Chapter 1

Bond Fundamentals

Risk management starts with the pricing of assets. The simplest assets to study are regular, fixed-coupon bonds. Because their cash flows are predetermined, we can translate their stream of cash flows into a present value by discounting at a fixed yield. Thus the valuation of bonds involves understanding compounded interest, discounting, and the relationship between present values and interest rates.

Risk management goes one step further than pricing, however. It examines potential changes in the price of an asset as the interest rate changes. In this chapter, we assume that there is a single interest rate, or yield, that is used to price the bond. This will be our fundamental risk factor. This chapter describes the relationship between bond prices and yield and presents indispensable tools for the management of fixed-income portfolios.

This chapter starts our coverage of quantitative analysis by discussing bond fundamentals. Section 1.1 reviews the concepts of discounting, present values, and future values. Section 1.2 then plunges into the price-yield relationship. It shows how the Taylor expansion rule can be used to relate movements in bond prices to those in yields. The Taylor expansion rule, however, covers much more than bonds. It is a building block of risk measurement methods based on local valuation, as we shall see later. Section 1.3 then presents an economic interpretation of duration and convexity.

The reader should be forewarned that this chapter, like many others in this handbook, is rather compact. This chapter provides a quick review of bond fundamentals with particular attention to risk measurement applications. By the end of this chapter, however, the reader should be able to answer advanced FRM questions on bond mathematics.
1.1 Discounting, Present Value, and Future Value

An investor considers a zero-coupon bond that pays $100 in 10 years. Say that the investment is guaranteed by the U.S. government and has no default risk. Because the payment occurs at a future date, the current value of the investment is surely less than an up-front payment of $100.

To value the payment, we need a discounting factor. This is also the interest rate, or more simply the yield. Define $C_t$ as the cash flow at time $t$ and the discounting factor as $y$. We define $T$ as the number of periods until maturity (e.g., number of years), also known as the tenor. The present value ($PV$) of the bond can be computed as

$$PV = \frac{C_T}{(1+y)^T} \quad (1.1)$$

For instance, a payment of $C_T = $100 in 10 years discounted at 6% is worth only $55.84 now. So, all else fixed, the market value of a zero-coupon bond decreases with longer maturities. Also, keeping $T$ fixed, the value of the bond decreases as the yield increases.

Conversely, we can compute the future value ($FV$) of the bond as

$$FV = PV \times (1+y)^T \quad (1.2)$$

For instance, an investment now worth $PV = $100 growing at 6% will have a future value of $FV = $179.08 in 10 years.

Here, the yield has a useful interpretation, which is that of an internal rate of return on the bond, or annual growth rate. It is easier to deal with rates of returns than with dollar values. Rates of return, when expressed in percentage terms and on an annual basis, are directly comparable across assets. An annualized yield is sometimes defined as the effective annual rate (EAR).

It is important to note that the interest rate should be stated along with the method used for compounding. Annual compounding is very common. Other conventions exist, however. For instance, the U.S. Treasury market uses semiannual compounding. Define in this case $y^S$ as the rate based on semiannual compounding. To maintain comparability, it is expressed in annualized form, that is, after multiplication by 2. The number of periods, or semesters, is now $2T$. The formula for finding $y^S$ is

$$PV = \frac{C_T}{(1+y^S/2)^{2T}} \quad (1.3)$$
For instance, a Treasury zero-coupon bond with a maturity of \( T = 10 \) years would have \( 2T = 20 \) semiannual compounding periods. Comparing with (1.1), we see that

\[
(1 + y) = (1 + y^S/2)^2 \tag{1.4}
\]

Continuous compounding is often used in modeling derivatives. It is the limit of the case where the number of compounding periods per year increases to infinity. The continuously compounded interest rate \( y^C \) is derived from

\[
PV = C_T \times e^{-y^C T} \tag{1.5}
\]

where \( e^{(\cdot)} \), sometimes noted as \( \exp(\cdot) \), represents the exponential function.

Note that in Equations (1.1), (1.3), and (1.5), the present values and future cash flows are identical. Because of different compounding periods, however, the yields will differ. Hence, the compounding period should always be stated.

**Example: Using different discounting methods**

Consider a bond that pays $100 in 10 years and has a present value of $55.8395. This corresponds to an annually compounded rate of 6.00% using \( PV = C_T/(1 + y)^{10} \), or \( (1 + y) = (C_T/PV)^{1/10} \).

This rate can be transformed into a semiannual compounded rate, using \( (1 + y^S/2)^2 = (1 + y) \), or \( y^S/2 = (1 + y)^{1/2} - 1 \), or \( y^S = ((1 + 0.06)^{1/2} - 1) \times 2 = 0.0591 = 5.91\% \). It can be also transformed into a continuously compounded rate, using \( \exp(y^C) = (1 + y) \), or \( y^C = \ln(1 + 0.06) = 0.0583 = 5.83\% \).

Note that as we increase the frequency of the compounding, the resulting rate decreases. Intuitively, because our money works harder with more frequent compounding, a lower investment rate will achieve the same payoff at the end.

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**Key concept:**

For fixed present value and cash flows, increasing the frequency of the compounding will decrease the associated yield.

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**Example 1-1: FRM Exam 2002—Question 48**

An investor buys a Treasury bill maturing in 1 month for $987. On the maturity date the investor collects $1,000. Calculate effective annual rate (EAR).

- a) 17.0%  
- b) 15.8%  
- c) 13.0%  
- d) 11.6%
Example 1-2: FRM Exam 2002—Question 51
Consider a savings account that pays an annual interest rate of 8%. Calculate the amount of time it would take to double your money. Round to the nearest year.
a) 7 years
b) 8 years
c) 9 years
d) 10 years

Example 1-3: FRM Exam 1999—Question 17
Assume a semiannual compounded rate of 8% per annum. What is the equivalent annually compounded rate?
a) 9.20%
b) 8.16%
c) 7.45%
d) 8.00%

1.2 Price-Yield Relationship

1.2.1 Valuation

The fundamental discounting relationship from Equation (1.1) can be extended to any bond with a fixed cash-flow pattern. We can write the present value of a bond $P$ as the discounted value of future cash flows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + y)^t}$$  \hspace{1cm} (1.6)

where

$C_t =$ cash flow (coupon or principal) in period $t$

$t =$ number of periods (e.g., half-years) to each payment

$T =$ number of periods to final maturity

$y =$ discounting factor per period (e.g., $y^{S/2}$)

A typical cash-flow pattern consists of a fixed coupon payment plus the repayment of the principal, or face value at expiration. Define $c$ as the coupon rate and $F$ as the face value. We have $C_t = cF$ prior to expiration, and at expiration, we have $C_T = cF + F$. The appendix reviews useful formulas that provide closed-form solutions for such bonds.
When the coupon rate $c$ precisely matches the yield $y$, using the same compounding frequency, the present value of the bond must be equal to the face value. The bond is said to be a **par bond**. 

