1 FUNDAMENTAL CONCEPTS OF TRANSMISSION LINES DERIVED FROM AC THEORY

1.1 INTRODUCTION

Almost everyone is familiar with radio and television waves which travel through air, but the concept of waves on an electrical line is not as commonly appreciated. Lights, television sets, radios, and numerous other appliances are used daily without the slightest thought being given to the manner in which the energy is transmitted to these devices through conductors, even though the underlying phenomenon is of fundamental importance. Consequently, the student who is being exposed for the first time to the concept of a transmission line usually finds the idea difficult to comprehend until certain questions have been answered in his own mind, the major question usually being just what is a transmission line and how does it differ from other lines or circuits. The answer is that any two conductors between which voltage is applied can be considered to be a transmission line. In general, there is no clear-cut distinction between the usual concept of a circuit and a transmission line except for extreme cases with usual circuits on one end and transmission lines on the other. Fortunately, many applications fall into one of these extreme cases, but this is occurring less and less as advances are being made in high-speed circuits and systems.

In order to understand the significance of this, let us consider a very simple, but nevertheless quite profound example, namely, the turning on of a light in a home which is supplied with 60 Hz power. When the switch is activated, the light appears instantaneously, for all practical purposes. If we wished to analyze the electrical system within the house itself, we would use simple ac circuit theory and would never consider whether or not the lines might be transmission lines. However, the power company which supplies the electricity is not so fortunate because the generating station may be located a long distance from the house, as is shown in Fig. 1-1. When a switch is turned on at station $A$, as in Fig. 1-1, for example, we know that the power does not appear instantaneously at station $B$, located $x$ miles
away. If the cables are suspended in air, then the power will travel at the velocity of light in air, which is about 186,000 miles per second or $3 \times 10^8$ meters per second. If $A$ is located 186 miles from $B$, the power will reach $B$ at a time of $10^{-3}$ seconds after the switch is closed. Suppose that the generator at station $A$ is supplying 60 Hz voltage and we close the switching at the exact instant that this voltage has reached peak value, as in Fig. 1-1(b) at time $t = 0$. This voltage of $V_m$ travels toward station $B$ at the speed of light and arrives 0.001 second later. In the meantime, the generator at $A$ has continued alternating the voltage so that at the time that $V_m$ arrives at station $B$, the voltage at $A$ has decreased to $0.93 \ V_m$. Thus, there is now a potential (voltage) difference along the wire itself as a result of the finite time required for the voltage to reach station $B$. Because of this voltage difference along the line, it is necessary to consider this case as a transmission line. The reason for this is that besides numerous other problems which must be considered, it is necessary to maintain the proper phase relations between various generators that are connected into a power distribution system. For instance, if there were another generator at a third station located a considerable distance away and connected by a similar line in series.
or parallel with generator A, it would be necessary that the two generators operate in phase; otherwise power would be transferred between the generators, a very undesirable situation.

Returning to the light switch in our homes, we see that the distances involved are so small that we need not worry about phase differences, since the voltage at one end of a given line is essentially the same as that at any other point on the line, exclusive, of course, of any series resistive, inductive, or capacitive voltage drop. If the frequency of the ac power were to be increased, for example, to 100 megacycles per second (typical FM radio frequency), then a voltage difference would exist along the lines in our homes and it would be necessary to consider them as transmission lines. Thus, from this very simple illustration, we can see that whether or not a line is to be considered as a transmission line depends on both the length of the line and the frequency of the applied voltage or, more specifically, it depends on the ratio of the length of line to the wavelength of the applied frequency. If the wavelength is very long compared to the line length, simple circuit analyses are applicable. The wavelength of 60 Hz in air is (from chart on frontispiece) $5 \times 10^6$ meters or over 3,100 miles, but it reduces to 3 meters at 100 megacycles per second. Thus, it is apparent that distances within the home are quite small compared to the wavelength at usual power frequencies, but distances between power stations are not. Furthermore, the wavelength of the FM signals is not large but instead is comparable to distances within the home, so that antenna and lead-in wire, for example, must be considered as transmission lines. This is a fundamental idea which will be encountered at various times and will be extended in Chap. 5 to cases where the applied voltage is a single pulse, thereby containing all frequencies.

In this chapter, we will usually be concerned with lines that are either infinitely long or that appear to be so. Thus, a voltage difference will appear on the conductors themselves because of the finite velocity of propagation of energy, as was shown previously, and it is necessary to treat them as transmission lines. In order to consider such lines from the more familiar circuit point of view, all the fundamental concepts of waves on transmission lines will be developed from simple ac theory as applied to circuits. But in order to use ac circuit analysis, it is first necessary to obtain the equivalent circuit representation of any two-conductor line. This will be done in the next section. Once this is obtained, the transmission-line properties such as characteristic impedance, phase shift, phase velocity, and attenuation are easily obtained in terms of these circuit parameters which can be measured or calculated (see Chap. 9).

