in the circuit. For this example, the inductance is small and one just sets $L = 0$.

Electromechanical Energy Conversion

As the back emf $\xi = Blv$ opposes the current i, electrical power is being absorbed by this back emf. Specifically, the electrical power absorbed by the back emf is $i\xi = iB\ell v$ while the mechanical power produced is $F_{\text{magnetic}}v = i\ell Bv$. That is, the electrical power absorbed by the back emf reappears as mechanical power, as it must by conservation of energy. Another way to view this is to note that $V_S i$ is the electrical power delivered by the source and, as $V_S - B\ell v = Ri$, one may write

$$V_S i = Ri^2 + i(B\ell v) = Ri^2 + F_{\text{magnetic}}v.$$

In words, the power from the source $V_S i$ is dissipated as heat in the resistance R while the rest is converted into mechanical power.

Equations of Motion for the Linear DC Machine

The equations of motion for the bar in the linear DC machine are now derived. With the inductance L of the circuit loop taken to be zero, m_ℓ the mass of the bar, f the coefficient of viscous friction, it follows that

$$\begin{aligned} V_S - B\ell v &= Ri \\ m_\ell \frac{dv}{dt} &= i\ell B - fv. \end{aligned}$$

Eliminating the current i, one obtains

$$m_\ell \frac{d^2x}{dt^2} = \ell B(V_S - B\ell v)/R - fv = -\left(\frac{B^2\ell^2}{R} + f\right)\frac{dx}{dt} + \frac{\ell B}{R}V_S$$

or

$$m_\ell \frac{d^2x}{dt^2} + \left(\frac{B^2\ell^2}{R} + f\right)\frac{dx}{dt} = \frac{\ell B}{R}V_S.$$

This is the equation of motion for the bar with V_S as the control input and the position x at the measured output.

1.3.4 Back Emf in the Single-Loop Motor

The back emf induced in the single loop motor by the external magnetic field of the permanent magnet is now computed. To do so, consider the flux surface for the rotor loop shown in Figure 1.13. The surface is a half-cylinder of radius $\ell_2/2$ and length ℓ_1 with the rotor loop as its boundary. The cylindrical surface is in the air gap, where the magnetic field is known to be radially directed and constant in magnitude, that is,

$$\vec{\mathbf{B}} = \begin{cases} +B\hat{\mathbf{r}} & \text{for } 0 < \theta < \pi \\ -B\hat{\mathbf{r}} & \text{for } \pi < \theta < 2\pi. \end{cases} \tag{1.1}$$

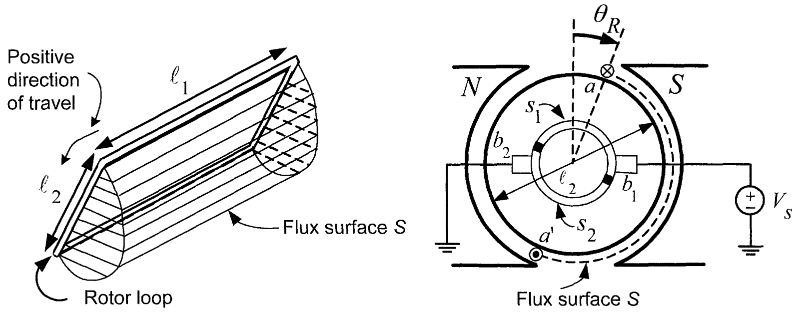

FIGURE 1.13. Flux surface for the single loop motor.

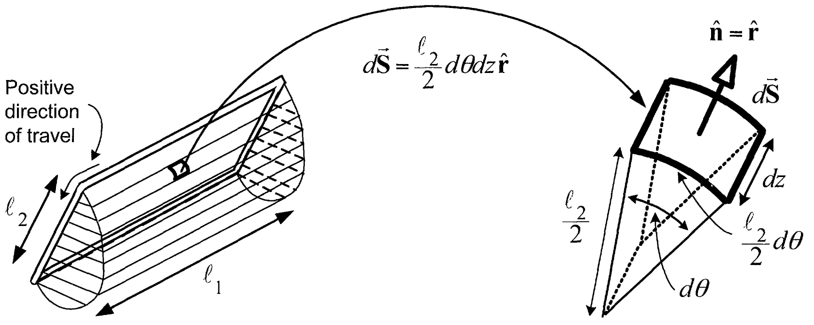

FIGURE 1.14. Surface element vector for the flux surface of Figure 1.13. The positive direction of travel around this surface is indicated by the curved arrow.

On the cylindrical part of the surface, the surface element is chosen as

$$d\vec{\mathbf{S}} = (\ell_2/2) d\theta dz \hat{\mathbf{r}}$$

which is directed outward from the axis of the cylinder as illustrated in Figure 1.14. The corresponding direction of positive travel is also indicated in Figure 1.14. On the two ends (half-disks) of the cylindrical surface, the $\vec{\mathbf{B}}$ field is quite weak making the flux through these two half-disks negligible.

Then, neglecting the flux through the two ends of the surface, the flux $\phi(\theta_R)$ for $0 < \theta_R < \pi$ is given by

$$\begin{aligned}
\phi(\theta_R) &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \\
&= \int_0^{\ell_1} \int_{\theta=\theta_R}^{\theta=\pi} (B\hat{\mathbf{r}}) \cdot (\frac{\ell_2}{2} d\theta dz \hat{\mathbf{r}}) + \int_0^{\ell_1} \int_{\theta=\pi}^{\theta=\pi+\theta_R} (-B\hat{\mathbf{r}}) \cdot (\frac{\ell_2}{2} d\theta dz \hat{\mathbf{r}}) \\
&= \int_0^{\ell_1} \int_{\theta=\theta_R}^{\theta=\pi} B\frac{\ell_2}{2} d\theta dz + \int_0^{\ell_1} \int_{\theta=\pi}^{\theta=\pi+\theta_R} -B\frac{\ell_2}{2} d\theta dz \\
&= \frac{\ell_1 \ell_2 B}{2}(\pi - \theta_R) - \frac{\ell_1 \ell_2 B}{2}\theta_R \\
&= -\ell_1 \ell_2 B \left(\theta_R - \frac{\pi}{2}\right).
\end{aligned} \tag{1.2}$$

This derivation is based on the fact that the $\vec{\mathbf{B}}$ field is directed radially outward over the length $(\ell_2/2)(\pi - \theta_R)$ and radially inward over the length $(\ell_2/2)\theta_R$ (see Figure 1.13). In problem 7, the reader is asked to show that

$$\phi(\theta_R) = -\ell_1 \ell_2 B \left(\theta_R - \pi/2 - \pi\right) \quad \text{for } \pi < \theta_R < 2\pi. \tag{1.3}$$

A plot of the flux versus the rotor angle θ_R is given in Figure 1.15.

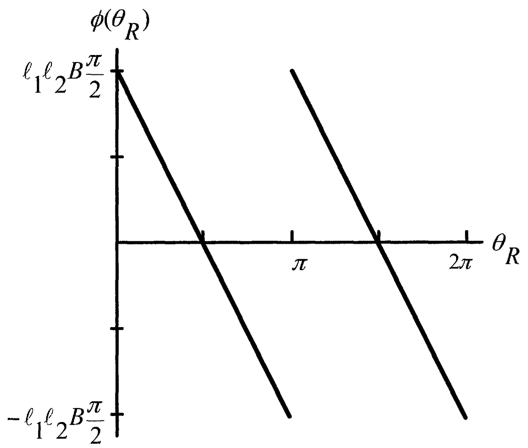

FIGURE 1.15. The rotor flux $\phi(\theta_R)$ due to the external magnetic field vs. θ_R.

Equations (1.2) and (1.3) may be combined into one expression as[5]

$$\phi(\theta_R) = -\ell_1 \ell_2 B \left(\theta_R \bmod \pi - \frac{\pi}{2}\right) \tag{1.4}$$

[5] $\theta_R \bmod \pi$ is the remainder after θ_R is divided by π. For example, $\theta_R = 5\pi/2 = 4 \times \pi + \pi/2$ so that $5\pi/2 \bmod \pi = \pi/2$.

which is a correct expression for any angle θ_R. By (1.2) and (1.3), the induced emf in the rotor loop is calculated as

$$\xi = -\frac{d\phi}{dt} = (\ell_1\ell_2 B)\frac{d\theta_R}{dt} = K_b\omega_R$$

where $K_b \triangleq \ell_1\ell_2 B$ is called the back emf constant.

The total emf in the rotor loop due to the voltage source V_S and external magnetic field is $V_S - K_b\omega_R$. How does one know to subtract ξ from the applied voltage V_S? As shown in Figure 1.14, the positive direction of travel around the loop is in *opposition* to V_S, so that if $\xi > 0$, it is opposing the applied voltage V_S. The standard terminology is to call $\xi \triangleq K_b\omega_R$ the *back emf* of the motor.

1.3.5 Self-Induced Emf in the Single-Loop Motor

The computation of the flux in the rotor loop produced by its own (armature) *current* is now done. To do so, consider the flux surface shown in Figure 1.16.

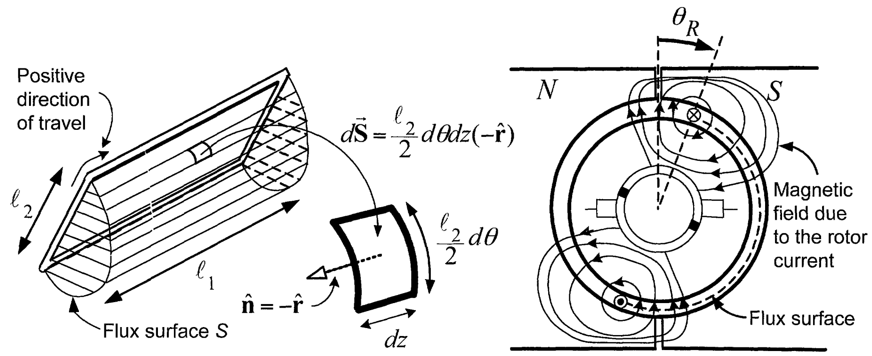

FIGURE 1.16. Computation of the inductance of the rotor loop. The surface element vector is $d\vec{\mathbf{S}} = -r_R d\theta dz\hat{\mathbf{r}}$ with a resulting positive direction of travel as indicated by the curved arrow. This direction coincides with the direction of positive current, that is, $i > 0$.

With reference to Figure 1.16, note that the magnetic field on the flux surface *due to the armature current* has the form

$$\vec{\mathbf{B}}(r_R, \theta - \theta_R, i) = iK(r_R, \theta - \theta_R)\,(-\hat{\mathbf{r}})$$

where

$$\begin{aligned} K(r_R, \theta - \theta_R) &> 0 \quad \text{for } 0 \le \theta - \theta_R \le \pi \\ K(r_R, \theta - \theta_R) &< 0 \quad \text{for } \pi \le \theta - \theta_R \le 2\pi. \end{aligned}$$

The exact expression for $K(r_R, \theta - \theta_R)$ is not easy to compute, but it is not needed for the analysis here. Rather, the point is that with $i > 0$, the magnetic field $\vec{\mathbf{B}}(r_R, \theta - \theta_R, i)$ due to the current in the rotor loop is radially in on the flux surface shown in Figure 1.16, that is, for $\theta_R \leq \theta \leq \theta_R + \pi$. For convenience, the surface element is chosen to be $d\vec{\mathbf{S}} = r_R d\theta dz(-\hat{\mathbf{r}})$ so that positive direction of travel around the surface coincides with the positive direction of the current i in the loop. The flux ψ in the rotor loop is then computed as[6]

$$
\begin{aligned}
\psi(i) &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_{\theta_R}^{\theta_R+\pi} \int_0^{\ell_1} iK(r_R, \theta - \theta_R)\,(-\hat{\mathbf{r}}) \cdot (-r_R d\theta dz \hat{\mathbf{r}}) \\
&= i \int_{\theta_R}^{\theta_R+\pi} \int_0^{\ell_1} K(r_R, \theta - \theta_R) r_R d\theta dz \\
&= Li
\end{aligned}
$$

where

$$L \triangleq \int_{\theta_R}^{\theta_R+\pi} \int_0^{\ell_1} K(r_R, \theta - \theta_R) r_R d\theta dz > 0. \tag{1.5}$$

This last equation just says the flux in the loop (due to the current in the loop) is proportional to the current i in the loop. The proportionality constant L is the called the *inductance* of the loop.[7] If $-d\psi/dt = -Ldi/dt > 0$, then the induced emf will force current into the page $\otimes$ on side a and out of the page $\odot$ of side a' in Figure 1.16. That is, this induced emf has the same sign convention as the armature current i and the source voltage V_S.

With the rotor locked at some angle θ_R so that the external magnetic field cannot induce an emf in the rotor loop, the equation describing the current i in the rotor loop is given by Kirchhoff's voltage law

$$V_S - Ri - L\frac{di}{dt} = 0$$

or

$$V_S = Ri + L\frac{di}{dt}.$$

Here R is the resistance of the loop and V_S is the source voltage applied to the loop. The loop and its equivalent circuit are shown in Figure 1.17.

---

[6] The notation ψ is used to distinguish this flux from the flux ϕ in the loop due to the *external* permanent magnet. However, the total flux using an inward normal would be $\psi - \phi$ as the *outward* normal was used to compute ϕ in Section 1.3.4.

[7] It appears from equation (1.5) and Figure 1.16 that L can vary with θ_R. However, in an actual motor, there are loops spread evenly around the complete periphery of the rotor and, due to symmetry, the total self-inductance does not depend on θ_R.

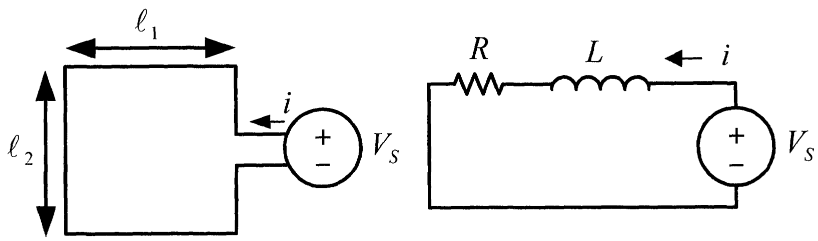

FIGURE 1.17. Left: Rotor loop. Right: Equivalent circuit.

The reader should convince himself/herself that Lenz's law holds as it must. For example, suppose a voltage $V_S > 0$ is applied to the loop resulting in both $i > 0$ and $di/dt > 0$, that is, the flux $\psi = Li$ is positive and increasing. The induced voltage is $-Ldi/dt < 0$ and opposes the current i producing the increasing flux $\psi = Li$. In this circumstance, the voltage source V_S is forcing the current i against this induced voltage $-Ldi/dt$ and the power absorbed by the induced voltage is $-iLdi/dt = -d\left(\frac{1}{2}Li^2\right)/dt$. This power is stored in the energy $\frac{1}{2}Li^2$ of the magnetic field surrounding the loop.

Arcing Between the Commutator and Brushes

Suppose the single-loop motor is rotating at constant speed ω_0 with a constant current i_0 in the rotating loop. Let L be the inductance of the loop. Now, every half-turn, the current in the loop reverses direction as shown in Figures 1.8(b)–(d). During this commutation, the current in the loop goes from i_0 to 0 to $-i_0$ (or vice versa) with a corresponding change in the loop's flux given by $\Delta\psi = L(-i_0) - Li_0 = -2Li_0$. By Faraday's law, the self-induced emf is then $-\Delta\psi/\Delta t = 2Li_0/\Delta t$ where Δt is the time for the current to change direction. Note that this time Δt decreases as the motor speed increases, so that, even if L is small, the induced emf in the loop (due to the reversal of current in the loop) can be quite large at high motor speeds. Large electric fields are produced by the induced voltage Ldi/dt when the loop is shorted which in turn ionizes the surrounding air. As the free electrons collide and recombine with the ionized air, light is given off and seen as arcing or sparking. These large voltages which cause the arcing between the slip rings and the brushes can damage the brushes as well as produce unwanted transient currents in the armature circuit.

1.4 Dynamic Equations of the DC Motor

Based on the simple single-loop DC motor analyzed above, the complete set of equations for a DC motor can be found. The total emf (voltage) in the loop due to the voltage source V_S, the external permanent magnet and the changing current i in the rotor loop is

$$V_S - K_b\omega_R - L\frac{di}{dt}.$$

This voltage goes into building up the current in the loop against the loop's resistance, that is,

$$V_S - K_b\omega_R - L\frac{di}{dt} = Ri$$

or

$$L\frac{di}{dt} = -Ri - K_b\omega_R + V_S.$$

This relationship is often illustrated by the equivalent circuit given in Figure 1.18. Recall that the torque τ_m on the loop due to the external magnetic field acting on the current in the loop is

$$\tau_m = K_T i$$

where $K_T \triangleq \ell_1\ell_2 B$ is called the torque constant. By connecting a shaft and gears to one end of the loop, this motor torque can be used to do work (lift weight, etc.). Let $-f\omega_R$ model the friction torque (due to the brushes, bearings, etc.) where f is the coefficient of viscous friction and let τ_L be the load torque (e.g., due to a weight being lifted).

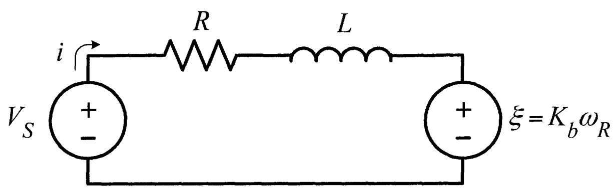

FIGURE 1.18. Equivalent circuit of the armature electrical dynamics.

