Come, let us *rely* on friends for once
Let us make life easy on us
Let us be lovers and loved ones
The earth shall be left to no one.
—Yunus Emre, the legendary mystic folk poet (1238–1320)

1

FUNDAMENTALS OF COMPONENT AND SYSTEM RELIABILITY AND REVIEW OF SOFTWARE RELIABILITY

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Nutshell 1.0

In this chapter we introduce some of the basic concepts, and mathematical and statistical functions used in hardware or software reliability and security evaluation. In the first section of this chapter we review some common statistical properties of density functions. In the second section we introduce some functions of importance in reliability. In the third section we introduce an extended list of statistical distributions that can be used in reliability, and we discuss how to generate random variables of interest using simulation methods. In the fourth section we study the testing of reliability in a variety of data forms. In the fifth section we move away from the components and start dealing with system reliability together with redundancy aspects and limitations. In the final section we introduce and review in depth some basic concepts and models used in software reliability.

1.1 FUNCTIONS OF IMPORTANCE IN RELIABILITY

In reliability theory there are a number of density functions of particular importance because of their theoretical and practical utilization and for their usefulness in illustrating statistical and reliability concepts. In this section, densities generally used are presented together with their more important characteristics. Some of the the most commonly used continuous density functions included are the exponential, normal, rectangular, Weibull, lognormal, and gamma. The

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characteristics presented are those normally considered important in reliability technology, including the reliability function $R(t)$, hazard function $h(t)$, mean, variance, mode, and region of definition.

The derivations of these characteristics are readily available in statistical texts and are not presented here. It will be an excellent exercise for the reader to verify these derivations. The following definitions are pertinent for the derivations. More detail on these definitions is provided throughout.

1. The *reliability:*

$$R(t) = \int_t^{t_u} f(t)\,dt \tag{1}$$

   where $f(t)$ is the probability density function and t_u is the upper bound on the region of definition of $f(t)$.

2. The *hazard function:*

$$h(t) = \frac{f(t)}{R(t)} \tag{2}$$

3. The *mean:*

$$\mu = \int_D t f(t)\,dt \tag{3}$$

   where D (domain) is the region of definition of $f(t)$.

4. The *variance:*

$$\sigma^2 = \int_D (t-\mu)^2 f(t)\,dt = \int_D t^2 f(t)\,dt - \mu^2 \tag{4}$$

5. The *mode* is that value of t (if it exists) such that $f(t)$ is a maximum there for densities with a single maximum.

6. M is the *median* or 50th percentile if

$$0.50 = \int_0^M f(t)\,dt \tag{5}$$

There are several functions of fundamental importance in modern reliability engineering [1]. Most of these are also important in applied and theoretical statistical studies. All are presented here for completeness. The first and fundamental function of importance is the *density function* (Figure 1.1). For a discrete variable (defined only at specific points $t_1, t_2, \ldots, t_n$), the density function gives the probability of occurrence of each point and is denoted $P(t)$. For a continuous variable (defined for all t in an interval I), the density function, denoted $f(t)$, gives the relative frequency with which the t-values occur. Characteristic of the density function is the fact that $\sum_{i=1}^{n} P(t_i) = 1$ for the discrete case and $\int_D f(t)\,dt = 1$

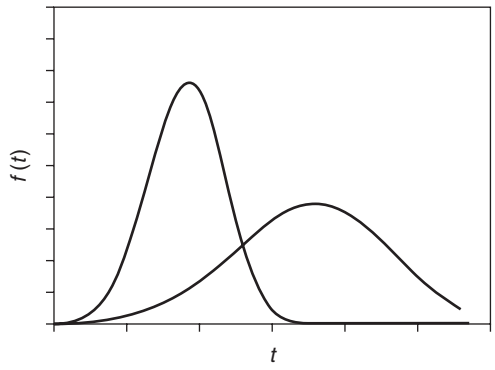


FIGURE 1.1 Probability density functions.

for the continuous case. D denotes the domain of definition or interval of integration. All other functions considered depend on the density function and its characteristics.

The second most important function from an estimation and interpretation standpoint is the *cumulative density function* (Figure 1.2). It is denoted $F(t)$ and is given as follows, where k is the number of discrete values and t_0 is the lower limit of domain D in the continuous case:

$$F(t_k) = \sum_{i=1}^{k} P(t_i) \tag{6}$$

$$F(t) = \int_{t_0}^{t} f(t)\, dt \tag{7}$$

The cumulative function thus gives the probability of a value less than or equal to t (or t_k) or by the fraction of values that are less than or equal to t (or t_k). By appropriate use of this function, it is impossible to evaluate a substantial

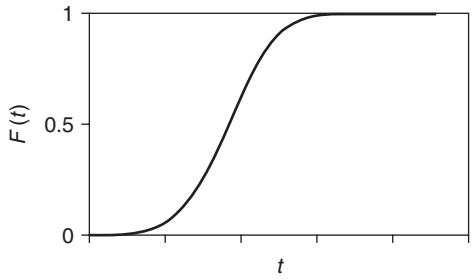


FIGURE 1.2 Cumulative distribution function.

number of probabilities of interest. For example,

$$P(t > t_0) = 1 - F(t_0) \tag{8}$$

$$P(t_a < t \leq t_b) = F(t_b) - F(t_a) \tag{9}$$

$$P(t \leq t_0) = F(t_0) \tag{10}$$

The *joint density function* of n independent random variables is given by

$$f(t_1, t_2, \ldots, t_n) = f_1(t_1) f_2(t_2) \cdots f_n(t_n) \tag{11}$$

This function is one of primary importance in estimation because it permits definition of the *likelihood function* for a random sample of size n. The likelihood function is defined as the joint density of the sample. If we have a random sample $t_1, t_2, \ldots, t_n$ and all t_i have the same density (as is the case in random sampling),

$$f(t_1, t_2, \ldots, t_n) = f(t_1) f(t_2) \cdots f(t_n) \tag{12}$$

The *marginal density* is defined when the density function is of higher order (a density function of more than one variable) as the function

$$f_X(x) = \int_{R_y} f(x, y)\, dy \tag{13}$$

where R_y is the range of the y's, $f(x, y)$ is the density of x and y, and $f_X(x)$ is the marginal density. Thus, the marginal density may be considered to be the result of eliminating random variables that are not of interest.

The *conditional density* is a density that describes a random variable (or variables) when other random variables are assigned specific values. Thus, if $f(x, y)$ is the joint density of x and y, the conditional density of $f(x \mid y)$ is a function of x at a specified y such that

$$F(a < x < b \mid y) = \int_a^b f(x \mid y)\, dx \tag{14}$$

Consider an arbitrary failure distribution given by $f(t)$ for $t \geq \gamma$, and 0 otherwise. In addition to the usual statistical functions, such as the cumulative density function; $F(t)$, which are of importance in reliability, the following are some of the others [2].

1. *Reliability function and reliable life*. The *reliability function*, $R(t)$ (Figure 1.3), is the probability that failure occurs after time t and is defined as

$$R(t) = \int_t^\infty f(x)\, dx = 1 - F(t) \tag{15}$$

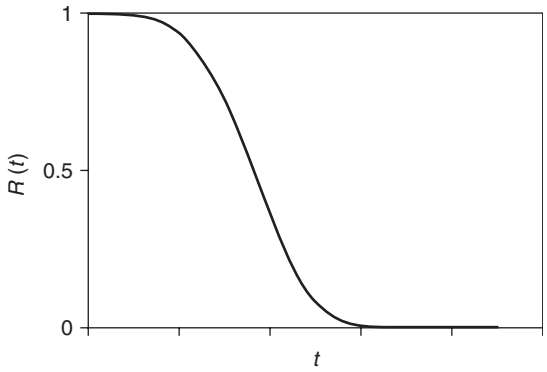


FIGURE 1.3 Reliability function.

The *reliable life*, ρ_R, sometimes called the *minimum life*, is defined for any specified R such that

$$R(t) = \int_{\rho_R}^{\infty} f(t)\,dt = R(\rho_R) \tag{16}$$

The reliable life, ρ_R, is the same as the qth quantile, where $q = 1 - R$. A special case is when $R = \frac{1}{2}$; ρ_R then becomes the median. Similarly, when $R = \frac{1}{4}$, ρ_R becomes the first quartile, or 25th percentile, and when $R = \frac{3}{4}$, ρ_R becomes the third quartile, or 75th percentile.