*For a review of ac theory, see [2] or any source which treats ac circuits.*
1.2 EQUIVALENT CIRCUIT OF A SIMPLE TRANSMISSION LINE

In order to analyze a transmission line in terms of ac circuit theory, it is necessary first to obtain the equivalent circuit of the line. Since any passive network can be composed only of combinations of resistive, capacitive, and inductive elements, the final circuit must contain only combinations of these. Let us consider two very long parallel wires which are suspended in air, as in Fig. 1-2, and let us examine a small subsection of the line included between the dotted lines in the figure. One restriction placed on our equivalent circuit is that the length $\ell$ of each subsection must be much smaller than the wavelength of the applied frequency, so that each subsection can be considered a circuit and the elements within the subsection can be precisely defined. When many subsections are tied together to form an infinitely long line, ac analysis can still be applied to each individual subsection, as will be seen. Another restriction is that the frequency must be sufficiently low and/or the conductor conductivity sufficiently high so that the series losses are small; otherwise, the resistive and inductive elements cannot be evaluated in any simple manner. This is discussed in greater detail in Sec. 4.6, with the

Fig. 1-2. Infinite parallel wire transmission line. (a) Subsection and notation used; (b) electric field and charge within subsection for an applied voltage; (c) magnetic field within subsection for an applied current.
restrictions given by Eqs. (4-90) and (4-91). These restrictions are usually of little concern in most applications and we will assume that all necessary line parameters can easily be evaluated.*

A voltage is applied across the wires such that a current $I$ flows in the top conductor and an equal and opposite current flows in the bottom conductor. This voltage can be supplied from a battery, i.e., dc source, an ac generator, or any other source. We want to determine the equivalent circuit for this small subsection of line. Since the wires have series resistance, the voltage $V_1$ at the input end of the small subsection will be larger than the voltage $V_2$ at the output end, as a result of the $IR$ drop through the wire. Thus, the small subsection must have a series resistance component in the equivalent circuit.

If the voltage across the line is not changing with time, then it is apparent that the voltage can be supported only by a static electric field since

$$V = \int E \cdot dl$$  \hspace{1cm} (1-1)

The presence of an electric field requires that there be free charges of opposite polarity on the two conductors, as in Fig. 1-2(b), since static electric fields can arise only from such free charges as are described by Coulomb's law

$$E = \frac{q}{4\pi r^2}$$  \hspace{1cm} (1-2)

with $E$ in volts per meter, $r$ in meters, $q$ in coulombs, and $\epsilon = 8.85 \times 10^{-12}$ farads per meter (in vacuum).

The free (stored) charge, accompanied by a voltage, represents a capacitor, since $C = q/V$. Thus, the equivalent circuit for the small subsection must contain a capacitive component.

In addition to the static electric field present between the conductors, there will also be a magnetic field or flux as a result of the current flow as given by either the Biot-Savart law or Ampere's law; these laws are, respectively

$$dB = \frac{\mu dl \times r}{4\pi r^3} \oint H \cdot dl = I$$  \hspace{1cm} (1-3)

*A further restriction occurs when the wavelength is comparable to or smaller than the cross-sectional dimensions at which time other modes of propagation can be excited or radiation can occur in open structures. These effects are briefly considered in Sec. 8.8. The restrictions imposed by series losses are usually more significant; thus these effects are ignored here.
with $B$ in webers per square meter, $dl$ and $r$ in meters, $I$ in amperes, $H$ in amperes per meter, and $\mu = 4\pi \times 10^{-7}$ henries per meter (in vacuum). The magnetic field associated with the current in the two parallel wires is illustrated in Fig. 1-2(c). If this magnetic flux linking the two wires is changing with time, then voltages $V_1$ and $V_2$ at the ends of the small subsection will differ not only by the resistive drop as described above, but also by the induced voltage as given by Faraday's law

$$V_1 - V_2 = \frac{d\phi}{dt}$$

where $\phi$ is the total flux within the subsection loop. Such an induced voltage, or voltage drop resulting from the time changing flux, is identified as inductance and is related by

$$e = L \frac{di}{dt} = V_1 - V_2$$

Thus, the equivalent circuit must contain an inductive component for the subsection of line.

One further component in the equivalent circuit remains to be identified, namely, that associated with any current flow across the insulator between the conductors. Such a current flow can result from ordinary conduction through the insulator or can result from losses associated with time changing electric and magnetic fields, e.g., dielectric or magnetic hysteresis losses. Generally speaking, the electronic conduction for common insulators used in transmission lines is very small and can be neglected, but as this is not always true, in order to be completely general, this effect should be included. Likewise the hysteresis losses associated with ordinary insulators are usually negligible, especially at low frequencies, but for very high frequencies, all insulators generally become more lossy; again, in order to be general, such terms must be included. Since both insulator conduction and other loss terms merely represent a current flow between the conductors which is in phase with the voltage, such mechanisms can be represented by a shunt resistor between the conductors.