Then, by Newton's law,

$$\tau_m - \tau_L - f\omega_R = J\frac{d\omega_R}{dt}$$

where J is the moment of inertia of the rotor (See the appendix of this chapter). The system of equations characterizing the DC motor is then

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega_R + V_S \\ J\frac{d\omega_R}{dt} &= K_T i - f\omega_R - \tau_L \\ \frac{d\theta_R}{dt} &= \omega_R. \end{aligned} \tag{1.6}$$

A picture of a DC motor servo system and its associated schematic is shown in Figure 1.19. In the schematic, R is the resistance of the rotor loop, L is the inductance of the rotor loop, $\xi = K_b\omega_R$ is the back emf, $\tau_m = K_T i$ is the motor torque, J is the rotor moment of inertia, and f is the coefficient of viscous friction. The positive directions for τ_m, θ_R, and τ_L are indicated by the curved arrows. The fact that the curved arrow for τ_L is opposite to that of τ_m just means that if the load torque is positive then it opposes a positive motor torque τ_m.

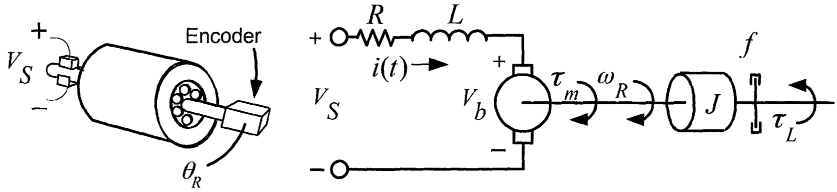

FIGURE 1.19. DC motor drawing and schematic.

Electromechanical Energy Conversion

The mechanical power produced by the DC motor is $\tau_m\omega_R = K_T i\omega_R = i\ell_1\ell_2 B\omega_R$ while the electrical power absorbed by the back emf is $i\xi = iK_b\omega_R = i\ell_1\ell_2 B\omega_R$. The fact that $K_T = K_b = \ell_1\ell_2 B$ must be for conservation of energy to hold. That is, the electrical power absorbed by the back emf equals (is converted to) the mechanical power produced. Another way to view this energy conversion is to write the electrical equation as

$$V_S = Ri + L\frac{di}{dt} + \xi.$$

The power out of the voltage source $V_S(t)$ is given by

$$\begin{aligned} V_S(t)i(t) &= Ri^2(t) + Li\frac{di}{dt} + iK_b\omega_R \\ &= Ri^2 + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + K_T i\omega_R \\ &= Ri^2 + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + \tau_m\omega_R. \end{aligned}$$

Thus the power $V_S(t)i(t)$ delivered by the source goes into heat loss in the resistance R, into stored magnetic energy in the inductance L of the loop and the amount $i\xi$ goes into the mechanical energy $\tau_m\omega_R$.

Remark *Voltage and Current Limits*

The amount of voltage V_S that may be applied to the input terminals T_1, T_2 of the motor is limited by capabilities of the amplifier supplying the voltage, that is, $|V_S| \le V_{\max}$. Let $V_c(t)$ be the voltage commanded to the amplifier, then the actual voltage V_S out of the amplifier to the motor is limited by $V_{\max}$ as illustrated in Figure 1.20.

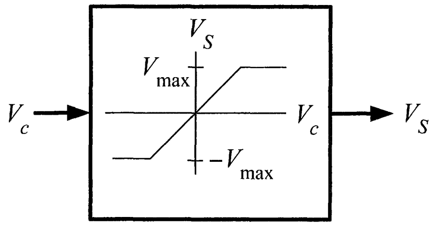

FIGURE 1.20. Saturation model of an amplifier.

In addition, there is a limit to the amount of current the rotating loop can handle before overheating or causing problems with commutation as previously mentioned. Typically there are two current limits (ratings), the *continuous* current limit $I_{\max\_\text{cont}}$ and the *peak* current limit $I_{\max\_\text{peak}}$. The continuous current limit $I_{\max\_\text{cont}}$ is the amount of current the motor can handle if left in use indefinitely. That is, the amount of heat dissipated in the rotor windings due to ohmic losses is equal to the amount of heat taken away by thermal conduction through the brushes and thermal convection with the air so as to be in a thermal equilibrium. The peak current limit $I_{\max\_\text{peak}}$ is the amount of current the motor can handle for short periods of time (typically only a few seconds).

1.5 Microscopic Viewpoint

Additional insight into the back emf ξ is found by calculating it from a microscopic point of view using the ideas given in Ref. [34] (page 887). To illustrate this approach, the back emf in the linear DC machine is recomputed from the microscopic point of view. To proceed, recall that the magnetic force on a charged particle q is $\vec{\mathbf{F}}_{\text{magnetic}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$, where $\vec{\mathbf{v}}$ is the velocity of the charge (see Ref. [34], page 816).

Example *A Linear DC Machine*

In this example, the linear DC machine is reanalyzed from the microscopic point of view. As before, $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$ where $B > 0$. Suppose the motor (bar) is moving to the right with a constant speed v_m. Each charge q in the sliding bar has total velocity $\vec{\mathbf{v}} = v_m\hat{\mathbf{x}} - v_d\hat{\mathbf{y}}$, where v_d is the drift speed of the charges down the wire. The magnetic force on the charge q is

$$\vec{\mathbf{F}}_{\text{magnetic}} = q\vec{\mathbf{v}}\times\vec{\mathbf{B}} = q(v_m\hat{\mathbf{x}} - v_d\hat{\mathbf{y}}) \times (-B\hat{\mathbf{z}}) = qv_mB\hat{\mathbf{y}} + qv_dB\hat{\mathbf{x}}.$$

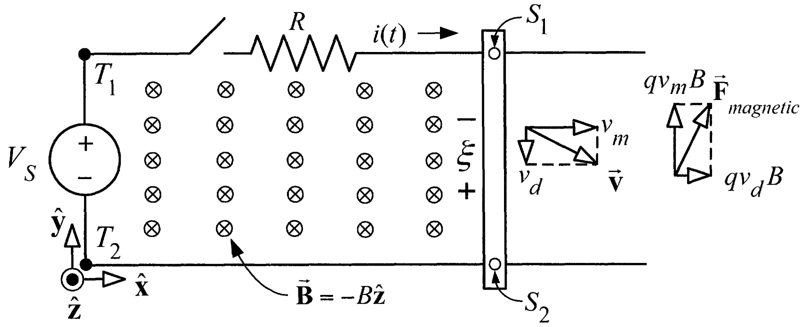

FIGURE 1.21. Linear DC machine.

Now, the component of force $qv_dB\hat{\mathbf{x}}$ perpendicular to the bar causes the bar to move to the right and the component $qv_mB\hat{\mathbf{y}}$ along the bar opposes the current flow. The source voltage V_S sets up an electric field $\vec{\mathbf{E}}_S$ in the bar to overcome the magnetic force $qv_mB\hat{\mathbf{y}}$ so as to make the current flow (setup the drift velocity v_d of the charge carriers against the resistance of the bar). In more detail, with T_1 and T_2 the upper and lower terminals of the source voltage, respectively, and S_1 and S_2 the upper and lower sliding contact points, respectively, the source voltage is given as

$$V_S = \int_{\mathrm{T_1-S_1-S_2-T_2}} \vec{\mathbf{E}}_S \cdot d\vec{\boldsymbol{\ell}}.$$

The quantity $q\vec{\mathbf{E}}_S$ is the force on each charge carrier and qV_S is the energy given to the charge carrier by the source voltage as the charge goes around the loop. There is also a component of the magnetic force on the charge carrier that opposes the electric field $\vec{\mathbf{E}}_S$. The energy per unit charge ξ that the magnetic force takes from the charge carrier as it goes down the bar from S_1 to S_2 is given by

$$\begin{aligned}\xi &= \frac{1}{q}\int_{S_1}^{S_2} \vec{\mathbf{F}}_{\mathrm{magnetic}} \cdot d\vec{\boldsymbol{\ell}} = \frac{1}{q}\int_{S_1}^{S_2} q(\vec{\mathbf{v}}\times\vec{\mathbf{B}}) \cdot d\vec{\boldsymbol{\ell}} = \int_0^{\ell} (v_dB\hat{\mathbf{x}}+v_mB\hat{\mathbf{y}}) \cdot (-d\ell\hat{\mathbf{y}}) \\ &= -v_mB\ell.\end{aligned}$$

The fact that ξ is *negative* just indicates that the magnetic force is taking energy out of the charge carrier as it goes down the bar from S_1 to S_2. (Note: The sign convention for ξ shown in Figure 1.21 is reversed from that of Figure 1.12.) This energy per unit charge ξ taken from each charge carrier by the magnetic force as it goes around the loop is called the induced emf. The voltage V_S was computed by integrating $\vec{\mathbf{E}}_S$ in the *clockwise* direction $\mathrm{T_1-S_1-S_2-T_2}$ around the loop, and ξ by integrating $\vec{\mathbf{v}}\times\vec{\mathbf{B}}$ also in the *clockwise* direction down the bar from S_1 to S_2; that is, they both have the same sign convention. This is in contrast to the macroscopic case where V_S and ξ had opposite sign conventions resulting in $\xi = v_m B\ell$ being positive. However, the same (physical) result occurs as in the macroscopic case.

In general, the emf in a loop is defined as the integral of the force per unit charge around the loop. The total emf is the sum of the source voltage and the induced emf. This total emf goes into producing the current, that is,

$$V_S + \xi = V_S - v_m \ell B = Ri$$

where an identical equation was found in the macroscopic case using Faraday's law. The *total* magnetic force on all the charge carriers in the bar in the $\hat{\mathbf{x}}$ direction is given by

$$q(NS\ell)v_d B\hat{\mathbf{x}}$$

where N is the number of charge carriers/volume and S is the cross sectional area of the sliding bar. That is, $NS\ell$ is the total number of charge carriers in the sliding bar each experiencing the force $qv_d B\hat{\mathbf{x}}$. As illustrated in Figure 1.22, in a time Δt, the charges in the volume $NS(v_d\Delta t)$ have moved along the conductor past the point P in Figure 1.22.

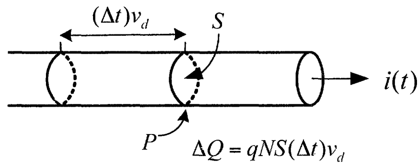

FIGURE 1.22. In the time Δt, the amount of charge $\Delta Q = qNS(v_d\Delta t)$ has moved past the point P resulting in the current $i = \Delta Q/\Delta t = qNSv_d$ in the bar.

That is, the amount of charge $\Delta Q = qNS(v_d\Delta t)$ has moved past the point P in the time Δt resulting in the current $i = \Delta Q/\Delta t = qNSv_d$ in the conductor. Consequently, the total magnetic force on the bar may be rewritten as

$$q(NS\ell)v_d B\hat{\mathbf{x}} = (qNSv_d)\ell B\hat{\mathbf{x}} = i\ell B\hat{\mathbf{x}}$$

which is identical to the expression derived from the macroscopic point of view.

1.5.1 Microscopic Viewpoint of the Single-Loop DC Motor

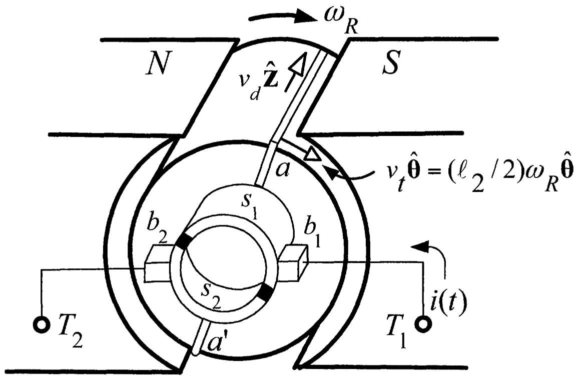

FIGURE 1.23. Single loop DC motor. Adapted from Ref. [21].

The back emf in the single-loop DC motor of Figure 1.23 is now computed from the microscopic point of view. With the loop rotating at the angular speed ω_R, the velocity of the charge carriers that make up the current is given by

$$\vec{\mathbf{v}} = \begin{cases} v_t\hat{\boldsymbol{\theta}} + v_d\hat{\mathbf{z}} & \text{for side } a \\ v_t\hat{\boldsymbol{\theta}} - v_d\hat{\mathbf{z}} & \text{for side } a' \end{cases}$$

where v_d is the drift speed of the charge carriers along the wire and $v_t = (\ell_2/2)\omega_R$ is the tangential velocity due to the rotating loop. Recall that the drift speed has the same sign as the current, that is, $v_d > 0$ for $i > 0$. Also recall that the angular velocity is written as $\vec{\boldsymbol{\omega}}_R = \omega_R\hat{\mathbf{z}}$ where $\hat{\mathbf{z}}$ is the axis the motor is turning about. The magnetic force per unit charge $\vec{\mathbf{F}}_{\text{magnetic}}/q$ on the charge carriers on the axial sides of the loop is

$$\vec{\mathbf{F}}_{\text{magnetic}}/q = \vec{\mathbf{v}}\times\vec{\mathbf{B}}$$

where

$$\vec{\mathbf{v}}\times\vec{\mathbf{B}} = \begin{cases} (v_t\hat{\boldsymbol{\theta}}+v_d\hat{\mathbf{z}}) \times (+B)\hat{\mathbf{r}} = -v_tB\hat{\mathbf{z}}+v_dB\hat{\boldsymbol{\theta}} & \text{for side } a \\ (v_t\hat{\boldsymbol{\theta}}-v_d\hat{\mathbf{z}}) \times (-B)\hat{\mathbf{r}} = +v_tB\hat{\mathbf{z}}+v_dB\hat{\boldsymbol{\theta}} & \text{for side } a' \end{cases}$$

or

$$\vec{\mathbf{v}}\times\vec{\mathbf{B}} = \begin{cases} v_dB\hat{\boldsymbol{\theta}} - \omega_R(\ell_2/2)B\hat{\mathbf{z}} & \text{for side } a \\ v_dB\hat{\boldsymbol{\theta}} + \omega_R(\ell_2/2)B\hat{\mathbf{z}} & \text{for side } a'. \end{cases}$$

The component $v_dB\hat{\boldsymbol{\theta}}$ is what produces the torque. In more detail, with N the number of charge carriers/unit volume, S the cross-sectional area of the wire loop, and ℓ_1 the axial length of the loop, the quantity $NS\ell_1$ is the total number of charge carriers of each side of the loop and the current due to the movement of these charges is $i = qNSv_d$ (see Figure 1.22). The total tangential forces on the axial sides of the rotor loop are given by

$$\begin{aligned}\vec{\mathbf{F}}_{\text{side } a} &= (qNS\ell_1)v_d B\hat{\boldsymbol{\theta}} = i(t)\ell_1 B\hat{\boldsymbol{\theta}}\\ \vec{\mathbf{F}}_{\text{side } a'} &= (qNS\ell_1)v_d B\hat{\boldsymbol{\theta}} = i(t)\ell_1 B\hat{\boldsymbol{\theta}}.\end{aligned}$$

The torque is then

$$\vec{\boldsymbol{\tau}} = \frac{\ell_2}{2}\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\text{side } a} + \frac{\ell_2}{2}\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\text{side } a'} = 2(\frac{\ell_2}{2}\hat{\mathbf{r}}) \times (i\ell_1 B\hat{\boldsymbol{\theta}}) = i\ell_1\ell_2 B\hat{\mathbf{z}}$$

which is the same result as in the macroscopic case.

It is now shown that the $\hat{\mathbf{z}}$ component of the magnetic force produces the back emf. The $\hat{\mathbf{z}}$ component of $\vec{\mathbf{F}}_{\text{magnetic}}$, that is, along the axial sides of the loop, is given by

$$(\vec{\mathbf{F}}_{\text{magnetic}}/q)_z\hat{\mathbf{z}} = \begin{cases} -\omega_R(\ell_2/2)B\hat{\mathbf{z}} & \text{for side } a\\ +\omega_R(\ell_2/2)B\hat{\mathbf{z}} & \text{for side } a'.\end{cases}$$

As shown in Figure 1.23, this component of the magnetic force per unit charge $(\vec{\mathbf{F}}_{\text{magnetic}}/q)_z$ opposes the electric field $\vec{\mathbf{E}}_S$ set up in the loop by the applied armature voltage V_S. The relationship between V_S and $\vec{\mathbf{E}}_S$ is

$$V_S = \int_{T_1}^{T_2} \vec{\mathbf{E}}_S \cdot d\vec{\boldsymbol{\ell}}$$

where

$$d\vec{\boldsymbol{\ell}} = \begin{cases} +d\ell\hat{\mathbf{z}} & \text{for side } a\\ -d\ell\hat{\mathbf{z}} & \text{for side } a'.\end{cases}$$

Then, with ξ denoting the integral of the magnetic force per unit charge from T_1 to T_2, one has

$$\begin{aligned}\xi &\triangleq \int_{T_1}^{T_2} (\vec{\mathbf{F}}_{\text{magnetic}}/q) \cdot d\vec{\boldsymbol{\ell}}\\ &= \int_{\text{side } a} (-\omega_R(\ell_2/2)B\hat{\mathbf{z}}) \cdot (d\ell\hat{\mathbf{z}}) + \int_{\text{side } a'} (\omega_R(\ell_2/2)B\hat{\mathbf{z}}) \cdot (-d\ell\hat{\mathbf{z}})\\ &= \int_{\ell=0}^{\ell=\ell_1} -\omega_R(\ell_2/2)Bd\ell + \int_{\ell=0}^{\ell=\ell_1} -\omega_R(\ell_2/2)Bd\ell\\ &= -\omega_R(\ell_2/2)B\ell_1 - \omega_R(\ell_2/2)B\ell_1\\ &= -\ell_1\ell_2 B\omega_R.\end{aligned}$$

In this example, the (back) emf ξ is due to the magnetic force while V_S is due to the electric field set up by the voltage source. As the induced emf ξ and the source voltage V_S have the same sign convention, the minus sign in the expression for ξ shows that it opposes the applied armature voltage V_S. The total emf (voltage) in the loop is

$$V_S + \xi = V_S - \ell_1\ell_2 B\omega_R.$$

Finally, the equation governing the current in the rotor loop is

$$L\frac{di}{dt} + Ri = V_S - \ell_1\ell_2 B\omega_R.$$

This is the same physical result as shown in the macroscopic case using Faraday's law. However, here the induced emf $\xi = -\ell_1\ell_2 B\omega_R$ is negative because it was chosen to have the same sign convention as V_S (i.e., both V_S and ξ are positive going from T_1 to T_2). This is in contrast to the macroscopic case in which they had opposite sign conventions so that $\xi = \ell_1\ell_2 B\omega_R$ was positive, but still opposed V_S.

