CHAPTER 1

IN THE BEGINNING . . .

1.1 Numeration

Our system of numeration, if not a machine, is machinery; without it (or something equivalent) every numerical problem involving more than a very limited number of units would be beyond the human mind.[1]

One of the first great intellectual feats of a young child is learning how to talk; closely following on this is learning how to count. From earliest childhood we have been so bound up with our system of numeration that it is a feat of imagination to consider the problems faced by early humans who had not yet developed this facility. Careful consideration of our system of numeration leads to the conviction that, rather than being a facility that comes naturally to a person, it is one of the great and remarkable achievements of the human race.

It is now impossible to learn the sequence of events that led to our developing a sense of number. Even the most backward tribe of humans ever found has had a system of numeration that, if not advanced, was sufficient for the tasks they had to perform. Our most primitive ancestors must have had little use for numbers; instead their considerations would have been more of the kind *Is this enough?* rather than *How many?* when they were engaged in food gathering, for example. When early humans first began to reflect about the nature of things around them, they discovered that they needed an idea of number simply to keep their thoughts in order. As they began to live a settled life, grow plants, and herd animals, the need for a sophisticated number system became paramount. How and when this ability at numeration developed we will never know, yet it is certain that numeration was well developed by the time humans had formed even semi-permanent settlements.

It is very popular, in works dealing with the early history of arithmetic, to quote *facts* about the so-called *primitive* peoples and their levels of numeration. These *facts* generally note that the primitive peoples of Tasmania are only able to count one, two, many; or the natives of South Africa count one, two, two and one, two two's, two two's and one, and so on. Although often correct in themselves, these statements do not explain that in realistic situations the number words are often accompanied by gestures to help resolve any ambiguity. For example, when using the *one, two, many* type of system, the word *many* would mean, *Look at my hands and see how many fingers I am showing you.* This type of system is limited in the range of numbers that it can express, but this range will generally suffice when dealing with the simpler aspects of human existence.

The lack of ability of some cultures to deal with large numbers is not really surprising. Our own European languages, when traced back to their earlier versions, are very poor in number words and expressions. The translation of the Gospels made by Bishop Ulfilas in the fourth century for the Goths uses the ancient Gothic word for ten, *tachund,* to express the number 100 as *tachund tachund,* that is, ten times ten. By the seventh century the word *teon* had become interchangeable with the *tachund* or *hund* of the Anglo-Saxon language, and the Gospels of that period denote 100 as *hund teontig,* or ten times ten. The average person alive in seventh-century Europe was not as familiar with numbers as we are today. The seventh-century Statute of Shrewsbury laid down the condition that, to qualify as a witness in a court of law, a man had to be able to count to nine. To apply such a condition today would seem ludicrous.

Perhaps the most fundamental step in developing a sense of number is not the ability to count, but rather the ability to see that a number is really an abstract idea instead of a simple attachment to a group of particular objects. It must have been within the grasp of primitive humans to conceive that four birds are distinct from two birds; however, it is not an elementary step to associate the number 4, as connected with four birds, to the number 4, as connected with four rocks. Associating a number as one of the qualities of a specific object is a great hindrance to the development of a true number sense. When the number 4 can be registered in the mind as a specific word, independent of the object being referenced, the individual is ready to take a first step toward the development of a notational system for numbers and, from there, to arithmetic. As was noted by Bertrand Russell:

> *It must have required many ages to discover that a brace of pheasants and a couple of days were both instances of the number two.*[2]

Traces of the very first stages in the development of numeration can be seen in several living languages today. Dantzig, in his book *Number*[3], describes the numeration system of the Thimshian language of a group of

British Columbia Indians. This language contains seven distinct sets of words for numbers: one for use when counting flat objects and animals, one for round objects and time, one for people, one for long objects and trees, one for canoes, one for measures, and one for counting when no particular object is being numerated. Dantzig conjectures that the last set of words is a later development while the first six groups show the relics of an older system. This diversity of number names is not confined to obscure tribal groups—it can be found, for example, in widely used languages such as Japanese.

Intermixed with the development of a number sense is the development of an ability to count. Counting is not directly related to the formation of a number concept because it is possible to count by matching the items being counted against a group of pebbles, grains of corn, or the counter's fingers. These counting aids must have been indispensable to very primitive people who would have found the process impossible without some form of mechanical aid. Such aids to counting, although in different form, are still used by even the best educated professionals today, simply because they are convenient. All counting ultimately involves reference to something other than the things being counted. At first it may have been grains of corn but now it is a memorized sequence of words that happen to be the names of the integers. This matching process could have been responsible for the eventual development of the various number bases that came into existence because the act of counting usually takes the form of counting into small groups, then groups of groups, and so on.

The maximum number of items that is easily recognizable by the human mind at one glance is small, say five or less. This may be why initial groups consisted of about five items, and would account for the large number of peoples whose number systems were of base five. The process of counting by matching with fingers undoubtedly led to the development of the different number systems based on ten.

By the time that a number system has developed to the point where a base such as 5, 10, 20, or even 60 has become obvious to an outside observer, the eventual development of its use in higher forms of arithmetic and mathematics has probably become fixed. If the base is too small, then many figures or words are required to represent a number and the unwieldy business of recording and manipulating large strings of symbols becomes a deterrent to attempting any big arithmetical problems. On the other hand, if the number base is too big, then many separate symbols are required to represent each number, and many rules must be learned to perform even the elementary arithmetical operations. The choice of either too small or too large a number base becomes a bar to the later development of arithmetic abilities.

A study of various forms of numeral systems yields the fact that a great variety of bases have been used by different peoples in various places over

the globe. By far the most common is, of course, the number systems based on ten. The most obvious conclusion is that a very large percentage of the human race started counting by reference to their ten fingers. Although 5, 10, or 20 was the most popular choice for a number base, it is not uncommon to find systems based on 4, 13, or even 18. It is easy to advance a possible explanation of systems based on 5 and 20 because of the anatomical fact of there being 5 fingers on each hand and a total of 20 such attachments on both hands and feet. It is not quite so easy to see how the other scales may have developed in such a natural way.

If modern linguistic evidence can be trusted, the scale of 20 must have been very widespread in ancient times. It has been pointed out by several authors that some Eskimo people have a well-developed system which uses terms such as *one-man* for the number 20, *two-men* for 40, and so forth. This obviously implies that a base 20 number system was in use at one time, but it is difficult to see how an early Eskimo could have had access to his toes for counting, unless he was inside a relatively warm igloo. This could imply that the Eskimos brought their number system with them when they moved into North America. In fact, it is possible to trace remnants of various number systems based on 20 all the way back to the very early people living in the tropical regions of southern Asia.

The European languages also show a marked preference for the grouping together of 20 objects. It is entirely possible that the general prevalence of base 10 numbers among the Indo-European peoples developed later from an original base 20 system. The most modern example of this is the French use of *quatre-vingt* (four-twenty) instead of *huitante* for 80, and *quatre-vingt-dix* (four-twenty-ten) instead of *nonante* for 90. If we are willing to look at forms of the language which died out in the thirteenth century, then the terms *six-vingt* (120) *sept-vingt* (140), *huit-vingt* (160), and even *quinze-vingt* (15-20 for 300) show that the base 20 system was no stranger. The English word score, as in the Biblical *three score and ten years* or Lincoln's famous *Four score and seven years ago . . .* is another modern example of Europeans dealing in groups of 20. The term score will be discussed further in the section dealing with tally sticks.

Once the sense of number, as an entity separate from the thing being counted, has developed, there arises a need to assign names to the integers. This sequence of names, once decided upon, often becomes one of the most stable parts of a people's language. Thus, although languages may evolve so that almost all the words acquire vastly different forms, the sequence of integer names remains almost intact. The original meanings of the names of the integers are likely, therefore, to be quite obscure. The similarity of number names across related languages is easily seen by examining a table of these names in the family of Indo-European languages. Although the actual names differ from one language to another, the basic form remains constant (languages are Ancient Greek, Latin, Sanskrit, French, German, and English).

1	en unus eka un eins one
2	duo duo dva deux zwie two
3	tri tres tri trois drei three
4	tetra quantour catur quatre vier four
5	pente quinque panca cinq fünf five
6	ex sex sas six sechs six
7	epta septem septa sept sieben seven
8	octa octo astra huit acht eight
9	ennea novem nava neuf neun nine
10	deca decem daca dix zehn ten

This stability of number names has been a good tool of the philologists. They have made extensive collections of the number names in many languages in an attempt to see which languages have common roots. Care must be taken in any exercise of this sort because a common set of names could have resulted from simple mixtures of languages, trade between two peoples, or simple coincidence.

George Peacock, in his *Arithmetic*[4], points out the situations that can arise by simple coincidence. He compares the number words of the Nanticocks (a tribe of Indians who used to reside on the south bank of the Chesapeake) with those of the Mandingoes (of Africa). The similarity is striking but can only be attributed to coincidence because any mixing of the languages due to geographic proximity or cultural contacts via trade is ruled out.

Number	*Nanticocks*	*Mandingoes*
1	killi	killim
2	filli	folla
3	sabo	sabba
4	nano	nani
5	turo	loolo
6	woro	soro
7	wollango	oronglo
8	secki	sec
9	collango	conanto
10	ta	tang

The opposite side of the coin can be shown by comparing the number words from languages which are known to be related. In Europe, the languages spoken by the Finnish and Hungarian people can be traced back to a common root, yet the number words in these languages have very little in common.

Number	*Finnish*	*Hungarian*
1	yksi	egy
2	kaksi	kettö
3	kolme	ha'rom
4	nelja	négy
5	viisi	öt
6	kuusi	hat
7	seitseman	hét
8	kahdeksan	nyolc
9	yhdeksan	kilenc
10	kymmenen	tiz

The ability to count must soon have led to the need to record the results of counting. This probably first took the form of setting aside the pebbles used in the matching process, but would soon lead to the use of tally sticks, knotted cords, and the actual invention of numerals.

1.2 Written Number Systems

There appear to be two distinct stages in the development of written record keeping systems, the pictorial stage and the symbolic stage. The inscriptions done by people in the pictorial stage will correspond with the earliest era in the development of a number sense. The inscriptions will have a picture of ten cattle to represent ten cows or ten tents may indicate ten family groups. This stage is most easily seen in some records made by North American Indians just about the time they came into contact with Europeans.

The symbolic stage, on the other hand, will show a picture of a cow followed by five strokes to represent five cattle. At this stage the strokes no longer represent objects but have become a determinative (a sign used to qualify the meaning of the object sign). In other words, the strokes have become adjectives rather than nouns. In many societies this step must have come at the same time as the development of numbers as abstract ideas. This stage is best illustrated by some early stone carvings left by the Egyptians.

Once a particular society has reached the symbolic stage of recording numbers, the notation system which it develops will be almost entirely dependent on cultural influences. The use of fingers or pebbles for counting, the number of items counted into a group before a new group is started (the number base), and the materials available for making a lasting record of the result are all culturally determined. Once these variables have been set, the result will lead to one of two different types of notational systems: the *additive* system or the *positional* system.