2. *Moments*. When the threshold γ is a finite value, the moments of the failure distribution may be found from $R(t)$. The kth *moment* of t is defined as

$$\mu_k' = \int_{-\infty}^{\infty} t^k f(t)\,dt = \gamma^k + \mathrm{k}\int_{\gamma}^{\infty} t^{k-1} R(t)\,dt \tag{17}$$

In particular, when $k = 1$ and $\gamma = 0$, the *mean time to failure* (MTTF) is given by

$$\begin{aligned}\mathrm{MTTF} = \mu &= \int_{\gamma}^{\infty} t f(t)\,dt = -\int_{\gamma}^{\infty} t\frac{dR}{dt}\,dt \\ &= -tR(t)\,|_0^{\infty} + \int_{\gamma}^{\infty} R(t)\,dt = \int_0^{\infty} R(t)\,dt\end{aligned} \tag{18}$$

3. *Failure rate, hazard rate, and retired life*. For a period of length δ, the *failure rate*, $G(t, \delta)$, is defined as

$$G(t,\delta) = \frac{1}{\delta}\int_t^{t+\delta} \frac{f(x)}{R(t)}\,dx = \frac{F(t+\delta) - F(t)}{\delta R(t)} = \frac{R(t) - R(t+\delta)}{\delta R(t)} \tag{19}$$

The *hazard rate* or *instantaneous failure rate*, $h(t)$, is the limit of $G(t, \delta)$ as δ approaches zero:

$$h(t) = \frac{f(t)}{1 - F(t)} \tag{20}$$

The *retired life*, or *replacement life*, ξ, is defined for any specified h and given by

$$h = \frac{f(\xi)}{R(\xi)} \tag{21}$$

4. *Life expectancy*. Suppose that an item has survived until time T. Then the expected additional *life expectancy*, $L(t)$, is given by

$$L(t) = \begin{cases} \mu - T, & T \leq \mu \\ \dfrac{1}{R(t)} \displaystyle\int_t^\infty R(t)\, dt, & T > \mu \end{cases} \tag{22}$$

5. *Probable life*. The *probable life*, $B(T)$, is the total expected life of an item of age T:

$$B(T) = L(T) + T \tag{23}$$

1.2 HAZARD RATE FUNCTIONS IN RELIABILITY

Sometimes in selecting the distribution of failure times, one must use empirical data. For nonsymmetrical probability density functions the major difference between densities will be in the long tail. However, due to limited sample sizes, we have sparse data for this tail. As an alternative, we may appeal to physical considerations to select the function, or as it is commonly called, the *hazard rate*. The hazard rate or *hazard function* is also interpreted as the instantaneous failure rate. It is also known as the *force of mortality* in actuarial science and as the *intensity function* in statistical extreme value theory. Let $F(t)$ be the cumulative distribution function (c.d.f.) of the time-to-failure variable, T, and let $f(x)$ be the corresponding probability density function (p.d.f.). Consider the probability that the item fails in the interval $(t, t + \Delta t]$ given that it has survived until time t:

$$P(t < T \leq t + \Delta t) \mid T > t) = \frac{F(t + \Delta t) - F(t)}{1 - F(t)} \tag{24}$$

Dividing this probability by the length of the interval, Δt, gives a "per unit time" value. The hazard function or instantaneous failure rate is given by (Figure 1.4)

$$h(t) = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t[1 - F(t)]} = \frac{f(t)}{R(t)} \tag{25}$$

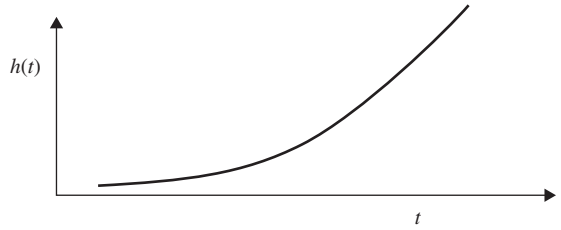


FIGURE 1.4 Hazard function.

by the definition of the derivative of $F(t)$ and since the reliability $R(t) = 1 - F(t)$. Now, on the basis of physical considerations we can choose a functional form for $h(t)$. Making use of the relationship (25) and that $f(t) = -R'(t)$, we can write

$$h(t)\,dt = \frac{1}{R(t)}\,dR(t) \tag{26}$$

Recalling that $R(0) = 1$, and integrating over the range $(0, t)$, we obtain

$$\int_0^t h(x)\,dx = -\int_1^{R(t)} \frac{1}{R(x)}\,dR(x) = -\ln R(t) \tag{27}$$

$$\ln R(t) = -\int_0^t h(x)\,dx \tag{28}$$

and finally, the general reliability equation,

$$R(t) = \exp\left[-\int_0^t h(x)\,dx\right] \tag{29}$$

There are three general types of hazard rates, as illustrated by the bathtub curve, detailed descriptions of which can be found in most textbooks on reliability (e.g., [3]). The first part of the curve represents initial failures and has a decreasing hazard rate. These failures correspond to infant mortalities, such as those caused by hereditary defects and childhood diseases. In reliability applications these failures are generally caused by poor workmanship (e.g., poorly soldered connections, nuts not tightened, untested equipment). The second part of the curve is the chance failure portion, usually represented by a constant hazard rate. Here failures are due to severe and unpredictable environmental conditions. For human mortality tales, this period would represent deaths by accidents or unusual diseases, for example. In reliability applications, failures occur because of unusual events such as shocks and sudden voltage surges. The final portion of the curve, which has an increasing hazard rate, corresponds to wear-out failures. In humans, these are failures due to heart diseases and deterioration of a body's organs. For physical processes these failures would be caused by wear, so that parts no longer fit, for example. Specifying a functional form for the hazard rate,

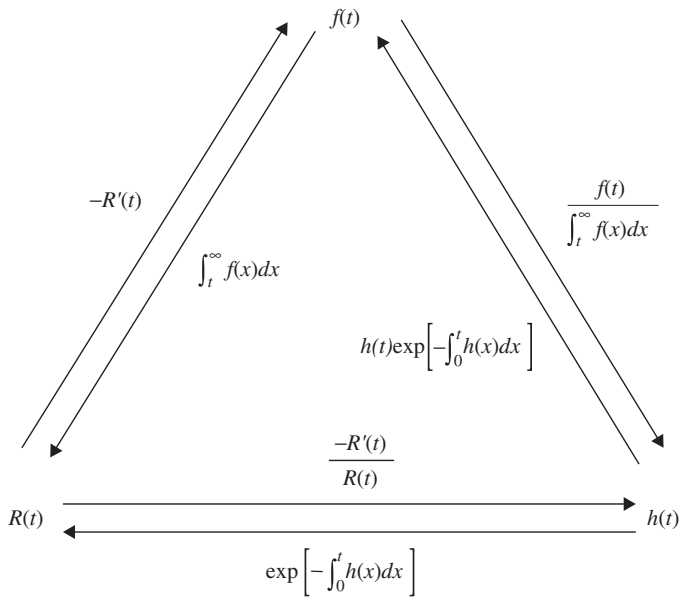


FIGURE 1.5 Reversible mathematical relationships among $f(t)$, $h(t)$, and $R(t)$.

we can, using equation (29), find the functional form of the reliability function. Figure 1.5 shows the relationships studied above [4].

1.3 COMMON DISTRIBUTIONS AND RANDOM NUMBER GENERATIONS

In this section we present some of the popular distributions used in reliability applications [5]. A summary of the distributions that receive most of the attention in reliability engineering is shown in Table 1.1. In the following sections we use the abbreviations p.d.f. for probability density function, p.m.f. for probability mass function, and c.d.f. for cumulative density function. The random number generations for each distribution for x (or t) will follow each [7,8].

1.3.1 Uniform (Rectangular) p.d.f

As the name implies, the uniform p.d.f. treats all values that have the same likelihood over the interval (a, b):

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases} \tag{30}$$

TABLE 1.1 Most Commonly Used Distributions and Reliability Functions[a]

Function	Exponential	Normal	Uniform	Weibull	Log Normal	Gamma
Density, $f(t)$	$\lambda \exp(-\lambda t)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2}$	$\frac{1}{b-a}$	$\frac{\beta t^{\beta-1}}{\alpha^\beta} e^{-t^\beta/\alpha^\beta}$	$\frac{1}{\sigma t\sqrt{2\pi}} e^{-(\ln t-\mu)^2/2\sigma^2}$	$\frac{t^{\alpha-1}e^{-t/\beta}}{\Gamma(\alpha)\beta^\alpha}$
Reliability, $R(t)$	$\exp(-\lambda t)$	$1-\Phi\left(\frac{t-\mu}{\sigma}\right)$	$\frac{b-t}{b-a}$	$e^{-t^\beta/\alpha^\beta}$	$1-\Phi\left(\frac{1}{w}\ln\frac{t}{t_0}\right)$	$\int_t^\infty f(t)\,dt$
Hazard, $h(t) = f(t)/R(t)$	Constant, λ	IFR	IFR	$\beta > 1$, IFR $\beta = 1$, CFR $\beta < 1$, DFR	Depends on $w = \exp(\sigma^2)$	$\alpha > 1$, IFR $\alpha = 1$, CFR $a < 1$, DFR
Mean	$\theta = \lambda^{-1}$	μ	$\frac{a+b}{2}$	$\alpha\Gamma\left(\frac{1}{\beta}+1\right)$	$\mu = \ln t_0 =$ $t_0 \exp\left(\frac{w^2}{2}\right)$	$\alpha\beta$
Variance	$\theta^2 = \lambda^{-2}$	σ^2	$\frac{(b-a)^2}{12}$	$\alpha^2\left[\Gamma\left(\frac{2}{\beta}+1\right) - \Gamma\left(\frac{1}{\beta}+1\right)^2\right]$	$t_0^2 \exp(w^2)\cdot$ $[\exp(w^2)-1]$	$\alpha\beta^2$
Mode, m	0	μ	None	$\alpha(1-\beta^{-1})^{1/\beta}$, $\beta > 1$	M/w	$(\alpha-1)\beta, \alpha > 1$
Median, $\int_0^M f(t)\,dt = 0.5$	$\theta \ln 2$	μ	$\frac{a+b}{2}$	$\alpha(\ln 2)^{1/\beta}$	$e^{-\left[(1/n)\sum_i^n \log x_i\right]}$	$\int_0^M f(t)\,dt = 0.5$
Range	$0 \le t \le \infty$	$-\infty \le t \le \infty$	$a \le t \le b$	$0 \le t \le \infty$	$0 \le t \le \infty$	$0 \le t \le \infty$
Notes	θ = rate	$M = m = \mu$, symmetric	Rectangular	α = scale parameter β = shape parameter	Geometric mean = t_0, Φ = normal c.d.f.	β = scale parameter α = shape parameter

Source: [2–6].