Thus, we see that the equivalent circuit contains a series resistance, series inductance, shunt capacitance, and shunt resistance. In the interest of clarity and consistency, these parameters will always be taken as the per-unit-length values, that is, $R$ in ohms per unit length, $L$ in henries per unit length, etc. Thus, it is necessary to multiply these by $\ell$, the length of the subsection, to get the total $R$, $L$, $C$, and $G$ of each subsection. We must now determine
how to connect these terms together to form the equivalent circuit, since various configurations are possible. Two such possibilities are shown in Fig. 1-3. Since, by definition, an equivalent circuit must give an exact representation regardless of what form it takes, then it does not matter which form is used. The two circuits of Fig. 1-3(a) and (b) are equivalent and we shall arbitrarily use that of (a) for the remainder of this chapter.

Fig. 1-3. Two possible equivalent circuits for a transmission line with subsections of length f.

1.3 IMPEDANCE OF AN IDEAL TRANSMISSION LINE

If a transmission line has no losses, it can be considered to be a repeated array of small inductors and capacitors in a ladder network. This could be obtained from Fig. 1-3 by letting $R = G = 0$. If the line is uniform such that all the incremental inductors are equal and all the capacitors are likewise equal, it is of interest to determine what the ac impedance looking into the terminals $a-b$ in Fig. 1-4 is.

In order to determine this input impedance, which we call $Z_I$, a number of techniques are possible; the one customarily used is to derive and solve the transmission-line equations, from which this impedance is automatically obtained. However, this method obscures the most essential idea associated with a transmission-line, namely, that the input impedance of a ladder network of reactive (nonresistive) elements looks like a pure resistance. In order to demonstrate this fundamental concept, we shall invoke nothing more than simple ac theory.
The line is assumed to extend to infinity toward the right. Let the impedance looking into the terminals \( a-b \) be \( Z_1 \), that into \( c-d \) be \( Z_2 \), that into \( e-f \) be \( Z_3 \), etc. From simple ac analysis, the input impedance is the impedance of the first inductor \( L \ell \) in series with the parallel combination of \( Z_2 \) and the impedance of capacitor \( C \ell \); thus

\[
Z_1 = j\omega L\ell + \frac{Z_2 (1/j\omega C\ell)}{Z_2 + (1/j\omega C\ell)} \quad (1-6)
\]

Now if the line is really infinitely long, then the same impedance should be seen looking into terminals \( a-b \) or \( c-d \) or \( e-f \), etc.; thus \( Z_1 = Z_2 \). Substituting this into Eq. (1-6) and collecting terms

\[
Z_1^2 - Z_1 j\omega L\ell - \frac{j\omega L}{j\omega C} = 0
\]

or

\[
Z_1^2 - j\omega L\ell Z_1 = \frac{L}{C} \quad (1-7)
\]

This equation contains a complex component which is frequency-dependent. This term can easily be eliminated by allowing \( \ell \), the length of our subsection of line, to become very small and by recognizing that the ratio \( L/C \) remains constant regardless of line length. The complex term can therefore be neglected and the impedance becomes

\[
Z_1 = \sqrt{\frac{L}{C}} \quad (1-8)
\]
Thus, the ac input impedance of this line has no reactive component but rather looks like a pure resistance of value given by Eq. (1-8). If this infinite line is broken at any point and is terminated in a resistor of this value, it would not be possible to distinguish this difference by any impedance measurement. We shall derive this same equation in Chap. 2 by a more involved technique.

Incidentally, the same result would have been obtained if we had chosen a different equivalent circuit, such as the T-network in Fig. 1-3(b) for the line. Since the circuits all represent the same structure, it is obvious that the results must by definition be the same.

It is interesting to note that this line, which is composed only of reactive components $L$ and $C$, looks like a pure resistor to the external world for ac or dc excitation. In other words, during any transient or at steady state, the line presents its characteristic impedance to the buildup of current; we shall consider such cases more fully in Chap. 5 since they are of fundamental importance in the pulse behavior of such lines.

1.4 IMPEDANCE OF A LINE WITH SERIES LOSSES

When series resistance is present in the conductors of the line, the equivalent circuit is modified to that of Fig. 1-5. The input impedance is now different from that of the previous section but can be obtained in a similar manner. If the line is assumed to be infinitely long, the impedance looking in at any point along the line must be the same, or, as in Fig. 1-5(b), $Z_{in}$ must equal $Z_0$, the characteristic impedance of the line. Proceeding as in Sec. 1.3

$$Z_{in} = Rl + j\omega L + \frac{Z_0/\omega CL}{Z_0 + (1/\omega CL)} = Z_0$$

(1-9)

Fig. 1-5. Equivalent circuit of a line with series losses.
Multiplying through and collecting terms

\[ Z_0^2 - Z_0(R + j\omega L)\ell = \frac{R + j\omega L}{j\omega C} \quad (1-10) \]

Once again, allowing the subsection length \( \ell \) to become very small, the characteristic impedance \( Z_0 \) must remain constant but the total series impedance of each subsection \( \ell(R + j\omega L) \) will become very small in comparison. Thus, neglecting this term, Eq. (1-10) becomes

\[ Z_0 = \sqrt{\frac{R + j\omega L}{j\omega C}} \quad (1-11a) \]