1.2.1 The Additive Number System

In this system there is a distinct symbol for each kind of group made during the counting process, and this symbol is repeated as often as necessary to indicate how many of each group are needed. The Egyptian number system is the classical example of an additive system; however, most people are more familiar with the additive system used by the Romans. In the Old Roman system (before the subtractive forms of IV for 4 and IX for 9 were in use) it was possible to express any number less than 5,000 by a sequence of symbols in which no individual sign need be repeated more than four times. For example, the number 2,976 would be represented as MMDCCCCLXXVI. Although it was the custom to write down the symbols in decreasing order of value (M = 1,000, D = 500, C = 100, L = 50, X = 10, V = 5, I = 1), this is by no means necessary. The value of a number in an additive system has nothing to do with the position of any symbol within the string representing the number. It is quite unambiguous to write the number 2,976 as ILVCMCXDMXCC but, of course, ease of reading always precluded its being written otherwise than in order of descending value.

The pure additive system is quite easy to use for simple calculations, although it does not appear so at first glance. Addition involves the two-step process of simply writing down the group symbols from each number, then collecting the sequences of smaller valued symbols to make larger valued ones so that the number regains its canonical form. For example:

$$\begin{array}{rcl} 2319 & = & \text{MM CCC X V IIII} \\ +821 & = & \text{D CCC XX I} \\ \hline 3140 & = & \text{MMDCCCCCCXXXVIIIII} \end{array}$$

The second step now takes over and, because IIIII = V, VV = X, CCCCC = D, DD = M, the final result is written as MMMCXXXX.

Multiplication, although slow, is not really difficult and only involves remembering multiples of 5 and 10. For example:

$$\begin{array}{rcl} 28 & = & \text{XXVIII} \\ \times 12 & = & \text{XII} \\ \hline 336 & & \end{array}$$

XXVIII times I = XXVIII
XXVIII times I = XXVIII
XXVIII times X = CCLXXX
CCLXXXXXXXVVIIIIII

which would be written as CCCXXXVI.

The operations of division and subtraction are a little more cumbersome; however, division was aided by operations (as was multiplication) which are no longer in use today. These operations, called *duplation* and *mediation*, are described in more detail in Section 1.3 on Egyptian arithmetic.

Although more cumbersome than systems of positional notation, the additive systems are not without their merits, and computation is not difficult once the rules are mastered. The modification of such a number system to include subtractive elements, such as the IV = 4 or IX = 9 of the Modern Roman system, tend to make matters very much more difficult as far as arithmetic is concerned. However, this development is a late modification of the Old Roman number system and is not to be found at all in most examples of additive notation. Examples of the use of IV for 4 or IX for 9 can be found as early as 130 A.D., but these symbols did not become popular until about 1600 A.D. when Roman numerals were no longer so important in everyday use.

1.2.2 The Positional System

In positional number systems, the counting groups are denoted entirely by the position of the symbol in the string of characters denoting the number. This means that it is usually necessary to have a zero symbol to show an empty group. It is possible to do away with the need for a zero symbol if the system has some regular way of positioning the other symbols to indicate that one position is empty. This latter method was used by the Chinese and will be discussed in detail in the section dealing with their numerals. The positional system used by some early cultures, for example the Babylonians prior to 400 B.C., did not have a zero symbol and thus their written numbers were ambiguous.

The rules of calculation in a positional system are more complex than those used with additive systems, and they usually require that the user memorize some form of multiplication table. Because everyone is familiar with the working of our own positional number system, no attempt will be made to describe it in detail. It will be more instructive to examine a small selection of some other number systems used by various cultures during the course of history.

The following description of various number systems and their written representations is not meant to be exhaustive. Indeed some of the world's most important and widespread numerical systems will not be mentioned at all. The intent is simply to heighten awareness of the different number systems and the numerals used among the world's peoples. The exception to this is the discussion of the European number system, which is included to provide some historical background on this very important development.

1.3 The Egyptians

Our knowledge of the Egyptian forms of numerical notation and arithmetical procedures stem from the translation of the many documents which have survived in the very dry air of the Nile valley. The earliest written records to have survived, which date from just before 3000 B.C., reveal a system of number representation that remained essentially the same from the third millennium B.C. right up to the last days of the ancient Egyptian culture. With such a vast time span to consider it is only natural that both the spoken and written languages, together with the arithmetic abilities and needs of the people using them, changed quite considerably. Yet because of the very conservative nature of the educated Egyptian, who generally belonged to the priestly class, the changes are not as vast as they might have been in other, less conservative, societies. The lack of money or coinage in the early centuries led to the development of very detailed accounting systems for the amounts of grain and beer stored in various places and paid out to the workers as salary. These accounts are one of the main sources of information concerning the forms of notation and the arithmetical operations in use during the first part of the Egyptian era. There also exist two major mathematical works, the Rhind papyrus and the Moscow papyrus, which give a very good indication of the level of mathematical knowledge of the scribes. They do not show all the processes the scribes might have used when doing the calculations, but they provide some answers to the basic questions of how the operations were sometimes carried out.

The earliest form of Egyptian writing is the hieroglyphic script so familiar to anyone who has ever read about Egypt. This type of picture writing is quite suitable for inscribing public monuments or temple walls (and was used in this capacity right up into the Christian era) but it is not very useful when attempting to take quick notes with paper and ink. When hieroglyphics are written quickly, they tend to become just sketches of the original figures with the lines drawn in a cursive and connected style. With time, these shorthand forms became standardized into what is now known as Hieratic script. The Hieratic was further shortened and modified until, about the fourth century B.C., it had developed into what is known as the Demotic script. Most of the arithmetical and mathematical works that have survived are written in Hieratic script on papyrus, a form of paper made from reeds. The Egyptian number system was strictly based on 10, but was additive rather than positional in nature. In later times some small changes were made to represent very large numbers by a multiplicative method, but these were a very late development. The seven basic signs shown in Figure 1-1 were all that were necessary to represent a number.

The symbol for one is a simple vertical stroke, likely representing a stick or reed used in the primitive counting process. The symbol for 10 is of unknown origin, but the figure also occurs in the written word for a horse's

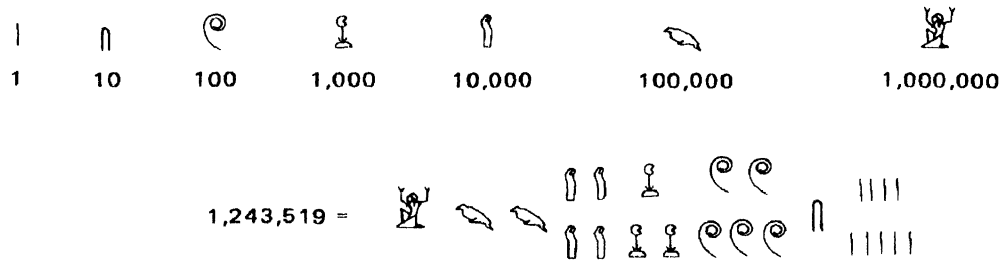


Figure 1-1
The classic Egyptian hieroglyphic additive number system.

heel and the two words are very much alike in their pronunciation (yes, we do know how to speak ancient Egyptian) so it is possible that the symbol was adopted simply because of a similarity of sound. Since modifications of the sign for 10 are also found in words representing sacks or baskets, the symbol might also be connected with a container holding ten objects. The symbol for 100 represents a coiled rope, perhaps of 100 units length; it is by association with this symbol, and the Egyptian practice of land measurement with ropes, that the Greeks referred to the Egyptian mathematicians as *rope stretchers*. The symbol for 1,000 represents a lotus plant, of which many grow along the banks of the Nile. A finger pointing at the sky (to the stars) is thought to be the derivation of the 10,000 figure. The Nile was full of tadpoles, so nothing could be more natural than to use such a symbol to represent a large number like 100,000. The final sign, that of the god Heh who held up the sky, for the number 1,000,000 was used in the early days but seems to have gone out of style during the later era of Egyptian civilization.

The construction of the numbers was very regular. The highest valued symbol always preceded the lower valued ones but, because Egyptian was written either right to left or left to right (and sometimes even top to bottom), it is not possible to predict which way the numbers would be written without consulting the text in which they are contained. It is unfortunate that we have very few examples of texts in which the names of the numbers are actually spelled out (that is, eight thousand twenty-seven instead of 8,027), so we have little real idea of how the Egyptians thought of their numbers.

The hieroglyphic text in Figure 1-2 shows a typical set of accounts relating to the Royal Court from about 1800 B.C. (transcribed from the Hieratic into Hieroglyphics). The first line reads *the harem, 45* (units of beer), the second line *the nursemaids, 61*, the third *members of the household, 38*, the fourth *the total 143*. If you are quick at mental addition you will see that this example contains the earliest arithmetic error known to the author. That scribe must turn in his grave when he sees us still remembering his error 3,700 years after he made it.



Figure 1-2
A redrawing of part of an Egyptian royal household account, circa 1,800 B.C., containing one of the earliest known errors in arithmetic.

In the later part of the Egyptian era the multiplicative device, mentioned above, was occasionally employed. As Figure 1-3 illustrates, this consisted of writing the number of times a symbol should appear beneath the symbol itself. This device has obvious advantages in cutting down the number of complex symbols used to represent a number, a large saving in effort if your writing medium is a sandstone block and a copper chisel.

As with Roman numerals, the operations of addition and subtraction are quite straightforward and easily implemented. The operation of multiplication was another matter. A scribe wishing to find the answer to the question, How much is 23 taken 13 times? would proceed as follows: He would write down the first number (23) in the first column and the number 1 in the second as shown in Figure 1-5. Beneath these, he would write down two times each number (an easy operation when you only have to write down each symbol twice as often as it occurred in the number above), and keep up this doubling process until he had gone far enough for his needs—generally until the number of the second column was about to become larger than the multiplier in use, 13 in this case. He would then inspect the second column and put a mark beside those numbers which would add up to the value of the multiplier (the first, third, and fourth in this case because 13 = 1+ 4 + 8). The scribe would then add up the corresponding numbers in the first column to obtain the desired result.

Several points are worth noting about this operation. First, the tick marks made by the scribe along the numbers in the second column are really just a mechanism of determining the binary representation of the value of the multiplier (13 in binary is 1101). The multiples of 23 which corresponded to binary "l" positions in the multiplier would then be added to

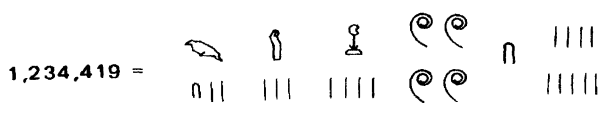


Figure 1-3
The later multiplicative method of representing numbers in Egypt.

Number	Hieroglyphic	Hieratic	Demotic
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
20			
30			
40			
50			
100			
200			
300			
400			
500			
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2000			
3000			
4000			
5000			

Figure 1-4
The hieroglyphic, hieratic, and demotic numbers.