Remark *Voltage and Emf*

The *electromotive force* or *emf* between two points in a circuit is the integral of the total force per unit charge along the circuit between those two points.[8] The force per unit charge can be due to an electric field, a magnetic field or both. The term voltage (drop) was originally reserved for the integral of the *electric field* between the two points. However, this distinction is usually not made and the two terms (voltage and emf) are used interchangeably.

1.5.2 Drift Speed

Above, it was shown that the drift speed of the charge carriers making up the current is given by $v_d(t) = i(t)/(qNS)$. As explained in Ref. [34] (page 781), this motion (drift speed) of the charges in the conductor is caused by the electric field setup in the conductor by the voltage source and/or induced emfs in the conducting circuit. This electric field is pushing the charges along the conductor against the internal resistance of the conductor. In metals, the outer valence electrons are free to move about the lattice of the metal and are called *conduction electrons.* For example, in copper there is one valence electron per atom and the other 28 electrons remain bound to the copper nucleus. Consequently, as there are 8.4×10^{22} atoms/cm^3 in copper, there are $N = 8.4 \times 10^{22}$ electrons/cm^3 that can move freely within the copper lattice to make up the current in the wire. To consider a simple numerical example, let $i = 10$ A, $S = 0.1$ cm^2 so that with $q = 1.6 \times 10^{-19}$ coulomb/electron, the corresponding drift speed is

$$\begin{aligned} v_d &= \frac{i(t)}{qNS} = \frac{10 \text{ A}}{(1.6 \times 10^{-19} \frac{\text{coulomb}}{\text{electron}})(8.4 \times 10^{22} \frac{\text{electrons}}{\text{cm}^3})(0.1 \text{ cm}^2)} \\ &= 0.74 \text{ cm/sec.} \end{aligned}$$

[8] Note that the electromotive force is *not* a force, but rather an energy per unit charge.

That is, it takes $1/(0.74\ \text{cm/sec}) = 1.35$ sec for the charge carrier to travel one centimeter. However, it should be noted that when a voltage/emf is applied to a circuit, the corresponding electric field is setup around the circuit at a speed close to the speed of light. This is analogous to applying pressure to a long tube of water. The pressure wave is transmitted down the tube rapidly (at the speed of sound in water) while the water itself moves much slower [34].

1.6 Tachometer for a DC Machine*[9]

A tachometer is a device for measuring the speed of a DC motor by putting out a voltage proportional to the motor's speed. A tachometer for the simple linear DC machine is considered first.

1.6.1 Tachometer for the Linear DC Machine

Figure 1.24 shows a tachometer added to the linear DC machine. The magnetic field in the DC motor is $\vec{\mathbf{B}}_1 = -B_1\hat{\mathbf{z}}$ with $B_1 > 0$ while in the tachometer it is $\vec{\mathbf{B}}_2 = -B_2\hat{\mathbf{z}}$ with $B_2 > 0$.

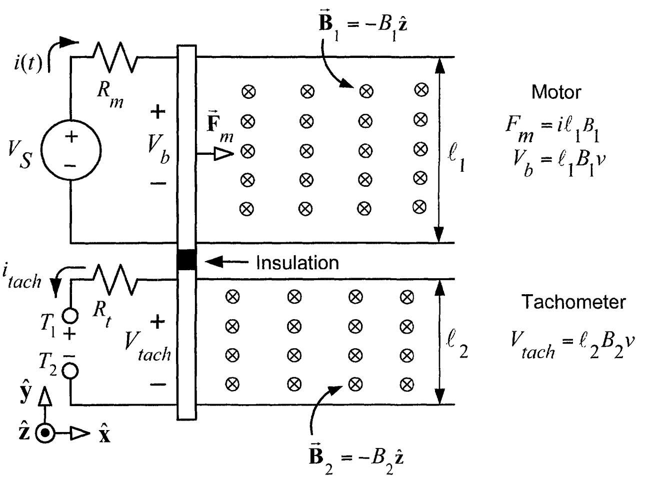

FIGURE 1.24. DC tachometer (generator).

The two bars are rigidly connected together by the insulating material. The motor force (the magnetic force on the upper bar) is $F_m = i\ell_1 B_1$, and

[9] Sections marked with an asterisk (*) may be skipped without loss of continuity.

the induced (back) emf in the motor is $\xi = V_b = B_1\ell_1 v$, where v is the speed of the motor (bar).

The induced (back) emf in the tachometer is given by $\xi = V_{tach} = B_2\ell_2 v$ so that by measuring the voltage between the terminals T_1 and T_2, the speed v of the motor can be computed. Note that the tachometer and motor have the same physical structure. In fact, the tachometer is nothing more than a generator putting out a voltage proportional to the speed.

1.6.2 Tachometer for the Single-Loop DC Motor

A tachometer for the single loop DC motor is constructed by attaching another loop to the shaft and rotating it an external magnetic field to act as a DC generator. That is, the changing flux in the tachometer loop produces (generates) an induced emf according to Faraday's law and this emf is proportional to the shaft's speed. To see this, consider Figure 1.25, where a motor loop is driven by a voltage V_S and, attached to the same shaft, is a second loop called a tachometer. Both loops rotate in an external radial magnetic field which is not shown in Figure 1.25, but is shown for the tachometer loop in Figure 1.26. It is important to point out that no voltage is applied to the terminals T_1 and T_2 of the tachometer as was the case for the motor. Instead, the voltage V_{tach} between the terminals T_1 and T_2 of the tachometer is measured (this voltage is proportional to the motor speed ω_R).

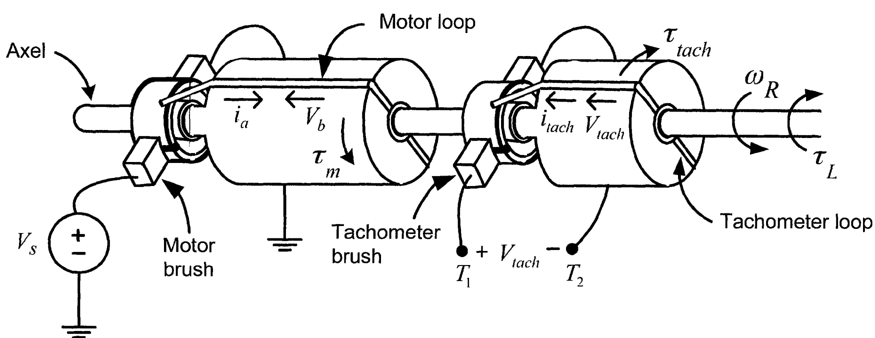

FIGURE 1.25. Single loop motor and tachometer. Drawn by Sharon Katz.

Specifically, in the same way the back emf was computed for the DC motor, one can calculate the flux in the loop of the tachometer due to the external magnetic field. This computation is (see Figure 1.26)

$$
\begin{aligned}
\phi &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \\
&= \int_0^{\ell_1} \int_{\theta_R}^{\pi} (B\hat{\mathbf{r}}) \cdot (\frac{\ell_2}{2} d\theta dz \hat{\mathbf{r}}) + \int_0^{\ell_1} \int_{\pi}^{\pi+\theta_R} (-B\hat{\mathbf{r}}) \cdot (\frac{\ell_2}{2} d\theta dz \hat{\mathbf{r}}) \\
&= \int_0^{\ell_1} \int_{\theta_R}^{\pi} B\frac{\ell_2}{2} d\theta dz + \int_0^{\ell_1} \int_{\pi}^{\pi+\theta_R} -B\frac{\ell_2}{2} d\theta dz \\
&= (\ell_1\ell_2 B/2)(\pi - \theta_R) - (\ell_1\ell_2 B/2)\theta_R \\
&= -\ell_1\ell_2 B\theta_R + (\ell_1\ell_2 B/2)\pi.
\end{aligned}
$$

The induced emf is then

$$V_{\text{tach}} = -d\phi/dt = (\ell_1\ell_2 B) d\theta_R/dt = K_{b\_\text{tach}}\omega_R$$

where $K_{b\_\text{tach}} = \ell_1\ell_2 B$ is a constant depending on the dimensions of the tachometer rotor and the strength of the external magnetic field of the tachometer. This shows that the voltage between the terminals T_1 and T_2 is proportional to the angular speed and therefore can be used to measure the speed.

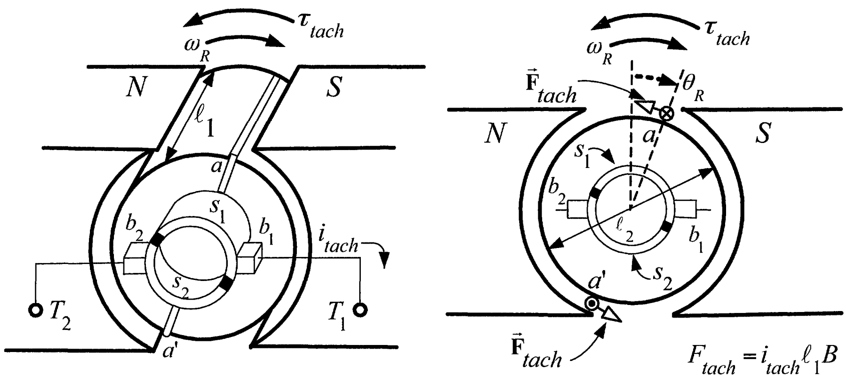

FIGURE 1.26. Cutaway view of the DC tachometer. From *Electromagnetic and Electromechanical Machines,* 3rd edition, L. W. Matsch and J. Derald Morgan, 1986. Reprinted by permisson of John Wiley & Sons.

1.7 The Multiloop DC Motor*

The above single loop motor of Figure 1.5 was used to illustrate the basic Physics of the DC motor. However, it is not a practical motor. The first thing that must be done is to add more loops to extract more torque from the machine. Further, in the single-loop motor, the magnetic field produced by the current in the *rotor* is an external magnetic field acting on the field windings. As the loop rotates, this magnetic field results in a changing flux in the field windings, which in turn induces an emf in the field windings. This emf is referred to as the *armature reaction* and makes it difficult to maintain a constant field current. (The term armature refers to the rotating current loop, and reaction refers to the induced emf in the field windings produced by the rotor current.) The problem of armature reaction can be alleviated by adding more loops to the motor.

1.7.1 Increased Torque Production

Figure 1.27 below shows the addition of several loops to the motor with each loop similar in form to the loop in Figure 1.5. As shown in the figure, there are now eight slots in the rotor with two loops placed in each pair of slots that are 180° apart for a total of eight loops.

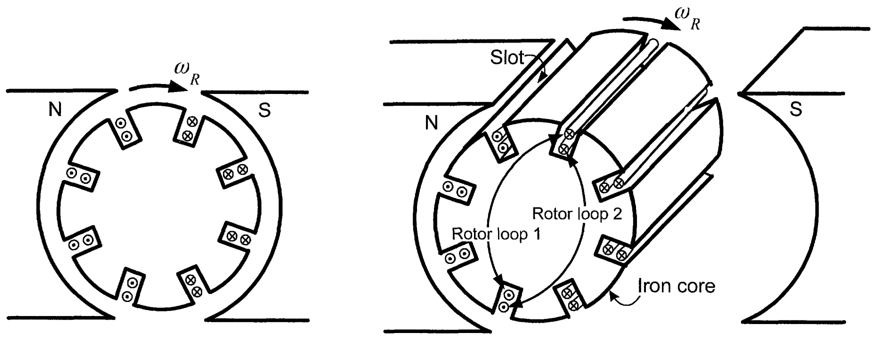

FIGURE 1.27. A multiloop armature for a DC motor.

The torque on the rotor is now $\tau_m = n\ell_1\ell_2 Bi$, where $n = 8$ is the number of rotor loops and B is the strength of the radial magnetic field in the air gap produced by the external magnetic field. Of course, some method must be found to ensure the current in each loop is reversed every half-turn so that (for positive torque) all the loop sides under the south pole face will have their current going into the page $\otimes$ and all the loop sides under the north pole face will have their current coming out of the page $\odot$. This process is referred to as *commutation* and is considered next.

1.7.2 Commutation of the Armature Current

As seen in Figure 1.27, as a rotor loop rotates clockwise past the vertical position, the current in the top side of the loop must change direction from coming out of the page to going into the page. That is, each rotor loop must have the current in it reversed every half-turn. This is done using a commutator which is illustrated in Figure 1.28 for the rotor shown in Figure 1.27. The commutator for this rotor consists of 8 copper segments (labeled a–h in 1.30(a)) which are separated by insulating material. By connecting each of the ends of the rotor loops of Figure 1.27 to the appropriate copper segments of the commutator, the current will be reversed every half-turn as it rotates past the vertical. To explain all of this, consider Figure 1.30(a), which shows explicitly how the ends of the rotor loops are connected to the segments of the commutator. The eight rotor loops of Figure 1.27 are labeled as $1-1'$,..., $8-8'$ in Figure 1.30(a) with the ends of each such loop electrically connected (soldered) to a particular pair of commutator segments. For example, the ends of loop $1-1'$ are connected to commutator segments a and b, respectively. The commutator and rotor loops all rotate together rigidly while the two brushes (labeled b_1 and b_2) remain stationary. The brushes are typically made of a carbon material and are mechanically pressed against the commutator surface, making electrical contact.[10] That is, as the commutator rotates, the particular segment that is rubbing against the brush makes electrical contact.

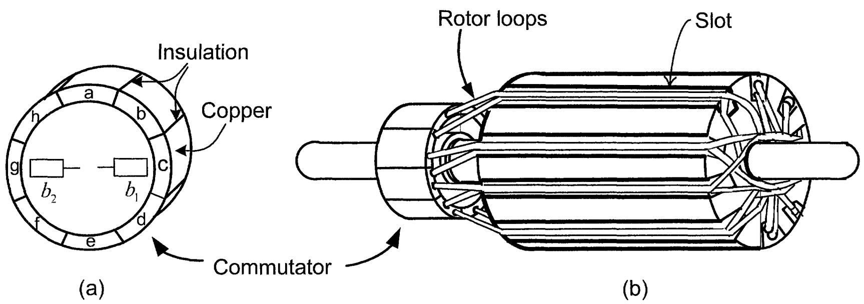

FIGURE 1.28. Commutator for the rotor in Figure 1.27.

Figure 1.29 is a photograph of the rotor of an actual DC motor with a tachometer.

[10] The figure shows a gap between the brushes and the commuator, but this was done for illustration and there is no gap in reality. Also, for illustrative purposes, the brushes are shown inside the commutator when in fact they are normally pressed against the commutator from the outside.

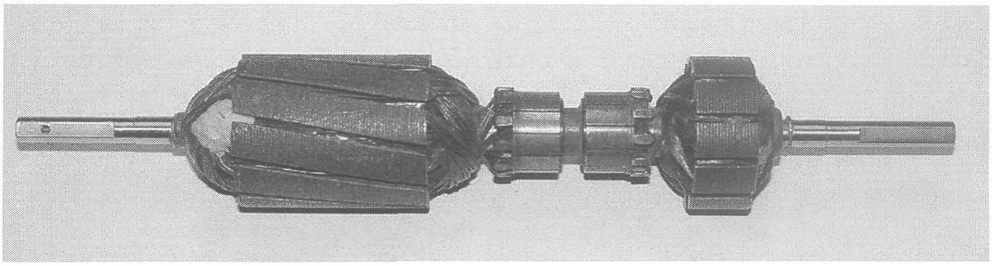

FIGURE 1.29. Photo of the rotor of a DC motor (left) and its tachometer (right). Note that the slots for the windings of the DC motor are skewed (see problem 9). Photo courtesy of Professor J. D. Birdwell of the University of Tennessee.

As previously explained, to obtain positive torque, it must be that whenever a side of a loop is under a south pole face, the current must be into the page ($\otimes$) and the other side of the loop, which is under the north pole face, must have its current out of the page ($\odot$). When the loop side rotates from being under one pole face to the other pole face, the current in that loop must be reversed (commutated). The mechanism of how this connection between the armature loops, the commutator and the brushes can reverse the current in each rotor loop every half turn is now explained.

With reference to Figure 1.30(a), the armature current i enters brush b_1 and into commutator segment c. By symmetry, half of this armature current (i.e., $i/2$) goes through loop $3-3'$ into commutator segment d, then through loop $4-4'$ into commutator segment e, then through loop $5-5'$ into commutator segment f, then through loop $6-6'$ into commutator segment g, and, finally, out through brush b_2. This path (circuit) of the current is denoted in **bold**. Similarly, there is a parallel path for the other half of the current armature current. Specifically, the other half of the armature current $i/2$ goes through loop $2'-2$ into commutator segment b, then through loop $1'-1$ into commutator segment a, then through loop $8'-8$ into commutator segment h, then into loop $7'-7$ into commutator segment g, and finally, out through brush b_2. This path (circuit) is denoted without bold. So, for the rotor in the position shown in Figure 1.30(a), there are two parallel circuits from b_1 to b_2 each made up of four loops connected in series and each circuit carries half of the armature current. The sides of the loops under the south pole face have their current into the page while the other side of these loops (which are under the north pole face) have their current out of the page so that positive torque is produced.