Equation (1.6) describes the relationship between the yield $y$ and the value of the bond $P$, given its cash-flow characteristics. In other words, the value $P$ can also be written as a nonlinear function of the yield $y$:

$$P = f(y)$$  \hspace{1cm} (1.7)

Conversely, we can set $P$ to the current market price of the bond, including any accrued interest. From this, we can compute the “implied” yield that will solve this equation.

Figure 1-1 describes the price-yield function for a 10-year bond with a 6% annual coupon. In risk management terms, this is also the relationship between the payoff on the asset and the risk factor. At a yield of 6%, the price is at par, $P = $100. Higher yields imply lower prices.

**FIGURE 1-1 Price-Yield Relationship**

Over a wide range of yield values, this is a highly nonlinear relationship. For instance, when the yield is zero, the value of the bond is simply the sum of cash flows, or $160 in this case. When the yield tends to vary large values, the bond price tends to
zero. For small movements around the initial yield of 6%, however, the relationship is quasi-linear.

There is a particularly simple relationship for consols, or perpetual bonds, which are bonds making regular coupon payments but with no redemption date. For a consol, the maturity is infinite and the cash flows are all equal to a fixed percentage of the face value, \( C_t = C = cF \). As a result, the price can be simplified from Equation (1.6) to

\[
P = cF \left[ \frac{1}{(1 + y)} + \frac{1}{(1 + y)^2} + \frac{1}{(1 + y)^3} + \cdots \right] = \frac{c}{y} F \quad (1.8)
\]
as shown in the appendix. In this case, the price is simply proportional to the inverse of the yield. Higher yields lead to lower bond prices, and vice versa.

**Example: Valuing a bond**

Consider a bond that pays $100 in 10 years and a 6% annual coupon. Assume that the next coupon payment is in exactly one year. What is the market value if the yield is 6%? If it falls to 5%?

The bond cash flows are \( C_1 = $6, C_2 = $6, \ldots, C_{10} = $106 \). Using Equation (1.6) and discounting at 6%, this gives the present values of cash flows of $5.66, $5.34, \ldots, $59.19, for a total of $100.00. The bond is selling at par. This is logical because the coupon is equal to the yield, which is also annually compounded. Alternatively, discounting at 5% leads to a price of $107.72.

**Example 1-4: FRM Exam 1998—Question 12**

A fixed-rate bond, currently priced at 102.9, has one year remaining to maturity and is paying an 8% coupon. Assuming the coupon is paid semiannually, what is the yield of the bond?

- a) 8%
- b) 7%
- c) 6%
- d) 5%

**1.2.2 Taylor Expansion**

Let us say that we want to see what happens to the price if the yield changes from its initial value, called \( y_0 \), to a new value, \( y_1 = y_0 + \Delta y \). Risk management is all about assessing the effect of changes in risk factors such as yields on asset values. Are there shortcuts to help us with this?
We could recompute the new value of the bond as $P_1 = f(y_1)$. If the change is not too large, however, we can apply a very useful shortcut. The nonlinear relationship can be approximated by a Taylor expansion around its initial value:\footnote{This is named after the English mathematician Brook Taylor (1685–1731), who published this result in 1715. The full recognition of the importance of this result came in 1755, when Euler applied it to differential calculus.}

$$P_1 = P_0 + f'(y_0) \Delta y + \frac{1}{2} f''(y_0) (\Delta y)^2 + \cdots \quad (1.9)$$

where $f'() = \frac{dP}{dy}$ is the first derivative and $f''() = \frac{d^2P}{dy^2}$ is the second derivative of the function $f()$ valued at the starting point.\footnote{This first assumes that the function can be written in polynomial form as $P(y + \Delta y) = a_0 + a_1 \Delta y + a_2 (\Delta y)^2 + \cdots$, with unknown coefficients $a_0, a_1, a_2$. To solve for the first, we set $\Delta y = 0$. This gives $a_0 = P_0$. Next, we take the derivative of both sides and set $\Delta y = 0$. This gives $a_1 = f'(y_0)$. The next step gives $2a_2 = f''(y_0)$. Note that these are the conventional mathematical derivatives and have nothing to do with derivatives products such as options.} This expansion can be generalized to situations where the function depends on two or more variables. For bonds, the first derivative is related to the duration measure, and the second to convexity.

Equation (1.9) represents an infinite expansion with increasing powers of $\Delta y$. Only the first two terms (linear and quadratic) are used by finance practitioners. This is because they provide a good approximation to changes in price relative to other assumptions we have to make about pricing assets. If the increment is very small, even the quadratic term will be negligible.

Equation (1.9) is fundamental for risk management. It is used, sometimes in different guises, across a variety of financial markets. We will see later that the Taylor expansion is also used to approximate the movement in the value of a derivatives contract, such as an option on a stock. In this case, Equation (1.9) is

$$\Delta P = f'(S) \Delta S + \frac{1}{2} f''(S) (\Delta S)^2 + \cdots \quad (1.10)$$

where $S$ is now the price of the underlying asset, such as a stock. Here the first derivative, $f'(S)$, is called delta, and the second, $f''(S)$, gamma.

The Taylor expansion allows easy aggregation across financial instruments. If we have $x_i$ units (numbers) of bond $i$ and a total of $N$ different bonds in the portfolio, the portfolio derivatives are given by

$$f'(y) = \sum_{i=1}^{N} x_i f'_i(y) \quad (1.11)$$
We will illustrate this point later for a three-bond portfolio.