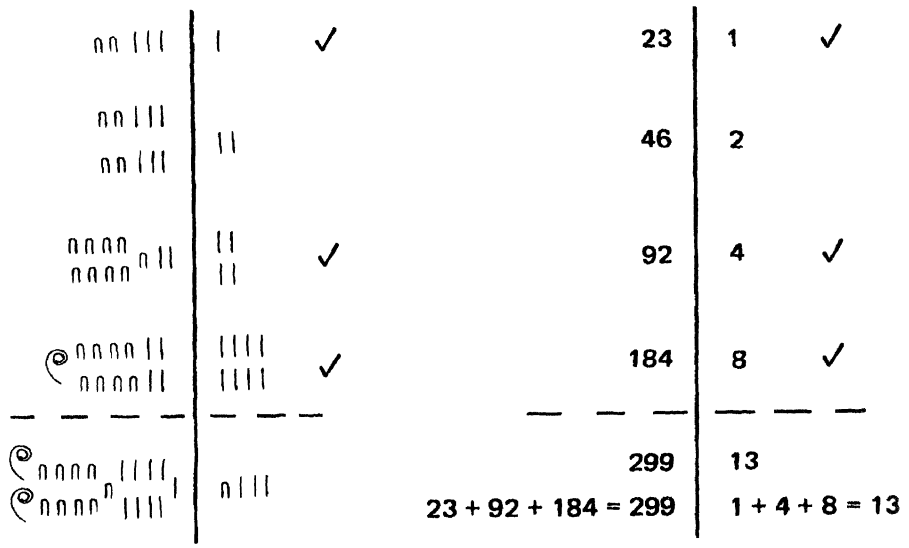


Figure 1-5
A sample of Egyptian multiplication.

form the final product. This is the same technique used in the shift and add binary multiplication scheme implemented on modern electronic computers. Although the Egyptians did not seem to follow through with any other results using binary notation for numbers, their methods of operation relied on it heavily. In addition, an intelligent scribe would occasionally throw in a multiple of 10 if he could see that it might be useful, as in Figure 1-6.

Doubling was the basic operation involved in this method of doing multiplication and, because the Greeks adopted the technique for use in their schools, this mechanism spread into Europe. Until the introduction of the Hindu-Arabic numerals, Europeans used the Egyptian doubling methods

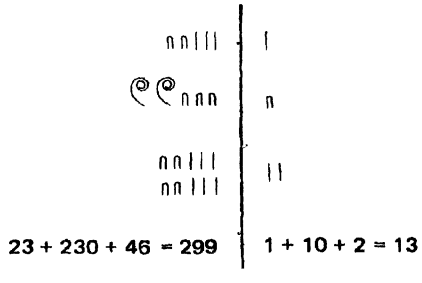


Figure 1-6
Modified Egyptian multiplication.

for multiplication and a similar halving technique for division. These methods were so much a part of the system that most of the early European textbooks on arithmetic instructed the readers that there were six fundamental arithmetic operations they had to learn: addition, subtraction, multiplication, division, duplation (doubling), and mediation (halving).

Division was the simple inverse operation to multiplication and its implementation was along similar lines. It did lead to the complication of fractions being needed to represent some of the results, and this led to the development of a strange and complex system of fractional notation. A complete discussion of Egyptian fractions is beyond the scope of this work, and the interested reader is referred to some of the more modern reference works on the subject, the work by Gillings[5] being an ideal place to start.

1.4 The Greeks

The Greek civilization has always been of interest to Europeans and its history has been quite well documented. The Greeks were some of the greatest mathematicians of the ancient world, at least in the fields of geometry and number theory. When it came to the consideration of arithmetic processes, they had a firm dividing line between the common methods of doing arithmetic, which they termed *logistic*, and the more advanced topics of number theory, which was called *arithmetic*. The theoretical side of the Greek studies, their *arithmetic*, was used both as a mental exercise and for telling of fortunes and other cabalistic processes. Because this area of study interested the early European mathematicians and historians, much is known about their theorems, while the simple knowledge of how they actually did their arithmetic operations, the *logistic*, has been largely ignored. Many Greek inscriptions contain numbers which are spelled out, and obviously the reason pertains to official documents where it is important that there be no confusion as to the exact meaning intended. But this system is impractical for everyday use. The Greeks eventually developed two separate forms of numerical notation which were both in use simultaneously, one form or another being dominant depending on the relative political influence of Athens and the other major city states.

The *Attic* system, sometimes called *Herodianic* because of its being found on a document attributed to Herodian, was in use from at least as early as 454 B.C. to about 95 B.C. It appears to have been adopted in Athens and, because of that city's influence, spread to other parts of the Greek world. The numerals appear to have been derived from the first letter of the names of the numbers, except the symbol for 1, which was represented in the Egyptian fashion by a simple line. The symbol for 5, Γ, was derived from the first letter of the Greek word *pente*, the symbol for 10, Δ, from *deca*, and so on. It is interesting to note that the system is strictly decimal with the excep-

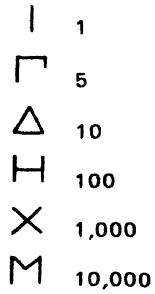


Figure 1-7
The Attic numerals.

tion of having a separate symbol for 5. This may well be a hold over from an early Greek number system based on 5 rather than on 10, and this view is supported by the fact that Homer[6] uses the term *to five* meaning *to count*.

The Attic symbols were combined in an additive system to form any number up to 50,000. The number of repetitions of each symbol was limited to four by combining the Γ with the others to produce a compound symbol for the numbers 50, 500, and 5,000. The higher value symbols always came first.

The second set of numerals, usually called the alphabetic numerals, likely developed in the Greek part of Asia Minor some time between 800 B.C. and 500 B.C. The earliest surviving inscription which uses this system dates from about 450 B.C., but evidence about the order in which the Greeks considered their letters puts the invention several centuries earlier. It is usually thought that the Greeks obtained their alphabet from the Phoenicians and that they changed some of the consonants to represent vowels, which were not written in Phoenician. They then assigned values to the various letters as shown in Figure 1-9.

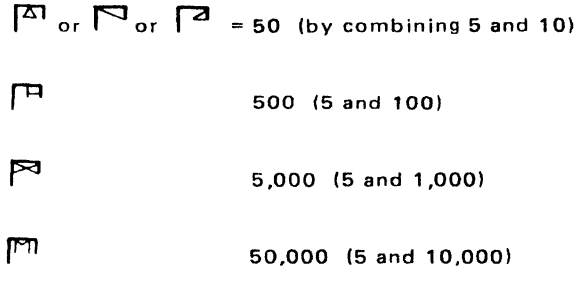


Figure 1-8
The combined numerals.

The Greek Alphabetic Numerals

Α = 1	Ι = 10	Ρ = 100
Β = 2	Κ = 20	Σ = 200
Γ = 3	Λ = 30	Τ = 300
Δ = 4	Μ = 40	Υ = 400
Ε = 5	Ν = 50	Φ = 500
ζ = 6	Ξ = 60	Χ = 600
Ζ = 7	Ο = 70	ψ = 700
Η = 8	Π = 80	Ω = 800
Θ = 9	φ = 90	ϡ = 900

Figure 1-9
The alphabetic numerals.

The system uses three letters that were not in the classical Greek alphabet of 24 letters, but appear to be older forms of letters or characters that have been borrowed from other alphabets for the purpose of the numeral system. This use of the alphabet to represent numerical quantities was not unique to the Greeks. In fact they may well have obtained the idea through trade contacts with other groups who used the same scheme. A similar technique is known to have been used by the Hebrews, early Arabs, Hindus, and many others.

The symbols were used in a strictly additive system, the resulting notation being very concise but not exactly easy to use. These alphabetic numerals eventually replaced the Attic system, partially because the concise nature of the notation allowed them to be used on coins and other places where the space is limited. Their first official use was on a coin, from 266 B.C., found at Alexandria and bearing the head of Ptolemy II. They eventually started showing up in official documents from mainland Greece about 200 B.C. and appear to have displaced the Attic notation entirely some time after 100 B.C. They lasted for a remarkably long period of time, being in use in different parts of the world long after the Greeks had given them up. In the early part of the fourteenth century, a monk named Barlaam wrote several books on geometry and arithmetic in Greek using these alphabetic numerals. Some of these works were published as late as 1564 in Strassburg.

In general, it was usual to write down a number with the higher valued symbol first, but examples are known, particularly from Asia Minor, where the reverse was true. A few examples are known where the scribe wrote the letters down in apparently random order. Any number from 1 to 999 can easily be represented by no more than three characters. This range would have sufficed for most of the problems encountered in everyday life. When larger numbers were needed, the system was simply repeated with an identifying mark added before or after the letter to indicate that it was now to be considered as 1,000 times larger than it had been before. For 10,000 and up, the alphabetic numerals were augmented by the character used in the

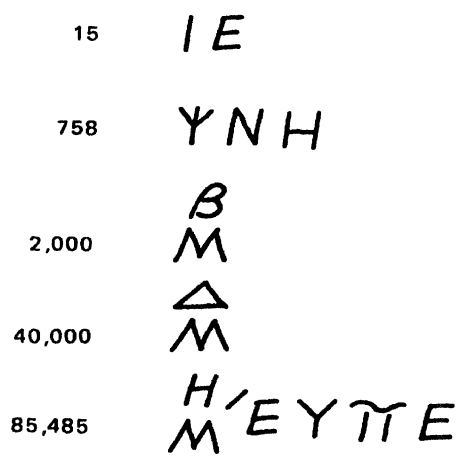


Figure 1-10
Examples of Greek alphabetic numerals.

Attic system, M (for *myriad*), which was placed either before, after, or underneath the letter to signify that it stood for 10,000 times as much as it normally would.

The two different numeral systems each had some slight advantages and disadvantages, but both were awkward to use when doing arithmetic, particularly the alphabetic numerals. Cantor, when describing the alphabetic numerals, sums up the situation by saying[7]:

> *Instead of an advance we have here to do with a decidedly retrograde step, especially so far as its suitability for the further development of the numeral system is concerned. If we compare the older "Herodianic" numerals with the later signs which we have called alphabetic numerals, we observe in the latter two drawbacks which do not attach to the former. There now had to be more signs, with values to be learnt by heart; and to reckon with them required a much greater effort of memory. The addition*
>
> $$\Delta\Delta\Delta + \Delta\Delta\Delta\Delta = \overline{\Gamma}\Delta\Delta \ (30 + 40 = 70$$
>
> *could be coordinated in one act of memory with that of*
>
> $$\mathrm{HHH} + \mathrm{HHHH} = \overline{\Gamma}\mathrm{HH} \ (300 + 400 = 700)$$
>
> *in so far as the sum of 3 and 4 units of the same kind added up to 5 and 2 units of the same kind. On the other hand* $\Lambda + \mathrm{M} = \mathrm{O}$ *did not at all immediately indicate that* $\mathrm{T} + \gamma = \psi$. *The new notation had only one advantage over the other, namely that it took less space. Consider, for instance, 849, which in the "Herodianic" form is* $\overline{\Gamma}\mathrm{HHH}\Delta\Delta\Delta\Delta\Gamma\mathrm{IIII}$, *but which in the alphabetic system is* $\Omega\mathrm{M}\Theta$. *The former is more self explanatory and, for reckoning with, has most important advantages.*

To present the other side of the coin, it does appear that people who have taken the trouble to learn the Greek alphabetic numerals, for example, those engaged in modern translations of older Greek works, report no real difficulty in performing simple arithmetic operations. Problems do arise, however, when users attempt to do slightly more advanced work which requires familiarity with multiplication tables. Perhaps the most damning indication that the alphabetic system was awkward to use comes from Ptolemy who, when producing the tables for his great work, *The Almagest*, opted to use the Babylonian number system rather than the Greek because it was easier to manipulate. It was his adoption of the Babylonian number system, which used base 60, that ultimately resulted in our having 60 seconds to the minute, 60 minutes to the hour, and angular measurement based on multiples of 60.