This equation is general and is valid for small or large values of \( R \). If the series losses are small but not negligible, they will have a small effect on the characteristic impedance. This effect can be obtained by writing Eq. (1-11a) as

\[ Z_0 = \sqrt{\frac{L}{C}} \left(1 + \frac{R}{j\omega L}\right)^{1/2} \quad (1-11b) \]

The square root can be expanded by using the identify* \((1 \pm \nu)^{1/2} = 1 \pm \nu/2\) for small \( \nu \). If the losses are small, then \( R/\omega L \) will be small and Eq. (1-11b) can be approximated by

\[ Z_0 = \sqrt{\frac{L}{C}} \left(1 - \frac{R}{2\omega L}\right) \quad (1-11c) \]

Since \( R/2\omega L \) will be much smaller than unity, it is apparent that small series losses have little effect on the characteristic impedance of the line. In subsequent sections, it will be seen that small series losses, and losses in general, do have other more significant effects on the line behavior which must be taken into account.

*See [1, p.2], [3, p. 98], or Eq. (1-25).
1.5 PHASE SHIFT AND PROPAGATION CONSTANT OF AN IDEAL LINE

The phase constant, or as it is more commonly known, the propagation constant, of a lossless line can be determined from simple theory as was done in the previous section. This phase constant is merely a measure of the small shift in phase between $V_1$ and $V_2$ in Fig. 1-5 or, in other words, the amount of phase shift introduced by each small subsection of the line. Included in this is also the determination of the change in amplitude, if any, introduced by the subsection in question.

It is desirable to determine this factor for an ideal line, i.e., assuming that there are no losses in the line. Also, we arbitrarily specify that we wish to determine the angle by which $V_1$ leads $V_2$, or, in other words, we wish to determine $V_1 / V_2$.

Referring to Fig. 1-5, if $R = 0$, then $Z_0 = \sqrt{L/C}$, and it is apparent that

$$V_2 = V_1 \frac{Z_C Z_0}{Z_C + Z_0} \frac{1}{Z_L + [Z_C Z_0/(Z_C + Z_0)]} \quad (1-12)$$

or

$$\frac{V_1}{V_2} = \frac{Z_L(Z_C + Z_0) + Z_C Z_0}{Z_C Z_0}$$

This simplifies to

$$\frac{V_1}{V_2} = 1 + Z_L \left( \frac{1}{Z_0} + \frac{1}{Z_C} \right) \quad (1-13)$$

Substituting the values for $Z_L$, $Z_C$, and $Z_0$

$$\frac{V_1}{V_2} = 1 - \omega^2 LC \ell^2 + j\omega \ell \sqrt{LC} \quad (1-14a)$$

Amplitude $\equiv \left| \frac{V_1}{V_2} \right| = \left( (1 - \omega^2 LC \ell^2)^2 + \omega^2 LC \ell^2 \right)^{1/2} \quad (1-14b)$

Phase angle $\equiv \tan \beta = \frac{\omega \ell \sqrt{LC}}{1 - \omega^2 LC \ell^2} \quad (1-14c)$
In order to evaluate the phase angle for this expression, it is desirable to make one further assumption, namely, that the subsection length $\ell$ is small enough so that the applied frequency is well below the resonant frequency $1/\ell \sqrt{LC}$ of each subsection; in other words

$$\omega^2 \ell^2 LC \ll 1 \quad (1-15)$$

Thus, this term can be neglected in Eq. (1-14c) and the phase angle between $V_1$ and $V_2$ is therefore given by

$$\tan \beta_\ell = \omega \ell \sqrt{LC}$$

From the assumption of Eq. (1-15), it is obvious that $\tan \beta_\ell$ must be very small and can be replaced by $\beta_\ell$ (in radians); in other words, $\tan \beta_\ell = \beta_\ell$ for small angles. Thus, we arrive at the fundamental relationship that the phase shift of each small subsection with no losses is

$$\beta_\ell = \omega \ell \sqrt{LC} \quad (1-16)$$

or the phase shift per unit length is

$$\beta = \frac{\beta_\ell}{\ell} = \omega \sqrt{LC} \quad (1-17)$$

The angle $\beta_\ell$ represents the amount by which the input voltage $V_1$ leads the output voltage $V_2$, or, conversely, the amount by which $V_2$ lags behind $V_1$. The amplitude ratio, given by Eq. (1-14b), can be simplified by using the above condition of small $\beta_\ell$

$$\left| \frac{V_1}{V_2} \right| = 1 \quad (1-18)$$

Thus, there is no decrease in amplitude of the voltage along the line, but only a shift in phase, an important fundamental feature, as we shall see later.