Example 1-5: FRM Exam 1999—Question 9
A number of terms in finance are related to the (calculus!) derivative of the price of a security with respect to some other variable. Which pair of terms is defined using second derivatives?

a) Modified duration and volatility
b) Vega and delta
c) Convexity and gamma
d) PV01 and yield to maturity

1.3 Bond Price Derivatives

For fixed-income instruments, the derivatives are so important that they have been given a special name. The negative of the first derivative is the dollar duration (DD):

$$f'(y_0) = \frac{dP}{dy} = -D^* \times P_0$$

(1.12)

where $D^*$ is called the modified duration. Thus, dollar duration is

$$DD = D^* \times P_0$$

(1.13)

where the price $P_0$ represents the market price, including any accrued interest. Sometimes, risk is measured as the dollar value of a basis point (DVBP),

$$DVBP = \left[ D^* \times P_0 \right] \times 0.0001$$

(1.14)

with 0.0001 representing one basis point (bp) or one-hundredth of a percent. The DVBP, sometimes called the DV01, measures can be easily added up across the portfolio.

The second derivative is the dollar convexity (DC):

$$f''(y_0) = \frac{d^2P}{dy^2} = C \times P_0$$

(1.15)

where $C$ is called the convexity.

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3Note that this chapter does not present duration in the traditional textbook order. In line with the advanced focus on risk management, we first analyze the properties of duration as a sensitivity measure. This applies to any type of fixed-income instrument. Later, we will illustrate the usual definition of duration as a weighted-average maturity, which applies for fixed-coupon bonds only.

For fixed-income instruments with known cash flows, the price-yield function is known, and we can compute analytical first and second derivatives. Consider, for example, our simple zero-coupon bond in Equation (1.1), where the only payment is the face value, $C_T = F$. We take the first derivative, which is

$$
\frac{dP}{dy} = \frac{d}{dy} \left[ \frac{F}{(1 + y)^T} \right] = (-T) \frac{F}{(1 + y)^{T+1}} = -\frac{T}{(1 + y)} P
$$

Comparing this with Equation (1.12), we see that the modified duration must be given by $D^* = T/(1 + y)$. The conventional measure of duration is $D = T$, which does not include division by $(1 + y)$ in the denominator. This is also called Macaulay duration. Note that duration is expressed in periods, such as $T$. With annual compounding, duration is in years. With semiannual compounding, duration is in semesters. It then has to be divided by 2 for conversion to years.

Modified duration is the appropriate measure of interest rate exposure. The quantity $(1 + y)$ appears in the denominator because we took the derivative of the present value term with discrete compounding. If we use continuous compounding, modified duration is identical to the conventional duration measure. In practice, the difference between Macaulay and modified duration is usually small.

Let us now go back to Equation (1.16) and consider the second derivative, which is

$$
\frac{d^2 P}{dy^2} = -(T + 1)(-T) \frac{F}{(1 + y)^{T+2}} = \frac{(T + 1)T}{(1 + y)^2} \times P
$$

Comparing this with Equation (1.15), we see that the convexity is $C = (T + 1)T/(1 + y)^2$. Note that its dimension is expressed in periods squared. With semiannual compounding, convexity is measured in semesters squared. It then has to be divided by 4 for conversion to years squared. So convexity must be positive for bonds with fixed coupons.

Putting together all these equations, we get the Taylor expansion for the change in the price of a bond, which is

$$
\Delta P = -[D^* \times P](\Delta y) + \frac{1}{2}[C \times P](\Delta y)^2 + \cdots
$$

Therefore, duration measures the first-order (linear) effect of changes in yield and convexity measures the second-order (quadratic) term.

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4This is because the conversion to annual terms is obtained by multiplying the semiannual yield $\Delta y$ by 2. As a result, the duration term must be divided by 2 and the convexity term by $2^2$, or 4, for conversion to annual units.
Example: Computing the price approximation
Consider a 10-year zero-coupon Treasury bond trading at a yield of 6%. The present value is obtained as $P = 100/(1 + 6/200)^{20} = 55.368$. As is the practice in the Treasury market, yields are semiannually compounded. Thus all computations should be carried out using semesters, after which final results can be converted to annual units.

Here, Macaulay duration is exactly 10 years, since $D = T$ for a zero-coupon bond. Its modified duration is $D^* = 20/(1 + 6/200) = 19.42$ semesters, which is 9.71 years. Its convexity is $C = 21 \times 20/(1 + 6/200)^2 = 395.89$ semesters squared, which is 98.97 years squared. Dollar duration is $DD = D^* \times P = 9.71 \times 55.37 = 537.55$. The DVBP is $DD \times 0.0001 = 0.0538$.

We want to approximate the change in the value of the bond if the yield goes to 7%. Using Equation (1.18), we have $\Delta P = -[9.71 \times 55.37](0.01) + 0.5[98.97 \times 55.37](0.01)^2 = -5.375 + 0.274 = -5.101$. Using the linear term only, the new price is $55.368 - 5.375 = 49.992$. Using the two terms in the expansion, the predicted price is slightly higher, at $55.368 - 5.375 + 0.274 = 50.266$.

These numbers can be compared with the exact value, which is $50.257$. The linear approximation has a relative pricing error of $-0.53\%$, which is not bad. Adding a quadratic term reduces this to an error of only $0.02\%$, which is minuscule given typical bid-ask spreads.

More generally, Figure 1-2 compares the quality of the Taylor series approximation. We consider a 10-year bond paying a 6% coupon semiannually. Initially, the yield is also at 6% and, as a result the price of the bond is at par, at $100$. The graph compares, for various values of the yield $y$,

- The actual, exact price $P = f(y_0 + \Delta y)$
- The duration estimate $P = P_0 - D^*P_0\Delta y$
- The duration and convexity estimate $P = P_0 - D^*P_0\Delta y + (1/2)CP_0(\Delta y)^2$
FIGURE 1-2 Price Approximation

The actual price curve shows an increase in the bond price if the yield falls and, conversely, a depreciation if the yield increases. This effect is captured by the tangent to the true-price curve, which represents the linear approximation based on duration. For small movements in the yield, this linear approximation provides a reasonable fit to the exact price.

**Key concept:**
Dollar duration measures the (negative) slope of the tangent to the price-yield curve at the starting point.

For large movements in price, however, the price-yield function becomes more curved and the linear fit deteriorates. Under these conditions, the quadratic approximation is noticeably better.

We should also note that the curvature is away from the origin, which explains the term *convexity* (as opposed to concavity). Figure 1-3 compares curves with different values for convexity. This curvature is beneficial since the second-order effect $0.5(C \times P)(\Delta y)^2$ must be positive when convexity is positive.
As the figure shows, when the yield rises, the price drops, but less than predicted by the tangent. Conversely, if the yield falls, the price increases faster than along the tangent. In other words, the quadratic term is always beneficial.