Of course, when you use alphabetic characters to represent the digits of your number system, it is entirely possible that an expression may be understood both as a number and a verbal concept. This ambiguity was generally avoided by marking the letters which should be considered a number, either by a line at the start and end of the letter group or, more commonly, by a line drawn over the top of the numbers. It was likely the use of these alphabetic numbers, in Greek and other cultures, that encouraged the development of the various cabalistic cults and fortune-telling schemes. The Hebrews used a similar alphabetic number scheme, and their religious beliefs show through quite clearly in the way some of their numbers were represented. The Hebrew letters for 10 and 5, for example, just happened to be the first two letters for Jehovah and, because it was not considered proper for God's name ever to be written down, even in abbreviated form, they had to represent the number 15 by (9 + 6) instead of (10 + 5). Of course, this led to some rather awkward rules which had to be remembered when doing arithmetic.

The actual process of doing arithmetic with either the Attic or the alphabetic numerals has never been adequately examined. This may well be because it was not considered to be all that interesting and, as a consequence, very few early Greek documents survive which describe the methods they used. It is known that they used the Egyptian methods of doubling and halving for performing multiplication and division but, if the average person had any other methods in addition to these, they have been lost. Certain mathematicians are known to have used other methods. Hero of Alexandria, for example, was known to do multiplications by factoring out powers of 10 whenever possible. For example, if he needed to multiply 13 by 18, he would note down the problem as (10 + 3) (10 + 8) which can be rewritten as 10 (10 + 8) + 3(10 + 8) which gives 100 + 80 + 30 + 24 = 234. By this method he could avoid the problems of having to know a huge multiplication table, and often could represent part of the problem by just doing a multiplication by 10, which was done by simply moving over to the next set of nine letters—almost as simple as our own method of just adding a

zero (see Figure 1-9). When he had to do an actual multiplication, it was likely done by simply looking up the answer in a table. The final additions would be done with the aid of an abacus. We know that the Greeks made use of a form of the table abacus to do a lot of their actual computation because there are a number of references to the device in Greek literature. This subject will be explored to greater depth in the section dealing with the use of the table abacus.

There was one further addition to the Greek system. Archimedes wrote an essay for Gelon, the king of Syracuse, in which he described how to extend the alphabetic number system. In its classical form, the system could be used to represent a myriad of myriads, or 10^8. By taking this as the new base unit and combining it with all the other previous symbols, a notation was developed for any number less than 10^{16}. He called the numbers between 10^8 and 10^{16} *the numbers of the second order.* Taking 10^{16} as the base for his *third order numbers* he was able to develop a representation for integers less than 10^{24}. He continued in this way until reaching numbers of the ninth order, or 10^{64}. The system does not seem to have been used in any way other than being an academic exercise for Archimedes and his readers.

1.5 The European Number System

Our own number system is so familiar to us that it is not worth describing. However, it is worth devoting some effort to an investigation of its origins. The Roman Empire spread through most of Europe and left its mark by leaving behind versions of the Latin language and the system of numeration used by the Roman soldiers and tradesmen. These Roman numerals were in such widespread use that they have lasted right up to the present day, at least for certain special functions such as marking the hours on clocks and numbering chapters in books, especially formal works such as the Bible.

The advent of what we call the Hindu-Arabic number system, together with its strict positional or place value scheme and the zero that must be used to show an empty place, was one of the great inventions of mankind. It enabled the development of mathematics and arithmetical methods which were far in advance of anything that had been known before. From our vantage point of being able to look back in history, it seems that the invention of a strictly positional number system, and a character to represent a zero, should have been an obvious extension of some of the early positional number systems. This idea can be easily put aside by noting that some of the greatest of all the ancient authors, even when they were aware of the shortcomings of their own number systems, did not conceive of the simple system we use today. The French mathematician Pierre-Simon Laplace (1749–1827) noted[8]:

How difficult it was to invent such a method one can infer from the fact that it escaped the genius of Archimedes and of Apollonius of Perga, two of the greatest men of antiquity.

The use of a zero was not uncommon in certain early positional number systems. The Babylonians used a zero-like character to represent an empty column in the middle of a number as early as 200 B.C., but it was not a consistent part of their system. If a number, such as 1024 required a zero to separate the digit 1 from the digit 2, they would use their zero symbol in the same way we do. If, however, they had to represent a number such as 1,000, they appeared to be unable to conceive of the fact that the zero symbol could be used to simply "pad out" the last several places, and they would represent the number as 1 and leave the reader to figure out that it meant 1,000 and not 10, 100, or even 1.

Exactly where and when humans first used our current positional number system, and the ten digits on which it relies, remains unclear. It certainly came into Europe from the Arabs, and it is quite clear that they in turn obtained it from the people on the Indian subcontinent. Where or when the Indians obtained the system is not known. It may have been an indigenous invention, it may have come from further east in Indo-China, or it may have been a development out of the Babylonian use of an *empty* column symbol. In early India, as in a lot of societies, the art of arithmetic was developed to a degree beyond that needed for commerce because of its importance to the local religion. All three of the early Indian religions (Jainism, Buddhism, and Hinduism) considered arithmetic important, as is shown by its being listed as one of the fundamental studies required for the priesthood.

The earliest use of Indian mathematics occurs in works written in verse form, in which complex literary devices were employed to represent numbers in order to preserve the rhyme and meter of the poems. Even the documents which use numerals to denote numbers are not always reliable guides to when this practice first appeared. It seems that, some time in the eleventh century, an attempt was made to rationalize the system of land ownership in part of India and this led to many people producing forged documents to aid their claims to various endowments. Of the 17 known inscriptions using numerals from before the tenth century, all but two have been shown to be forgeries. The earliest undoubted occurrence of a zero in a written inscription in India was in 876 A.D. with the numbers 50 and 270 being represented in the local version of the Indian digits.

The history of our number system becomes much clearer about the ninth century A.D. In the seventh century, when the dynasty of the Caliphs was founded in Baghdad, the learning of the adjacent countries was being absorbed into the newly expanded Arabic culture. As the Arabs conquered a country, they would often take over the mode of writing, particularly the notation of numerals, of the conquered people and search out the sources of knowledge in the literature which may have survived the ravages of war.

The great leader of the Eastern part of the Arab empire, the Caliph Al-Mamun, established an academic center known as the House of Wisdom in Baghdad some time about the year 800. This House of Wisdom was a combination of library, university, and institute for translating foreign works into Arabic. It was likely the most important such school since the destruction of the one in Alexandria. This school acquired and translated works from their original Greek, Persian, Sanskrit, and Coptic sources into Arabic. In fact our only knowledge of several of the early scientific works in these languages comes from their Arabic translations (or the later Latin versions) because the originals have long since been lost. These translations were first undertaken under the Caliph Al-Rashid, but the majority were done during the reign of Al-Mamun. Unfortunately, history repeated itself when the House of Wisdom, like its predecessor in Alexandria, was completely destroyed in 1258 by the grandson of Genghis Khan, Hulagu Khan, who destroyed Baghdad together with all its libraries and museums.

One of the scholars attached to Al-Mamun's House of Wisdom, sometime early in the ninth century was Mohammed ibn Musa Al-Khowarizmi (Mohammed, the son of Moses, from Khowarizm—the modern city of Khiva in southern Russia). Al-Khowarizmi, as he is usually known, had contact with Indian astronomers who had been brought to the court of Al-Mamun to provide the Arabs with the resources of Indian astronomy and mathematics. Al-Khowarizmi wrote textbooks on both arithmetic and algebra and calculated astronomical tables for use in Baghdad. In his *Arithmetic*, he distinctly indicates that the system came to him from the Indians. Although none of the original Arabic versions is extant, we do have copies of translations that were made into other languages. The *Arithmetic* begins with the subject of numeration, then discusses the digits and the use of the zero at great length, and follows this with a systematic discussion of the fundamental operations upon integers. Al-Khowarizmi's work was very influential, since it was evidently copied many times and later Arabic writers often based their own mathematical works on Al-Khowarizmi's model. The use of the Hindu numerals quickly spread through out the entire Arab empire.

The eventual spread of these Hindu numerals into Europe is most easily explained by the contacts generated between Arabs and Europeans by trade and war. It is likely that the Italian traders would know of the accounting systems of their trade partners, and the soldiers and priests returning from the Crusades would also have had ample opportunity to have made contact with the Arabic system of notation and arithmetic. The oldest known European manuscript to contain the Hindu-Arabic numerals was written in the Albelda Cloister in Spain in 976 A.D. The new numerals are also found in another Spanish manuscript of 992 A.D., a tenth-century manuscript found in St. Gall, and a Vatican document of 1077 A.D. Their use was very rare during this early period, however, and it is likely that very few people would have understood the system before the middle of the thirteenth century.

The first great attempt to introduce this new form of notation was made by Leonardo of Pisa (1175–1250?), better known by the name of Fibonacci (which came from filius Bonaccio, the son of Bonaccio), who ranks as one of the best European mathematicians of the Middle Ages. During Fibonacci's time, Pisa was one of the great trading cities of Italy, and it had connections all through the Mediterranean area. His father was the head of one of Pisa's overseas custom houses, in Bugia on the coast of North Africa. Bugia was an important center for merchants and scholars so when, at the age of 12, Fibonacci was sent to join his father, he had a first-class opportunity to observe the methods of the Arabs. He certainly obtained part of his schooling while in Bugia, and legend has it that he was taught Arabic and arithmetic by a local grocer. He later visited Egypt, Syria, Greece, and France where he took pains to inquire into the local systems of arithmetic. He found all these numerical systems so inferior to those of the Arabs that, when he returned to Pisa, he wrote a book to explain the Arabic system of numerals and computation. This book, entitled *Liber Abaci* (The Book of the Abacus) was first published in 1202, and revised and extended in 1228. It was a massive tome for its day, consisting of 459 pages divided into 15 chapters. Chapters 1 through 7 introduced the Arabic notation and the fundamental operations on integers, chapters 8 to 11 dealt with various applications while the remainder were devoted to methods of calculation involving series, proportion, square and cube roots, and a short discussion of geometry and algebra. It was in one of these later chapters that he introduced the famous rabbit problem and the series of numbers which now bears his name.

The *Liber Abaci* was not as influential as it might have been because it was rather large, and thus difficult to copy in the days before printing. It also contained advanced material suitable only for scholars so that it was known only to a few people, none of whom seemed to have had much influence on the methods of calculation used in everyday transactions.

Although the efforts of Fibonacci were of little success, the idea of the Hindu-Arabic numerals gradually spread into Europe. The main source of information was the various translations, or partial translations, of Al-Khowarizmi's *Arithmetic*. The fact that Arabic was completely different from any European language was a great bar to the dissemination of Arabic scientific ideas. In order to learn Arabic it was usually necessary to travel to an Arabic-speaking country, and this was a dangerous task as some Arabs did not take kindly to Christian visitors (and vice versa). This problem was partially solved when, in 1085, Alphonso VI of Leon recaptured Toledo from the Moors and a large Arabic-speaking population came into the sphere of European influence. The majority of the early translators, or at least the people who aided the translators, came from this population.

The two main works which spread the knowledge of Hindu-Arabic arithmetic through Europe were the *Carmen de Algorismo* (The Poem of Algorism) by Alexander De Villa Dei from about 1220 and the *Algorismus*



Figure 1-11
A hypothetical portrait of Leonardo of Pisa (Fibonacci).