[a]IFR, increasing failure rate; CFR, constant failure rate; DFR, decreasing failure rate.

The c.d.f. of the uniform random variable X is

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases} \tag{31}$$

Note that median $= E(x) = (a+b)/2$ and $\text{Var}(x) = (b-a)^2/12$. No mode exists.

How to Generate Random Numbers from a Uniform p.d.f.

Using the inverse transform technique, $u = (x-a)/(b-a)$ yields $x^* = a + (b-a)u_i$, where $u_i \sim U(0,1)$ are generated through software or from a random number table such as Appendix 1A.

1.3.2 Triangular p.d.f.

A triangular p.d.f. models a process if only minimum (a), maximum (c), and mode (b) values are known, such as the times for life testing a product until defective units are detected. It has the p.d.f.

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)}, & a \le x \le b \\ \dfrac{2(c-x)}{(c-b)(c-a)}, & b \le x \le c \\ 0, & \text{otherwise} \end{cases} \tag{32}$$

$E(x) = (a+b+c)/3$, $\text{Var}(x) = [a(a-b) + c(c-a) + b(b-c)]/18$, and mode $= 3E(x) - (a+c)$. The c.d.f. is

$$F(x) = \begin{cases} 0, & x \le a \\ \dfrac{(x-a)^2}{(b-a)(c-a)}, & a \le x \le b \\ 1 - \dfrac{(c-x)^2}{(c-a)(c-b)}, & b \le x \le c \\ 1, & x \ge c \end{cases} \tag{33}$$

How to Generate Random Numbers from a Triangular p.d.f.

Generation of a triangular p.d.f. deviates by the inverse transform technique solving for $u = F(x)$:

$$x^* = \begin{cases} a + \sqrt{[(b-a)(c-a)u]}, & 0 \le u < \dfrac{b-a}{c-a} \\ c - \sqrt{[(c-a)(c-b)u]}, & \dfrac{b-a}{c-a} \le u \le 1 \end{cases} \tag{34}$$

Therefore, first generating u_i, check if it is between the limits 0 and $(b-a)/(c-a)$. If yes, use the first row in the random number equation above. Otherwise, use the second row.

1.3.3 Negative Exponential p.d.f., Pareto, and Power Functions

One of the most common distributions used in reliability is the *negative exponential distribution*. One characteristic that has contributed to its popularity is the ease with which the distributions of statistics under various sampling plans can be derived. It is also used widely in queuing theory, since the time between arrivals may be negative exponentially distributed. Hence, for the exponential distribution, a device that is T_0 units old but still operating has the same reliability as that of a new device. Obviously, this reliability function is not suitable for the infancy and wear-out portions of the bathtub hazard rate.

How to Generate Random Numbers from a Negative Exponential p.d.f.

Using the inverse transform technique, where $0 < u_i \sim U(0,1) < 1$ is the uniform random number,

$$u_i = F(x) = 1 - e^{-\lambda x}, \qquad 0 \le F(x) \le 1, \qquad \lambda > 0 \tag{35}$$

$$e^{-\lambda x} = 1 - u_i \tag{36}$$

$$\ln(e^{-\lambda x}) = -\lambda x = \ln(1 - u_i) \tag{37}$$

$$x^* = -\frac{\ln(1-u_i)}{\lambda} \quad \text{or} \quad x^* = -\frac{\ln(u_i)}{\lambda} \tag{38}$$

is the negative exponential random deviate, since if u_i is uniform, so is $1 - u_i$. Sometimes called *antithetical variables* in simulation studies, $1 - u_i$ are used to help reduce the sampling error.

If the time spent in each stage, X_1 and X_2, is independent and exponentially distributed with respective parameters λ_1 and λ_2, where $\lambda_1 \neq \lambda_2$, the overall time $Y = X_1 + X_2$ is hypoexponentially distributed [i.e., $Y \sim \text{Hypo}(\lambda_1, \lambda_2)$] with $f(t)$ and $R(t)$ given by

$$f(t) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}), \qquad t \ge 0 \tag{39}$$

$$R(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}, \qquad t \ge 0 \tag{40}$$

Similarly, for k sequential stages with unequal rates, and if $0 < \lambda_i$, $\sum_{i=1}^{k} p_i = 1$, and $0 < p_i < 1$, then $Y = X_1 + X_2 + \cdots + X_k$ becomes k-phase hyperexponentially distributed with the p.d.f.

$$f(t) = \sum_{i=1}^{k} p_i \lambda_i e^{-\lambda_i t}, \qquad t \geq 0 \tag{41}$$

$$F(t) = \sum_{i=1}^{k} p_i (1 - e^{-\lambda_i t}) = \sum_{i=1}^{k} p_i - \sum_{i=1}^{k} p_i e^{-\lambda_i t}$$

$$= 1 - \sum_{i=1}^{k} p_i e^{-\lambda_i t}, \qquad t \geq 0 \tag{42}$$

$$h(t) = \frac{\sum_{i=1}^{k} p_i \lambda_i e^{-\lambda_i t}}{\sum_{i=1}^{k} p_i e^{-\lambda_i t}}, \qquad t \geq 0 \tag{43}$$

$$R(t) = \frac{f(t)}{h(t)} = \sum_{i=1}^{k} p_i e^{-\lambda_i t} = 1 - F(t) \tag{44}$$

Note that $h(t)$ is a decreasing failure rate. The hyperexponential distribution has more variability than the negative exponential distribution. Therefore, if the product is manufactured in several locations or stages, the failure density of the overall product is hyperexponential, known to be a mixture distribution. For the special case of $k = 2$, $Y \sim \text{Hyper}(\lambda_1, \lambda_2)$, $f(t)$ and $h(t)$ are

$$f(t) = p_1 \lambda_1 e^{-\lambda_1 t} + p_2 \lambda_2 e^{-\lambda_2 t} \tag{45}$$

$$h(t) = \frac{p_1 \lambda_1 e^{-\lambda_1 t} + p_2 \lambda_2 e^{-\lambda_2 t}}{p_1 e^{-\lambda_1 t} + p_2 e^{-\lambda_2 t}} \tag{46}$$

If $\lambda_1 = \lambda_2$, then $Y \sim$ two-stage Erlang ($k = 2$) (see Section 1.3.4). On the other hand, the *double exponential distribution*, also called as the *Pareto* or *hyperbolic* or *power-law distribution*, used to model the amount of central processing unit (CPU) time by an arbitrary process or the thinking time of a Web browser, to cite a few examples, has the following values:

$$f(x) = cx^{-c-1}, \qquad 1 \leq x \leq \infty, \quad c > 0 \tag{47}$$

$$F(x) = 1 - x^{-c} \tag{48}$$

$$R(x) = x^{-c}, \quad c > 0 \quad \text{and} \quad R(t) = 0, \quad c < 0 \tag{49}$$

$$E(x) = \frac{c}{c-1}, \qquad c > 1 \tag{50}$$

$$\text{Var}(x) = \frac{c}{c-2} - \left(\frac{c}{c-1}\right)^2, \qquad c > 2 \tag{51}$$

$$h(t) = \frac{c}{x} \tag{52}$$

A function very similar to the Pareto is the power function, where $f(x) = cx^{c-1}$, the power of x being positive c, not negative. The mode is 1 for $c > 1$ and 0 for $c < 1$.

$$F(x) = x^c \tag{53}$$

$$R(x) = 1 - x^c \tag{54}$$

$$M(\text{median}) = (0.5)^{1/c} \tag{55}$$

$$E(x) = \frac{c}{c+1} \tag{56}$$

$$\text{Var}(x) = \frac{c}{(c+2)(c+1)^2} \tag{57}$$

How to Generate Random Numbers from the Pareto and Power Functions

By the inverse transform technique, $U(0, 1) = F(x)$, that is, $1 - x^{-c} = u$, $1 - u = x^{-c}$, $(1 - u)^{-1} = x^c$, $(1 - u)^{-1/c} = x$, and therefore for the Pareto function,

$$x^* = \left(\frac{1}{1-u_i}\right)^{1/c} \tag{58}$$

Again using the inverse transform technique, $x^c = u$, and then taking the $(1/c)$th root of both sides, we obtain for the power function distribution

$$x^* = (u_i)^{1/c} \tag{59}$$

1.3.4 Gamma, Erlang, and Chi-Square p.d.f.'s

The general form of the *gamma distribution* is

$$f(t) = \begin{cases} \dfrac{t^{\alpha-1}e^{-t/\beta}}{\beta^a \Gamma(\alpha)}, & t \geq 0, \quad \alpha, \beta \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{60}$$

When α is an integer, this distribution is also known as the *Erlangian distribution*, and when $\alpha = 1$, the gamma density reduces to an exponential distribution with $\beta = \lambda^{-1}$. For example, a computer network fails when a mainframe computer

and two backup servers fail, and each may have a time to failure that is negative exponentially distributed.