Since $\beta_\ell$ merely represents the phase shift of the subsection, it is apparent that we could write

$$\frac{V_1}{V_2} = e^{j\beta_\ell} \quad (1-19)$$
From the definition of a natural logarithm, it is clear that taking the ln of both sides of Eq. (1-19) gives

\[ j\beta L = \ln \frac{V_1}{V_2} \quad (1-20) \]

A number of fundamental concepts concerning the nature of transmission lines can be deduced from these simple expressions for the amplitude and phase angle of Eqs. (1-16) and (1-18). One important idea concerns the frequency limit of this transmission line. In order to derive Eq. (1-16), it was assumed that the applied frequency was well below the resonant frequency of the subsection as given by Eq. (1-15). If the applied frequency is increased such that this is no longer true, then not only will the amount of phase shift change, but there will be a change in amplitude of the voltage along the line as given by Eq. (1-14b). Fortunately, it is a simple matter to further divide the subsection into small subsections, i.e., make \( \ell \) smaller, for which the applied frequency is once again well below the resonant frequency. In principle, it is possible to subdivide the line until the subsections are vanishingly small so that any frequency can be applied and the voltage at points along the line will have the same amplitude but will differ only in phase. Unfortunately, as the frequency is increased, losses and other difficulties appear which cannot be avoided,\(^*\) so that practical transmission lines have this ideal behavior only at the lower frequencies where losses are negligible. Nevertheless, this concept of constant amplitude and shift in phase along the line is often true to a large extent and is important in understanding traveling waves along a transmission line. We shall see in Chap. 2 that this progressive phase shift along the line really represents a wave traveling down the line with a velocity determined by the inverse of the phase shift per section. In particular, the velocity is

\[ a = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \quad (1-21) \]

and is independent of the applied frequency, as might be expected since all the line parameters were assumed to be independent of frequency. It should be noted that the larger the phase-shifting ability of the line, i.e., the larger \( L \) and \( C \), the smaller will be this velocity of propagation.

\(^*\)See Sec. 4.6.
1.6 PROPAGATION CONSTANT FOR A LINE WITH SMALL SERIES LOSSES

In the previous section, we saw that for a lossless transmission line \((R = G = 0)\), the voltages at various points along the line have the same amplitude but differ only in phase with respect to each other, progressively lagging the applied voltage as one proceeds down the line. When small losses are present due to series wire resistance, for instance, a similar phenomenon exists in that the voltage experiences phase shift along the line. However, in addition, the amplitude of the voltage no longer remains constant but decreases in value as one proceeds down the line. This is a result of the small voltage drop across the series resistors, as can easily be understood if one considers the equivalent circuit of Fig. 1-5.

We wish to derive both the phase shift and the decrease in amplitude introduced by each small subsection of the line. The phase angle can be determined in a manner analogous to that used in Sec. 1.5 by evaluation of \(V_1/V_2\). For the decrease in amplitude, or as it is more commonly labeled, attenuation, it is desirable to determine the voltage at a given point on the line as a fraction of the applied voltage. This would be given by the amplitude of \(V_2/V_1\), which can be determined from an ac analysis of Fig. 1-5 in a manner analogous to that of Sec. 1.5. The voltage ratio is obtained simply by adding \(Rl\) to \(Z_L\) in Eq. (1-13)

\[
\frac{V_1}{V_2} = 1 + (Rl + Z_L)\left(\frac{1}{Z_0} + \frac{1}{Z_C}\right)
\]  

(1-22)

If the subsection of line is again allowed to become very small, then \(1/Z_C\) will approach zero while \(Z_0\) remains constant. Thus, the last term in Eq. (1-22) can be neglected

\[
\frac{V_1}{V_2} = 1 + \frac{Rl + Z_L}{Z_0}
\]  

(1-23)

In Sec. 1.4, we derived the characteristic impedance of a line with losses to be that of Eq. (1-11). Even though we are presently considering a line with small losses, it is necessary to retain the loss term in the characteristic impedance of Eq. (1-11) in order to obtain the correct answer. This results from the fact that \(Z_0\) is multiplied by \(R + j\omega L\), and neglecting the loss term in \(Z_0\) would neglect important cross-product terms.
Substitution of Eq. (1-11) into Eq. (1-23) yields

$$\frac{V_1}{V_2} = 1 + \frac{RL + Z_L}{[Z_C (RL + Z_L)]^{1/2}} = 1 + [(RL + j\omega LC)\lambda]\frac{1}{2} \quad (1-24)$$

In order to evaluate this, it is necessary to express the term under the radical in a binomial expansion. This can best be done by using the series expansion

$$(1 \pm \nu)^{1/2} = 1 \pm \frac{\nu}{2} - \frac{\nu^2}{2 \cdot 4} + \frac{3\nu^3}{2 \cdot 4 \cdot 6} - \text{etc.} \quad (1-25)$$

$$= 1 \pm \frac{\nu}{2} \quad \text{for} \quad \nu \ll 1$$

The term under the radical can be rewritten

$$\ell[-(\omega^2 LC - j\omega RC)]^{1/2} = j\ell\omega \sqrt{LC} \left(1 - \frac{jR}{\omega LC}\right)^{1/2} \quad (1-26)$$