The sides of the loops in Figure 1.30(a) are 45° apart. Figure 1.30(b) shows the rotor turned 45°/2 with respect to Figure 1.30(a). In this case, brush b_1 shorts the two commutator segment b and c together while the brush b_2 shorts together the two commutator segments f and g. The ends of loop $2-2'$ are connected to commutator segments b and c (which are now shorted together) so that the current in this loop is now zero. Similarly, the ends of loop 6–6′ are connected to commutator segments f and g and the current in this loop is also zero. For the remaining loops, $i/2$ goes through loop 3–3′ into commutator segment d, then through loop 4–4′ into commutator segment e, then into loop 5–5′ into commutator segment f, and finally, out brush b_2. These loops are denoted in **bold** in the figure. Similarly, $i/2$ goes through loop 1–1′ into commutator segment a, then through loop 8′–8 into commutator segment h, then into loop 7′–7 into commutator segment g, and finally out brush b_2.

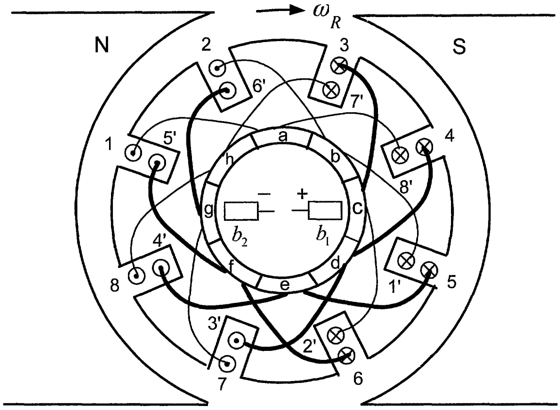

FIGURE 1.30. (a). Rotor loops and commutator for 4 sets of rotor loops. Brushes remained fixed in space, that is, they do not rotate. From *Electric Machinery Fundamentals,* 2nd edition by S. J. Chapman, McGraw-Hill 1991. Reproduced with permission of McGraw-Hill Companies.

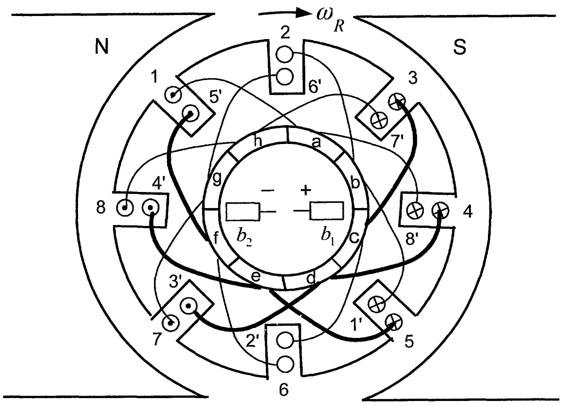

Figure 1.30(b). Rotor turned 45°/2 with respect to Figure 1.30(a). Adapted from Chapman [19].

The motor continues to rotate and consider it now after it has moved additional $45°/2$ so that it has the position shown in Figure 1.30(c). In this case, the current enters brush b_1 and into commutator segment b. By symmetry, half the current $i/2$ goes through loop $2{-}2'$ into commutator segment c, then through loop $3{-}3'$ into commutator segment d, then through loop $4-4'$ into commutator segment e, then through loop $5{-}5'$ into commutator segment f, and finally out through brush b_2. This path (circuit) of the current is denoted in **bold**. Similarly, the other half of the current goes through loop $1'{-}1$ into commutator segment a then through loop $8'{-}8$ into commutator segment h then through loop $7'{-}7$ into commutator segment g then into loop $6'{-}6$ into commutator segment f and finally, out through brush b_2. This path (circuit) is denoted without using bold.

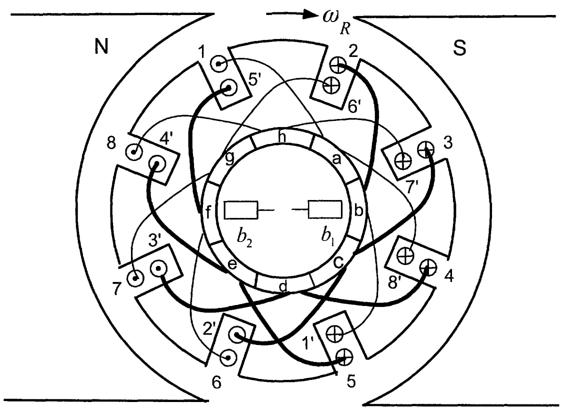

Figure 1.30(c). Rotor turned 45° with respect to Figure 1.30(a). Adapted from Chapman [19].

As the sequence of figures 1.30(a)–(c) show, the current in loops $2{-}2'$ and $6{-}6'$ were reversed as these two loops rotated past the vertical position. In summary, there are two parallel paths, each consisting of four loops, and when any loop goes to the vertical position, the current in that loop is reversed. In this way, all sides of the loop under the south pole have their current going into the page and all sides under the north pole have their current coming out of the page for positive torque production.

Remark The scheme for current commutation presented here is from [19]. However, there are many other schemes and the reader is referred to Refs. [19], [21], [24], and [26] for an introduction to other schemes. See Ref. [24] for a discussion of how commutation is often carried out in small PM DC motors.

Stator Iron Construction

A more realistic depiction of the stator for a (single pole pair) permanent magnet DC motor is shown in Figure 1.31. The radial magnetic field in the air gap is produced by the two semicircular-shaped permanent magnets.

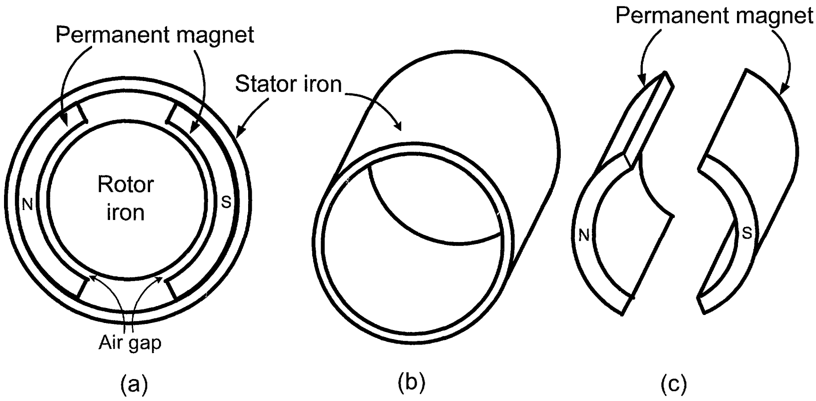

FIGURE 1.31. (a) Stator of a PM DC motor. (b) Stator iron. (c) Stator permanent magnet.

In the case where the DC motor has a field winding, a more realistic depiction of the stator iron (single pole pair) is shown in Figure 1.32 (compare with Figure 1.7).

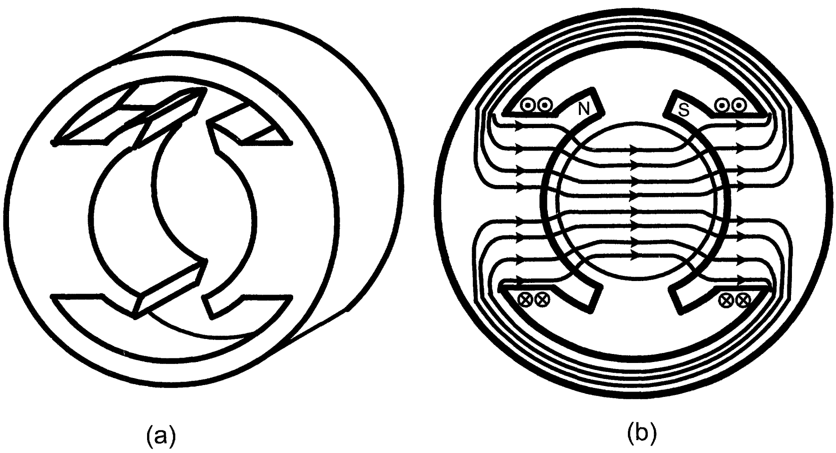

FIGURE 1.32. (a) Stator iron for a wound field DC motor. (b) Cross-sectional view of the stator iron with the field windings.

1.7.3 *Armature Reaction*

Figure 1.33 shows current in a field winding which is used to magnetize the iron in a fashion similar to that shown in Figure 1.7. Closed curves are drawn in Figure 1.33 to show the magnetic field distribution in the iron and air gap due to just the field current. Note that the magnetic field tends to be in the horizontal direction as it goes inside the field windings. Figure 1.34 shows the magnetic field distribution in the iron and air gap due to just the armature current. By having many equally spaced loops on the rotor, the magnetic field distribution due to the armature current will be as shown in Figure 1.34 for *any* rotor position. Note that this magnetic field tends to be vertical in the iron core inside the field windings (the field windings are not shown in Figure 1.34). In other words, in the stator iron core within the field windings, the magnetic field due to the armature current is perpendicular to the flux surfaces of the field windings. As a result, any changing magnetic field due to a changing armature current will not induce voltages inside the field windings. *Armature reaction* refers to the voltage induced in the field winding by the magnetic field of the armature current. (This is undesirable as such a voltage would cause the current in the field winding to change, and therefore not be constant as the analysis up to this point has assumed.) The symmetric placement of rotor loops around the periphery of the rotor essentially eliminates the armature reaction.

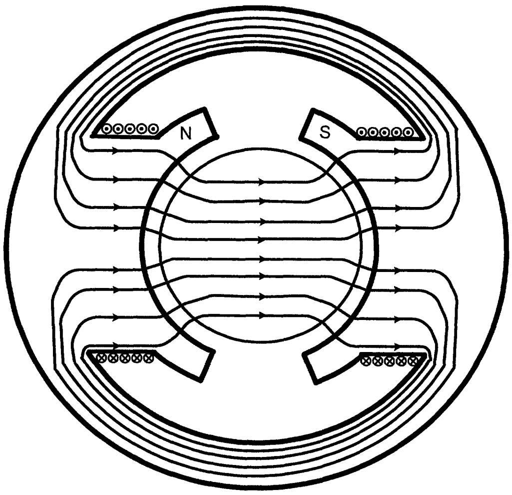

FIGURE 1.33. Magnetic field due to the field current. Adapted from Ref. [23].

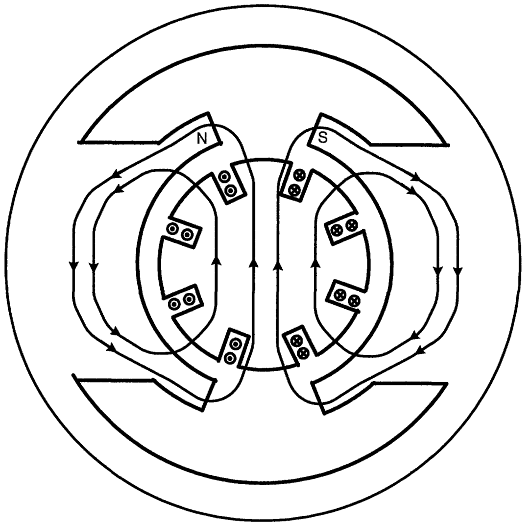

FIGURE 1.34. Magnetic field distribution due to the armature current. Adapted from Ref. [23].

Finally, Figure 1.35 shows the sum of the two magnetic fields.

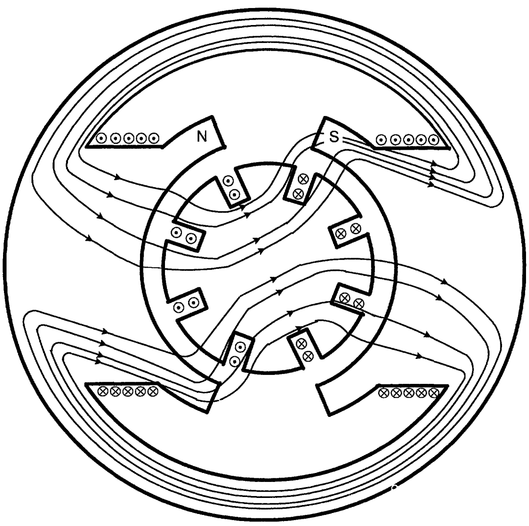

FIGURE 1.35. Magnetic field due to field and armature currents. Adapted from Ref. [23].

1.7.4 Field Flux Linkage and the Air Gap Magnetic Field

In the separately excited DC motor of Figure 1.36, it is convenient to have an expression relating the radial magnetic field B in the air gap to the total flux in the field windings.

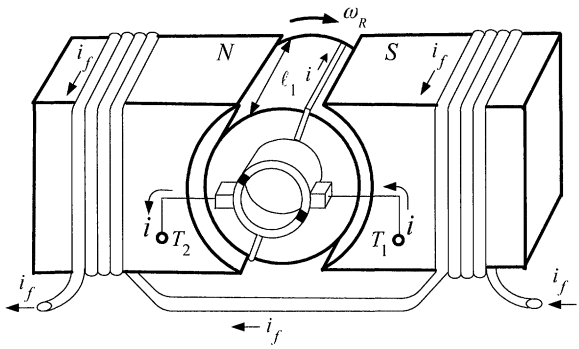

FIGURE 1.36. Separately excited DC motor. Adapted from Ref. [21].

The magnetic field lines are shown in Figure 1.4. Figure 1.37 shows a closed flux surface which is now used to derive a relationship between the radial magnetic field B in the air gap and the field flux linkage λ_f.

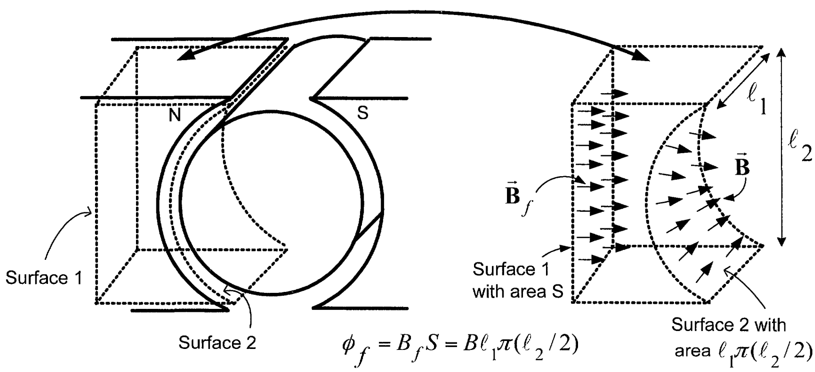

FIGURE 1.37. The flux through surface 1 is $B_f S$ and the flux through surface 2 is $\ell_1\pi(\ell_2/2)B$. To a good approximation, these two fluxes are equal.

The field flux *linkage* λ_f is defined by

$$\lambda_f(i_f) = N_f\phi_f = N_f S B_f(i_f)$$

where N_f is the total number of field windings, S is the cross-sectional area of the iron core of the field winding and $B_f(i_f)$ is the magnetic field produced *inside the magnetic material* of the field circuit due to the field current i_f. With this definition of flux linkage, the total induced emf in the field windings produced by a changing B_f is given by

$$\xi_{\text{fieldwinding}} = -\frac{d\lambda_f(i_f)}{dt} = -N_f\frac{d\phi_f}{dt}.$$

With the two flux surfaces shown in Figure 1.37, it turns out that *conservation of flux*[11] implies the flux through the surface 1 in the iron core equals the flux through the half cylindrical surface 2 in the air gap. That is,

$$\phi_f = SB_f(i_f) = \ell_1\pi(\ell_2/2)B(i_f)$$

where $\ell_2/2$ is the radius of the rotor, ℓ_1 is the axial length of the rotor and $B(i_f)$ is the radial magnetic field *in the air gap* due to the field current i_f. Consequently,

$$B(i_f) = \frac{SB_f(i_f)}{\pi\ell_1\ell_2/2} = \frac{\lambda_f(i_f)}{N_f\pi\ell_1\ell_2/2}$$

is an expression for the radial magnetic field $B(i_f)$ *in the air gap* in terms of the flux linkage $\lambda_f(i_f) = N_f\ell_1\ell_2B_f(i_f)$ in the field windings.

1.7.5 Armature Flux Due to the External Magnetic Field

In what follows, the multiloop motor of Figure 1.30 is considered in which the armature circuit consists of two parallel circuits each having n loops. Let θ_R be referenced relative to loop $1-1'$ so that $\theta_R = 0$ corresponds to loop $1-1'$ being vertical. (Recall in Figure 1.5 that $\theta_R = 0$ corresponded to loop $a-a'$ being vertical.) For the single loop motor, the flux $\phi_{sl}(i_f, \theta_R)$ in the loop $1-1'$ of the rotor due to the external magnetic field $B(i_f)$ is

$$\phi_{sl}(i_f, \theta_R) = -\ell_1\ell_2B(i_f)(\theta_R \bmod \pi - \pi/2).$$

In general, let $\theta_k = 0$ correspond to loop $k-k'$ being vertical so that the position of the kth rotor loop may be written as

$$\theta_k = \theta_R + (k-1)\pi/n \qquad \text{for } k = 1, ..., n$$

where n is the number of loops connected in series in each parallel circuit ($n = 4$ in Figure 1.30). The flux in the kth rotor loop is [see equation (1.4)]

$$\begin{aligned}\phi_k(i_f, \theta_R) &= -\ell_1\ell_2B(i_f)\left(\theta_k \bmod \pi - \frac{\pi}{2}\right) \\ &= -\ell_1\ell_2B(i_f)\left(\left\{\theta_R + (k-1)\frac{\pi}{n}\right\} \bmod \pi - \frac{\pi}{2}\right).\end{aligned}$$

[11] See problem 6 and Chapter 3.