The Erlang distribution may be obtained as the distribution of the sum of α many independent identically distributed (i.i.d.) exponential random variables. Suppose that the failure of a device occurs when the kth shock arrives if a Poisson process with a parameter λ generates the shocks. Let the random variable T denote the arrival time of the kth shock. Then $T = \sum_{i=1}^{k} T_i$, where T_i is the time between the $(i-1)$st shock and the ith shock. Then, from the property given above,

$$f(t) = \begin{cases} \dfrac{\lambda^k t^{k-1} e^{-\lambda t}}{\Gamma(k)}, & t \geq 0, \quad \Gamma(k) = (k-1)! \\ 0, & \text{otherwise} \end{cases} \tag{61}$$

$$R(t) = \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \qquad t \geq 0, \quad \lambda > 0 \tag{62}$$

This is also called the *k-stage Erlang* and is the same formula as equation (60) with $\beta = \lambda^{-1}$ and $\alpha = k$.

How to Generate Random Numbers from the Erlang Distribution

To generate an Erlang random deviate, we take advantage of the fact that the negative exponential is a special case of the Erlang distribution with the shape parameter $\alpha = 1$. Therefore, we can generate an Erlang deviate by summing α negative exponential random deviates with mean $\beta = \lambda^{-1}$ as follows:

$$t^* = \sum_{k=1}^{\alpha} \frac{-\ln(1-u_i)}{\lambda} \tag{63}$$

Another use of the gamma distribution is for a parallel standby system with n identical devices. In this type of system, the system fails when all devices fail, and the system operates when only one device is operating at a time. Then the time to system failure, or the sum of the n failure times, has a gamma distribution with $\beta = \lambda^{-1}$ and $\alpha = n$. If $X \sim \text{Gamma}(\alpha, \beta)$, $\sum_{i=1}^{m} X_i$ is distributed as Gamma$(m\alpha, \beta)$.

How to Generate Random Numbers from the Gamma Distribution (Johnk's Rejection Technique)

Let α be a noninteger shape parameter, $\alpha_1 = [\alpha]$, the largest truncated integer of α, and u_i be the ith uniform random number, $0 \leq u_i \leq 1$. Then:

1. Let $x = -\ln \prod_{i=1}^{\alpha_1} u_i$.
2. Set $A = \alpha - \alpha_1$, $B = 1 - A$.

a. Set $j = 1$.
b. Generate random number u_j and set $y_1 = (u_j)^{1/A}$.
c. Generate random number u_{j+1} and set $y_2 = (u_{j+1})^{1/B}$.
d. If $y_1 + y_2 \leq 1$, go to f.
e. Set $j = j + 2$ and go to b.
f. Let $z = y_1/(y_1 + y_2)$, which is a beta random deviate with parameters A and B.

3. Generate the random number u_N and set $w = -\ln(u_N)$.
4. The random deviate desired for a gamma p.d.f. is then $G = (x + zw)\beta$.

If $X \sim \text{Gamma}(\alpha = 0.5, \beta = n/2)$, X is said to have a *chi-square distribution* with n degrees of freedom. Then $E(x) = n$, $\text{Var}(x) = 2n$, and mode $= n - 2$, $n \geq 2$ [10].

How to Generate Random Numbers from the Chi-Square Distribution

A chi-square random variable with n degrees of freedom is the sum of squares of n independent normally distributed random variables with $\mu = 0$ and $\sigma^2 = 1$, that is, $\sum_{i=1}^{n} [N(0, 1)]^2$.

Case 1. For n even:

$$x^* = -0.5 \ln \left(\prod_{i=1}^{n/2} u_i \right) \tag{64}$$

Case 2. For n odd:

$$x^* = -0.5 \ln \left(\prod_{i=1}^{(n-1)/2} u_i \right) + \sum_{i=1}^{n} [N(0, 1)]^2 \tag{65}$$

Another important analysis useful in standby redundancy is having two identical components, X and Y, each with a negative exponential time to failure with parameter λ. Only one component is required to be working for the system to operate. The second, spare component is a "cold standby," inactive unless called for. Then $Z = X + Y$ has a gamma density using the convolution formula, where $f(z) = \lambda^2 z e^{-\lambda z}$, $z > 0$. Hence, $Z \sim \text{Gamma}(\alpha = 2, \beta = \lambda^{-1})$, which is a two-stage Erlang distribution, whose reliability function is given by equation (41) as

$$R(t) = (1 + \lambda t)\mathrm{e}^{-\lambda t}, \qquad t > 0 \tag{66}$$

For the sake of comparison of simplex active parallel and standby parallel, the reliability values for any given λ are larger (favorable) for the standby than for the simplex. As a consequence of this theorem, the reliability expression for a standby redundant system with a total of n components each of which has a negative exponentially distributed lifetime is shown as in (62) by

$$R_{\text{standby}}(t) = \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \qquad t \geq 0, \quad \lambda > 0 \tag{67}$$

1.3.5 Student's t-Distribution

Let X be $N(0,1)$ and Y be χ_ν^2, independent of each other. Then set $T = X/\sqrt{Y/\nu}$, which is defined as Student's t distributed with ν degrees of freedom (d.f.) and denoted by t_ν, which has the p.d.f.

$$f_T(t) = \frac{\Gamma[\left(\frac{1}{2}\right)(\nu+1)][1 + (\nu^2/r)]^{-(1/2)(\nu+1)}}{\sqrt{\pi\nu}\Gamma(\nu/2)}, \qquad -\infty < t < \infty \tag{68}$$

Student's t, which is popularly used in confidence intervals and hypothesis testing for small samples ($n < 15$) with an unknown variance (σ) is symmetric. Mode = median = $\mu = 0$.

$$\text{Var}(T) = \frac{\nu}{\nu - 2}, \qquad \nu > 2 \tag{69}$$

How to Generate Random Numbers from the t-Distribution

Provided that a source of random deviates from a chi-square distribution (χ_ν^2) with ν degrees of freedom, one generates a t-deviate with $\nu - 1$ degrees of freedom as follows:

$$x_{\nu-1}^* = \frac{N(0,1)}{\sqrt{\chi_\nu^2/\nu}} \tag{70}$$

1.3.6 Fisher's F-Distribution

Let X be $\chi_{\nu_1}^2$ and Y be $\chi_{\nu_2}^2$ independently, and set $F_{\alpha,\nu_1,\nu_2} = \chi_{\nu_1}^2 \nu_2 / \chi_{\nu_2}^2 \nu_1$. The random variable F_α^{-1} has a p.d.f. with ν_2 and ν_1 degrees of freedom interchanged compared to the original at a level of $1 - \alpha$ (i.e., $F_{1-\alpha,\nu_2,\nu_1}$):

$$f_F(f) = \frac{\Gamma[(1/2)(\nu_1+\nu_2)](\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \times \frac{f^{(\nu_1/2)-1}}{[1 + (\nu_1/\nu_2)f]^{(1/2)(\nu_1+\nu_2)}}, \quad f > 0 \quad \text{and} \quad f_F(f) = 0, \quad f = 0 \tag{71}$$

$$E(F) = \frac{\nu_2}{\nu_2 - 2}, \qquad \nu_2 > 2 \tag{72}$$

$$\text{mode} = \frac{\nu_2(\nu_1 - 2)}{\nu_1(\nu_2 + 2)}, \qquad \nu_1 > 1 \tag{73}$$

$$\text{Var}(F) = \frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)}, \qquad \nu_2 > 4 \tag{74}$$

The F-distribution is in common use in statistical estimation theory for comparing variances in ANOVA (analysis of variance) to compare group means. Note that $(1/F_{\alpha,\nu_1,\nu_2}) \sim F_{1-\alpha,\nu_2,\nu_1}$ [10].

How to Generate Random Numbers from an F-Distribution

Similarly, given two sources of random deviates from a chi-square distribution (χ^2_ν) with ν_1 and ν_2 degrees of freedom, one generates an F-deviate as follows:

$$x^*_{\nu_1,\nu_2} = \frac{\chi^2_{\nu_1}\nu_2}{\chi^2_{\nu_2}\nu_1} \tag{75}$$

1.3.7 Two- and Three-Parameter (Sahinoglu–Libby) Beta p.d.f.'s

The *beta distribution* is quite useful in the theory of statistics and has a wide variety of applications in applied engineering and quality control problems when the random variable of interest varies between 0 and 1. This p.d.f. is very flexible and is used to model bounded random variables with fixed upper and lower limits. This variable can be shifted away from zero by adding a constant. It can also have a larger range than 0 to 1 by scaling it with a constant larger than 1. It is used in forming confidence intervals for tolerance limits in distribution functions. It often occurs as both a prior and a posterior p.d.f. in the Bayes estimation, where it is treated as a conjugate for binomial density. It is useful to establish a relationship between the gamma and beta distributions. (See Figure 1.6 on page 32.) If a random variable y is distributed as Gamma(α_1, β) and z is distributed as Gamma(α_2, β), then $x = y/(y + z)$ is distributed as Beta(A, B), where it is expressed as [10]

$$E(x) = \frac{A}{A + B} \tag{76}$$

$$\text{Var}(x) = \frac{AB}{(A + B + 1)(A + B)^2} \tag{77}$$

$$\text{mode} = \frac{A - 1}{A + B - 2}, \qquad A > 1, \qquad B > 1 \tag{78}$$

$$f(x) = \begin{cases} \dfrac{\Gamma(A+B)}{\Gamma(A)\Gamma(B)} x^{A-1}(1-x)^{B-1} \\ \quad = \dfrac{(A+B-1)!}{(A-1)!(B-1)!} x^{A-1}(1-x)^{B-1}, \\ \qquad\qquad 0 \le x \le 1, \qquad A \text{ and } B \text{ integers} \qquad (79) \\ \dfrac{1}{B(A,B)} x^{A-1}(1-x)^{B-1}, \qquad A \text{ and } B \text{ nonintegers} \qquad (80) \end{cases}$$

where for A and B nonintegers, $B(A, B)$ is the beta function:

$$B(A, B) = \frac{\Gamma(A+B)}{\Gamma(A)\Gamma(B)} = \int_0^1 u^{A-1}(1-u)^{B-1} du \qquad (81)$$

How to Generate Random Numbers from the Beta Distribution

Assume random variables, $y \sim$ Gamma$(A, 1)$ and $z \sim$ Gamma$(B, 1)$; then $y = -\ln \prod_{i=1}^{A} u_i$ and $z = -\ln \prod_{i=A+1}^{A+B} u_i$, so $x^* = y/(y+z)$ is the Beta(A, B) deviate, where u_i is a number from a uniform random generator and A and B are integers. For A and B nonintegers, use the algorithm to obtain the beta deviate in item 2f given in "*How to Generate Random Numbers from the Gamma Distribution"* in Section 1.3.4.