Since $R/\omega L$ was assumed to be much smaller than 1, then Eq. (1-24) becomes, using the first two terms of the series expansion

$$\frac{V_1}{V_2} = 1 + j\ell\omega \sqrt{LC} - \frac{j^2 R}{2\omega L} \sqrt{LC}$$

$$= 1 + \frac{\ell R}{2\sqrt{LC}} + j\ell\omega \sqrt{LC} \quad (1-27)$$

The phase shift or phase angle between $V_1$ and $V_2$ is obtained by dividing the imaginary by the real part

$$\tan \beta_t = \frac{\ell\omega \sqrt{LC}}{1 + (\ell R/2\sqrt{LC})} \quad (1-28)$$

If the subsection length $\ell$ is chosen to be small enough, then the denominator equals unity, since the second term can be neglected. Also, the phase shift for the subsection will be small

$$\tan \beta_{\ell} = \beta_t = \omega \ell \sqrt{LC} \quad (1-29)$$
or the phase shift per unit length is

\[ \beta = \frac{\beta_L}{\ell} = \omega \sqrt{LC} \]  

(1-30)

This is the same expression as was obtained for a lossless line; it may thus be
collapsed that small series losses have no effect on the phase constant to a
first-order approximation.

The attenuation is determined by the peak amplitude ratio, which is
given by Eq. (1-14b) as the square root of the sum of the squares of the
real and imaginary parts of Eq. (1-14a). However, since we have assumed
that the losses are small, the imaginary part of Eq. (1-27) is very small com-
pared to the real part (as evidenced by the small value of phase shift). Thus,
the magnitude of the voltage ratio is approximately

\[ \frac{V_1}{V_2} = 1 + \frac{1}{2} \frac{RL}{\sqrt{LC}} \]  

(1-31)

It was initially assumed that \( LR \) was very small compared with \( \sqrt{LC} \). In
order to get this into exponential form, we can make use of the identity

\[ e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \text{etc.} \]

or for \( u \ll 1 \)

\[ e^u = 1 + u \]  

(1-32)

Then it follows that

\[ \frac{V_1}{V_2} = e^{\alpha_L} \quad \text{or} \quad \frac{V_2}{V_1} = e^{-\alpha_L} \]  

(1-33)

where

\[ \alpha_L = \frac{RL}{2\sqrt{LC}} \]
The attenuation constant per unit length is then

\[ \alpha = \frac{a_t}{L} = \frac{R}{2\sqrt{L/C}} \]  

(1-34)

In addition to the attenuation, we could also have obtained the phase shift directly from the identity of Eq. (1-32). For instance, from Eq. (1-27)

\[ \frac{V_1}{V_2} = 1 + \left( \frac{R\ell}{2\sqrt{L/C}} + j\omega L/C \right) \]  

(1-35)

Using Eq. (1-32)

\[ \frac{V_1}{V_2} = \exp \left[ \frac{R\ell}{2\sqrt{L/C}} + j\omega L/C \right] \]  

(1-36a)

or

\[ \frac{V_2}{V_1} = \exp \left[ -\frac{R\ell}{2\sqrt{L/C}} - j\omega L/C \right] \]  

(1-36b)

The imaginary exponent represents the phase shift and the real exponent represents the attenuation for a line with small series losses.

It becomes apparent from the above analysis that the phase and attenuation constants could both have been obtained directly from the natural log of \( V_1/V_2 \). This results from the identity

\[ \ln(1 + \nu) = \nu - \frac{\nu^2}{2} + \frac{\nu^3}{3} - \text{etc.} \]  

(1-37a)

\[ = \nu \quad \text{for small } \nu \]  

(1-37b)

The voltage ratio of Eq. (1-35) is of the form \( 1 + \nu \), so that for small \( \nu \) (or in this case small \( \ell \))

\[ \ln \frac{V_1}{V_2} = \ln e^{a_t + j\beta \ell} = a_t + j\beta \ell \]  

(1-38)
In other words, the total propagation constant is

\[ \gamma_t = \alpha_t + j\beta_t = \ln \frac{V_1}{V_2} \]  

(1-39)

where \( \gamma_t \) can be expressed as Eq. (1-19)

\[ V_2 = V_1 e^{-\gamma_t} \]  

(1-40)

Of course, the same result could have been obtained by taking the natural log of both sides of Eq. (1-36). These two methods are, in fact, equivalent.

It should be understood that \( \alpha_t \) and \( \beta_t \) are the total attenuation constant and the phase constant, respectively, per subsection. If the length of the subsection is changed, these parameters will also change, since a longer section will obviously have more series resistance and therefore more attenuation than a shorter section. Similarly, a longer section will have more phase shift. However, \( \alpha \) and \( \beta \), the per-unit-length parameters, are constant, independent of \( \ell \).

Even though in all the above analyses, we have been concerned with cases where the losses are small, it is generally true that even for large losses, where higher-order terms cannot be neglected, the propagation constant is given by Eq. (1-39). Proof of this is left as an exercise to the reader.