**Key concept:**
Convexity is always positive for regular coupon-paying bonds. Greater convexity is beneficial for both falling and rising yields.

The bond’s modified duration and convexity can also be computed directly from numerical derivatives. Duration and convexity cannot be computed directly for some bonds, such as mortgage-backed securities, because their cash flows are uncertain. Instead, the portfolio manager has access to pricing models that can be used to reprice the securities under various yield environments.

We choose a change in the yield, \( \Delta y \), and reprice the bond under an up-move scenario, \( P_+ = P(y_0 + \Delta y) \), and a down-move scenario, \( P_- = P(y_0 - \Delta y) \). Effective duration is measured by the numerical derivative. Using \( D^* = -(1/P)dP/dy \), it is estimated as

\[
D^E = \frac{[P_- - P_+]}{(2P_0\Delta y)} = \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{(2\Delta y)P_0}
\]  

(1.19)
Using \( C = (1/P)d^2P/dy^2 \), **effective convexity** is estimated as

\[
C^E = \frac{[D_- - D_+]}{\Delta y} = \left[ \frac{P(y_0 - \Delta y) - P_0}{(P_0 \Delta y)} - \frac{P_0 - P(y_0 + \Delta y)}{(P_0 \Delta y)} \right] / \Delta y \quad (1.20)
\]

To illustrate, consider a 30-year zero-coupon bond (illustrated in Figure 1-4) with a yield of 6%, semiannually compounded. The initial price is $16.9733. We revalue the bond at 5% and 7%, with prices shown in Table 1-1. The effective duration in Equation (1.19) uses the two extreme points. The effective convexity in Equation (1.20) uses the difference between the dollar durations for the up-move and down-move. Note that convexity is positive if duration increases as yields fall, or if \( D_- > D_+ \).

**FIGURE 1-4** Effective Duration and Convexity

The computations in Table 1-1 show an effective duration of 29.56. This is very close to the true value of 29.13, and it would be even closer if the step \( \Delta y \) were smaller. Similarly, the effective convexity is 869.11, which is close to the true value of 862.48.
**TABLE 1-1 Effective Duration and Convexity**

<table>
<thead>
<tr>
<th>State</th>
<th>Yield (%)</th>
<th>Bond Value</th>
<th>Duration Computation</th>
<th>Convexity Computation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial $y_0$</td>
<td>6.00</td>
<td>16.9733</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Up $y_0 + \Delta y$</td>
<td>7.00</td>
<td>12.6934</td>
<td>Duration up: 25.22</td>
<td></td>
</tr>
<tr>
<td>Down $y_0 - \Delta y$</td>
<td>5.00</td>
<td>22.7284</td>
<td>Duration down: 33.91</td>
<td></td>
</tr>
</tbody>
</table>

| Difference in values | $-10.0349$ | 8.69       |
| Difference in yields  | 0.02       | 0.01       |
| Effective measure     | 29.56      | 869.11     |
| Exact measure         | 29.13      | 862.48     |

Finally, this numerical approach can be applied to estimate the duration of a bond by considering bonds with the same maturity but different coupons. If interest rates decrease by 1%, the market price of a 6% bond should go up to a value close to that of a 7% bond. Thus we replace a drop in yield of $\Delta y$ by an increase in coupon $\Delta c$ and use the effective duration method to find the **coupon curve duration**:\(^5\)

$$D^{CC} = \frac{[P_+ - P_-]}{\Delta c} = \frac{P(y_0; c + \Delta c) - P(y_0; c - \Delta c)}{2\Delta c P_0} \quad (1.21)$$

This approach is useful for securities that are difficult to price under various yield scenarios. It requires only the market prices of securities with different coupons.

**Example: Computation of coupon curve duration**

Consider a 10-year bond that pays a 7% coupon semiannually. In a 7% yield environment, the bond is selling at par and has modified duration of 7.11 years. The prices of 6% and 8% coupon bonds are $92.89 and $107.11, respectively. This gives a coupon curve duration of $(107.11 - 92.89)/(0.02 \times 100) = 7.11$, which in this case is the same as modified duration.

**Example 1-6: FRM Exam 1998—Question 22**

What is the price impact of a 10-basis-point increase in yield on a 10-year par bond with a modified duration of 7 and convexity of 50?

a) $-0.705$

b) $-0.700$

c) $-0.698$

d) $-0.690$

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\(^5\)For an example of a more formal proof, we could take the pricing formula for a consol at par and compute the derivatives with respect to $y$ and $c$. Apart from the sign, these derivatives are identical when $y = c$. 

CHAPTER 1. BOND FUNDAMENTALS

Example 1-7: FRM Exam 1998—Question 17
A bond is trading at a price of 100 with a yield of 8%. If the yield increases by 1 basis point, the price of the bond will decrease to 99.95. If the yield decreases by 1 basis point, the price of the bond will increase to 100.04. What is the modified duration of the bond?

a) 5.0
b) −5.0
c) 4.5
d) −4.5

Example 1-8: FRM Exam 1998—Question 20
Coupon curve duration is a useful method for estimating duration from market prices of a mortgage-backed security (MBS). Assume the coupon curve of prices for Ginnie Maes in June 2001 is as follows: 6% at 92, 7% at 94, and 8% at 96.5. What is the estimated duration of the 7s?

a) 2.45
b) 2.40
c) 2.33
d) 2.25

Example 1-9: FRM Exam 1998—Question 21
Coupon curve duration is a useful method for estimating convexity from market prices of an MBS. Assume the coupon curve of prices for Ginnie Maes in June 2001 is as follows: 6% at 92, 7% at 94, and 8% at 96.5. What is the estimated convexity of the 7s?

a) 53
b) 26
c) 13
d) −53

1.3.1 Interpreting Duration and Convexity

The preceding section showed how to compute analytical formulas for duration and convexity in the case of a simple zero-coupon bond. We can use the same approach for coupon-paying bonds. Going back to Equation (1.6), we have

\[
\frac{dP}{dy} = \sum_{t=1}^{T} \frac{-tC_t}{(1+y)^{t+1}} = - \left[ \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t} \right] / P \times \frac{P}{(1+y)} = - \frac{D}{(1+y)} P 
\]  (1.22)
which defines duration as

\[ D = \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t} / P \]  

(1.23)

The economic interpretation of duration is that it represents the average time to wait for each payment, weighted by the present value of the associated cash flow. Indeed, replacing \( P \), we can write

\[ D = \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t} = \sum_{t=1}^{T} t \times w_t \]  

(1.24)

where the weights \( w \) represent the ratio of the present value of cash flow \( C_t \) relative to the total, and sum to unity. This explains why the duration of a zero-coupon bond is equal to the maturity. There is only one cash flow and its weight is 1.