The total flux linkage $\lambda(i_f, \theta_R)$ in the n rotor loops is then

$$\begin{aligned}\lambda(i_f, \theta_R) &= \sum_{k=1}^{n} \phi_k(i_f, \theta_R) \\ &= -\sum_{k=1}^{n} \ell_1 \ell_2 B(i_f) \left(\left\{ \theta_R + (k-1)\frac{\pi}{n} \right\} \bmod \pi - \frac{\pi}{2} \right). \quad (1.7)\end{aligned}$$

In Figures 1.30(a)–(c), *each* of the two parallel sets of n rotor loops has the flux linkage $\lambda(i_f, \theta_R)$ in it.

Recall that the sign convention for the fluxes $\phi_k(i_f, \theta_R)$ is such that if $-d\phi_k(i_f, \theta_R)/dt > 0$, then it is acting in the direction opposite to positive current flow, that is, its sign convention is opposite to that of applied voltage V_S. As a consequence, the sum

$$V_S - \left(-\frac{d\lambda(i_f, \theta_R)}{dt} \right) = V_S + \frac{d\lambda(i_f, \theta_R)}{dt}$$

is the total emf in the loop due to the applied voltage and the external magnetic field.

For $n = 4$, the flux linkage $\lambda(i_f, \theta_R)$ is plotted as a function of the rotor position θ_R in Figure 1.38. Note that $\partial\lambda(i_f, \theta_R)/\partial\theta_R = -n\ell_1\ell_2 B(i_f)$ which is proportional to the number of rotor loops n and to the strength of the external magnetic field strength $B(i_f)$ in the air gap.

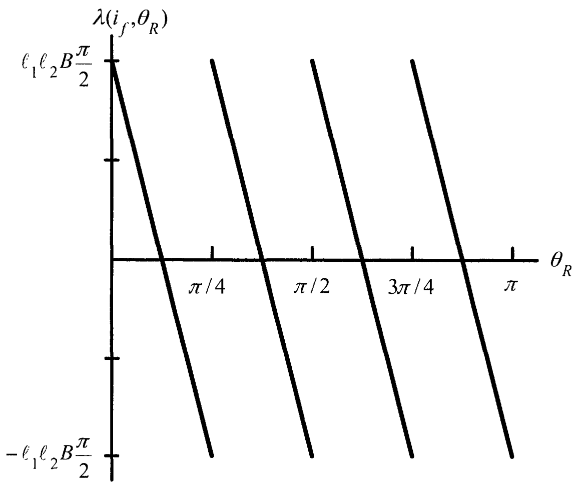

FIGURE 1.38. Flux linkage $\lambda(i_f, \theta_R)$ versus θ_R in radians with $n = 4$ sets of rotor loops and $0 \le \theta_R \le \pi$. The slope is $\partial\lambda(i_f, \theta_R)/\partial\theta_R = -n\ell_1\ell_2 B(i_f)$.

In the case of a permanent magnet stator, the armature flux linkage is simply given by

$$\begin{aligned}\lambda(\theta_R) &= \sum_{k=1}^{n}\phi_k(\theta_R)\\ &= -\sum_{k=1}^{n}\ell_1\ell_2 B\Big(\Big\{\theta_R+(k-1)\frac{\pi}{n}\Big\}\operatorname{mod}\pi-\frac{\pi}{2}\Big) \qquad (1.8)\end{aligned}$$

where $B > 0$ is the strength of the radial magnetic field in the air gap produced by the permanent magnet of the stator.

1.7.6 Equations of the PM DC Motor

In the multiloop motor considered here, the armature circuit consists of two parallel circuits each having n loops. That is, there are a total of $2n$ loops on the rotor as each of the two parallel circuits has a loop at the same location on the rotor. Let L denote the total inductance of the n rotor loops making up either of the two parallel circuits. Then, if a current $i/2$ is in each parallel circuit, each circuit will have a flux linkage of $Li/2$ due to its current and an *additional* flux linkage of $Li/2$ due to the current in the other circuit. This is simply because the two sets of parallel loops (windings) are wound around the rotor core together so that they can be considered to be perfectly (magnetically) coupled.

The total flux linkage in the n loops making up either of the parallel circuits is now computed. To proceed, let i be the current into the armature so that $i/2$ is the current in the n loops of each parallel circuit. The quantity $Li/2$ is the flux linkage due to the current $i/2$ in the loops and an additional flux linkage of $Li/2$ is produced in these same loops by the current $i/2$ in the *other* parallel circuit for a total flux of Li. By equation (1.8), $\lambda(\theta_R)$ is the flux linkage in the n loops due to the *external* magnetic field produced by the permanent magnet of the stator. Recall that the sign convention for the induced emf $-d\lambda(\theta_R)/dt$ is opposite to that of $-d(Li)/dt$. Specifically (see Section 1.3.5), the normal to the flux surface was taken to be *radially in* to compute the flux Li while in Section 1.3.4 the surface normal was taken to be *radially out* to compute the flux $\lambda(\theta_R)$. Simply writing $-\lambda(\theta_R)$ then gives the flux due to the external magnetic field with the surface normal radially in and the induced voltages by both of these changing fluxes will then have the same sign convention as the applied armature voltage. Using the same sign convention as the applied armature voltage V_S, the total flux linkage in each parallel circuit of the armature may be written as

$$Li - \lambda(\theta_R).$$

Let R_1 denote the resistance of the n loops connected in series making up either of the two parallel circuits. The equation describing the electrical dynamics of the current in each of the parallel circuits is found by applying Kirchhoff's voltage law to obtain

$$-\frac{d}{dt}\Big(Li - \lambda(\theta_R)\Big) - R_1 i/2 + V_S = 0$$

where V_S is the applied voltage to the armature. Finally, defining $R \triangleq R_1/2$, the equation describing the electrical dynamics of the armature circuit is

$$L\frac{di}{dt} = -Ri + \frac{d\lambda(\theta_R)}{dt} + V_S. \tag{1.9}$$

The quantity $d\lambda(\theta_R)/dt$ can be expanded to obtain

$$\frac{d\lambda(\theta_R)}{dt} = \frac{\partial\lambda(\theta_R)}{\partial\theta_R}\omega_R = -n\ell_1\ell_2 B\omega_R = -K_b\omega_R \tag{1.10}$$

where $K_b \triangleq n\ell_1\ell_2 B$.

Each loop carries the current $i/2$ so that the torque produced by the two sides of each loop is $2\,(\ell_2/2)\,(i/2)\ell_1 B$. As there are $2n$ loops, the total torque is

$$\tau_m = 2n\ell_1\ell_2 B(i/2) = n\ell_1\ell_2 Bi = K_T i \tag{1.11}$$

where $K_T \triangleq n\ell_1\ell_2 B = K_b$. Finally, using (1.9) and (1.11), the complete set of equations for the PM DC motor is then

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega_R + V_S & (1.12)\\ J\frac{d\omega_R}{dt} &= K_T i - f\omega_R - \tau_L & (1.13)\\ \frac{d\theta_R}{dt} &= \omega_R & (1.14)\end{aligned}$$

where V_S is the applied armature voltage, τ_L is the load torque on the motor, f is the coefficient of viscous friction and J is the rotor's moment of inertia.

1.7.7 Equations of the Separately Excited DC Motor

The separately excited DC motor has an additional equation compared to the PM DC motor which describes the flux/current in the field winding. Similar to the case of the PM DC motor, the total flux linkage in either parallel circuit of the armature windings is given by

$$Li - \lambda(i_f, \theta_R)$$

where $\lambda(i_f, \theta_R)$ now depends on i_f as given by (1.7), and L is the inductance the n loops connected in series making up either parallel circuit. Again, R_1 denotes the resistance of either circuit and Kirchhoff's voltage law gives

$$-\frac{d}{dt}\Big(Li - \lambda(i_f, \theta_R)\Big) - R_1 i/2 + V_S = 0$$

where V_S is the applied voltage to the armature. Finally, defining $R \triangleq R_1/2$, the equation describing the electrical dynamics of the armature circuit is

$$L\frac{di}{dt} = -Ri + \frac{d\lambda(i_f, \theta_R)}{dt} + V_S. \tag{1.15}$$

The quantity $d\lambda(i_f, \theta_R)/dt$ can be expanded to obtain

$$\frac{d\lambda(i_f, \theta_R)}{dt} = \frac{\partial\lambda}{\partial i_f}\frac{di_f}{dt} + \frac{\partial\lambda}{\partial\theta_R}\omega_R = \frac{\partial\lambda}{\partial i_f}\frac{di_f}{dt} - n\ell_1\ell_2 B(i_f)\omega_R. \tag{1.16}$$

Here $B(i_f)$ is the strength of the radial magnetic field *in the air gap* produced by the current in the field windings. The flux linkage in the field windings is $\lambda_f(i_f) = N_f S B_f(i_f)$ where $B_f(i_f)$ is the magnetic field strength *in the iron core* of the field, N_f is the number of field windings, and S is the cross sectional area of this iron core. The flux $SB_f(i_f)$ in each of the field windings goes through the air gap and, by conservation of flux (see Section 1.7.4), $SB_f(i_f) = \ell_1(\pi\ell_2/2)B(i_f)$ so that the radial magnetic field in the air gap produced by the field current is

$$B(i_f) = \frac{N_f}{N_f}\frac{SB_f(i_f)}{\pi\ell_1\ell_2/2} = \frac{\lambda_f(i_f)}{N_f\pi\ell_1\ell_2/2}.$$

Consequently,

$$n\ell_1\ell_2 B(i_f) = n\ell_1\ell_2\frac{\lambda_f(i_f)}{N_f\pi\ell_1\ell_2/2} = K_m\lambda_f(i_f)$$

where

$$K_m \triangleq \frac{2}{\pi}\frac{n}{N_f}.$$

As a result, equation (1.16) may be rewritten as

$$\frac{d\lambda(i_f, \theta_R)}{dt} = \frac{\partial\lambda}{\partial i_f}\frac{di_f}{dt} - K_m\lambda_f(i_f)\omega_R. \tag{1.17}$$

Each loop carries the current $i/2$ so that the torque produced on each of the two sides of the loop is $2\,(\ell_2/2)\,(i/2)\ell_1 B(i_f)$. As there are $2n$ loops, the total torque is

$$\tau_m = 2n\ell_1\ell_2 B(i_f)(i/2) = n\ell_1\ell_2 B(i_f)i = K_m\lambda_f(i_f)i \tag{1.18}$$

using the above expressions for K_m and $\lambda_f(i_f)$. Finally, using (1.15), (1.16), and (1.18), the mathematical model of the separately excited *DC* motor is given by

$$\begin{aligned} L\frac{di}{dt} &= -Ri + \frac{\partial\lambda}{\partial i_f}\frac{di_f}{dt} - K_m\lambda_f(i_f)\omega_R + V_S & (1.19)\\ J\frac{d\omega_R}{dt} &= K_m\lambda_f(i_f)i - \tau_L & (1.20)\\ \frac{d\lambda_f(i_f)}{dt} &= -R_f i_f + V_f & (1.21) \end{aligned}$$

where V_f is the applied voltage to the field, R_f is the resistance of the field winding, τ_L is the load torque on the motor and J is the rotor's moment of inertia.

If the iron in the field is not in saturation, then one may write

$$\lambda_f(i_f) = L_f i_f$$

and the dynamic model simplifies to

$$L\frac{di}{dt} = -Ri + \frac{\partial\lambda}{\partial i_f}\frac{di_f}{dt} - K_m L_f i_f \omega_R + V_S \tag{1.22}$$

$$J\frac{d\omega_R}{dt} = K_m L_f i_f i - \tau_L \tag{1.23}$$

$$L_f\frac{di_f}{dt} = -R_f i_f + V_f. \tag{1.24}$$

Under normal operating conditions, the term $\dfrac{\partial\lambda}{\partial i_f}\dfrac{di_f}{dt}$ is typically negligible and the model then reduces to[12]

$$L\frac{di}{dt} = -Ri - K_m L_f i_f \omega_R + V_S \tag{1.25}$$

$$J\frac{d\omega_R}{dt} = K_m L_f i_f i - \tau_L \tag{1.26}$$

$$L_f\frac{di_f}{dt} = -R_f i_f + V_f. \tag{1.27}$$

A typical mode of operation is to use the field voltage V_f to hold the field current i_f constant at some value. Then the field flux is of course constant and $L_f di_f/dt = 0$. However, in the first equation (1.19) (or 1.25), it is seen that the back emf is $-K_m\lambda_f\omega_R$ (or $-K_m L_f i_f \omega_R$) which increases in proportion to the speed. The input voltage V_S must be at least large enough to overcome the back emf in order to maintain the armature current. To have the motor achieve higher speeds within the voltage limit $|V_S| \le V_{\max}$, *field weakening* is employed. This is accomplished by using the input voltage V_f to decrease the field flux $\lambda_f = L_f i_f$ at higher speeds usually according to a flux reference of the form

$$\lambda_{ref} = \begin{cases} \lambda_{f0} & |\omega_R| \le \omega_{base} \\ \lambda_{f0}\dfrac{\omega_{base}}{|\omega_R|} & |\omega_R| \ge \omega_{base}. \end{cases} \tag{1.28}$$

This is illustrated in Figure 1.39. For $\omega > \omega_{base}$, field weakening results in the back emf $-K_m\lambda_f\omega_R = -K_m\lambda_{f0}\frac{\omega_{base}}{|\omega_R|}\omega_R = -K_m\lambda_{f0}\omega_{base}\mathrm{sign}(\omega_R)$ being constant. The trade-off is that the torque $K_m\lambda_f i = K_m\lambda_{f0}\frac{\omega_{base}}{|\omega_R|} i$ is now

[12] It will be seen later that field-oriented control of an induction motor results in a mathematical model of the induction motor that looks similar to these equations!

less for the same armature current i due to the decrease in the field flux linkage λ_f. If the armature resistance is negligible and i_f is constant, then the *base speed* ω_{base} is defined to be the speed satisfying $K_m \lambda_f \omega_{base} = V_{\max}$. Otherwise, the base speed is chosen to be somewhat smaller to account for the Ri and $\frac{\partial \lambda}{\partial i_f} \frac{di_f}{dt}$ voltage drops [see equation (1.19)].

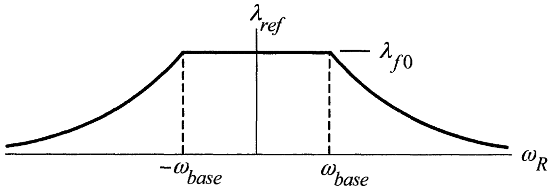

FIGURE 1.39. Flux reference for field weakening.

Appendices

Rotational Dynamics

The equations of motion of a rigid body that is constrained to rotate about a fixed axis are reviewed here briefly. Consider a cylinder which is constrained to rotate about a fixed axis as shown below.

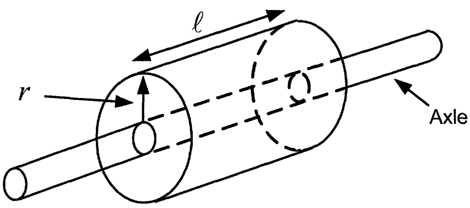

FIGURE 1.40. Cylinder constrained to rotate about a fixed axis.

The approach here is to obtain the equations of motion of the cylinder by first computing obtaining an expression for its kinetic energy. To do so, denote the angular speed of the cylinder by ω and the mass density of the material making up the cylinder by ρ. Then consider the cylinder to be made up of n small pieces of material Δm_i where the ith piece has mass

$$\Delta m_i = \rho r \Delta\theta \Delta\ell \Delta r.$$

This is illustrated in Figure 1.41.

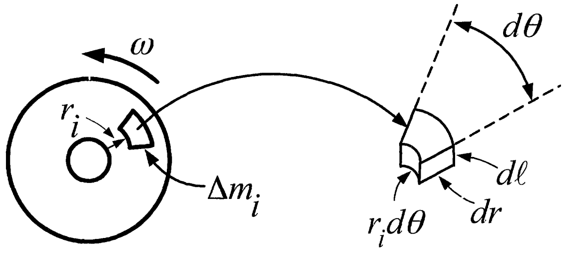

FIGURE 1.41. Cylinder is considered to be made up of small masses Δm_i. Drawn by Sharon Katz.

Each piece of mass Δm_i is rotating at the same angular speed ω so that the *linear* speed of Δm_i is $v_i = r_i\omega$ where r_i is the distance of Δm_i from the axis of rotation. The kinetic energy KE_i of Δm_i is given by

$$KE_i = \frac{1}{2}\Delta m_i v_i^2 = \frac{1}{2}\Delta m_i (r_i\omega)^2 .$$

The total kinetic energy is then

$$KE = \sum_{i=1}^{n} (KE)_i = \sum_{i=1}^{n} \frac{1}{2}\Delta m_i v_i^2 = \sum_{i=1}^{n} \frac{1}{2}\Delta m_i (r_i\omega)^2 = \frac{1}{2}\omega^2 \sum_{i=1}^{n} \Delta m_i r_i^2 .$$

Dividing the cylinder into finer and finer pieces so that $n \to \infty$ and $\Delta m_i \to 0$, the sum

$$\sum_{i=1}^{n} \Delta m_i r_i^2$$

becomes the integral

$$J = \iiint_{cylinder} r^2 dm.$$

The quantity J is called the *moment of inertia* and the kinetic energy of the cylinder may now be written as

$$KE = \frac{1}{2} J\omega^2 .$$

Assuming the axle radius is zero, the moment of inertia of the cylinder is computed to be

$$J = \int_0^R \int_0^\ell \int_0^{2\pi} r^2 \rho r d\theta d\ell dr = \frac{1}{2}(\pi R^2 \ell \rho) R^2 = \frac{1}{2} M R^2$$

where M is the total mass of the cylinder.

The above expression for the kinetic energy is now used to derive a relationship between torque and angular acceleration. Recall from elementary mechanics that the work done on a mass by an external force equals the change in its kinetic energy. In particular, consider an external force $\vec{\mathbf{F}}$ acting on the cylinder as shown in Figure 1.42.