1.7 TOTAL PROPAGATION CONSTANT FOR MANY IDENTICAL SUBSECTIONS

For cascaded subsections, numbered 1 through \( n \) as in Fig. 1-6, it is apparent that the ratio of input voltage to the voltage at the \( n \)th section is

\[ \frac{V_0}{V_n} = \frac{V_0}{V_1} \frac{V_1}{V_2} \cdots \frac{V_{n-1}}{V_n} \]  

(1-41)

The total propagation constant of \( n \) sections is the natural logarithm of the ratio of input to output voltage or the natural logarithm of Eq. (1-41). Taking the logarithm of both sides

\[ \ln \frac{V_0}{V_n} = \ln \frac{V_0}{V_1} + \ln \frac{V_1}{V_2} + \cdots + \ln \frac{V_{n-1}}{V_n} \]
Subsections

\[ V_n \]

Fig. 1-6. Identical cascaded subsections.

\[ y_T = y_1 + y_2 + y_3 + \cdots + y_n = n y_1 \quad (1-42) \]

Thus, the phase shift of \( n \) sections in series is just \( n \) times the phase shift of each individual subsection of line. Similarly, the attenuation constant is the attenuation of one subsection multiplied by the number of subsections.

1.8 GENERAL NETWORK EQUATIONS FOR A UNIFORM LINE

In all the preceding sections, we have considered cases for which the conductance \( G \) always equaled zero and the series resistance was either small (but finite) or zero. This is usually the case for practical lines, since otherwise large losses cause serious distortion and attenuation and are therefore avoided in the design. As can also be seen from the preceding analyses, the assumption of small or negligible losses greatly facilitates the evaluation of line characteristics in terms of simple, closed-form expressions. If these simplifying assumptions are not made, it is still possible to analyze the network as before, but the expressions become very involved. We shall now derive some general expressions for the general line and then attempt to derive the various parameters without making simplifying assumptions.

Consider the general case of the uniform, repeated array of Fig. 1-7(a) with series impedance elements of

\[ Z_s = (R + j \omega L) \ell \quad (1-43) \]

and parallel admittance of

\[ Y_p = (G + j \omega C) \ell = \frac{1}{Z_p} \quad (1-44) \]
Since the line is assumed to be infinitely long and uniform, the impedance seen looking into any subsection must be the same as that of all other subsections, just as in Sec. 1.4. Thus, Fig. 1-7(a) can be simplified to the form of (b) where $Z_T$ is the characteristic impedance of the line and $Z_{in}$ must equal $Z_T$.

It is easily seen that

$$Z_{in} = Z_s + \frac{Z_p Z_T}{Z_p + Z_T} = Z_T$$

Multiplying through by $Z_p + Z_T$ and collecting terms

$$Z_T^2 - Z_s Z_T - Z_s Z_p = 0$$

The solution to this equation can easily be obtained with the aid of the quadratic formula

$$Z_T = \frac{Z_s \pm \sqrt{Z_s^2 + 4Z_s Z_p}}{2}$$

Using the definition of parameters given by Eqs. (1-43) and (1-44), the above becomes

$$Z_T = \frac{1}{2} \ell (R + j\omega L) \pm \frac{1}{2} \sqrt{\ell^2 (R + j\omega L)^2 + 4 \frac{R + j\omega L}{G + j\omega C}}$$

In order to obtain a simplified, closed-form solution for $Z_T$, it is necessary to recognize that as the size of the subsection is reduced, all the parameters
\( \ell R, \ell L, \ell G, \) and \( \ell C \) decrease in the same proportion since \( R, L, G, \) and \( C \) are constant. Because of this, the ratio \( (R + j\omega L)/(G + j\omega C), \) that is, the second term under the radical in Eq. (1-47), will remain constant as the subsection length \( \ell \) is reduced. Thus, the subsection size can be reduced sufficiently so that both terms involving \( (R + j\omega L)\ell \) are negligible compared to the constant ratio, and then Eq. (1-47) becomes

\[
Z_T = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{Z_s Z_p}{Y_p}} = \sqrt{\frac{Z_s}{Y_p}} \quad (1-48)
\]

This equation represents a simple expression for the characteristic impedance of the general line where the losses are not small. It can be seen that this impedance will vary with the applied frequency, having real and imaginary components for the general case.

In the derivation of Eq. (1-48), it was necessary to choose a subsection small enough to neglect certain terms. In the derivation of the differential equations of a transmission line in Chap. 2, we will let the size of the subsection approach 0 and therefore expect the same result. This will be found to be the case. As a matter of fact, Eq. (1-48) will naturally fall out of the equations without any need to neglect certain terms, as might be expected.