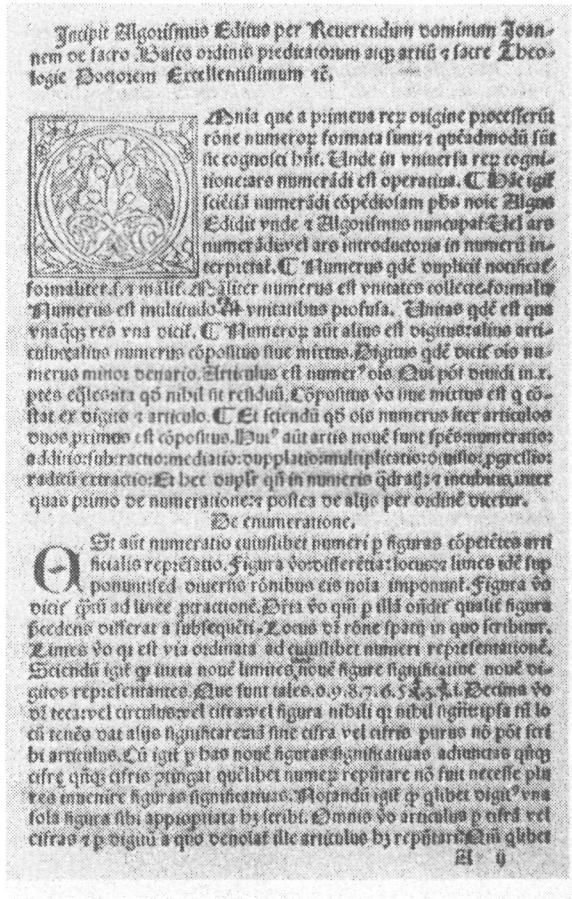
Vulgaris (Common Algorism) by John of Halifax, better known as Sacrobosco, from about 1250 A.D. Both of these works were based, at least in part, on the works of Al Khowarizmi or one of his successors. They were both designed for use in the European universities then starting up in places such as Paris and Oxford and were not meant to be complete explanations of the system; rather they simply gave the basics so that a lecturer could explain them, line by line, to his students. The *Carmen de Algorismo* was particularly difficult to follow, especially in the discussion on the calculation of roots, because it was written in hexameter verse. Despite this it became very popular, being copied many times in its Latin original and even being translated into English, French, and Icelandic. Part of its popularity was due to the fact that Alexander de Villa Dei (?-1240), who was a native of Normandy and wrote and taught in Paris, was already very well known for a Latin Grammar, also in verse, which was widely used in schools of the period. Also, being only 284 lines long, it was easily copied by scribes, and a hundred copies could be made and distributed in the time it would take to make one copy of *Liber Abaci*. The same was true of the *Algorismus Vulgaris* which was only about 4,000 words long.

Sacrobosco, who also taught in Paris during the first half of the thirteenth century, was well known for a work on astronomy, and this undoubtedly contributed to the success of his *Algorismus Vulgaris*, which continued

to be used as a university text on arithmetic even after the invention of printing. Printed editions are known from as late as the fifteenth and sixteenth centuries.

One of the first translators of Al-Khowarizmi's work was Adelard of Bath who, about the year 1120, produced a Latin text whose first words *Dixit Algorismi* . . . (Thus said Algorismi. . .) resulted in this new science being known as algorism. This term, and the various corruptions of it generated by different authors, eventually spread through every European language to the point where the process of doing arithmetic with the Hindu-Arabic numerals was termed *algorism*, and this in turn has given us the term *algorithm* which is so familiar to students of computer science. Before long the connection of *Algorismi* with Al-Khowarizmi was lost and various authors invented different people, such as King Algor, to whom the origin of these methods could be attributed .

The copiers of these translations were very faithful in rendering exact copies down through the centuries. One example of this comes from an English translation of Sacrobosco's work in which the ten digits are described as:


Incipit Algorismus Editus per Reuerendum dominum Joannem de sacro Busco ordinis predicatorum atq; artiū q sacre Theologie Doctorem Excellentissimum ꝛc.

Omnia que a primeus rez origine processerūt rōne numerox formata sunt: q quēadmodū sūt sic cognosci hūt. Unde in vniuersa rez cognitione ars numerādi est operatiua. ¶ Hoc igit sciēdū numerādi cōpēdiosam pbō nois Algo Edidit vnde q Algorismus nuncupat: Hec ars numerādi vel ars introductoria in numerū interpretat. ¶ Numerus qdē dupliciter notificat formaliter. q n illic. Aliter numerus est vnitatis collecta forma ꝛ Numerus est multitudo ex vnitatibus profusa. Unitas qdē est qua vnaqq; rez vna dicit. ¶ Numerox aūt alius est digitalis articularis numerus cōpositus siue mixtus. Digitus qdē dicit ois numerus minor denario. Articulus est numer? ois. Qui pōt diuidi in .x. ptes eqles sine qō nihil sit relictū. Cōpositus vo siue mixtus est q cōstat ex digito q articulo. ¶ Et sciendū qō ois numerus iter articulos vnos primos est cōpositus. Duo aūt artis nouē sunt spēs numeratio: addītio: subtractio: mediatio: duplatio: multiplicatio: diuisio: progressio: radicū extractio: Et hec oēs qs in numeris qdrātis q incubis inter quas primo de numeratione: q postea de aliis per ordinē dicetur.

De enumeratione.

Q Uit aūt numeratio cuiuslibet numeri p figuras cōpetētes artificialis repetitio. Figura dicit differētia: locus q; iuncte idē sup ponunt sed diuersis rōnibus eis nota imponunt. Figura vo dicit' q; ad lineā practicā. Ordo vo qm p illā ordīat' qualis figura precedens offerat a subsequēti. Locus vo rōne sparū in quo scribunt. Limes vo qi est via ordinata ad cuiuslibet numeri representationē. Sciendū igit qr tanta nouē limites, nouē figure significatiue nouē digitos representantes. Que sunt tales. 0. 9. 8. 7. 6. 5. 4. 3. 2. 1. Decima vo dicit' nec vel circulus vel cifra vel figura nichili q nichil significat ipsa nil locū tenēs dat alijs significare nā sine cifra vel cifris purus nō pōt scribi articulus. Cū igit p tres nouē figuras significatiuas adiuncta q; cifrę q; cifris pungat quilibet numer? reputare nō fuit necesse plures inuenire figuras significatiuas. Notandū igit q; quilibet digit? vna sola figura sibi appropriata describit. Omnis vo articulus q cifrā vel cifras a p digitū a quo denotat ille articulus dicit reputari. Cū quilibet

A ij

Figure 1-12
The first page of the 1501 edition of Sacrobosco's Algorismus. Note the later corrections to the form of the numerals on the eighth last line.



which is just opposite to the order in which any European would have written them down. They had obviously been copied faithfully, through version after version, from the order in which they appeared in the original Arabic (which would have been read right to left) several centuries earlier.

The switch from the additive Roman numerals to the positional system of the Hindu-Arabic numerals came slowly over a period of centuries. It was not easy for the Europeans to comprehend the use of the zero which, while representing nothing itself, could magically make other digits dramatically increase in value. The Hindu word for the zero sign was *sunya* which, appropriately enough, meant *empty* or *blank*. When the system was adopted by the Arabs, they used their own word for empty which is generally written as *sifr* in our alphabet. This Arabic word was simply transliterated into Latin either as *zephirum*, from which we get our word *zero*, or as *cipher*, from which we derive the older English form *to cipher*, meaning to do arithmetic. Some of the mystery with which the new system was considered can be observed by the fact that this same root word has given rise to terms that have been involved in magic and secret writing, such as to *cipher* or *decipher* a coded text. This air of mystery was heightened by the attitude of some people who, after they knew algorism, thought that it was knowledge to be kept for a select few and not explained to the common people. This attitude is illustrated in several pictures from the late Middle Ages, one of which is reproduced here. The picture, Figure 1-13, which dates from the late 1400s and was used in a book designed to teach arithmetic to the Queen of Spain, shows two individuals doing arithmetic, one using the old methods of the table abacus while the other, hiding the work from the first, was using the new algorism.

By 1375 the Hindu-Arabic numerals had a firm hold on Europe. They began to appear in many different documents and were well enough known for Chaucer, who called them *the figures newe* to write a poem explaining their power[9]:

The Wedde

Shortly it was so full of bestes
That though Argus the noble countour
Y sate to rekin in his contour
And rekin with his figures ten.
For by the figures newe all ken.
If they be crofty, reckin and nombre
And tell of every thing the nombre
Yet shull he fail to reckin even
The wonders we met in my sweven.

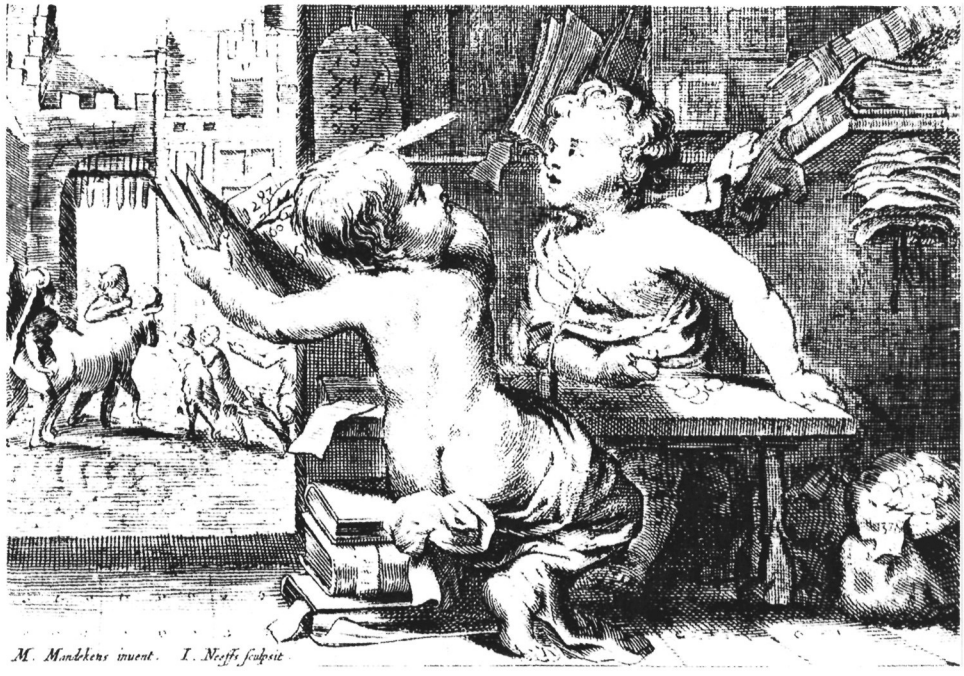


Figure 1-13
Both the abacus and algorism being used to do arithmetic. (*Anusis Ferdinandea, c 1430.*)

Even though they were becoming well known, there was still a great deal of resistance to adopting the new numbers. In 1299 the city of Florence issued an ordinance which prohibited the use of Hindu-Arabic numerals on the grounds that they were very easy to alter or forge, for example, changing a 0 to a 6 or a 9 would have been quite easy. The merchants had developed various tricks to prevent this sort of thing with Roman numerals; for example, XII would be written as Xij so that an extra i could not be added to the end without arousing suspicion and, by making the first character a capital, they prevented anyone from adding characters to the left-hand side of a number. It would only be a short time before the same sorts of devices were developed for the new figures but, even as late as 1594 the merchants of Antwerp were warned not to use them in contracts or bank drafts.