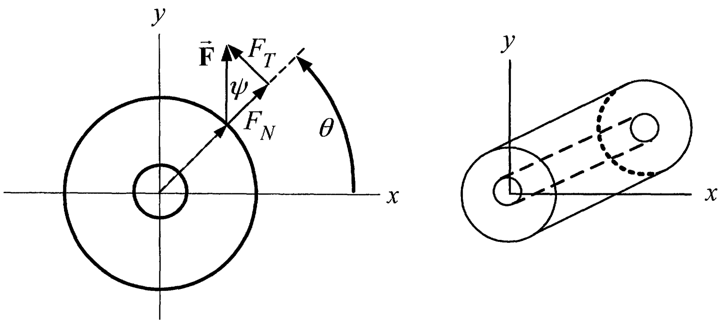

FIGURE 1.42. Force $\vec{\mathbf{F}}$ applied to the cylinder is resolved into a normal and tangential component. Drawn by Sharon Katz.

The cylinder is on an axle and therefore constrained to rotate about the z axis. Figure 1.42 shows the force $\vec{\mathbf{F}}$ applied to the cylinder at the position (r, θ) (in polar coordinates) resolved into a tangential component F_T and a normal component F_N so that $\vec{\mathbf{F}} = F_N\hat{\mathbf{r}} + F_T\hat{\boldsymbol{\theta}}$. The torque is defined as

$$\vec{\boldsymbol{\tau}} \triangleq \vec{\mathbf{r}} \times \vec{\mathbf{F}} = r\hat{\mathbf{r}} \times \left(F_N\hat{\mathbf{r}} + F_T\hat{\boldsymbol{\theta}}\right) = rF_T\hat{\mathbf{z}} = rF\sin(\psi)\hat{\mathbf{z}} \tag{1.29}$$

where ψ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$. (By definition, the magnitude of the cross product $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$ is defined as $rF\sin(\psi)$ and the direction of $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$ is perpendicular to both $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ along the axis of rotation determined by the right hand rule[13]). As $F_T = F\sin(\psi)$ is the tangential component, the torque may be rewritten as

$$\vec{\boldsymbol{\tau}} = \tau\hat{\mathbf{z}} = rF_T\hat{\mathbf{z}}$$

or, in scalar form, as

$$\tau = rF_T.$$

The motivation for the definition of torque as given by (1.29) is that it is the cause of *rotational* motion. Specifically, the rotational motion about an axis is caused by the applied tangential force F_T and the further away from the axis of rotation that the tangential force F_T is applied, the easier it is to get rotational motion. That is, the torque (cause of rotational motion)

[13]Using your right hand, curl your fingers in the direction from the first vector $\vec{\mathbf{r}}$ to the second vecotor $\vec{\mathbf{F}}$. Then your thumb points in the direction of $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$.

increases if either r or F_T increases which corresponds to one's experience (e.g., opening doors).

To summarize, $\vec{\boldsymbol{\tau}}$ is a vector pointing along the axis of rotation and the magnitude is given by

$$|\vec{\boldsymbol{\tau}}| = |\tau| = |rF_T|.$$

(Recall that the angular velocity vector $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{z}}$ also points along the axis of rotation where ω is the angular speed.)

With $d\vec{\mathbf{s}} \triangleq ds\hat{\boldsymbol{\theta}} = rd\theta\hat{\boldsymbol{\theta}}$, the change in work done on the cylinder by the external force $\vec{\mathbf{F}}$ is

$$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = F_T r d\theta = \tau d\theta$$

where $\tau \triangleq rF_T$. Dividing by dt, the power (rate of work) delivered to the cylinder is given by

$$\frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega.$$

As the rate of work done equals the rate of change of kinetic energy, it follows that

$$\frac{dW}{dt} = \frac{d}{dt}\left(\frac{1}{2}J\omega^2\right) = \tau\frac{d\theta}{dt}$$

or

$$J\omega\frac{d\omega}{dt} = \tau\omega.$$

This gives the fundamental relationship between torque and angular acceleration:

$$\tau = J\frac{d\omega}{dt}.$$

That is, the applied torque equal the moment of inertia times the angular acceleration. This is the basic equation for rigid body rotational dynamics.

Viscous Rotational Friction

Almost always there are frictional forces, and therefore, frictional torques acting between the axle and the bearings.[14] This is illustrated in Figure 1.43.

[14] An interesting exception are magnetic bearings where the axel is levitated by magnetic fields so that there is no mechanical contact.

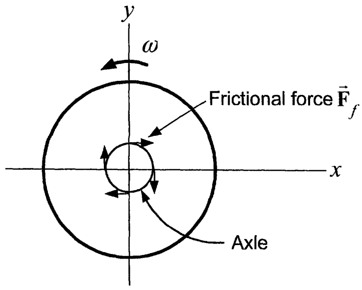

FIGURE 1.43. Viscous friction torque.

Often the frictional force is proportional to the angular speed and this model of friction is called *viscous friction* which is expressed mathematically as

$$\vec{\tau} = -f\vec{\omega} = -f\omega\hat{\mathbf{z}}$$

or, in scalar form,

$$\tau = -f\omega$$

where $f > 0$ is the *coefficient of viscous friction.*

Sign Convention for Torque

Suppose the axis of rotation is along the z axis. Recall that the definition of torque is

$$\vec{\tau} \triangleq \vec{\mathbf{r}} \times \vec{\mathbf{F}} = rF\sin(\psi)\hat{\mathbf{z}} = rF_T\hat{\mathbf{z}}$$

where ψ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$. The magnitude of the cross product $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$ is $rF\sin(\psi)$ and the direction of $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$ is perpendicular to both $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ along the z axis. In engineering applications, the systems are designed so that the applied force is tangential to the rotational motion, i.e., $\psi = \pi/2$, $F = F_T$, and

$$\vec{\tau} \triangleq \tau\hat{\mathbf{z}} = rF\hat{\mathbf{z}}.$$

If $\tau = rF > 0$ then the torque will cause the cylinder to rotate around the z axis in the direction indicated by the curved arrow. On the other hand, if $\tau = rF < 0$ then the torque will cause the cylinder to rotate around the z axis in the direction *opposite* to that indicated by the curved arrow. Typically in engineering texts, the sign convention for torque is indicated by a curved arrow as shown in Figure 1.44. (Physics texts prefer to write $\vec{\tau} \triangleq \tau\hat{\mathbf{z}}$.)

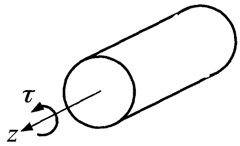

FIGURE 1.44. Sign convention for torque.

Gears

Using the elementary rigid body dynamics developed in the previous appendix, the model of a two gear system illustrated in Figure 1.45 below is now developed.

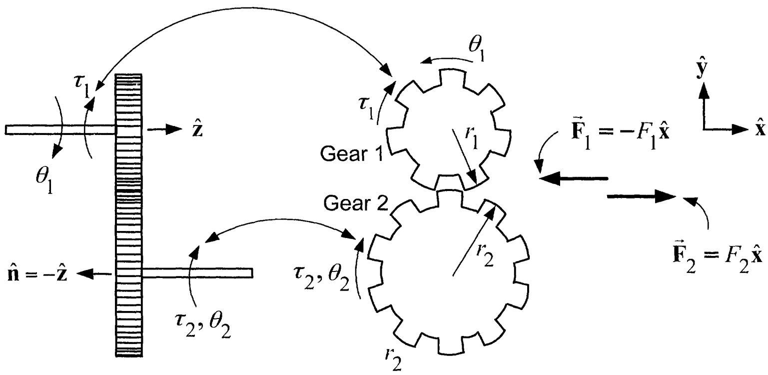

FIGURE 1.45. Two gear system. Drawn by Sharon Katz.

This presentation is from that given in Professor Ogata's book [37]. In Figure 1.45,

τ_1 is the torque exerted on gear 1 by gear 2.
F_1 is the force exerted on gear 1 by gear 2.
τ_2 is the torque exerted on gear 2 by gear 1.
F_2 is the force exerted on gear 2 by gear 1.
θ_1 is the angle rotated by gear 1.
θ_2 is the angle rotated by gear 2.
n_1 is the number of teeth on gear 1.
n_2 is the number of teeth on gear 2.
r_1 is the radius of gear 1.
r_2 is the radius of gear 2.

Let $\vec{\mathbf{F}}_1 \triangleq F_1(-\hat{\mathbf{x}})$ so that if $F_1 > 0$, the force is in the $-\hat{\mathbf{x}}$ direction as shown in Figure 1.45. Also, let $\vec{\mathbf{r}}_1 \triangleq r_1(-\hat{\mathbf{y}})$ so that $\vec{\boldsymbol{\tau}}_1 = \vec{\mathbf{r}}_1 \times \vec{\mathbf{F}}_1 =$

$r_1F_1(-\hat{\mathbf{y}})\times(-\hat{\mathbf{x}}) = r_1F_1(-\hat{\mathbf{z}}) = r_1F_1\hat{\mathbf{n}}$. That is, $\vec{\boldsymbol{\tau}}_1 = \tau_1\hat{\mathbf{n}}$ where $\tau_1 = r_1F_1$ and $\hat{\mathbf{n}} \triangleq -\hat{\mathbf{z}}$ is a unit vector. Similarly, let $\vec{\mathbf{F}}_2 \triangleq F_2\hat{\mathbf{x}}$ so that if $F_2 > 0$, the force is in the $\hat{\mathbf{x}}$ direction. Writing $\vec{\mathbf{r}}_2 \triangleq r_2\hat{\mathbf{y}}$ it follows that $\vec{\boldsymbol{\tau}}_2 = \vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}_2 = r_2F_2(-\hat{\mathbf{z}}) = -\tau_2\hat{\mathbf{z}} = \tau_2\hat{\mathbf{n}}$ with $\tau_2 = r_2F_2$. The reason that $\vec{\mathbf{r}}_1, \vec{\boldsymbol{\tau}}_1, \vec{\mathbf{F}}_1$ are referred to the basis vectors $-\hat{\mathbf{x}}, -\hat{\mathbf{y}}, -\hat{\mathbf{z}}$ while $\vec{\mathbf{r}}_2, \vec{\boldsymbol{\tau}}_2, \vec{\mathbf{F}}_2$ are referred to the basis vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ is so that there will no minus signs in the gear relationships to be derived below.

Algebraic Relationships Between Two Gears

There are three important algebraic relationships between the gears.

1. The gears have different radii, but the teeth on each gear are the same size so that they will mesh together properly. Consequently, the number of teeth on the surface of gears is proportional to the radius of the gears so that, for example, if $r_2 = 2r_1$, then $n_2 = 2n_1$. In general,
$$\frac{r_2}{r_1} = \frac{n_2}{n_1}.$$
2. By Newton's third law, the forces $\vec{\mathbf{F}}_1 = F_1(-\hat{\mathbf{x}}), \vec{\mathbf{F}}_2 \triangleq F_2\hat{\mathbf{x}}$ are equal in magnitude, but opposite in direction so that $F_2 = -(-F_1) = F_1$. Thus, as $\tau_1 = r_1F_1$ and $\tau_2 = r_2F_2$ it follows that
$$\frac{\tau_2}{\tau_1} = \frac{r_2}{r_1}.$$
3. As the teeth on each gear are meshed together at the point of contact, the distance traveled along the surface of the gears is the same. In other words, $\theta_1 r_1 = \theta_2 r_2$ or
$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2}$$

The first two algebraic relationships can be summarized as
$$\frac{\tau_2}{\tau_1} = \frac{r_2}{r_1} = \frac{n_2}{n_1}$$
and these ratios are easily remembered by thinking of gear 2 as larger in radius than gear 1. Then the number of teeth on gear 2 must also be larger (because its circumference is larger) and the torque on gear 2 is also larger (because its radius is larger).

The last algebraic relationship is summarized as
$$\frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} = \frac{\omega_2}{\omega_1}$$
but it is more easily remembered by writing $\theta_1 r_1 = \theta_2 r_2$ which just states the distance traveled along the surface of each gear is the same as they are meshed together.

Dynamic Relationship Between Two Gears

Consider the two gear system shown in Figure 1.46 below. The motor torque τ_m acts on gear 1 and the torque τ_L is a load torque acting on gear 2.

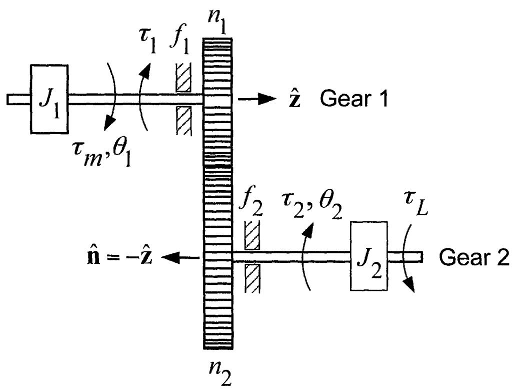

FIGURE 1.46. Dynamic equations for a two gear system. Drawn by Sharon Katz.

In Figure 1.46,

J_1 is the moment of inertia of the motor shaft.
J_2 is the moment of inertia of the output shaft.
f_1 is the viscous friction coefficient of the motor shaft.
f_2 is the viscous friction coefficient of the output shaft.
θ_1 is the angle rotated by gear 1.
θ_2 is the angle rotated by gear 2.
ω_1 is the angular speed of gear 1.
ω_2 is the angular speed of gear 2.
τ_1 is the torque exerted on gear 1 by gear 2.
τ_2 is the torque exerted on gear 2 by gear 1.

The sign conventions for the torques $\tau_m, \tau_1, \tau_2, \tau_L$ are indicated in Figure 1.46. In particular, if $\tau_m > 0, \tau_1 > 0$ then they oppose each other and similarly, if $\tau_2 > 0, \tau_L > 0$ then these two torques oppose each other. A load torque is illustrated in Figure 1.47 in which the load torque on gear 2 is $\tau_L = r_2 mg$ with r_2 the radius of the pick up reel (gear 2).

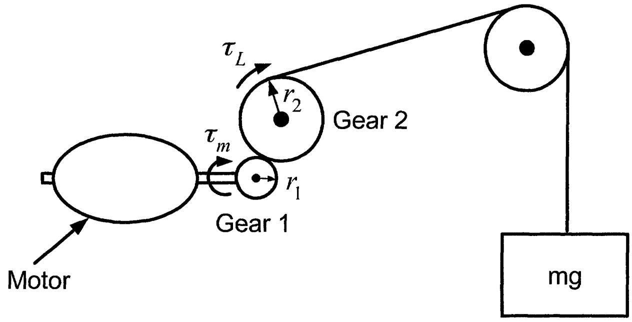

FIGURE 1.47. Illustration of load torque. Drawn by Sharon Katz.

The above development is now put together to write down differential equations that characterize the dynamic behavior of the gears. Recall that the fundamental equation of rigid body dynamics is given by

$$\tau = J\frac{d\omega}{dt}$$

where τ is the *total* torque on the rigid body, J is the moment of inertia of the rigid body and $d\omega/dt$ is its angular acceleration about the fixed axis of rotation. Applying this relationship, the equations of motion for the two gears are then

$$\begin{aligned} \tau_m - \tau_1 - f_1\omega_1 &= J_1\frac{d\omega_1}{dt} \\ \tau_2 - \tau_L - f_2\omega_2 &= J_2\frac{d\omega_2}{dt}. \end{aligned} \tag{1.30}$$

Typically, the input (motor) torque τ_m is known and, the output position θ_2 and speed ω_2 are measured. Consequently, the variables τ_1, τ_2, ω_1 need to be eliminated which is done as follows:

$$\begin{aligned} \tau_2 &= \frac{n_2}{n_1}\tau_1 \\ &= \frac{n_2}{n_1}\left(\tau_m - f_1\omega_1 - J_1\frac{d\omega_1}{dt}\right) \\ &= \frac{n_2}{n_1}\left(\tau_m - f_1\left(\frac{n_2}{n_1}\omega_2\right) - J_1\frac{d}{dt}\left(\frac{n_2}{n_1}\omega_2\right)\right) \\ &= \frac{n_2}{n_1}\tau_m - \left(\frac{n_2}{n_1}\right)^2 f_1\omega_2 - \left(\frac{n_2}{n_1}\right)^2 J_1\frac{d\omega_2}{dt}. \end{aligned} \tag{1.31}$$

Substituting this expression for τ_2 into the second equation of (1.30) results in

$$\frac{n_2}{n_1}\tau_m - \left(\frac{n_2}{n_1}\right)^2 f_1\omega_2 - \left(\frac{n_2}{n_1}\right)^2 J_1\frac{d\omega_2}{dt} - \tau_L - f_2\omega_2 = J_2\frac{d\omega_2}{dt}$$

Rearranging, the desired result is

$$\frac{n_2}{n_1}\tau_m = \Big(J_2 + (n_2/n_1)^2 J_1\Big)\frac{d\omega_2}{dt} + \Big(f_2 + (n_2/n_1)^2 f_1\Big)\omega_2 + \tau_L. \quad (1.32)$$

Let $n = n_2/n_1$ denote the *gear ratio*, $J \triangleq J_2 + n^2 J_1$ denote the total inertia *reflected to output shaft* and $f \triangleq f_2 + n^2 f_1$ denote the total viscous friction coefficient *reflected to the output shaft*, equation (1.32) can be written succinctly as

$$n\tau_m = J\frac{d\omega_2}{dt} + f\omega_2 + \tau_L. \quad (1.33)$$

The net effect of the gears is to increase the motor torque from τ_m on the motor shaft to $n\tau_m$ on the output shaft, to add the quantity $n^2 J_1$ to the inertia of the output shaft and to add $n^2 f_1$ to the viscous friction coefficient of the output shaft.