Figure 1-5 lays out the present value of the cash flows of a 6% coupon, 10-year bond. Given a duration of 7.80 years, this coupon-paying bond is equivalent to a zero-coupon bond maturing in exactly 7.80 years.

**FIGURE 1-5 Duration as the Maturity of a Zero-Coupon Bond**

![Figure 1-5 Duration as the Maturity of a Zero-Coupon Bond](image)

For bonds with fixed coupons, duration is less than maturity. For instance, Figure 1-6 shows how the duration of a 10-year bond varies with its coupon. With a zero coupon, Macaulay duration is equal to maturity. Higher coupons place more weight on prior payments and therefore reduce duration.
CHAPTER 1. BOND FUNDAMENTALS

Duration can be expressed in a simple form for consols. From Equation (1.8), we have \( P = \frac{c}{y}F \). Taking the derivative, we find

\[
\frac{dP}{dy} = cF\left(-\frac{1}{y^2}\right) = \left(-1\right)\frac{1}{y} \left[ \frac{c}{y} F \right] = \left(-1\right)\frac{1}{y}P = -\frac{DC}{(1 + y)}P \tag{1.25}
\]

Hence the Macaulay duration for the consol \( D_C \) is

\[
D_C = \frac{(1 + y)}{y} \tag{1.26}
\]

This shows that the duration of a consol is finite even if its maturity is infinite. Also, it does not depend on the coupon.

FIGURE 1-6 Duration and Coupon

This formula provides a useful rule of thumb. For a long-term coupon-paying bond, duration must be lower than \( (1 + y)/y \). For instance, when \( y = 6\% \), the upper limit on duration is \( D_C = 1.06/0.06 \), or 17.7 years. In this environment, the duration of a par 30-year bond is 14.25, which is indeed lower than 17.7 years.

**Key concept:**
The duration of a long-term bond can be approximated by an upper bound, which is that of a consol with the same yield, \( D_C = (1 + y)/y \).
Figure 1-7 describes the relationship between duration, maturity, and coupon for regular bonds in a 6% yield environment. For the zero-coupon bond, $D = T$, which is a straight line going through the origin. For the par 6% bond, duration increases monotonically with maturity until it reaches the asymptote of $D_C$. The 8% bond has lower duration than the 6% bond for fixed $T$. Greater coupons, for a fixed maturity, decrease duration, as more of the payments come early.

**FIGURE 1-7 Duration and Maturity**

![Duration and Maturity Diagram](image)

Finally, the 2% bond displays a pattern intermediate between the zero-coupon and 6% bonds. It initially behaves like the zero, exceeding $D_C$ initially and then falling back to the asymptote, which is the same for all coupon-paying bonds.

Now taking the second derivative in Equation (1.22), we have

$$\frac{d^2P}{dy^2} = \sum_{t=1}^{T} \frac{t(t + 1)C_t}{(1 + y)^{t+2}} = \left[ \sum_{t=1}^{T} \frac{t(t + 1)C_t}{(1 + y)^{t+2}} \right] / P \times P$$

(1.27)

which defines convexity as

$$C = \sum_{t=1}^{T} \frac{t(t + 1)C_t}{(1 + y)^{t+2}} / P$$

(1.28)
Convexity can also be written as

$$C = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^{2}} \times \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^{2}} \times w_t \quad (1.29)$$

Because the squared $t$ term dominates in the fraction, this basically involves a weighted average of the square of time. Therefore, convexity is much greater for long-maturity bonds because they have payoffs associated with large values of $t$. The formula also shows that convexity is always positive for such bonds, implying that the curvature effect is beneficial. As we will see later, convexity can be negative for bonds that have uncertain cash flows, such as mortgage-backed securities (MBSs) or callable bonds.

Figure 1-8 displays the behavior of convexity, comparing a zero-coupon bond with a 6% coupon bond with identical maturities. The zero-coupon bond always has greater convexity, because there is only one cash flow at maturity. Its convexity is roughly the square of maturity—for example, about 900 for the 30-year zero. In contrast, the 30-year coupon bond has a convexity of only about 300.

**FIGURE 1-8 Convexity and Maturity**

As an illustration, Table 1-2 details the steps of the computation of duration and convexity for a two-year, 6% semiannual coupon-paying bond. We first convert the annual coupon and yield to semiannual equivalents, $3 and 3\%$. The $PV$ column then
reports the present value of each cash flow. We verify that these add up to $100, since the bond must be selling at par.

Next, the Duration Term column multiplies each $PV$ term by time, or more precisely, the number of half-years until payment. This adds up to $382.86$, which divided by the price gives $D = 3.83$. This number is measured in half-years, and we need to divide by 2 to convert to years. Macaulay duration is 1.91 years, and modified duration $D^* = 1.91 / 1.03 = 1.86$ years. Note that, to be consistent, the adjustment in the denominator involves the semiannual yield of $3\%$.

Finally, the rightmost column shows how to compute the bond’s convexity. Each term involves $PV_t$ times $t(t + 1)/(1 + y)^2$. These terms sum to 1,777.755 or, divided by the price, 17.78. This number is expressed in units of time squared and must be divided by 4 to be converted to annual terms. We find a convexity of $C = 4.44$, in years squared.