For the three-parameter beta, otherwise known as the Sahinoglu–Libby p.d.f. for the FOR [forced outage rate = uptime/(uptime + downtime) = (failure rate)/(failure rate + repair rate) = $\lambda/(\lambda+\mu)$] with a Bayesian approach. See Chapter 5 and Appendix 5A for the derivation [11]. Let

a = number of occurrences of operative (up) times sampled

x_T = total sampled uptime for a up occurrences

b = number of occurrences of inoperative (down) times sampled

y_T = total sampled downtime for b down occurrences

c = shape parameter of a gamma prior for component failure rate λ

ξ = inverse scale parameter of a gamma prior for component failure rate λ

d = shape parameter of a gamma prior for component recovery rate μ

η = inverse scale parameter of a gamma prior for component recovery rate μ

In using the distribution function technique, the p.d.f. of FOR, denoted as $q = \lambda/(\lambda+\mu)$, is obtained first by deriving its c.d.f. $G_Q(q) = P(Q \le q) =$

$P(\lambda/(\lambda+\mu) \le q)$ and then taking the derivative of $g_Q(q)$ for $0 \le q \le 1$.

$$g_Q(q) = \frac{\Gamma(a+b+c+d)}{\Gamma(a+c)\Gamma(b+d)} \frac{(\xi+x_T)^{a+c}(\eta+y_T)^{b+d}(1-q)^{b+d-1}q^{a+c-1}}{[\eta+y_T+q(\xi+x_T-\eta-y_T)]^{a+b+c+d}}$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} (1-q)^{\beta-1}q^{a-1}\left[\frac{1}{1+q(L-1)}\right]^{\alpha+\beta} L^{\alpha} \tag{82}$$

where $\alpha = a+c$, $\beta = b+d$, $\beta_1 = \xi + x_T$, and $\beta_2 = \eta + y_T$. If $L = \beta_1/\beta_2 = 1$ or $\beta_1 = \beta_2$, the conventional two-parameter beta p.d.f. is obtained. An alternative expression is

$$g_Q(q) = \frac{L^{\alpha+c}q^{\alpha+c-1}(1-q)^{b+d-1}}{B(b+d,a+c)[1-(1-L)q]^{a+b+c+d}} \tag{83}$$

where

$$B(b+d,a+c) = \frac{\Gamma(a+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \quad \text{and} \quad L = \frac{\xi+x_T}{\eta+y_T} \tag{84}$$

How to Generate Random Numbers from the Sahinoglu–Libby p.d.f.

Assume the random variables $y \sim \text{Gamma}(\alpha_1 = a+c, \beta_1 = \xi + x_T)$ and $z \sim \text{Gamma}(\alpha_2 = b+d, \beta_2 = \eta + y_T)$, where the random variable $q = y/(y+z)$ has the p.d.f.

$$g_Q(q) = \frac{\Gamma(m'+n')}{\Gamma(m')\Gamma(n')} a'^{m'} b'^{n'} \frac{(1-q)^{m'-1}q^{n'-1}}{[a'+q'(b'-a')]^{m'+n'}} \tag{85}$$

and the c.d.f.

$$G_Q(q) = 1 - G_{F_{2m',2n'}}\left[\frac{a'n'}{b'm'}q^{-1}-1)\right]$$
$$= P\left[F_{2m',2n'} > C_1 = \frac{a'n'}{b'm'}(q^{-1}-1)\right] \tag{86}$$

Resubstituting for $n' = a+c$, $m' = b+d$, $b' = \xi + x_T$, and $a' = \eta + y_T$, we obtain for equation (85)

$$g_Q(q) = \frac{\Gamma(a+b+c+d)}{\Gamma(a+c)\Gamma(b+d)}(\eta+y_T)^{b+d}(\xi+x_T)^{a+c}$$
$$\times \frac{(1-q)^{b+d-1}q^{a+c-1}}{[\eta+y_T+q'(\xi+x_T-\eta-y_T)]^{a+b+c+d}} \tag{87}$$

where Snedecor's F-distribution in equation (86) has been given by equation (71). By the inverse transform approach, find the constant $C_1 =$ inverse of $F_{2m',2n'}(1-u_i)$ as in equation (86):

$$C_1 = \frac{a'n'}{b'm'}(q^{-1}-1) \rightarrow q^* = \frac{a'n'}{a'n' + C_1 b'm'}, 0 < q^* < 1 \tag{88}$$

is the SL($\alpha = a + c$, $\beta = b + d$, $L = \beta_1/\beta_2$) random deviate, where u_i is a random uniform.

1.3.8 Poisson p.m.f.

The p.m.f. for a Poisson-distributed X is given by

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0, 1, 2, 3, \ldots, \infty \quad \text{and} \quad \lambda > 0 \tag{89}$$

The Poisson distribution is used to approximate the binomial distribution by allowing the number of trials $n \rightarrow \infty$ for $p \gg 0$. This function is used to describe the number of events to occur within some designated period of time, such as in inventory and quality control, and queuing models, where arrival rates are often considered to be Poisson distributed. The expected value of the Poisson random variable is $E(X) = \lambda$. Its variance is $\text{Var}(X) = \lambda$. Note that $\text{Var}(X) = E(X)$. The Poisson rate λ is defined as the number of occurrences expected per unit time. Then the time between arrivals is negative exponentially distributed with mean $\theta = \lambda^{-1}$. If $\text{Var}(X) > E(X)$, it is a compound Poisson distribution [12–14].

How to Generate Random Numbers from a Poisson Distribution

The relationship between the negative exponential and Poisson distributions can be used to generate Poisson random numbers. A Poisson x can be defined as

$$\sum_{i=1}^{x} y_i \leq 1 \leq \sum_{i=1}^{x+1} y_i \tag{90}$$

where, using the inverse transform method, $y_i = \lambda^{-1} \ln u_i = \theta \ln u_i$ are variates from a negative exponential distribution of mean λ^{-1}. Cumulative sums of y_i for $i = 1, \ldots, k$ are generated until the inequality (90) holds. Then $x^* = x$ is the Poisson deviate that terminates the summation and satisfies (90).

1.3.9 Bernoulli, Binomial, and Multinomial p.m.f.'s

A discrete random variable $Y = \{0, 1\}$ is distributed as a *Bernoulli p.m.f.* if failure probability $P(Y = 0) = q$ and success probability $P(Y = 1) = p$ add to unity

(i.e., $p + q = 1$). A sequence of n identical and independent Bernoulli trials, where $X = \sum_{i=1}^{n} Y_i$, will result in the $Binomial(X; n, p)$ p.m.f. The expected value for the binomial random variable is $E(X) = np$. Its variance is $\text{Var}(X) = npq$, where $\text{Var}(X) < E(X)$. For nonindependent (first-order Markov-dependent) Bernoulli random variables in a sequence, the limit of the sum leads to a *compound Poisson*, useful in systems such as electric power or computer networks [15–17]:

$$B(X = k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k}, \qquad 0 \le k \le n, \quad p > 0, \quad p + q = 1 \tag{91}$$

A similar method for generating random *multinomial* vectors involves an extension of the binomial. Suppose, for example, that for a quadrinomial p.m.f., the line segment is drawn as

$$0|_____p_1_____|_____p_2_____|_____p_3_____|_____p_4_____|1 \tag{92}$$

where $0 < u_i < p_1$, $p_1 < u_i < p_1 + p_2$, and in general, $\sum_{m=1}^{j-1} p_m < u_i < \sum_{m=1}^{j} p_m$, $p_0 = 0$, $\sum_{m=1}^{k} p_m = 1$, where $m = 4$:

$$M\left(X_1 = k_1, X_2 = k_2, \ldots, X_m = k_m; n, \sum_{m=1}^{k} p_m = 1\right)$$
$$= \binom{n}{k_1, k_2, k_3} p_1^{k_1} p_2^{k_2} p_3^{k_3} (1 - p_1 - p_2 - p_3)^{n-k_1-k_2-k_3} \tag{93}$$

How to Generate Random Numbers from the Bernoulli, Binomial, and Multinomial Distributions

Generate a uniform number u_i, and if $0 < u_i < p$, it is a Bernoulli success; otherwise, it is a failure (i.e., $p \le u_i < 1$). Next, to generate a binomial random sample of $x^* = k$ successes in n trials, draw n random uniform numbers and count those less than the p that is given. Moreover, to generate a multinomial random sample of $x_m^* = k_m$ successes in n trials for a given m, draw n random uniform numbers and count the numbers falling in each of the m classes, yielding to $n_1, n_2, \ldots, n_m$. It would be n_1, n_2, n_3, n_4 as above in the example of $m = 4$ classes.

