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Introduction

This book serves as a source/reference book for engineers and scientists working with measurement errors in one-, two-, and three-dimensional space, as well as for people who desire to obtain a clear understanding of the concepts of the various error standards and their interrelationships.

Through MATLAB, this book also introduces and provides a convenient tool for computation and comparison so that table look-up and "guestimation" can be avoided.

Since the book is application oriented, only the important, relevant results from Probability and Statistics are used. For proofs of theorems and derivations, the reader can refer to excellent textbooks such as [**1**, **2**].

The reader is assumed to have had an introductory course in Probability and Statistics, and to have a fairly good working knowledge of Differential and Integral Calculus.

We will be concerned mainly with normally distributed random variables. The normal (Gaussian) distribution is useful because it seems to describe the random observations of most experiments. It also describes the distribution associated with the parameter estimation for most probability distributions.

The notation to be used throughout this book is defined in Section 1.1. All problems in error analysis can be cast into two categories: direct problems and inverse problems. In Sections 1.2 and 1.3 we shall define these two types of problems in order to set a pattern for discussion in the chapters to come. Section 1.4 shows how to use the author-generated programs to solve problems in navigation accuracy analysis.

1.1 NOTATION

We will write random variables symbolically in boldface type as, for example, the random variable $\mathbf{x}$. Often, it is necessary to find the probability that the value of a random variable $\mathbf{x}$ is less than or equal to some real number x; this we write as

$$P\{\mathbf{x} \leq x\}$$

The notation shown in the following list will be used throughout this book, and any deviation from it will be noted immediately to avoid any confusion.

Let $\triangleq$ stand for "equal by definition."

- $[\ldots] \triangleq$ row vector
- $[\ldots]' \triangleq$ column vector
- $\vec{V}_1 = [x]' \triangleq$ position vector of one component
- $\vec{V}_2 = [x, y]' \triangleq$ position vector of two components
- $\vec{V}_3 = [x, y, z]' \triangleq$ position vector of three components
- $g(t) \triangleq$ probability density function of a random variable $\mathbf{t}$
- $G(t) \triangleq$ cumulative distribution function of $g(t)$, $G(t) = \int_{-\infty}^{t} g(u)\,du$
- $P\{\vec{V} \in A\} \triangleq$ probability that the random vector $\vec{V}$ falls into region A
- $N(\mu, \sigma^2) \triangleq$ normal distribution with mean μ and variance σ^2

1.2 DIRECT PROBLEMS

Let the vectors $\vec{V}_1 = [\mathbf{x}]'$, $\vec{V}_2 = [\mathbf{x}, \mathbf{y}]'$, and $\vec{V}_3 = [\mathbf{x}, \mathbf{y}, \mathbf{z}]'$ represent one-, two-, and three-dimensional random vectors, with corresponding probability density functions, $f_1(x)$, $f_2(x, y)$, and $f_3(x, y, z)$, respectively.

The direct problem consists of finding the probability $p = P\{\vec{V} \in A\}$ when the region A is specified. Variables μ_x, μ_y, μ_z, and R are defined in Section 1.3.

For one-dimensional problems, A could be an interval specified as

$$|x - \mu_x| \leq R \tag{1.1}$$

For two-dimensional problems, A could be a circular region:

$$(x - \mu_x)^2 + (y - \mu_y)^2 \leq R^2 \tag{1.2}$$

or A could be an elliptical region with **elliptical scale** k

$$\frac{(x - \mu_x)^2}{a^2} + \frac{(y - \mu_y)^2}{b^2} \leq k^2 \tag{1.3}$$

For three-dimensional problems, A could be a spherical region:

$$(x - \mu_x)^2 + (y - \mu_y)^2 + (z - \mu_z)^2 \leq R^2 \tag{1.4}$$

or A could be an ellipsoidal region with **ellipsoidal scale** k

$$\frac{(x - \mu_x)^2}{a^2} + \frac{(y - \mu_y)^2}{b^2} + \frac{(z - \mu_z)^2}{c^2} \leq k^2 \tag{1.5}$$

Thus, the relationship between $f_1(x)$ and $p = P\{\vec{V}_1 \in A\}$ is

$$p = P\{\vec{V}_1 \in A\} = P\{x \in (\mu - R, \mu + R)\} = \int_{\mu-R}^{\mu+R} f_1(x)\,dx$$

Furthermore, the relationship between $f_2(x, y)$ and $p = P\{\vec{V}_2 \in A\}$ is

$$p = P\{\vec{V}_2 \in A\} = P\{(x, y) \in A\} = \iint_A f_2(x, y)\,dx\,dy$$

and the relationship between $f_3(x, y, z)$ and $p = P\{\vec{V}_3 \in A\}$ is

$$p = P\{\vec{V}_3 \in A\} = P\{(x, y, z) \in A\} = \iiint_A f_3(x, y, z)\,dx\,dy\,dz$$

The direct problem becomes "given the dimension of A, find $p = P\{\vec{V} \in A\}$," where the dimension of A is specified in terms of R or k in Equations (1.1) through (1.5).