The Italians, quick to see the usefulness of the system for mercantile purposes, led Europe in the general adoption of the place value system. Algorism was known throughout Europe by about 1400, but the more conservative merchants kept their records in Roman numerals until about 1550, and many monasteries and colleges used them until the middle of the seventeenth century. A number of interesting documents show the change actually taking place. A set of family records in the British Museum shows the first child being born in Mijc.Lviii (1258, the *ijc* being a form of writing 200), the second was born in Mijc.Lxi, the third in Mijc.63, while the fourth and fifth children were born in 1264 and 1266.[10] Obviously this family accom-

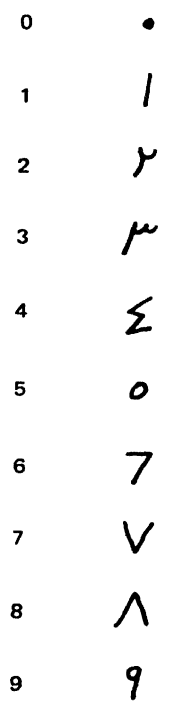


Figure 1-14
A table of the standard European and Arabic numeral forms.

modated to the new system in only three or four years. Even as late as 1681 we find evidence that the new system was not entirely understood. A book published in that year had its chapters numbered[11] as: 1, 2, 3, 4, 5, 6, 7, 8, 9, X, X1, X2, X3, X4, . . ., XXX, XXX1, 302, 303, . . ., XXXX, 401, 402, . . .

This hanging onto the old methods of notation was caused, not by a lack of knowledge of the new, but by the fear that, without a complete understanding of the system, something might go wrong. This type of fear is seldom seen today, but two examples from the seventeenth century might help to illustrate this phenomenon. William Oughtred, whom we will meet again in the discussion about the slide rule, would often calculate $ab + ac$ rather than $a(b + c)$ because of a fear that some sort of error might enter into an abbreviated system. In 1673 John Wallis from Oxford wrote to John Collins and admitted that he had doubts about writing $\sqrt{12}$ as $2\sqrt{3}$ simply because no one had ever done it that way before and, although he knew it to be the same value, he *durst not without precedent, . . . introduce a new way of notation.*[12]

Many different forms of the numerals have been used in Europe, some of which would not be recognized for what they are today. The manuscript versions of the old works are particularly difficult to read because the author and copyist used the forms of numerals with which they were most familiar. Books produced a few years, or even a few miles, apart would use different characters for at least some of the numerals. It was the invention of the printing press which standardized them into the forms we now know,

although even today the forms of 5 and 7 vary slightly between some Europeans and Americans. It is interesting to note that, despite the fact that the Europeans obtained the system from the Arabs, the two cultures use remarkably different forms of the numerals today. The European tourist is constantly having problems with the fact that the circle is used in Arabic countries as the symbol for the digit five and something resembling a dot is used as the symbol for zero.

1.6 The Far East

Although there is a lot known about the old forms of numerals and their computational use in China, there is great difficulty in establishing exact dates for the introduction of various concepts and practices. This difficulty stems from the fact that the ancient Chinese had a high veneration for the works of very old authors and, much like the European Medieval tradition of considering Aristotle to be the ultimate authority, they would confine their research to elaborating the knowledge set down by the ancients. If anyone did propose a new method of computation, it was almost always attributed to one of the ancient authors whose writings were, of course, no longer available. Several authors have been taken in by this practice of attributing results to the ancients and have credited the Chinese with a very sophisticated arithmetic system by as early as 3500 B.C. Although some evidence exists to show that abilities in arithmetic existed that early, the major developments in this area did not take place until about 600 B.C.

Most of our knowledge of early Chinese numbers comes from monuments and coins because the written works on arithmetic have not survived the ravages of either human or natural disaster. One of the worst of these ravages was perpetrated by Chi Hwang-ti, the first emperor of the Tsin dynasty, in 213 B.C. when he ordered all the books in China to be burned and the scholars buried alive. He was, unfortunately, reasonably successful in this undertaking. There are some texts which survived the purge, dating from about 540 B.C., which indicate that the Chinese had a well-developed decimal place value notation that had been in use for several hundred years. The best estimate is that they used a positional decimal number system from about 1300 B.C., possibly even earlier.

It has been conjectured that the development of a place value notational system depends on the style of writing adopted by a people. Those civilizations which had an alphabetic writing system tended to use the alphabetic symbols to represent integers and, like the Greeks, saw no real reason to stop equating symbols with integers when the digits from 1 to 9 had been assigned their signs. The Chinese idiographic writing system did not tempt one to assign individual symbols to numbers greater than nine because there would have been no built-in compulsion to use up all of the symbols

in an alphabet. It is likely that the Chinese were the first people to use a completely formed decimal representation for numbers.

The characters used by the Chinese undoubtedly were simple pictures of the sticks they used for the primitive counting process. Like the Egyptian system, the integers from 1 to 5 were just represented by simple vertical strokes but, for representing numbers from 6 to 9, they employed the shorthand device of a horizontal stroke representing five together with enough vertical strokes to make up the required digit. The system is illustrated in Figure 1-15.

During computation, the numbers were represented by a special set of calculating rods which would be laid down in such a way as to form the ideographs of the required numerals. When attempting to represent a number such as 24 (see Figure 1-15), it is very easy to confuse the rods belonging to one digit with those belonging to its neighbor. In order to overcome this problem the technique of rotating every second digit through 90 degrees was evolved. This meant that the digit 1 would be represented as | if it was used in the units, hundreds, ten thousands, or millions position, and by — if it was used in the tens, thousands, or hundred thousands position as illustrated in Figure 1-16.

In later years it became the practice to perform arithmetic on a large squared board, much like a checker board, with each digit of a number occupying a separate square. This eliminated the need to rotate adjacent digits by 90 degrees but, because of the lack of any device to indicate an empty column (zero) when writing down the result, the orientation of the numerals was still vital to their understanding. An arithmetic text[13] from the third century A.D. starts off with:

> *In making a calculation we must first know the positions. The units are vertical and tens horizontal, the hundreds stand while the thousands lie down; thousands and tens therefore look the same, as also the ten thousands and the hundreds.*

A representation for zero was added to the system, probably from contact with India, about the year 800 A.D.

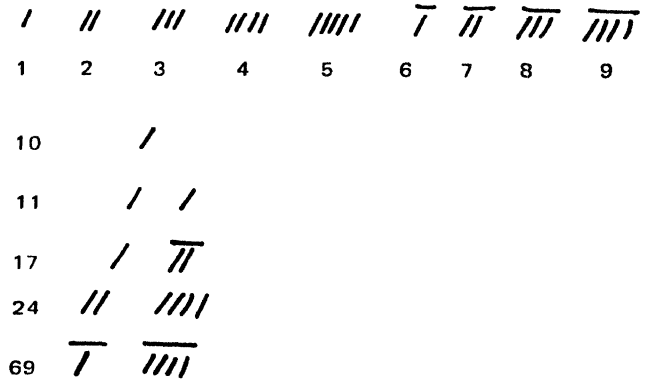


Figure 1-15
Examples of early Chinese numerals.

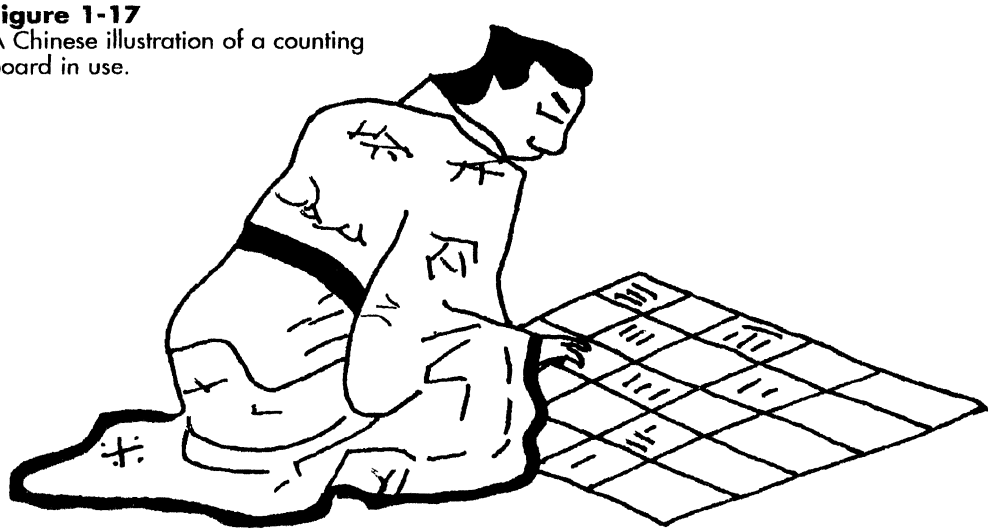
198,617 = 𝍠 𝍡 𝍢 𝍣 𝍤 𝍥

Figure 1-16
An example of 90-degree rotation of early Chinese digits.

The rods and squared board provided a substitute for the modern use of a pencil and paper. Two numbers to be added together were first laid out on the board, one above the other, then the result of adding together the various columns of digits would be represented by another line of rod numerals below the first two. Subtraction was done by placing the rod numeral answer above the two lines which represented the given problem, in effect creating the complementary addition problem. Multiplication and division, while not exactly equivalent to our pencil and paper methods, would be readily understood by a modern individual. The conjecture has been made that, because the Chinese always wrote from top to bottom, their horizontal use of the calculating board indicates that it may have been introduced from another culture. There is no real evidence to support this conjecture, but it is not beyond the realm of possibility that the system was adopted from either the Indian or Tai people. It is interesting to note that the Tai number words are very similar to the Chinese number words but that the two languages have almost no other words in common.

One indication of the sophistication of this system is that the early Chinese had some notion of negative numbers. The computing rods came in two colors, red for use when representing positive quantities and black for negative ones. There is also some indication that, at least in later years, the calculating rod system was modified by having numbers engraved upon the individual rods. Almost nothing is known of the use of this sys-

Figure 1-17
A Chinese illustration of a counting board in use.



tem but it is possible that it was a primitive attempt at the use of decimal representation, much like Gerbert's *apices* described in Section 2.3.

A complete set of traditional calculating rods contained 271 pieces, each of which was about 0.1 in. in diameter and 6 in. long. Initially they were constructed from bamboo, but after 400 B.C. they were often constructed from ivory, bone, or horn, and some time between 600 A.D. and 900 A.D. they were fashioned from cast iron. One very old way of indicating the wealth or status of a man was to imply that he was so rich he could afford to *calculate with ivory rods.*

Calculating rods continued to be used in China up to about the year 1600 when, due to the increasing popularity of the wire and bead abacus, no further references are found in the literature. The wire and bead abacus is, undoubtedly, easier and quicker to use than the calculating rods on a squared board, but the rods were a useful tool in the hands of a trained manipulator. In the eleventh century A.D. an astronomer was described as being able to

> *move his counting rods as though they were flying, so quickly that the eye could not follow their movements before the results were obtained.*[14]

The date of the introduction of the wire and bead abacus is not known for certain, but it is known that a calendar reform done in the year 1281 A.D. was accomplished by the calculations being done with calculating rods while another calendar reform in 1384 was aided by the astronomers using a wire and bead abacus. Thus the calculating rods must have been displaced by the more modern abacus starting some time about the year 1300 A.D.