1.3.10 Geometric p.m.f.

The *geometric p.m.f.* of X is defined as the number of failures in a sequence of Bernoulli trials until the first success occurs. Geometric p.m.f. is often regarded as the only discrete equivalent of the continuous negative exponential distribution with the memoryless or forgetfulness property. Letting $p + q = 1$, the geometric

p.m.f. can be denoted in one of two ways, either (94) or (95):

$$P(x) = \begin{cases} pq^x, & x = 0, 1, 2, 3, \ldots \quad (94) \\ pq^{x-1}, & x = 1, 2, 3, \ldots \quad (95) \end{cases}$$

$$E(x) = \frac{q}{p}, \qquad \mathrm{Var}(x) = \frac{q}{p^2} \quad (96)$$

How to Generate Random Numbers from a Geometric p.m.f.

Employing the inverse transform method, x^* is the desired geometric deviate for $P(X = x) = pq^{x-1}$, $x = 1, 2, 3, \ldots$, where $u_i \sim U(0,1)$:

$$x_i^* = \frac{\ln u_i}{\ln q} \quad (97)$$

For the other alternative, $P(X = x) = pq^x$, $x = 0, 1, 2, 3, \ldots$, it is slightly different (round up X_i^* to the next larger integer):

$$x_i^* = \frac{\ln u_i}{\ln q} - 1 \quad (98)$$

1.3.11 Negative Binomial and Pascal p.m.f.'s

If we observe the number of trials until the kth success occurs, X is the *negative binomial* (NB) *random variable*, which is when we stop. If we now observe the number of trials up to and including when the kth success occurs, X is the *Pascal random variable*. The NB random variable models the number of trials required to achieve $k - 1$ successes: for example, the number of integrated circuits used in computer hardware to find $k = 5$ defective chips. The expected value of the NB random variable is $E(X) = nqp^{-1}$. Its variance $\mathrm{Var}(X) = nqp^{-2}$. Note that $\mathrm{Var}(X) > E(X)$. To calculate, observe $n - 1$ trials with $k - 1$ successes. At the nth trial, the long-awaited kth success occurs and the experiment is terminated. Then X is the number of failures before the occurrence of the kth success for the NB random variable.

$$\mathrm{NB}(X = k; n, p) = \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k}, \quad 0 \le k \le n, \quad p > 0, \quad p + q = 1 \quad (99)$$

$$\mathrm{Pascal}(X = k; n, p) = p\binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} = \binom{n+k-1}{k-1} p^k (1-p)^{n-k}, \quad 0 \le k \le n, \quad p > 0, \quad p + q = 1 \quad (100)$$

When $k = 1$ is the special case, the NB p.m.f. above reduces to the geometric p.m.f. for the first success:

$$\text{NB}(k = 1; n, p) = \binom{n}{0} pq^{n-1}, \qquad \text{where}$$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1, \qquad n = 1, 2, 3, \ldots \tag{101}$$

How to Generate Random Numbers from the Negative Binomial and Pascal p.m.f.'s

One way to generate random numbers from the negative binomial and Pascal p.m.f.'s is to generate a sequence of Bernoulli random numbers, as in the case of the binomial. Draw n random uniform numbers and record those less or more than the p given. When a number greater than p reaches the prescribed number of k successes, stop and identify that number as the NB random deviate. Since NB is the sum of $k - 1$ geometric random variates, an alternative way to generate an NB random number is

$$x_i^* = \sum_{i=1}^{k} \left(\frac{\ln u_i}{\ln q} - 1 \right) = \sum_{i=1}^{k} \frac{\ln u_i}{\ln q} - k \tag{102}$$

where we round up to the next-larger integer. For a Pascal random deviate, simply sum k geometric deviates, $x_i^* = \sum_{i=1}^{k} (\ln u_i / \ln q)$, to round up to the next-larger integer.

1.3.12 Weibull p.d.f.

In recent years the *Weibull distribution* has become more popular as a reliability function. It is named after the Swedish scientist Waloddi Weibull, who used it to analyze the breaking strength of solids. A chief advantage of the Weibull distribution is that as in the bathtub curve, its hazard rate function may be decreasing for $\beta < 1$, constant for $\beta = 1$, or increasing for $\beta > 1$. When $\beta = 2$, the Weibull is called the *Rayleigh p.d.f.* The hazard rate function is given by

$$h(t) = \frac{\beta t^{\beta-1}}{\alpha^{\beta}}, \qquad t \geq 0, \quad \alpha, \beta > 0 \tag{103}$$

The Weibull density and reliability functions are, respectively,

$$f(t) = \frac{\beta t^{\beta-1}}{\alpha^{\beta}} e^{-(t/\alpha)^{\beta}}, \qquad t \geq 0, \quad \alpha, \beta > 0 \tag{104}$$

$$R(t) = e^{-(t/\alpha)^{\beta}}, \qquad t \geq 0, \quad \alpha, \beta > 0 \tag{105}$$

The Weibull family of distributions is a member of the family of extreme value distributions discussed later.

The Weibull distribution is probably the most widely used family of failure (e.g., electronic component, mechanical fatigue) distributions, mainly because by

proper choice of its shape parameter β, it can be used as an IFR (increasing failure rate), DFR (decreasing failure rate), or CFR (constant failure rate, as in the negative exponential case). Often, a third parameter, known as the *threshold* or *location parameter*, t_0, is added to obtain a three-parameter Weibull, where

$$R(t) = e^{-(t-t_o/\alpha)^{\beta}}, \qquad t \geq t_0 > 0, \quad \alpha, \beta > 0 \tag{106}$$

How to Generate Random Numbers Using the Weibull Distribution

Employing the inverse transform method and solving the equation $u = F(x) = e^{(-x/\alpha)^{\beta}}$ for x, we obtain $u = 1 - e^{(-x/\alpha)^{\beta}}$, and the random deviate for the Weibull p.d.f.

$$x_i^* = \beta[-\ln(1-u_i)]^{1/\alpha}, \qquad i = 1, 2, \ldots \tag{107}$$

As in the bathtub curve explained in Section 1.2, a component or, equally, a system may possess three modes of failure, based on its decreasing, constant, and increasing failure rates. Again, the bathtub curve of an individual unit may be assumed to denote the sum of Weibull p.d.f.'s, as in the overall integral for the overall reliability:

$$h(t) = \int_0^t \lambda(t')\,dt' = \left(\frac{t}{\alpha_1}\right)^{\beta_1} + \left(\frac{t}{\alpha_2}\right)^{\beta_2} + \left(\frac{t}{\alpha_3}\right)^{\beta_3} \tag{108}$$

$$R(t) = \exp\left[-\int_0^t h(t)\,dt\right] = \exp\left[-\left(\frac{t}{\alpha_1}\right)^{\beta_1}\right] + \exp\left[-\left(\frac{t}{\alpha_2}\right)^{\beta_2}\right] + \exp\left[-\left(\frac{t}{\alpha_3}\right)^{\beta_3}\right] \tag{109}$$

where, $\beta_1 < 1$ refers to the infancy or commissioning period, $\beta_2 = 1$ corresponds to the constant failure or useful life period, and $\beta_3 > 1$ symbolizes the wear-out period. For such a system, preventive maintenance practices must be chosen carefully for those intervals where the wear-out period is in effect. Preventive maintenance therefore decreases reliability adversely in the infancy period; that is, $\beta_1 < 1$ is in effect. Needless to say, it has no effect on the useful life period. Aside from preventive maintenance practices to increase instantaneous reliability, corrective maintenance plays a significant role in reducing the number of failures and the time required to make repairs. Then the notion of *availability*, $A(t)$, the proportion of time that a system or component is in an operational state, proves useful. *Maintainability* is a degree or measure of how quickly a system can be brought back into a repaired state following the failures experienced. If a component or system cannot be repaired, its availability is equal to its reliability

at that point. Therefore, in general terms,

$$A(t) = \frac{1}{T}\int_0^T R(t)\,dt \tag{110}$$

Hence, as the period T increases to infinity, the numerator integral approaches the MTTF. However, the denominator becomes infinite, and the long-run availability of a nonrepairable component or system becomes zero, since all fail with no repair action, yielding an average availability of zero. This case is shown as

$$A(\infty) = \frac{\text{MTTF}}{\text{MTTF} + \text{MTTR}} = 0 \tag{111}$$

where MTTR is the mean time to repair ($\cong \infty$).