If exact integration is possible, p can be expressed as a closed-form formula; otherwise p can be obtained through numerical integration of a single, double, or triple integral.

1.3 INVERSE PROBLEMS

For the inverse problem, the probability p is given, and one is to find, depending on dimensionality, the half-length R of the interval centered at the mean μ_x for the one-dimensional problem; the radius R or the scale k of the circle or ellipse centered at the mean (μ_x, μ_y) for the two-dimensional problem; and the radius R or the scale k of the sphere or ellipsoid centered at the mean (μ_x, μ_y, μ_z) for the three-dimensional problem, such that $p = P\{\vec{V} \in A\}$.

That is, given p, find the dimension of A in terms of R or k in Equations (1.1) through (1.5) such that

$$\int_{\mu-R}^{\mu+R} f_1(x)\,dx = p$$

or

$$\iint_A f_2(x, y)\,dx\,dy = p$$

or

$$\iiint_A f_3(x, y, z)\,dx\,dy\,dz = p$$

Two approaches are used in this book to solve the general inverse problem: given p, find R such that $F(R) = p$.

The first approach is via numerical trial-and-error. We make an initial guess R_0 of the true solution and compute $p_0 = F(R_0)$. If $p_0 > p$, decrease R_0; if $p_0 < p$, increase R_0. The process is repeated until adequate resolution is attained.

The second approach uses Newton–Raphson's method [**3**] to find the root of the equivalent problem

$$G(R) = F(R) - p = 0$$

We start with an initial guess R_0, and we iterate according to

$$R_{i+1} = R_i - G(R_i)/G'(R_i), \quad \text{for } i = 0, 1, 2, \ldots \tag{1.6}$$

where G' is the derivative of G, until a certain accuracy criterion is satisfied. This process is implemented in the M-file **newton.m**.

1.4 USE OF AUTHOR-GENERATED M-FILES

All the programs used in this book are MATLAB M-files. Each has been tested thoroughly with MATLAB Version 4.2c and Version 5.2 on various PC platforms (Pentium [166 MHz, 66 MHz], 486-60 MHz computers).

In order to install and use these M-files, the reader should follow these steps:

1. Create a subdirectory c:\mfile
   md c:\mfile
2. Copy all files from a:\mfile to c:\mfile
   copy a:\mfile*.* c:\mfile
3. (for Version 5.2) Start MATLAB.
   Click path browser to include c:\mfile in the MATLAB path.
   (for Version 4.2c) Include c:\mfile in the path of MATLAB by adding the line 'c:\mfile' in the file c:\matlab**matlabrc.m** through an editor. Start MATLAB.
4. Enter $nf2a(1)$. The user should see the numerical result

   ```
   ans = 0.6827
   ```

   produced with an accompanying graph (see Figure 1.1) which describes the geometric meaning of the number 0.6827. This indicates a successful installation of the M-files.

The names and functions of these author-generated M-files are listed in the last section of each chapter. Those who wish to jump right in for a hands-on experience are encouraged to do so by referring to these sections in Chapters 1 through 8.

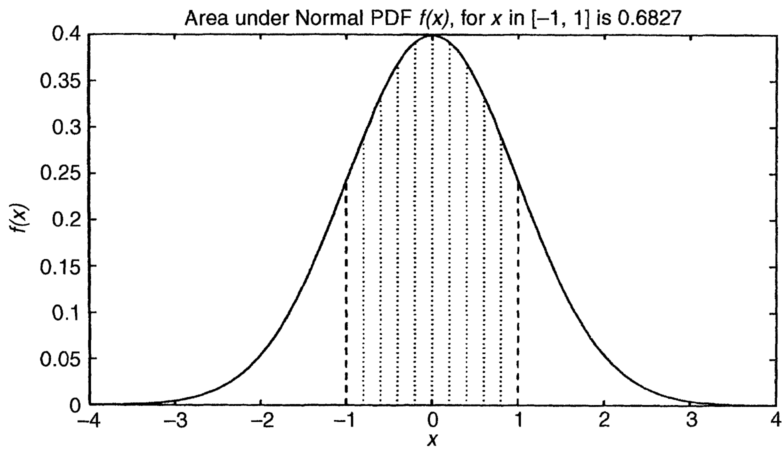

Figure 1.1 Results of the MATLAB Command $nf2a(1)$.

1.5 SUMMARY OF M-FILES

The M-files used or generated in this chapter are summarized in Table 1.1.

TABLE 1.1 M-files Used in Chapter 1

File Name	MATLAB Command
nf2a.m	$p = nf2a(r)$
nf1.m	$y = nf1(x)$
nf2.m	$p = nf2(r)$
pf3.m	$pf3('fname')$
newton.m	$x = newton(x_0, 'fun', 'dfun', tol)$

The three files **nf1.m**, **nf2.m**, and **pf3.m** are called within the test program **nf2a.m**.