In later years the Chinese used three distinct numeral systems. The first were those that occur in normal written material, the second were a special set of numerals for use in legal documents because of the ease with which the first set could be forged or altered, while the third set (the counting rod numerals) were those which, because of their simple construction, were normally used for written arithmetic or accounting work. The second, or legal, set of numerals were the ideographs for certain words which, because of historical and poetic connotations, came to be associated with certain numbers:

The numeral for	*Means*
1	perfection
2	to assist
3	an accusation
4	to expose publicly
5	to associate
6	a mound of earth
7	a tree
8	to divide
9	a stove
10	to collect

From China the use of the calculating rods, and the numeral representation based on them, spread to Korea. Here they took a much firmer hold than they did in China, not even being completely displaced when the wire and bead abacus was introduced. Even as late as the nineteenth century the use of the calculating rods was still so widespread that they were obvious to foreigners. In 1881 the French missionaries at Yokohama published a book about Korea, *Grammaire Coreene*, in which they stated:

> *The Coreans, to make their calculations, use small rods in nonfixed number, which are placed from right to left, isolated or in groups, in order to represent the units, tens, hundreds, etc. Disposed vertically they indicate each unit of the order of the units or of the tens etc.; disposed horizontally, they are worth five units of the same order.*[15]

Shortly after the introduction of Buddhism into Japan, in 552 A.D., two Korean scholars came to Japan in order to teach more about this new faith. They not only brought the teachings of Buddha but were also responsible for the introduction of the Chinese calendar and the Chinese system of astrology. With these came the knowledge of the calculating rods and the rod numerals. By 600 A.D., when the Japanese adopted the Chinese system of measures, the use of the calculating rods was well established.

One of the annoying problems of doing arithmetic with the calculating rods is their tendency to roll around, particularly if the calculating board is

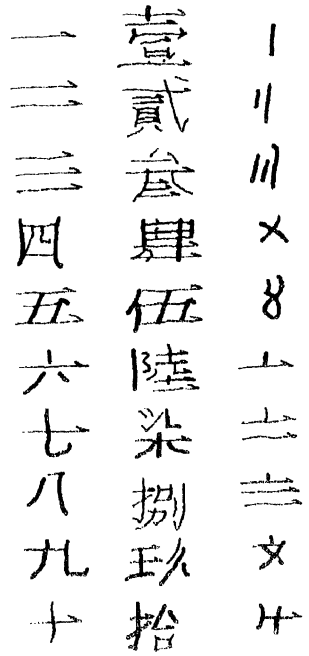


Figure 1-18
A table of the three different Chinese numerical character sets.

not perfectly flat. The Japanese solved this problem by making their rods with a square cross section. This rather obvious improvement does not seem to have been passed back to the Koreans or Chinese, perhaps because the circular rods were already too firmly embedded in these other two cultures, or it may simply be that, because the mainland peoples considered the Japanese to be barbarians, they were reluctant to accept any of their ideas.

1.7 Other Forms of Notation

Earlier sections of this chapter have illustrated several of the different ways that people used writing to record the results of their computations. It should be kept in mind, however, that the majority of the early societies in this world were not fortunate enough to have access to reasonable sources of writing materials. Clay or wax tablets, as used by the Babylonians and Greeks, were not always available due to different climate or geological conditions. Even when the raw materials for these media could be readily obtained, they were often not used because of the need to bake the clay or keep the wax cool in order to preserve the records which had been so carefully inscribed upon them. What to us would seem like more reasonable modes of recording information, by the use of pen and ink on paper or parchment, was again not suitable because these materials represent relatively late technological developments. Papyrus was not known outside of Egypt until about 600 B.C., and parchment was not available until about 400 B.C. Even those societies that had items like paper or parchment would not often use them—except for recording important data—because they were difficult to produce and consequently quite expensive. Because of these factors, large sections of the world's population had to develop alternate physical representations for information. A variety of different materials was used, but knotted cords or notches cut into *tally sticks* were the most common forms.

1.7.1. Knotted Cords for Record Keeping

All cultures used threads or ropes for everyday tasks. It would only be natural for them to think of using knots in these ropes to represent elementary forms of data. The knotted rope was superior to the other obvious alternative, that of keeping a bag of pebbles as a number record, because it was not subject to the same type of information corruption that could take place if the recording device were accidentally dropped. A knotted cord would also have been an obvious choice as a simple memory tool for many societies because they were already familiar with it from its use in the contexts of magic and religion—although which association would come first is a debatable point. The knot has had a long history as a tool of the witch doc-

tor. The association has survived right up to modern times in such cross-cultural expressions as to *bind* or *untie* a spell.

Although there are many historical records to indicate that knotted strings were used for remembering information, this technique seems to have been almost lost by modern societies except for the occasional ceremonial use, mainly by religious groups. For example, the ancient Hebrews were known to have used knotted strings as a memory device, and Deuteronomy 6:6-8 exhorts them to tie cords around the hands and head (seen today in the Jewish phylacteries or frontlets) to remind them of the laws. According to tradition, God showed Moses the exact form of tying these knots to make them the most suggestive. A similar device is the fringe of white and blue cords, worn on Orthodox Jewish clothing, tied with knots at different distances to remind them of the Commandments. Other Biblical references to knots can be seen in some of the Old Testament descriptions. For example, Ezekiel 40:3 records the prophet's vision of a man *with a line of flax and a measuring reed in his hand* who took down the exact dimensions of the future temple. It has long been supposed that the *line of flax* (the word *cord* would be a more accurate translation from the Hebrew) was being used to record these measurements by means of a series of knots. The fact that the cord was being used as a recording device would have been obvious to people in Ezekiel's time, but the connection is lost on the modern reader.

Even as late as the third century A.D. the tax gatherers in Roman Palestine were using knotted strings both for their own records and for receipts. Even when this mechanism was replaced by writing, the receipts were still referred to as *publican's knots*. The literary references to the use of knots is not just confined to Biblical sources. There are countless references in old Arabic sources which refer to digits as knots. Although it is highly likely that the Arabic references stem from an early use of knotted strings as recording devices, some care must be taken with this statement because many of the early references reserve the term knot only for multiples of 10 (that is 10, 20, 30, and so on). This may be some indication that the term knot, meaning number, arose from sources other than the use of knotted strings, or that knots were only used to represent multiples of 10 while the units digits were represented by some other mechanism, for example, the number of strands in the cord, or some form of marking on the lower sections of the cords.

China, too, has historical records indicating knotted cords were once in use as recording devices. In an appendix to a historical work, Confucius writes:

> *In great antiquity knotted cords served them for the administration of affairs. During the following generation, the saintly man Fou-Hi replaced these by writing.*[16]

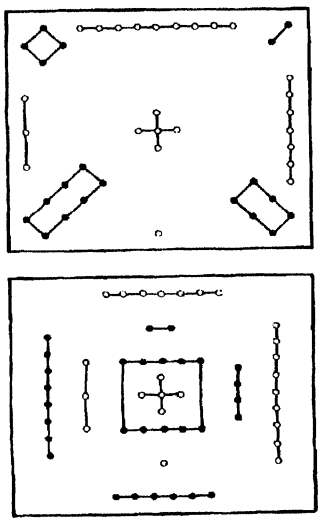
Tradition places Fou-Hi about 2800 B.C. This passage would indicate that the use of knots died out very early in Chinese history, but it is likely

that this is just one more example of the corruption of the historical record by crediting discoveries to the ancients. Other Chinese references refer to the use of knotted cords until the start of the Han dynasty (about 300 B.C.). Even after writing materials were in common use, some old customs concerning the representation of numbers by knots managed to survive: generally involved with magic or religion. The magical or divinatory work known as the *I Ching* contains two diagrams of undoubted antiquity. According to legend, they were presented to emperor Yu the Great by two magic animals from the Yellow River. The diagrams are a 3 by 3 magic square (in which the sum across any row, column, or diagonal adds up to 15) and a square in which are represented all the integers from 1 to 10. The numbers, in each case, are represented by cords with knots tied in them; the cords representing even (yin) numbers are black while those representing odd (yang) numbers are white.

Not all examples of the use of knots to record information come from the distant past. In Germany, up to the end of the nineteenth century, millers would use knots to record the amount and kind of flour they delivered to the bakers. The strings used to close the flour sacks would be knotted in specific ways to record the sack's contents. One measure of flour was indicated by a simple overhand knot, two measures by an overhand knot with an extra string passed through it, and more complex knots were used to record five and ten measures of flour. The type of flour was recorded by a special series of loops tied in the ends of the cords.

A further modern example comes from India. During the census of 1872 the illiterate among the census takers would use four different colors of cord to record the population of rural villages. They would tie one knot for each

Figure 1-19
The Chinese magic squares represented by knot numerals.



individual, using a black cord for a man, a red cord for a woman, white for a boy, and yellow for a girl.

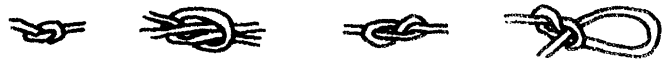
The practice of using knotted records was also common among various groups of American Indians from the interior of British Columbia to the middle of South America. The Incas, from the northwest coast of South America, were the only ones to really develop the system beyond the simple recording of elementary items such as the number of days that had elapsed from a given event. The Incas were an anomaly; they were a highly civilized people with a sophisticated government and yet had little in the way of a written language.

Their use of knotted strings reached its highest development in the fifteenth and sixteenth-century Inca empire device known as a quipu (quipu was the Inca word for knot). The quipu had two basic functions: it would record purely numerical information, and it also would act as an aide-memoire for the record keepers in recalling historical traditions, poems, lists of kings, and so on.

The quipu's method of operation was based upon the decimal system of notation, and different kinds of records were indicated by the thickness or color of the strings. Most of the quipus in existence today were discovered in graves along the Pacific coast of Peru and, through improper handling (both before and after their owners' demise) tend to resemble nothing more than a well-worn and tangled string mop. One of the best early descriptions of the quipu comes from a Spaniard (Pedro de Ciera de Leon, born in Seville in 1519) who went to the new world when he was 14. Unlike most of the other early Europeans in the Americas, he was quite careful to record the life style of the Incas and, when talking about their records, he states[17]:

> *The Peruvian quipus are of twisted wool, and consist of a large cord with finer threads attached to it by knots. These fringes contained the contents of the quipu, which were denoted either by single knots or by artificial intertwinings. The different colours of the threads had different meanings, and not only was the colour and mode of intertwinings of the knots to be considered in reading a quipu, but even the mode of twisting the thread, the distance of the knots from each other and from the main cord. The registers of tribute, the enrollment of the tribes, lists of arms and troops, inventories of the contents of storehouses; all of these were the primary uses of the quipus. They were also available for recording the most striking events and thus supplied the place of the chronicles!*

Figure 1-20
German miller's knots.