### TABLE 1-2 Computing Duration and Convexity

<table>
<thead>
<tr>
<th>Period (half-year) $t$</th>
<th>Yield (%)</th>
<th>$PV$ of Payment $C_t/(1 + y)^t$</th>
<th>Duration Term $t(t + 1)PV_t/(1 + y)^2$</th>
<th>Convexity Term $tPV_t \times (1/(1 + y)^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>3.00</td>
<td>2.913</td>
<td>5.491</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3.00</td>
<td>2.828</td>
<td>5.656</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3.00</td>
<td>2.745</td>
<td>8.236</td>
</tr>
<tr>
<td>4</td>
<td>103</td>
<td>3.00</td>
<td>91.514</td>
<td>1725.218</td>
</tr>
<tr>
<td>Sum:</td>
<td></td>
<td></td>
<td>100.00</td>
<td>382.861</td>
</tr>
<tr>
<td>(half-years)</td>
<td></td>
<td></td>
<td></td>
<td>1777.755</td>
</tr>
<tr>
<td>(years)</td>
<td></td>
<td></td>
<td></td>
<td>1.91</td>
</tr>
<tr>
<td>Modified duration</td>
<td></td>
<td></td>
<td></td>
<td>1.86</td>
</tr>
<tr>
<td>Convexity</td>
<td></td>
<td></td>
<td></td>
<td>4.44</td>
</tr>
</tbody>
</table>

### Example 1-10: FRM Exam 2001—Question 71

Calculate the modified duration of a bond with a Macaulay duration of 13.083 years. Assume market interest rates are 11.5% and the coupon on the bond is paid semiannually.

a) 13.083  
b) 12.732  
c) 12.459  
d) 12.371
Example 1-11: FRM Exam 2002—Question 118
A Treasury bond has a coupon rate of 6% per annum (the coupons are paid semiannually) and a semiannually compounded yield of 4% per annum. The bond matures in 18 months and the next coupon will be paid 6 months from now. Which number is closest to the bond’s Macaulay duration?

a) 1.023 years
b) 1.457 years
c) 1.500 years
d) 2.915 years

Example 1-12: FRM Exam 1998—Question 29
A and B are two perpetual bonds; that is, their maturities are infinite. A has a coupon of 4% and B has a coupon of 8%. Assuming that both are trading at the same yield, what can be said about the duration of these bonds?

a) The duration of A is greater than the duration of B.
b) The duration of A is less than the duration of B.
c) A and B have the same duration.
d) None of the above.

Example 1-13: FRM Exam 1997—Question 24
Which of the following is not a property of bond duration?

a) For zero-coupon bonds, Macaulay duration of the bond equals its years to maturity.
b) Duration is usually inversely related to the coupon of a bond.
c) Duration is usually higher for higher yields to maturity.
d) Duration is higher as the number of years to maturity for a bond selling at par or above increases.

Example 1-14: FRM Exam 1999—Question 75
Suppose that your book has an unusually large short position in two investment-grade bonds with similar credit risk. Bond A is priced at par yielding 6.0% with 20 years to maturity. Bond B also matures in 20 years with a coupon of 6.5% and yield of 6%. If risk is defined as a sudden and large drop in interest rate, which bond contributes greater market risk to the portfolio?

a) Bond A.
b) Bond B.
c) Bond A and bond B will have similar market risk.
d) None of the above.
Example 1-15: FRM Exam 2001—Question 104
When the maturity of a plain coupon bond increases, its duration increases:

a) Indefinitely and regularly
b) Up to a certain level
c) Indefinitely and progressively
d) In a way dependent on the bond being priced above or below par

Example 1-16: FRM Exam 2000—Question 106
Consider the following bonds:

<table>
<thead>
<tr>
<th>Bond Number</th>
<th>Maturity (yrs)</th>
<th>Coupon Rate</th>
<th>Frequency</th>
<th>Yield (ABB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>6%</td>
<td>1</td>
<td>6%</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>6%</td>
<td>2</td>
<td>6%</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>0%</td>
<td>1</td>
<td>6%</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>6%</td>
<td>1</td>
<td>5%</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>6%</td>
<td>1</td>
<td>6%</td>
</tr>
</tbody>
</table>

How would you rank the bonds from shortest to longest duration?

a) 5-2-1-4-3
b) 1-2-3-4-5
c) 5-4-3-1-2
d) 2-4-5-1-3

1.3.2 Portfolio Duration and Convexity

Fixed-income portfolios often involve very large numbers of securities. It would be impractical to consider the movements of each security individually. Instead, portfolio managers aggregate the duration and convexity across the portfolio. A manager with a view that rates will increase should shorten the portfolio duration relative to that of the benchmark. Say the benchmark has a duration of five years. The manager shortens the portfolio duration to one year only. If rates increase by 2%, the benchmark will lose approximately $5 \times 2\% = 10\%$. The portfolio, however, will only lose $1 \times 2\% = 2\%$, hence “beating” the benchmark by 8%.

Because the Taylor expansion involves a summation, the portfolio duration is easily obtained from the individual components. Say we have $N$ components indexed by $i$. Defining $D_p^*$ and $P_p$ as the portfolio modified duration and value, the portfolio dollar duration (DD) is

$$D_p^* P_p = \sum_{i=1}^{N} D_i^* x_i P_i$$

(1.30)

where $x_i$ is the number of units of bond $i$ in the portfolio. A similar relationship holds.
for the portfolio dollar convexity (DC). If yields are the same for all components, this equation also holds for the Macaulay duration.

Because the portfolio total market value is simply the summation of the component market values,

\[ P_p = \sum_{i=1}^{N} x_i P_i \]  

we can define the portfolio weight as \( w_i = x_i P_i / P_p \), provided that the portfolio market value is nonzero. We can then write the portfolio duration as a weighted average of individual durations:

\[ D_p^* = \sum_{i=1}^{N} D_i^* w_i \]  

Similarly, the portfolio convexity is a weighted average of convexity numbers:

\[ C_p = \sum_{i=1}^{N} C_i w_i \]

As an example, consider a portfolio invested in three bonds as described in Table 1-3. The portfolio is long a 10-year and a 1-year bond, and short a 30-year zero-coupon bond. Its market value is $1,301,600. Summing the duration for each component, the portfolio dollar duration is $2,953,800, which translates into a duration of 2.27 years. The portfolio convexity is \(-76,918,323/1,301,600 = -59.10\), which is negative due to the short position in the 30-year zero, which has very high convexity.