Remark

Everything could have been referred to the motor shaft instead of the output (load) shaft. To do so, simply substitute $\omega_2 = (n_1/n_2)\omega_1$ into (1.32) to obtain

$$\frac{n_2}{n_1}\tau_m = \left(J_2 + \left(\frac{n_2}{n_1}\right)^2 J_1\right)\frac{d\left(\dfrac{n_1}{n_2}\omega_1\right)}{dt} + \left(f_2 + \left(\frac{n_2}{n_1}\right)^2 f_1\right)\left(\frac{n_1}{n_2}\omega_1\right) + \tau_L$$

Multiply both sides by n_1/n_2 results in

$$\tau_m = \left(J_2 + \left(\frac{n_2}{n_1}\right)^2 J_1\right)\left(\frac{n_1}{n_2}\right)^2\frac{d\omega_1}{dt} + \left(f_2 + \left(\frac{n_2}{n_1}\right)^2 f_1\right)\left(\frac{n_2}{n_1}\right)^2\omega_1 + \frac{n_1}{n_2}\tau_L$$

or, finally, the desired form is

$$\tau_m = \left(\left(\frac{n_1}{n_2}\right)^2 J_2 + J_1\right)\frac{d\omega_1}{dt} + \left(\left(\frac{n_1}{n_2}\right)^2 f_2 + f_1\right)\omega_1 + \frac{n_1}{n_2}\tau_L. \quad (1.34)$$

In this formulation, the load torque on the input shaft is *reduced* by n_1/n_2 from that on the output shaft, and $(n_1/n_2)^2 J_2$ has been added to the inertia of the motor shaft and $(n_1/n_2)^2 f_2$ has been added to the viscous friction coefficient of the motor shaft.

Problems

Faraday's Law and Induced Electromotive Force (emf)

Problem 1 Faraday's Law

Consider Figure 1.48 where a magnet is moving up *into a square planar loop of copper wire.*

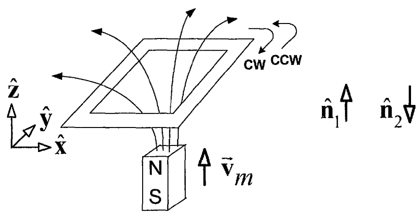

FIGURE 1.48. Induced emf in a loop due to a moving magnet.

(a) Using the normal $\hat{\mathbf{n}}_1$*, is the flux in the loop produced by the magnet increasing or decreasing?*

(b) Using the normal $\hat{\mathbf{n}}_1$*, what is the direction of positive travel around the surface whose boundary is the loop (clockwise or counterclockwise)?*

(c) What is the direction of the induced current in Figure 1.48 (clockwise or counterclockwise)? Does the induced current produce a change in the flux in the loop that opposes the change in flux produced by the magnet?

(d) Using the normal $\hat{\mathbf{n}}_2$*, is the flux increasing or decreasing?*

(e) Using the normal $\hat{\mathbf{n}}_2$*, what is the direction of positive travel around the surface whose boundary is the loop (clockwise or counterclockwise)?*

(f) What is the direction of the induced current in Figure 1.48 (clockwise or counterclockwise)? Does the induced current produce a change in the flux in the loop that opposes the change in flux produced by the magnet?

Problem 2 Faraday's Law

Consider Figure 1.49 where a magnet is moving down *away from a square planar loop of copper wire.*

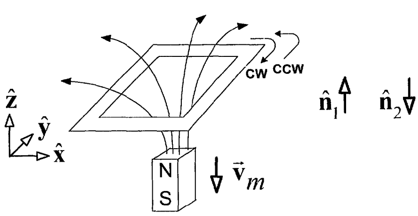

FIGURE 1.49. Induced emf in a loop due to a moving magnet.

(a) Using the normal $\hat{\mathbf{n}}_1$, is the flux in the loop produced by the magnet increasing or decreasing?

(b) Using the normal $\hat{\mathbf{n}}_1$, what is the direction of positive travel around the surface whose boundary is the loop (clockwise or counterclockwise)?

(c) What is the direction of the induced current in Figure 1.49 (clockwise or counterclockwise)? Does the induced current produce a change in the flux in the loop that opposes the change of flux produced by the magnet?

(d) Using the normal $\hat{\mathbf{n}}_2$, is the flux increasing or decreasing?

(e) Using the normal $\hat{\mathbf{n}}_2$, what is the direction of positive travel around the surface whose boundary is the loop (clockwise or counterclockwise)?

(f) What is the direction of the induced current in Figure 1.49 (clockwise or counterclockwise)? Does the induced current produce a change in the flux in the loop that opposes the change of flux produced by the magnet?

Problem 3 The Linear DC Motor

Consider the simple linear DC motor of Figure 1.3. Take the normal to the surface enclosed by the loop taken to be $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$.

(a) What is the flux through the surface?

(b) What is the direction of positive travel around this flux surface?

(c) What is the induced emf ξ in the loop in terms of B, ℓ and the speed v of the bar?

(d) Do V_S and ξ have the same sign convention? Explain why ξ is now negative. With R the resistance of the circuit and the inductance $L = 0$, write down the differential equations for the current i in the machine and the speed v of the bar.

Problem 4 The Linear DC Motor

Consider the simple linear motor in Figure 1.50 where the magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{z}}$ $(B > 0)$ is up out of the page.

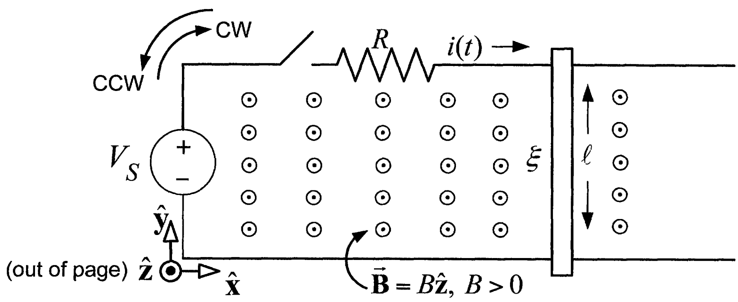

FIGURE 1.50. Linear DC machine with $\vec{\mathbf{B}} = B\hat{\mathbf{z}}$, $B > 0$.

Closing the switch causes a current to flow in the wire loop.

(a) What is the magnetic force $\vec{\mathbf{F}}_{magnetic}$ on the sliding bar in terms of B, i, and ℓ? Give both the magnitude and direction of $\vec{\mathbf{F}}_{magnetic}$.

(b) Take the normal to the surface enclosed by the loop to be $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. *What is the flux through the surface?*

(c) What is the induced emf ξ *in the loop in terms of* B, ℓ, *and the speed* v *of the bar?*

(d) What is the sign convention for the induced emf ξ *drop around the loop? (That is, if* $\xi > 0$, *would it act to push current in the clockwise or counterclockwise direction?)*

(e) Do V_S *and* ξ *have the same sign convention? Draw + and − signs above and below* ξ *to indicate the sign convention for* ξ.

Problem 5 Back Emf in the Single-Loop Motor

Consider the single loop motor with the flux surface as indicated in Figure 1.51. A voltage source connected to the brushes is forcing current down side a ($\otimes$) *and up side* a' ($\odot$).

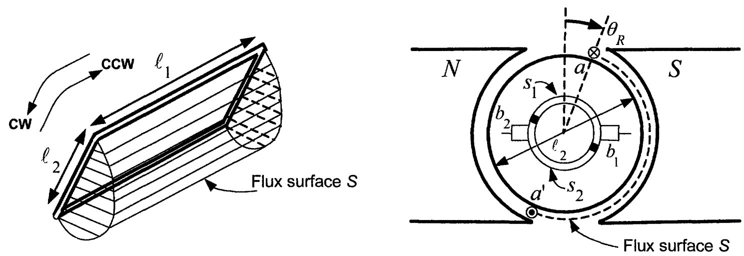

FIGURE 1.51. Computing the flux with $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$.

(a) With the motor at the angular position θ_R *shown, that is, with* $0 < \theta_R < \pi$, *and using the* inward *normal* ($\hat{\mathbf{n}} = -\hat{\mathbf{r}}$), *compute the flux through the surface in terms of the magnitude* B *of the radial magnetic field in the air gap, the axial length* ℓ_1 *of the motor, the diameter* ℓ_2 *of the motor and the angle* θ_R *of the rotor.*

(b) What is the positive direction of travel around the flux surface S *(CW or CCW)?*

(c) What is the emf ξ *induced in the rotor loop? What is the sign convention for the induced emf* ξ *drop around the loop? (That is, if* $\xi > 0$, *would it act to push current in CW direction or the CCW direction?) Do* V_S *and* ξ *have the same sign convention? Explain why* ξ *is now negative. Draw an equivalent circuit of the form in Figure 1.18 for the rotor loop current.*

Problem 6 Gauss's Law and Conservation of Flux

The flux surface in Figure 1.13 was chosen as the half-cylindrical surface with two half disks at either end because the B *field is known on the cylindrical surface being given by equation (1.1) and can be taken to be zero on the two half disks. If the flux surface had been taken to be a flat planar surface with the rectangular loop as its boundary, then it would not be clear* *how to compute the flux on this surface as the B field there is unknown. Show, using Gauss's law*

$$\phi = \oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \equiv 0,$$

that both surfaces give the same flux. In general, by Gauss's law, one can compute the flux using any surface as long as its boundary is the loop.

Problem 7 Flux in the Single-Loop DC Motor

Figure 1.52 shows the rotor loop with $\pi < \theta_R < 2\pi$.

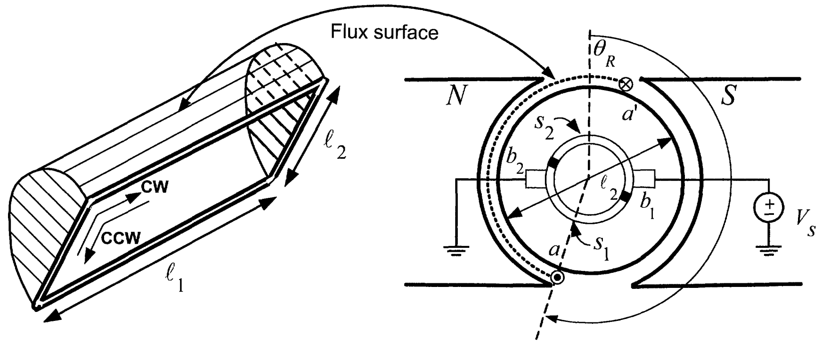

FIGURE 1.52. Rotor loop where $\pi < \theta_R < 2\pi$.

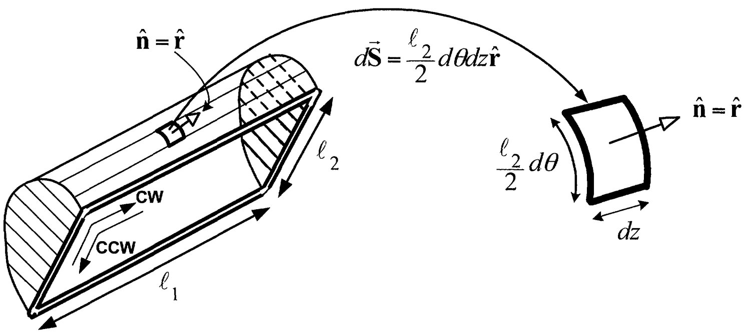

FIGURE 1.53. Flux surface with the normal radially out.

(a) Using the flux surface shown in Figure 1.53 with $d\vec{\mathbf{S}} = (\ell_2/2)d\theta dz\hat{\mathbf{r}}$, show that $\phi(\theta_R) = (\theta_R - \pi/2 - \pi)\ell_1\ell_2 B$ for $\pi < \theta_R < 2\pi$. Plot $\phi(\theta_R)$ for $0 < \theta_R < 2\pi$ (note that $\phi(\theta_R)$ for $0 < \theta_R < \pi$ is computed in the text). Compute the back emf ξ and give its sign convention, that is, if $\xi = -d\phi(\theta_R)/dt > 0$, will it force current in the CW or the CCW direction?

Do ξ and V_S have the same sign convention? (Yes! Explain!) Draw an equivalent circuit diagram of the form of Figure 1.18 to illustrate the sign convention.

(b) Using the flux surface shown in Figure 1.54 with $d\vec{\mathbf{S}} = (\ell_2/2)d\theta dz\,(-\hat{\mathbf{r}})$, *show that* $\phi(\theta_R) = -(\theta_R - \pi/2 - \pi)\,\ell_1\ell_2 B$ *for* $\pi < \theta_R < 2\pi$.

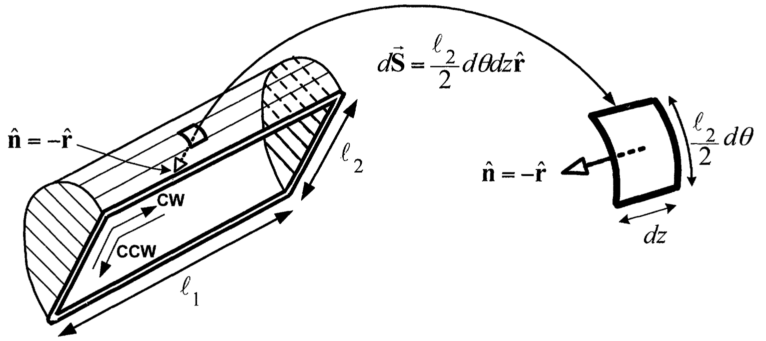

FIGURE 1.54. Flux surface with the normal radially in.

Compute the back emf and give its sign convention, that is, if $-d\phi(\theta_R)/dt > 0$ *will it force current in the CW or the CCW direction. Do ξ and V_S have the same sign convention? (No, they are opposite! Explain!) Draw an equivalent circuit diagram of the form of Figure 1.18 to illustrate the sign convention.*

(c) Note that in part (b) the normal to the flux surface is taken to be radially in while with $0 < \theta_R < \pi$ *it was taken to be radially out (see Figure 1.14). Explain why reversing the normal of the flux surface each half turn in this way results in an equivalent circuit of the form shown in Figure 1.18 which is valid for all rotor angular positions*

Hint: Note that the + side of V_S is now electrically connected to side a' of the loop through brush b_1, while when $0 < \theta_R < \pi$ in Figure 1.14, the + side of V_S was electrically connected to side a of the loop through brush b_1. That is, the sign convention for V_S in the loop changes every half-turn. Thus it is also necessary to change the sign convention for the flux and therefore for ξ every half-turn in order that the sign conventions for V_S and ξ have the same relationship to each other for all θ_R.

(d) Show by taking $d\vec{\mathbf{S}} = (\ell_2/2)d\theta dz\hat{\mathbf{r}}$ *for* $0 < \theta_R < \pi$ *and* $d\vec{\mathbf{S}} = (\ell_2/2)d\theta dz\,(-\hat{\mathbf{r}})$ *for* $\pi < \theta_R < 2\pi$, *that* $\phi(\theta_R) = -(\theta_R \bmod \pi - \pi/2)\,\ell_1\ell_2 B$ *for all* θ_R *and that* $\xi = -d\phi(\theta_R)/dt$ *has the sign convention given in Figure 1.18 for all* θ_R. *Verify that the plot of* $\phi(\theta_R)$ *in Figure 1.15 is correct.*

Problem 8 Simulation of the DC Motor

Let $V_{max} = 40$ V, $I_{max} = 5$ A, $K_b = K_T = 0.07$ V/rad/sec (= N-m/A), $J = 6\times10^{-5}$ kg-m^2, $R = 2$ ohms, $L = 2$ mH, and $f = 0.0004$ N-m/rad/sec. Develop a simulation of the DC motor that includes the motor model given by (1.6) and a voltage saturation model of the amplifier as illustrated in Figure 1.20. Put a step input of $V_S(t) = 10$ V into the motor and plot out (a) $\theta(t)$, (b) $\omega(t)$, (c) $i(t)$, and (d) $V_S(t)$.

Multiloop Motor

Problem 9 Skewing of the Rotor Sides

Figure 1.29 shows the sides of the rotor loops for the motor (but not the tachometer!) are not straight in the axial direction, but instead are skewed. Can you think of a reason why this is done?

Problem 10 Neutral Plane and Brush Shifting

In the commutation scheme for the multiloop motor, it was shown that when a rotor loop was perpendicular to the brushes, the current in the loop was shorted out (brought to zero). Consequently, it is highly desirable that the total induced voltage in that loop be as close to zero as possible to prevent arcing. Figure 1.35 shows the total *B field distribution in the DC machine. If the armature current were zero, then the field would be horizontal as in Figure 1.33. At very high armature currents (e.g., in large machines used in heavy industry), the field is skewed as shown in Figure 1.35. The neutral plane is the plane cutting through the axis of the rotor for which the* total *B field is perpendicular to the plane. Let $\psi_k(i, i_f, \theta_R)$ be the total flux in the* kth *rotor loop due to* both *the field current and the armature current.*

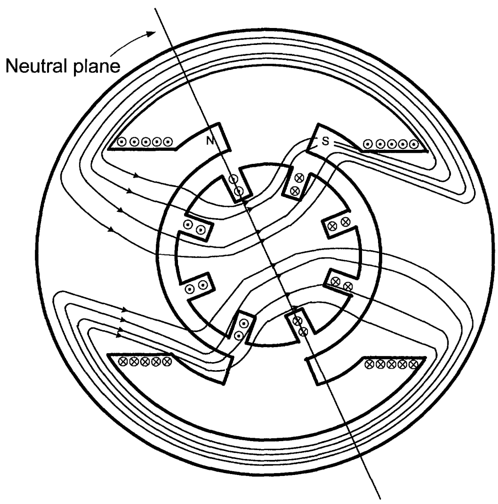

FIGURE 1.55. Magnetic field due to both the field and armature currents.