The propagation constant can also be determined for this general case with the aid of the relationship given by Eq. (1-39). The ratio of input to output voltage for the circuit of Fig. 1-7(b) can be found from

\[
V_2 = V_1 \frac{Z_T Z_p}{Z_T + Z_p} \left[ \frac{1}{Z_T Z_p/(Z_T + Z_p)} \right] + [Z_T Z_p/(Z_T + Z_p)]
\]

Multiply through and collect terms

\[
\frac{V_1}{V_2} = 1 + \frac{Z_s}{Z_p} + \frac{Z_s}{Z_T} \quad (1-49)
\]

\( Z_T \) is the characteristic impedance given by Eq. (1-48). This equation is identical in form to Eq. (1-13), except that now we have included \( R \) and \( G \) in the impedance terms.
Making use of Eq. (1-39)

\[ \gamma_t = a_t + j\beta_t = \ln \left( 1 + Z_s \left( \frac{1}{Z_p} + \frac{1}{Z_p} \right) \right) \]  

(1-50)

Substituting Eq. (1-48) for \( Z_T \) and \( Y_p \) for \( 1/Z_p \)

\[ \gamma_t = \ln \left( 1 + Z_s \left( Y_p + \sqrt{\frac{Y_p}{Z_s}} \right) \right) \]  

(1-51)

Once again we will allow the subsection to become small enough so that \( Y_p \) becomes negligible compared to \( \sqrt{Y_p/Z_s} \), the latter remaining constant as the subsection is decreased. Thus, Eq. (1-51) reduces to

\[ \gamma_t = \ln \left( 1 + \sqrt{Z_s Y_p} \right) \]  

(1-52)

Making use of Eq. (1-37a) for the series expansion of the natural log

\[ \gamma_t = \sqrt{Z_s Y_p} - \frac{(\sqrt{Z_s Y_p})^2}{2} + \ldots \text{ etc.} \]

Since we have already specified that the subsection is allowed to become very small, then \( \sqrt{Z_s Y_p} \) will be much less than 1, so that the higher-order terms in the above equation can be neglected to yield

\[ \gamma_t = \sqrt{Z_s Y_p} = \ell \sqrt{(R + j\omega L)(G + j\omega C)} \]  

(1-53)

for each subsection. The propagation constant per unit length is thus

\[ \gamma = \frac{\gamma_t}{\ell} = \sqrt{(R + j\omega L)(G + j\omega C)} \]  

(1-54)
with all parameters taken per unit length. If \( R \) and \( G \) are allowed to approach zero, the attenuation terms must be zero with the result that Eq. (1-53) becomes

\[
y = j\beta = j\omega \sqrt{L/C}
\]

which is identical to Eq. (1-16), as was expected.

This propagation constant will be derived in Chap. 2 from the differential equation. It will be seen there that no simplifying assumptions are necessary, primarily because the subsection length must approach zero in order to arrive at the differential equations.

Thus far, we have only been concerned with the derivation of some fundamental parameters of transmission lines and have not used the results to examine their behavior in any detail. This has been done purposely in order to show the fundamental aspects and how they evolved quite naturally from elementary ac circuit theory. These concepts will be applied to numerous situations in succeeding chapters. It is, however, important for the student to have a basic understanding of the fundamentals before we proceed to other aspects of transmission lines. We shall derive these fundamental parameters once again in a more convenient form through the use of differential equations from which other basic characteristics of transmission lines can easily be obtained as well.

It is possible to analyze the entire behavior of transmission lines in terms of ac theory applied to small subsections of the line. However, this representation is that of a lumped-parameter transmission line, that is, discrete and separated \( R, L, G, \) and \( C \), whereas in reality such a line is a distributed-parameter line, that is, \( R, L, G, \) and \( C \) are not discrete and distinguishably separated from each other. Thus the differential-equation approach is somewhat more realistic. Furthermore, the differential equations simplify the mathematical analysis and allow one to derive many important phenomena in a more convenient form. However, both the ac-theory and differential-equation approach are really approximate representations of wave propagation on an electrical line. Fortunately, these approximations are quite adequate for most applications. The exact solution for one general case of a strip line is detailed in Sec. 4.5 and the limitations of the approximate representations are presented in Sec. 4.6.

PROBLEMS

1-1. Calculate the resistance of an infinite ladder of 1-ohm resistors connected as in Fig. 1-3 with \( L = C = 0. \)

\textit{Answer:} 1.62 ohms.
1-2. Determine the input impedance in Fig. 1-3(a) when the shunt resistance losses \( G \) are not negligible.

1-3. Show that the circuit of Fig. 1-3(b) gives the same value for input impedance as that of (a) when \( G \) is not negligible.

1-4. Show that \( e^{j\theta} \) is a vector of amplitude unity, inclined at an angle \( \theta \).

1-5. Show that \( V_0 e^{j\omega t} \) is equivalent to a sinusoidal vector of amplitude \( V_0 \) and argument \( \omega t \).

1-6. Given a line for which the series resistance \( R \) is zero and shunt conductance \( G \) is small but not negligible, determine the expression for \( \alpha \) and \( \beta \).

Answer: See Sec. 2.6.

1-7. Derive the input impedance of a simple series \( R, L, C \) circuit. Show that the resonant frequency is \( 1/\sqrt{LC} \) and that at this frequency, the input impedance equals \( R \).

1-8. Derive the input impedance of a simple parallel \( R, L, C \) circuit. Show that the resonant frequency is \( 1/\sqrt{LC} \) and that at resonance, the input impedance equals \( R \).

1-9. Show that at resonance, a series \( LC \) circuit has zero impedance, while a parallel \( LC \) circuit has infinite impedance.

REFERENCES