**TABLE 1-3 Portfolio Dollar Duration and Convexity**

<table>
<thead>
<tr>
<th></th>
<th>Bond 1</th>
<th>Bond 2</th>
<th>Bond 3</th>
<th>Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity (years)</td>
<td>10</td>
<td>1</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Coupon</td>
<td>6%</td>
<td>0%</td>
<td>0%</td>
<td></td>
</tr>
<tr>
<td>Yield</td>
<td>6%</td>
<td>6%</td>
<td>6%</td>
<td></td>
</tr>
<tr>
<td>Price ( P_i )</td>
<td>$100.00</td>
<td>$94.26</td>
<td>$16.97</td>
<td></td>
</tr>
<tr>
<td>Modified duration ( D_i^* )</td>
<td>7.44</td>
<td>0.97</td>
<td>29.13</td>
<td></td>
</tr>
<tr>
<td>Convexity ( C_i )</td>
<td>68.78</td>
<td>1.41</td>
<td>862.48</td>
<td></td>
</tr>
<tr>
<td>Number of bonds ( x_i )</td>
<td>10,000</td>
<td>5,000</td>
<td>-10,000</td>
<td></td>
</tr>
<tr>
<td>Dollar amounts ( x_i P_i )</td>
<td>$1,000,000</td>
<td>$471,300</td>
<td>-$169,700</td>
<td>$1,301,600</td>
</tr>
<tr>
<td>Weight ( w_i )</td>
<td>76.83%</td>
<td>36.21%</td>
<td>-13.04%</td>
<td>100.00%</td>
</tr>
<tr>
<td>Dollar duration ( D_i^* P_i )</td>
<td>$744.00</td>
<td>$91.43</td>
<td>$494.34</td>
<td></td>
</tr>
<tr>
<td>Portfolio DD: ( x_i D_i^* P_i )</td>
<td>$7,440,000</td>
<td>$457,161</td>
<td>-$4,943,361</td>
<td>$2,953,800</td>
</tr>
<tr>
<td>Portfolio DC: ( x_i C_i P_i )</td>
<td>68,780,000</td>
<td>664,533</td>
<td>-146,362,856</td>
<td>-$76,918,323</td>
</tr>
</tbody>
</table>

Alternatively, assume the portfolio manager is given a benchmark, which is the first bond. The manager wants to invest in bonds 2 and 3, keeping the portfolio duration equal to that of the target, 7.44 years. To achieve the target value and dollar duration, the manager needs to solve a system of two equations in the numbers $x_1$ and $x_2$:

Value: $100 = x_1 \times 94.26 + x_2 \times 16.97$

Dol. duration: $7.44 \times \$100 = 0.97 \times x_1 \times 94.26 + 29.13 \times x_2 \times 16.97$

The solution is $x_1 = 0.817$ and $x_2 = 1.354$, which gives a portfolio value of $100 and modified duration of 7.44 years. The portfolio convexity is 199.25, higher than the index. Such a portfolio consisting of very short and very long maturities is called a **barbell portfolio**. In contrast, a portfolio with maturities in the same range is called a **bullet portfolio**. Note that the barbell portfolio has much greater convexity than the bullet because of the payment in 30 years. Such a portfolio would be expected to outperform the bullet portfolio if yields move by a large amount.

In sum, duration and convexity are key measures of fixed-income portfolios. They summarize the linear and quadratic exposure to movements in yields. This explains why they are routinely used by portfolio managers.

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**Example 1-17: FRM Exam 2002—Question 57**

A bond portfolio has the following composition:
1) Portfolio A: price $90,000, modified duration 2.5, long position in 8 bonds
2) Portfolio B: price $110,000, modified duration 3, short position in 6 bonds
3) Portfolio C: price $120,000, modified duration 3.3, long position in 12 bonds

All interest rates are 10%. If the rates rise by 25 basis points, then the bond portfolio value will:
- a) Decrease by $11,430
- b) Decrease by $21,330
- c) Decrease by $12,573
- d) Decrease by $23,463

---

6This can be obtained by first expressing $x_2$ in the first equation as a function of $x_1$ and then substituting back into the second equation. This gives $x_2 = (100 - 94.26x_1)/16.97$, and $744 = 91.43x_1 + 494.34x_2 = 91.43x_1 + 494.34(100 - 94.26x_1)/16.97 = 91.43x_1 + 2913.00 - 2745.79x_1$. Solving, we find $x_1 = (-2169.00)/(-2654.36) = 0.817$ and $x_2 = (100 - 94.26 \times 0.817)/16.97 = 1.354$. 

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Example 1-18: FRM Exam 2000—Question 110

Which of the following statements are true?
I. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 10-year, 6% bond.
II. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 6% bond with a duration of 10 years.
III. Convexity grows proportionately with the maturity of the bond.
IV. Convexity is positive for all types of bonds.
V. Convexity is always positive for “straight” bonds.

a) I only
b) I and II only
c) I and V only
d) II, III, and V only

1.4 Answers to Chapter Examples

Example 1-1: FRM Exam 2002—Question 48

a) The EAR is defined by \( FV/PV = (1 + \text{EAR})^T \). So \( \text{EAR} = (FV/PV)^{1/T} - 1 = (1,000/987)^{1/12} - 1 = 17.0\% \).

Example 1-2: FRM Exam 2002—Question 51

c) The time \( T \) relates the current and future values such that \( FV/PV = 2 = (1 + 8\%)^T \). Taking logs of both sides, this gives \( T = \ln(2)/\ln(1.08) = 9.006 \).

Example 1-3: FRM Exam 1999—Question 17

b) This is derived from \( (1 + y^S/2)^2 = (1 + y) \), or \( (1 + 0.08/2)^2 = 1.0816 \), which gives 8.16\%. This makes sense because the annual rate must be higher due to the less frequent compounding.

e) First derivatives involve modified duration and delta. Second derivatives involve
convexity (for bonds) and gamma (for options).

**Example 1-6: FRM Exam 1998—Question 22**
c) Since this is a par bond, the initial price is $P = 100. The price impact is $\Delta P = -D^* P \Delta y + (1/2)CP(\Delta y)^2 = -(7 \times 100)(0.001) + (1/2)(50 \times 100)(0.001)^2 = -0.70 + 0.0025 = -0.6975$. The price falls slightly less than predicted by duration alone.

**Example 1-7: FRM Exam 1998—Question 17**
c) This question deals with effective duration, which is obtained from full repricing of the bond with an increase and a decrease in yield. This gives a modified duration of $D^* = -\Delta P / \Delta y / P = -(99.95 - 100.04) / 0.0002 / 100 = 4.5$.

**Example 1-8: FRM Exam 1998—Question 20**
b) The initial price of the 7s is 94. The price of the 6s is 92; this lower coupon is roughly equivalent to an up-move of $\Delta y = 0.01$. Similarly, the price of the 8s is 96.5; this higher coupon is roughly equivalent to a down-move of $\Delta y = 0.01$. The effective modified duration is then $D^E = (P_0 - P_+) / (2 \Delta y P_0) = (96.5 - 92) / (2 \times 0.01 \times 94) = 2.394$.