1.3.13 Normal p.d.f.

The *normal distribution* is sometimes used as the wear-out density function. During the wear-out phase of the bathtub hazard curve, component life follows a normal distribution. It should be familiar to anyone who has studied statistical procedures. This p.d.f. models the distribution of a process as the sum of a large number of processes, although it cannot be used for negative times. The density function of the random variable of the time to failure, T, is given as

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right], \qquad -\infty < t < \infty, \quad \mu, \sigma > 0 \tag{112}$$

If we denote the standard normal distribution of $z = (t-\mu)/\sigma$ with $\mu = 0$ and $\sigma = 1$ by

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right), \qquad -\infty < z < \infty \tag{113}$$

and its c.d.f. by

$$\Phi(z) = \int_{-\infty}^{z} \phi(u)\,du \tag{114}$$

its reliability function is given by

$$R(t) = \int_t^\infty f(u)\,du = \int_z^\infty \phi(z)dz = 1 - \Phi(z),$$
$$z = \frac{t-\mu}{\sigma}, \quad \mu = 0, \quad \sigma = 1 \tag{115}$$

One difficulty with the normal distribution as a reliability function is that it allows for negative values of the random variable. If $\Phi(-\mu/\sigma)$ is negligible, this causes no trouble. If it cannot be ignored, the truncated distribution

$$f(t) = \left[\Phi\left(\frac{-\mu}{\sigma}\right)\right]^{-1}\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right], \qquad t \geq 0, \quad \mu, \sigma > 0 \tag{116}$$

should be used. Finally, by using the central limit theorem (CLT), the mean of a sample of n mutually independent random variables with a finite mean and variance is normally distributed asymptotically as $n \to \infty$. Measurement errors often have this distribution, as in the case of all the (positive and negative) deviations from Greenwich–London time for all the clocks around the globe. The CLT also works for a sequence of nonindependent and nonidentical variables in a system given certain statistical assumptions [17–22].

How to Generate Random Numbers from a Normal Distribution

Method 1: Law of Large Numbers An earlier method of generating exact standard normal $N(0, 1)$ deviates that takes advantage of the law of large numbers is to generate a random sample of N uniform deviates and then calculate:

$$\overline{X} = \sum_{i=1}^{n} \frac{u_i}{n} \tag{117}$$

For large n, the distribution of $\overline{X}$ (= average of X) approaches normality by the CLT. Actually, this happens for as small a value as $n = 12$; and it gets better with increasing n. Since $U(0, 1)$ variables have a mean of 0.5 and variance of $\frac{1}{12}$, we can reformulate equation (117) to obtain the mean and variance by taking $n = 12$ to get $\overline{X} \sim N(0, 1)$:

$$\overline{X} = \sqrt{\frac{12}{n}} \sum_{i=1}^{n} (u_i - 0.5) = \sum_{i=1}^{12} u_i - 6 \tag{118}$$

This method of generating $N(0, 1)$ is faster, but the characteristics of the statistical distribution generated are not quite as good as those of the next method.

Method 2: Mathematical Derivation Technique This method, also called the *Box–Muller method* (1958), combines the inverse transformation technique and the polar coordinates method. It generates two random deviates from the standard normal distribution upon generating a pair of uniform random numbers (u_1, u_2):

$$X_1 = \sqrt{(-2 \ln u_2)} \sin 2\pi u_1 \tag{119}$$

$$X_2 = \sqrt{(-2 \ln u_2)} \cos 2\pi u_1 \tag{120}$$

1.3.14 Lognormal p.d.f.

A distribution useful in maintainability and certain fracture problems is the *lognormal distribution*, which models the distribution of a process that can be considered as the product (as compared to normal distribution, which goes for the

sum) of a number of processes. For example, the rate of return on a compound-interest investment is the product of the returns for a given number of periods. It is used primarily for the wear-out region of the bathtub curve, where the wear on a system may be proportional to the product of the magnitudes of the demands that have been exerted on it. If the random variable $T = t_1 t_2 t_3 \ldots t_n$ has the lognormal distribution (whereas previously $T = \sum t_i$ was, by CLT, normally distributed), the variable $y = \ln T = \sum \ln t_i$ is normally distributed. If the variables x_1 and x_2 have lognormal distributions, the product random variable $q = x_1 x_2$ is also lognormally distributed. The p.d.f. of T is

$$f(t) = \frac{1}{\sigma t \sqrt{2\pi}} \exp\left[-\frac{(\ln t - \mu)^2}{2\sigma_x^2}\right], \qquad t \geq 0, \quad \mu, \sigma > 0 \tag{121}$$

If we let $\mu = \ln t_0$, and $\sigma_x = w$, then

$$f(t) = \frac{1}{wt\sqrt{2\pi}} \exp\left[-\frac{\ln(t/t_0)^2}{2w^2}\right], \quad t \geq 0, \quad \text{and} \quad t_0, w > 0 \tag{122}$$

The corresponding c.d.f. is obtained by integrating over t with a lower limit of $t = 0$. The results can be expressed in terms of the standardized normal integral as

$$F_Y(y) = \Phi\left(w^{-1} \ln \frac{t}{t_0}\right) \tag{123}$$

For small values of w, the lognormal and normal distributions are similar in shape. When the t_0 is the median of the random variable T, the mean and variance of the lognormal p.d.f. for T are

$$\mu = t_0 \exp\left(\frac{w^2}{2}\right) \tag{124}$$

$$\sigma^2 = t_0^2 \exp(w^2)[\exp(w^2) - 1] \tag{125}$$

The lognormal distribution may be derived by the following argument. Consider a certain process where failure is due to fatigue cracking. Let $X_1 < X_2 < \cdots < X_n$ be a sequence of random variables denoting the size of the crack at successive stages of growth. Assume that the growth is proportional to the size of the crack. $X_i - X_{i-1}$ is randomly proportional to X_{i-1}. Failure occurs when the crack size is X_n. The proportionality factors, Π_i, are independent but not necessarily identically distributed random variables. Hence, skipping the intermediate steps and equations yields

$$\ln X_n = \sum \Pi_i + \ln X_0 \tag{126}$$

where X_0 is the initial size of minute flows, voids, and so on. By the CLT (Section 1.3.13), the Π_i converge in distribution to the normal distribution, and hence $\ln X_n$ is asymptomatically normally distributed.

How to Generate Random Numbers from a Lognormal p.d.f.

Method 1: Law of Large Numbers One follows the same steps for the normal random number generator presented above. The relationship of a lognormal variable with median $m = \exp(\mu)$ and shape parameter σ to the standard normal deviate $N(0, 1)$, and the relationship of $N(0, 1)$ to the uniform random number u_i will give

$$x^* \sim me^{\sigma N(0,1)} = m \exp \sigma \left(\sum_{i=1}^{12} u_i - 6 \right) \tag{127}$$

Method 2 See method 2 for the normal generator. One can apply the same derivation for the lognormal deviate y, where $y = \ln x$, $y \sim$ normal, and $x \sim$ lognormal.

1.3.15 Logistic p.d.f.

X is *logistic* if its p.d.f.

$$f(x) = \frac{\exp[-(x-a)/k]}{k\{1 + \exp[-(x-a)/k]\}^2} = \frac{\sec \mathrm{h}^2[-(x-a)/k]}{4k} \tag{128}$$

and its c.d.f., reliability, and hazard functions are

$$F(x) = 1 - [1 + e^{-(x-a)/k}]^{-1} \tag{129}$$

$$R(x) = [1 + e^{-(x-a)/k}]^{-1} \tag{130}$$

$$h(x) = \frac{e^{-(x-a)/k}}{k[1 + e^{-(x-a)/k}]} \tag{131}$$

Also, $E(x) = \text{mode} = \text{median} = a$ and $\text{Var}(x) = b^2 = k^2\pi^2/3$. For a standard logistic, if $a = 0$, $k = 1$,

$$b = 3^{-1/2}k\pi \tag{132}$$

How to Generate Random Numbers from a Logistic p.d.f.

By the inverse transform technique, $u_i = R(x) = [1 + e^{-(x-a)/k}]^{-1}$ leads to $[(1 - u_i)/u_i] = e^{-(x-a)/k}$. Then taking the ln of both sides, $\ln[(1 - u_i)/u_i] = -(x - a)/k$; we get $x_i^* = a - k \ln[(1 - u_i)/u_i]$.

1.3.16 Cauchy p.d.f.

X is *Cauchy* if

$$f(x) = \frac{1}{\pi b\{[(x-a)^2/b] + 1\}}, \qquad -\infty < x < \infty \tag{133}$$

as the ratio of two independent $N(0, 1)$ random variables with a (location parameter) and b (scale parameter). The standard Cauchy is

$$f(x) = \frac{1}{\pi(x^2 + 1)}, \qquad a = 0, \quad b = 1 \tag{134}$$

Symmetrical about $x = 0$, odd moments about the origin are zero and the mode is at $x = 0$. The reciprocal of a Cauchy(a, b) random variable is also Cauchy(a', b'), where $a' = a/(a^2 + b^2)$ and $b' = b/(a^2 + b^2)$.

How to Generate Random Numbers from a Cauchy p.d.f.

A Cauchy(0, 1) uses the ratio of two independent $N(0, 1)$ standard random variables:

$$x^* = \frac{\sum_{i=1}^{12} u_i - 6}{\sum_{j=1}^{12} u_j - 6} \tag{135}$$

where u_i and u_j are independent uniform variates, $i,\ j = 1, 2, 3, \ldots, 12$.

1.3.17 Hypergeometric p.m.f.

The probability of x many successes in a sample size of n out of X many successes in a population of total N many elements is *hypergeometric* with a p.m.f. given as

$$P(x) = \frac{\binom{X}{n}\binom{N-X}{n-x}}{\binom{N}{n}} \tag{136}$$

with mean and variance

$$E(x) = \frac{nX}{N} \tag{137}$$

$$\mathrm{Var}(x) = \frac{(nX/N)(1 - X/N)(N-n)}{N-1} \tag{138}$$

How to Generate Random Numbers from a Hypergeometric p.m.f.