(a) Explain why $\psi_k(i, i_f, \theta_R)$ is a maximum (or minimum) as a function of the rotor position θ_R when the kth *loop coincides with the neutral plane, or equivalently, why $-\partial\psi_k(i, i_f, \theta_R)/\partial\theta_R = 0$ when the* kth *loop coincides with the neutral plane. (This is seen more easily if one takes the flux surface to be the plane of the loop rather than the cylindrical surface.)*

(b) As explained in Section 1.7.2, the current in a rotor loop is commutated when the plane of the loop is perpendicular to the line containing the two brushes. Consider the situation in which the brushes of Figure 1.30(a) are rotated counterclockwise (shifted) so that the plane *of the loop undergoing commutation is coincident with the neutral plane as shown in Figure 1.56. (After the brushes are rotated, they are held fixed at that position.)*

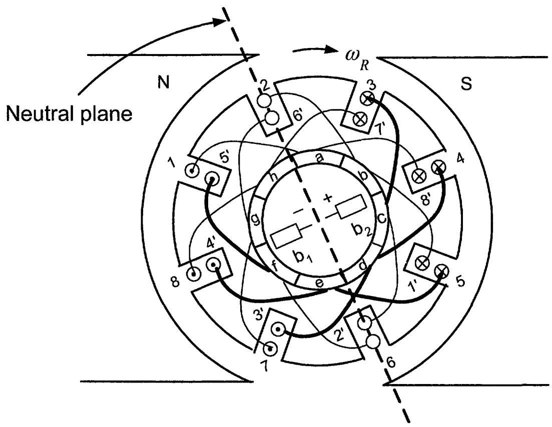

FIGURE 1.56. Rotating the brushes so that commutation in the loop occurs when the loop is aligned with the neutral plane.

Show then that the induced voltage in the kth *loop when it undergoes commutation is*

$$-\frac{d\psi_k(i, i_f, \theta_R)}{dt} = -\frac{\partial\psi_k(i, i_f, \theta_R)}{\partial i}\frac{di}{dt} - \frac{\partial\psi_k(i, i_f, \theta_R)}{\partial i_f}\frac{di_f}{dt}$$

(c) Explain why shifting the brushes alleviates arcing.

(d) Does the amount that the brushes are to be rotated depend on the amount of armature current?

Problem 11 Flux in the Multiloop Motor

With $\lambda(\theta_R)$ given by (1.8) and $n = 4$, write a program to plot $\lambda(\theta_R)/\left(\ell_1\ell_2 B\right)$ versus θ_R to obtain a figure similar to that of Figure 1.38.

Separately Excited, Series and Shunt DC Machines

Problem 12 Conservation of Energy in the Separately Excited DC Motor

Using the equations (1.19), (1.20) and (1.21) of the separately excited DC motor show that energy conservations holds. Give a physical interpretation to the various expressions.

Problem 13 Series DC Motor [1]

Consider a separately excited DC motor in which the terminal T_1 *of the armature is connected to the terminal* T_2' *of the field circuit and a single voltage source is applied between the two remaining terminals* T_1' *and* T_2*. This configuration is referred to as a* series DC motor *and is illustrated in Figure 1.57. This type of connection is often used when the motor is employed as a traction drive, that is, for subways, trolley cars, and so on. An equivalent circuit for the series DC motor is given in Figure 1.58 where* $\lambda_f(i_f) = L_f i_f$ *is used.*

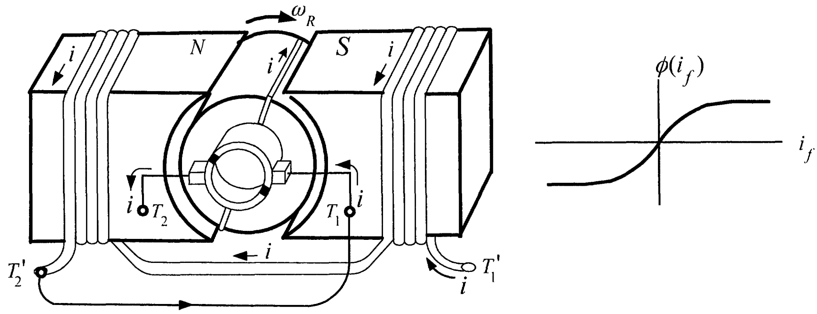

FIGURE 1.57. Series-connected DC motor. Adapted from Ref. [21].

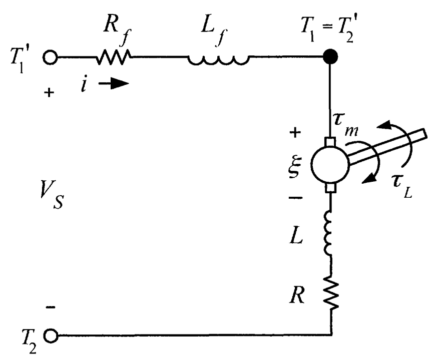

FIGURE 1.58. Equivalent circuit for a series DC motor.

(a) Starting from the model (1.22), (1.23), (1.24) of the separately excited DC motor, derive the set of differential equations that characterize the series connected DC motor.

(b) Show that the torque cannot change sign, i.e., it is always positive or always negative.

(c) What must be done to change the sign of the torque?

Problem 14 Shunt DC Motor [3]

Consider a separately excited DC motor in which the terminal T_1 of the armature is connected to the terminal T_1' of the field circuit and similarly, the other two terminals T_2, T_2' are connected together as indicated in Figure 1.59. This configuration is referred to as a shunt DC motor *for which an equivalent circuit is shown in Figure 1.60. The resistance R_{adj} is an adjustable resistance added in series with the field winding and aids in the control of the motor.*

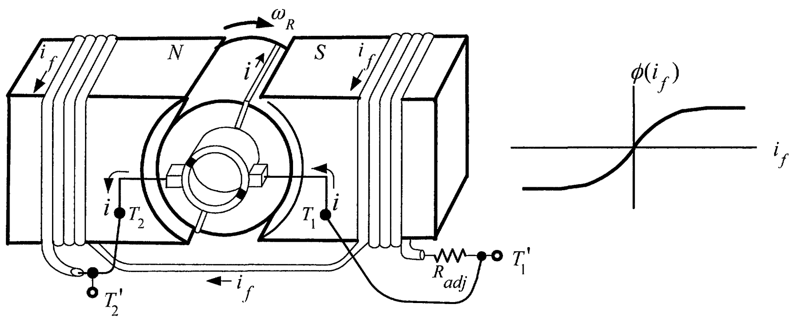

FIGURE 1.59. Shunt connected DC motor. Adapted from Ref. [21].

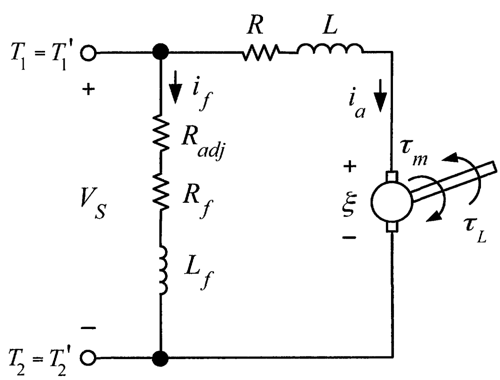

FIGURE 1.60. Equivalent circuit for a shunt DC motor.

Let $\lambda(i_f) = L_f i_f$ and using the model (1.22), (1.23), (1.24) of the separately excited DC motor, derive the complete set of equations that characterize the shunt DC motor.

Simple AC Generators

Problem 15 A Three-Phase Generator [38]
Consider a simplified model of a three-phase generator shown in Figures 1.61 and 1.62.

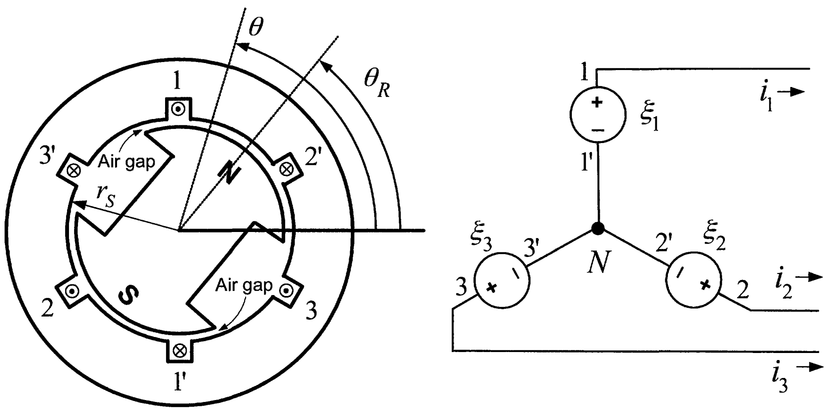

FIGURE 1.61. Simplified three-phase permanent magnet generator.

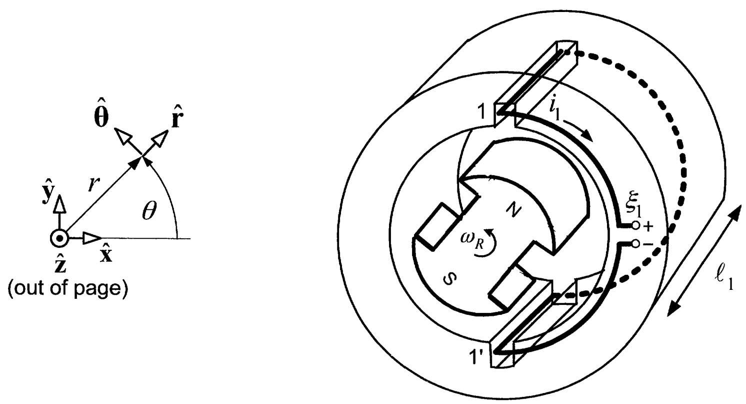

FIGURE 1.62. Half-cylindrical-shaped winding of phase 1.

The rotor consists of a two-pole (one north and one south) permanent magnet with the pole faces shaped so that the radial magnetic field in the *air gap due to the rotor's permanent magnet is given by*

$$\vec{\mathbf{B}}(r,\theta) = B_{R\max}\frac{r_S}{r}\cos(\theta-\theta_R)\hat{\mathbf{r}}$$

where $B_{R\max} > 0, r_S = \ell_2/2 + g$ *is the radius of the inside surface of the stator iron and* θ, θ_R *are defined as in Figure 1.61. One of the stator phases (consisting of a single loop) is shown in Figure 1.62 where the rotor and stator sizes are distorted for expository reasons.*

(a) Using a half-cylinder flux surface whose boundary is the stator loop and whose surface normal is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, *compute the flux* ϕ_1 *in stator loop 1 and the voltage* $\xi_1 = -d\phi_1/dt$ *induced in this loop.*

(b) Also, compute the fluxes ϕ_2, ϕ_3 *and the induced voltages* ξ_2, ξ_3 *in phases 2 and 3, respectively.*

(c) With the ends $1', 2', 3'$ *tied together (called the neutral point), show that if the machine is rotating at constant angular speed* ω_R, *this results in a three-phase generator producing three sinusoidal voltages which are identical except being* $120°$ *out of phase with each other.*

(d) It is shown in Chapters 4 *and* 5 *that the magnetic field due to the rotating permanent magnet rotor produces an* axial electrical field *in the air gap given by*

$$\vec{\mathbf{E}}_R\left(\theta-\theta_R\right) = \omega_R B_{R\max} r_S \cos\left(\theta-\theta_R\right)\hat{\mathbf{z}}.$$

Show that

$$\xi_1 = \int_{1'}^{1} \vec{\mathbf{E}}_R(\theta-\theta_R)\cdot d\vec{\boldsymbol{\ell}}.$$

Problem 16 A Two-Phase Four-Pole Generator

To introduce a two-phase four-pole *generator, consider first Figure 1.63(a), which shows a two-phase* two-pole *generator. Its rotor is a two-pole (one north and one south) permanent magnet whose pole faces are shaped so that the radial air gap magnetic field is given by*

$$\vec{\mathbf{B}}(r,\theta) = B_{R\max}\frac{r_S}{r}\cos(\theta-\theta_R)\hat{\mathbf{r}}$$

where $B_{R\max} > 0, r_S = \ell_2/2 + g$ *is the radius of the inside surface of the stator iron, and* θ, θ_R *are defined as in Figure 1.63(a). The stator phases are wound in a similar fashion to those of problem 15, and the voltages induced in the stator phases are computed similarly.*

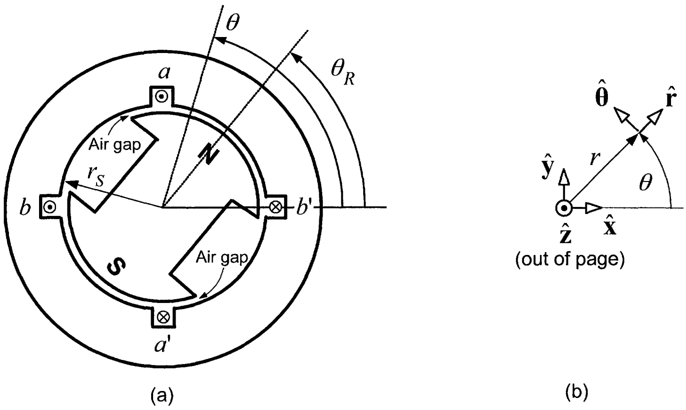

FIGURE 1.63. (a) A two-phase two-pole machine. (b) Polar coordinate system.

Now consider a two-phase four-pole *generator illustrated in Figure 1.64(a). The rotor is a* four-pole *(two north poles and two south poles) permanent magnet whose pole faces are shaped so that the radial air gap magnetic field is given by*

$$\vec{\mathbf{B}}(r,\theta) = B_{R\max}\frac{r_S}{r}\cos\left(n_p(\theta-\theta_R)\right)\hat{\mathbf{r}}.$$

Here $B_{R\max} > 0$, $r_S = \ell_2/2 + g$ *is the radius of the inside surface of the stator iron,* θ_R *is defined as in Figure 1.64(a), and* n_p $(= 2$ *in Figure 1.64)* *is the number of* pole pairs. *Figure 1.64(c) is a perspective view showing how phase a is wound while Figure 1.64(d) is a cross-sectional view of the same phase.*

(a) Using an outward normal for the flux surface ($\hat{\mathbf{n}} = \hat{\mathbf{r}}$)*, compute the flux linkage* λ_a *in phase a due to the permanent magnet rotor. (The flux linkage is the sum of the fluxes in the two loops* $\mathrm{a}_1{-}\mathrm{a}_1'$ *and* $\mathrm{a}_2{-}\mathrm{a}_2'$ *making up phase a.)*

(b) Does the positive direction of travel around each of the two flux surfaces of phase a coincide with the positive direction of current?

(c) Compute the voltage ξ_a *induced in phase a by the permanent magnet rotor.*

(d) Using the expression for ξ_a *in phase a from part (c), and the fact that phase b is rotated* $\pi/(2n_p)$ *radians from phase a, give the expression for* ξ_b.

(e) It is shown in Chapters 4 *and* 5 *that the magnetic field due to the rotating permanent magnet rotor produces an* axial electrical field *in the air gap given by*

$$\vec{\mathbf{E}}_R\left(\theta-\theta_R\right) = \omega_R B_{R\max} r_S \cos\left(n_p(\theta-\theta_R)\right)\hat{\mathbf{z}}.$$

Show that

$$\xi_a = \int_{a'}^{a} \vec{\mathbf{E}}_R(\theta - \theta_R) \cdot d\vec{\boldsymbol{\ell}}.$$

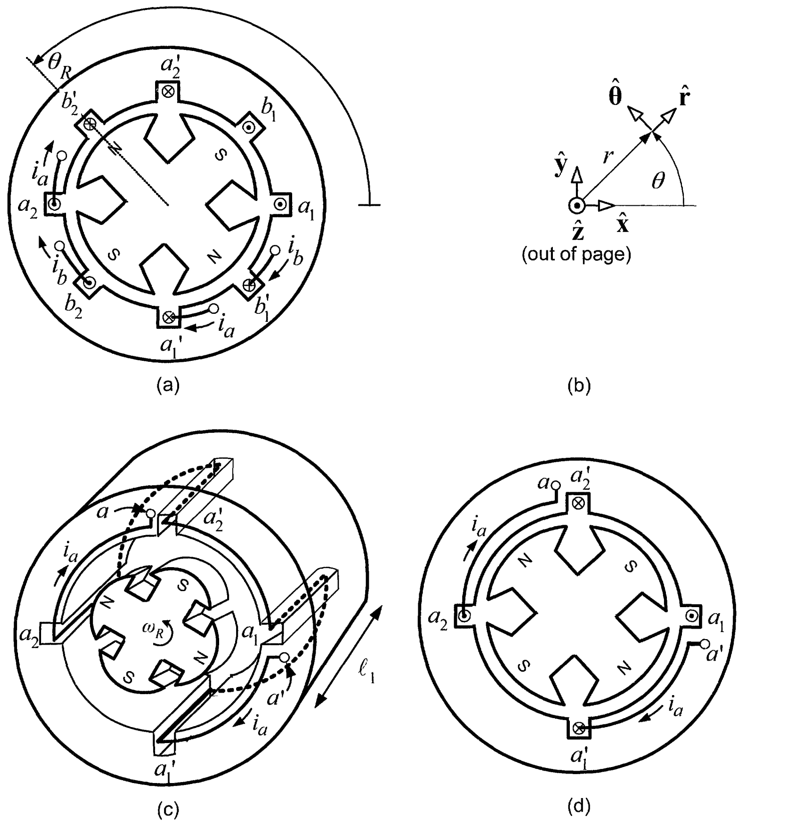

FIGURE 1.64. (a) A two-phase four-pole machine. (b) Polar coordinate system. (c) A perspective view of the phase a winding. (d) Cross-sectional view of the phase a winding.

Problem 17 Generator/Motor

Do problem 6 in Chapter 5. (The background in this chapter is sufficient to work this problem!)