**Example 1-9: FRM Exam 1998—Question 21**
a) We compute the modified duration for an equivalent down-move in $y$ as $D_- = (P_- - P_0) / (\Delta y P_0) = (96.5 - 94) / 0.01 \times 94) = 2.6596$. Similarly, the modified duration for an up-move is $D_+ = (P_0 - P_+) / (\Delta y P_0) = (94 - 92) / 0.01 \times 94) = 2.1277$. Convexity is $C^E = (D_- - D_+) / (\Delta y) = (2.6596 - 2.1277) / 0.01 = 53.19$. This is positive because modified duration is higher for a down-move than for an up-move in yields.

**Example 1-10: FRM Exam 2001—Question 71**
d) Modified duration is $D^* = D / (1 + y/200)$ when yields are semiannually compounded. This gives $D^* = 13.083 / (1 + 11.5/200) = 12.3716$.

**Example 1-11: FRM Exam 2002—Question 118**
b) For coupon-paying bonds, Macaulay duration is slightly less than the maturity, which is 1.5 years here. So b) would be a good guess. Otherwise, we can compute duration exactly.

**Example 1-12: FRM Exam 1998—Question 29**
c) Going back to the duration equation for the consol, Equation (1.26), we see that it
does not depend on the coupon but only on the yield. Hence, the durations must be the same. The price of bond A, however, must be half that of bond B.

Example 1-13: FRM Exam 1997—Question 24

c) Duration usually increases as the time to maturity increases (Figure 1-7), so d) is correct. Macaulay duration is also equal to maturity for zero-coupon bonds, so a) is correct. Figure 1-6 shows that duration decreases with the coupon, so b) is correct. As the yield increases, the weight of the payments further into the future decreases, which *decreases* (not increases) the duration. So c) is false.

Example 1-14: FRM Exam 1999—Question 75

a) Bond B has a higher coupon and hence a slightly lower duration than Bond A. Therefore, it will react less strongly than bond A to a given change in yields.

Example 1-15: FRM Exam 2001—Question 104

b) With a fixed coupon, the duration goes up to the level of a consol with the same coupon. See Figure 1-7.

Example 1-16: FRM Exam 2000—Question 106

a) The nine-year bond (number 5) has shorter duration because the maturity is shortest, at nine years, among comparable bonds. Next, we have to decide between bonds 1 and 2, which differ only in the payment frequency. The semiannual bond (number 2) has a first payment in six months and has shorter duration than the annual bond. Next, we have to decide between bonds 1 and 4, which differ only in the yield. With lower yield, the cash flows further in the future have a higher weight, so that bond 4 has greater duration. Finally, the zero-coupon bond has the longest duration. So, the order is 5-2-1-4-3.

Example 1-17: FRM Exam 2002—Question 57

a) The portfolio dollar duration is 
   
   \[ D^\ast P = \sum x_i D_i^\ast P_i = +8 \times 2.5 \times 90,000 - 6 \times 3.0 \times 110,000 + 12 \times 3.3 \times 120,000 = 4,572,000. \]

   The change in portfolio value is then
   
   \[ -(D^\ast P)(\Delta y) = -4,572,000 \times 0.0025 = -11,430. \]

Example 1-18: FRM Exam 2000—Question 110

c) Because convexity is proportional to the square of time to payment, the convexity of a bond will be driven by the cash flows far into the future. Answer I is correct because the 10-year zero has only one cash flow, whereas the coupon bond has several others that reduce convexity. Answer II is false because the 6% bond with 10-year duration...
must have cash flows much further into the future, say 30 years, which will create greater convexity. Answer III is false because convexity grows with the square of time. Answer IV is false because some bonds—for example, MBSs or callable bonds—can have negative convexity. Answer V is correct because convexity must be positive for coupon-paying bonds.

**Appendix: Applications of Infinite Series**

When bonds have fixed coupons, the bond valuation problem often can be interpreted in terms of combinations of infinite series. The most important infinite series result is for a sum of terms that increase at a geometric rate:

\[
1 + a + a^2 + a^3 + \cdots = \frac{1}{1 - a}
\]

(1.34)

This can be proved, for instance, by multiplying both sides by \((1 - a)\) and canceling out terms.

Equally important, consider a geometric series with a finite number of terms, say \(N\). We can write this as the difference between two infinite series:

\[
1 + a + a^2 + a^3 + \cdots + a^{N-1} = (1 + a + a^2 + a^3 + \cdots) - a^N (1 + a + a^2 + a^3 + \cdots)
\]

(1.35)

such that all terms with order \(N\) or higher will cancel each other.

We can then write

\[
1 + a + a^2 + a^3 + \cdots + a^{N-1} = \frac{1}{1 - a} - a^N \frac{1}{1 - a}
\]

(1.36)

These formulas are essential to value bonds. Consider first a consol with an infinite number of coupon payments with a fixed coupon rate \(c\). If the yield is \(y\) and the face value \(F\), the value of the bond is

\[
P = cF \left[ \frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots \right]
\]

\[
= cF \frac{1}{(1+y)} [1 + a^2 + a^3 + \cdots]
\]

\[
= cF \frac{1}{(1+y)} \left[ \frac{1}{1-a} \right]
\]

\[
= cF \frac{1}{(1+y)} \left[ \frac{1}{(1-1/(1+y))} \right]
\]

\[
= cF \frac{1}{(1+y)} \left[ \frac{(1+y)}{y} \right]
\]

\[
= \frac{c}{y} F
\]
Similarly, we can value a bond with a \textit{finite} number of coupons over $T$ periods at which time the principal is repaid. This is really a portfolio with three parts:

(1) A long position in a consol with coupon rate $c$
(2) A short position in a consol with coupon rate $c$ that starts in $T$ periods
(3) A long position in a zero-coupon bond that pays $F$ in $T$ periods.

Note that the combination of (1) and (2) ensures that we have a finite number of coupons. Hence, the bond price should be

\[
P = \frac{c}{y}F - \frac{1}{(1+y)^T} \frac{c}{y}F + \frac{1}{(1+y)^T}F = \frac{c}{y}F \left[1 - \frac{1}{(1+y)^T}\right] + \frac{1}{(1+y)^T}F \tag{1.37}
\]

where again the formula can be adjusted for different compounding methods.

This is useful for a number of purposes. For instance, when $c = y$, it is immediately obvious that the price must be at par, $P = F$. This formula also can be used to find closed-form solutions for duration and convexity.