Another Spaniard (Garcilasso de la Vega) implied that the quipus were used for actually doing arithmetic[18]:

> *They knew a great deal of arithmetic and had an admirable method by knots made on strings of different colours . . . they added up, and multiplied by these knots, and to know what portions referred to each village, they divided the strings by grains of maize or small stones, so that their calculation might be without confusion.*

It is likely that de la Vega was confused in this regard for, from other sources, it appears as though the Incas actually used a form of abacus to do their calculations. This abacus used grains of corn or small stones in its operation, and it is possible that de la Vega became confused as to the actual role of each device when he witnessed an Inca using an abacus to do calculations involving data stored on a quipu.

The most common form of quipu consisted of a main cord with the various pendant strings looped around it. These pendant cords were generally tied together by top cords which acted as grand totals of the information on the pendant strings. The information would be recorded either by single or multiple knots, and the distance from the main cord indicated its decimal position, with the units digit being the furthest from the main cord. Figure 1-21 shows a quipu which might have been used to record the number of pages in the first edition of Knuth's *The Art of Computer Programming*. Reading from left to right, the pendant strings indicate that there are 225 pages in Chapter 1 and 229 pages in Chapter 2, giving a combined total of 454 pages in the first volume, as seen in the top string. Volume II (438 pages) is made up of 155 pages in Chapter 3 and 283 pages in Chapter 4. Volume III is made up of 379 pages in Chapter 5 and 162 pages in Chapter 6, giving a total of 541 pages. The last pendant string, which would be a different color from the others, indicates that there is a grand total of 1433 pages in the first three volumes.

1.7.2 Tally Sticks

Another obvious device which could be used to record information is the notched stick. Again, this form of data recording has been used in almost every civilization, although most modern cultures show little or no sign of its once having been of importance. The earliest example that has come to light is the so-called Ishango bone found on the shore of Lake Edward in Zaire (Congo). This small bone, about 8,000 years old, has a little piece of quartz stuck in a groove at one end and has three columns of notches cut into the sides of the bone itself. Microscopic examination of the bone indicates that the notches were made by some 39 different tools, thus it likely represents a record of events rather than a random decoration scratched into the bone. There is some limited evidence that suggests the markings may represent a record of some activity based upon a lunar calendar. If that

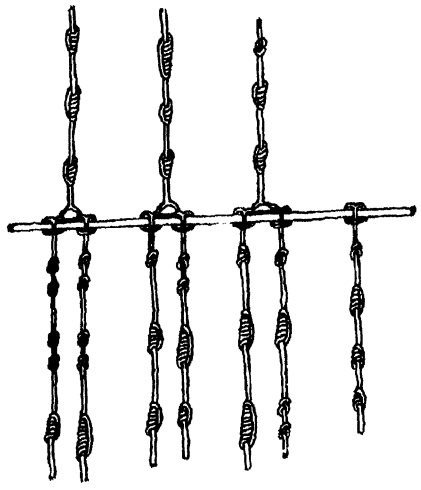


Figure 1-21
Knuth's table of contents in the form of a quipu.



Figure 1-22
A drawing of an Inca quipu keeper, as illustrated in one of the early Spanish documents.

is the case, then the notations record a series of events over a time span of almost six months. If the Ishango bone is really a numerical record rather than just a decoration, then traces of this type of recording system can be found from as early as the Upper Palaeolithic cultures of 30,000 B.C.

There are so many examples of the use of notched sticks in different parts of the world that it is impossible to describe them all. The stick is almost universal in its availability and the cutting tools would be at hand in every culture that was sufficiently advanced to need its record-keeping functions. Most ethnographic museums contain several examples of its use in one form or another. As typical examples of the device, we need only examine them in the forms found in China and in England.

The Chinese used tally sticks about the same time that they were using knotted strings. The inhabitants of the Liu-Chhiu islands have some cultural traditions which likely date back to the time when the use of knotted strings and tally sticks were common all over China. The people of Liu-Chhiu were basically isolated from their neighbors and, being very conservative, were slow to adopt new cultural ideas even when contact was established. These people used a very simple set of tally sticks to keep records of rice, money, and so on. The sticks were from 1 to 2.5 ft. long and 1 to 1.5 in. thick. On the sticks were scratched a primitive set of numerals, much like the old rod numerals of mainland China, but they showed a marked preference for the base 5 system, much like the traditional numerals of Korea.

One obvious remnant remains of tally stick usage in mainland China. The old written character for an agreement or contract is made up of three radicals, two on the top and one on the bottom. The first top radical is a stick with cuts shown across it, the second is a pictorial representation of a knife, and the lower radical is the character meaning *large* or *big*; thus a contract is literally *a large tally stick.*

Figure 1-23
A diagram of the Ishango bone—an 8,000 year old tally stick.

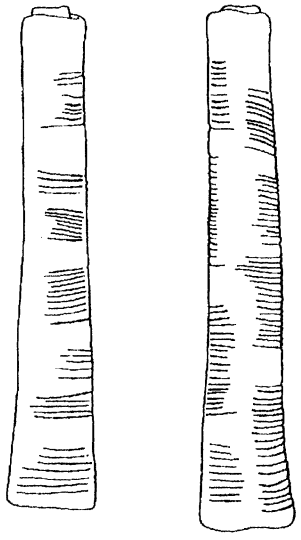
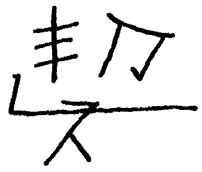


Figure 1-24
The old Chinese character meaning *contract.*



The high point of the use of tally sticks occurred in the British Exchequer during the thirteenth to nineteenth centuries. Any debts or payments were recorded on lengths of elm wood by means of special notches cut into the sides of the stick. The stick was then split lengthwise into two pieces, a thin piece called the *foil* (from the French word for leaf), and a thick piece called the *stock*. The Exchequer kept the foil and the other party kept the stock as an invoice or receipt. When further payments were to be made, the two parts of the original tally were compared to ensure that no forging of receipts had taken place by either party. This practice was so widespread in British commerce that it has resulted in many words which reflect this heritage. For example, if a person deposited money in a bank a tally would be cut of the amount, the bank would keep the foil and the depositor would keep the stock, thus becoming literally a *stock holder* in the bank. Today we no longer trade stocks in our stock markets—we have evolved to the point where we only trade stock certificates! Phrases such as *to tally up the bill* or the *counter foil* one fills out to record the issue of a cheque come from this same source.

The Exchequer tallies were customarily cut as follows: The notch for 1000 pounds was to be as large as the hand is wide; for 100 pounds the notch was to be as wide as the thumb with a rounded bottom, for 20 pounds it was to be as wide as the little finger, for 1 pound it was to be the size of a grain of barley, for 1 shilling it was to be just large enough to be seen as a notch, and for 1 pence it was to be a simple cut with no wood removed. If it was necessary to represent half of any of these quantities, as, for example, when recording 10 pounds, then the notch was to be made with one straight and one slanted side. The notch for 20 shillings (one pound) was the first time that any substantial amount of wood had to be removed (or scored) from the tally, and this led to the adoption of the English term *score* for 20.

Charles Dickens was very interested in administrative reform and his description of the eventual fate of the Exchequer tallies is rather interesting[19]:

Figure 1-25
A diagram of an English tally stick recording the sum 1253 pounds, 5 shillings, 3 and one-half pence.



Ages ago a savage mode of keeping accounts on notched sticks was introduced into the Court of the Exchequer and the accounts were kept much as Robinson Crusoe kept his calendar on the desert island. In the reign of George III an inquiry was made by some revolutionary spirit whether, pens and paper, slates and pencils being in existence, this obstinate adherence to an obsolete custom ought to be continued, and whether a change ought not to be effected. All the red tape in the country grew redder at the bare mention of this bold and original conception, and it took until 1826 to get these sticks abolished. In 1834 it was found that there was a considerable accumulation of them; and the question then arose, what was to be done with such worn-out, worm-eaten, rotten old bits of wood? The sticks were housed in Westminster, and it would naturally occur to any intelligent person that nothing could be easier than to allow them to be carried away for firewood by the miserable people who lived in that neighbourhood. However, they never had been useful, and official routine required that they should never be, and so the order went out that they were to be privately and confidentially burned. It came to pass that they were burned in the stove in the House of Lords. The stove, over-gorged with these preposterous sticks, set fire to the panelling; the panelling set fire to the House of Commons; the two houses were reduced to ashes; the architects were called in to build others; and we are now in the second million of the cost thereof.

Thus the modern House of Commons and the House of Lords, so easily recognized in tourist photographs of central London, owe their existence to the British practice of keeping tax records on tally sticks.

1.7.3 Other Methods of Numerical Notation

There were, of course, lots of other methods used to record and manipulate numerical information during the long span of human history. Special situations would require some unique system to be constructed, for example, the invention of the Braille system of numbers and letters to enable the blind to both read and record information. In many cases the invention of a new system was intended to enable a single individual, or a small group of individuals, with some form of handicap to function in society. A typical example of this type of system was the one developed by Dr. Nicholas Saunderson (1682-1739) who contracted smallpox when he was only one year old, which left him totally blind. When he was a young boy, he invented a notational system for helping him in doing the usual school arithmetic exercises. His system consisted of a smooth thin board about one foot square, upon which was drawn a grid of horizontal and vertical lines about 0.1 in. apart, a small hole being drilled at the intersection of each horizontal and vertical line. It is possible to consider this board as being covered by a multitude of tiny three by three grid patterns as shown in Figure 1-26.

Saunderson considered that each of the small grids represented a single decimal digit, and would indicate which digit was being represented by sticking a pin in one of the holes. The center hole indicated a 1, the hole

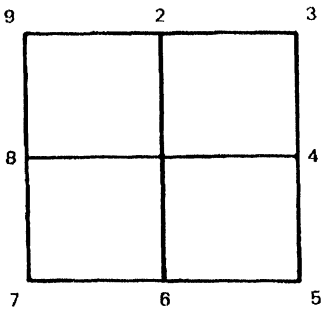
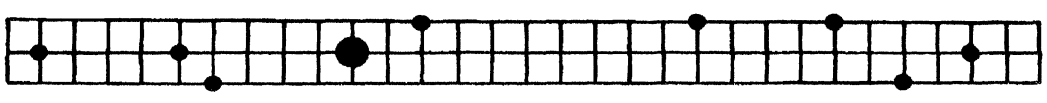


Figure 1-26
Saunderson's numerical grid.

immediately above it a 2, and so on, as indicated in Figure 1-26. In order to indicate a zero, a special pin with an extra large head was placed in the center hole. The units, tens, and hundreds digits were arranged beside one another just as we would do with pencil and paper. Saunderson kept a large stock of pins, with their points cut off to avoid pricking his fingers, in two separate boxes, one for the normal small-headed pins and one for the large-headed pins used to represent a zero. In order to locate any particular digit of any number recorded on the instrument, he placed a series of notches along two sides of the square, the notches marking the center of each three by three grid. He was said to have acquired great facility and quickness in the use of this board for performing the standard arithmetic operations.

Although he remained totally blind, Saunderson became such a good mathematician that he was eventually made Lucasian Professor of Mathematics at Cambridge in 1711, created a Doctor of Laws in 1728, and became a Fellow of the Royal Society in 1736. It is interesting to note that, while at Cambridge, he became very famous for his advanced course on optics. It is rather remarkable that a man, totally blind since he was an infant, could become famous for his lectures on such optical phenomena as the use of a prism for creating rainbows.

Figure 1-27
The numbers 14,709 and 2,368 as they would be represented on Dr. Saunderson's board.



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