Select n rectangular (uniform) independent random numbers, $i = 1, 2, \ldots, n$, in sequence. If $u_i < p_i$, record a success, and sum the successes to obtain x^*, where

$$p_1 = \frac{X}{N}, \qquad N_1 = N, \qquad p_{i+1} = (N_i p_i - d)(N - i); \text{ and}$$
$$d = 1 \text{ if } u_i \geq p_i, \quad d = 0 \text{ if } u_i < p_i \tag{139}$$

1.3.18 Extreme Value (Gumbel) p.d.f.'s

1. *Smallest extreme value*. Consider a sample of n independent identically distributed random variables from a distribution with c.d.f. $F(x)$. Then the c.d.f. of $x = \min X_i$, $i = 1, \ldots, n$ is

$$G(x) = 1 - [1 - F(x)]^n \tag{140}$$

As n gets larger there are three possible resulting distributions (as in a series system based on the principle that the system cannot get stronger than its weakest or minimum element):

a. *Type I distribution*. If $f(x)$ tends to zero exponentially as $x \to \infty$ (i.e., normal distribution), then

$$G(x) = 1 - exp\left[-\exp\left(\frac{x-\gamma}{\alpha}\right)\right], \qquad -\infty < x < \infty, \quad \gamma, \alpha > 0 \tag{141}$$

b. *Type II distribution*. If the range of x is unlimited from below and if for some α, $\beta > 0$, $\lim\ (-x)^\alpha F(x) = \beta$, then

$$G(x) = 1 - \exp\left(\frac{x-\gamma}{\alpha}\right)^{-\beta}, \qquad -\infty < x \leq \gamma, \quad \alpha, \beta > 0 \tag{142}$$

c. *Type III distribution*. If the range of x is bounded from below, that is, $F(x) = 0$, $x \leq \gamma < \infty$, and $F(x)$ behaves like $\alpha(x - \gamma)^\beta$ for some α, $\beta > 0$, as $x \to \gamma$, which denotes uniform, exponential, and Weibull p.d.f.'s, then

$$G(x) = 1 - \exp\left(\frac{x-\gamma}{\alpha}\right)^{\beta}, \qquad \gamma \leq x < \infty, \quad \alpha, \beta > 0 \tag{143}$$

Type I asymptotic distribution of the smallest extreme results from defining the hazard rate to be of the form e^x. The type I function may also be used as a failure model for a series system when the underlying distribution is exponential. Type II distribution is not very useful in reliability since it is also defined in the

negative domain. Type III functions include the Weibull distribution. If $x = \ln T$, where T has the Weibull distribution, X has a type I extreme value distribution.

2. *Largest extreme value*. Consider a sample of n many i.i.d. (independent identical distributed) random variables each having a c.d.f. of $F(x)$. The distribution of the largest is

$$U(x) = [F(x)]^n \tag{144}$$

As n gets large, there are again three possible resulting distributions (as in a parallel system based on the principle that the system cannot get weaker than its strongest or maximum element):

a. *Type I distribution.* If $f(x)$ tends to zero exponentially as $x \to \infty$ (i.e., exponential), then

$$U(x) = \exp\left[-\exp\left(\frac{x-\gamma}{\alpha}\right)\right], \qquad -\infty < x < \infty, \quad \gamma, \alpha > 0 \tag{145}$$

b. *Type II distribution.* If the range of x is unlimited from below and if for some α, $\beta > 0$, $\lim (x)^{\alpha}[1 - F(x)] = \beta$, then

$$U(x) = \exp\left(\frac{x-\gamma}{\alpha}\right)^{-\beta}, \qquad x \geq \gamma \tag{146}$$

c. *Type III distribution.* If the range of x is bounded above $F(x) = 1, x \geq \alpha$ and for finite γ, $1 - F(x)$ behaves like $\alpha(x-\gamma)^{\beta}$ as $x \to \gamma$ (i.e., uniform), then

$$U(x) = \exp\left(\frac{x-\gamma}{\alpha}\right)^{\beta}, \qquad x \leq \gamma, \quad \alpha, \beta > 0 \tag{147}$$

Type I asymptotic distribution of the largest value may be used for corrosive processes and for time to failure in parallel systems.

How to Generate Random Numbers from the Smallest Type I Extreme Value

Using the inverse transform technique for the smallest extreme value, $x_i^* = \gamma + \alpha \ln(\ln u_i)$, $\infty < x < \infty$. For the largest extreme, the sign of x will be reversed.

1.3.19 Summary of the Distributions and Relationships Most Commonly Used

Figure 1.6 shows the relationships among the distributions in Chapter 1.

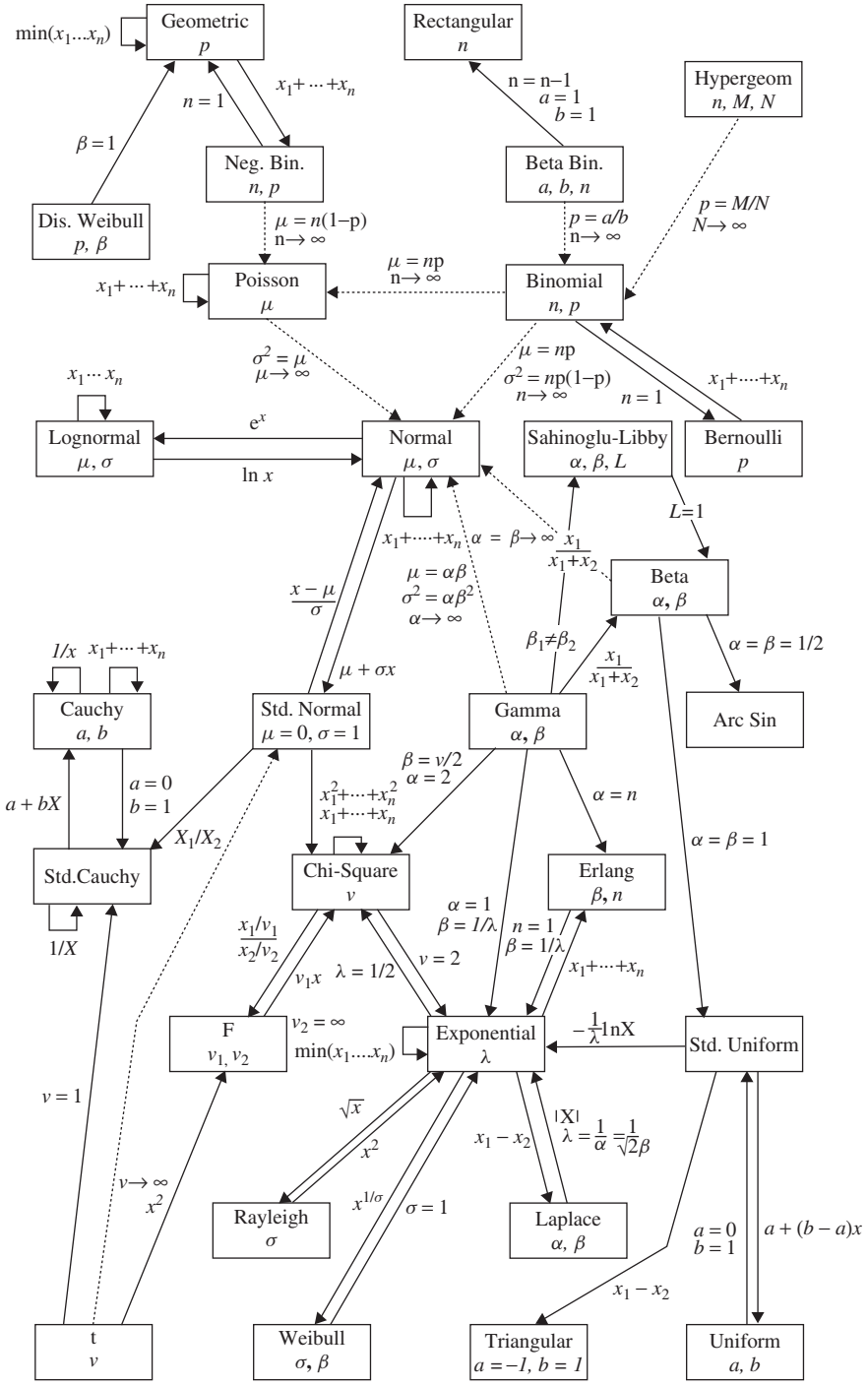


FIGURE 1.6 Relationship among most statistical distributions in Chapter 1.

1.4 LIFE TESTING FOR COMPONENT RELIABILITY

If the random variable X is for the lifetime or time to failure of a unit, then study of this variable is defined to involve life testing or reliability theory. The reliability of a component surviving until time X_0 is defined as the reliability of the component at time X_0 and is denoted as $R(X_0)$. Even though the random variable is referred to as representing time, one may have other variables, such as distance in the case of a vehicle. However, the same concepts would apply [24].

1.4.1 Estimation Methods for Complete Data

We consider next the estimation of reliability and the parameters of the failure distributions. There will be two situations: (1) when the failure distribution is not known (i.e., the nonparametric case), and (2) when the failure distribution is known (i.e., the parametric case). In each case, there will be another two situations: (1) when the exact failure times are known, and (2) when the only information available is the number of survivors at different points in time [1,2]. The data used in the estimation procedures are obtained primarily from life tests. Life testing is a procedure in which the failure data are obtained from a sample of N items put into the projected operating environment. If all units fail before the test is terminated, it is a complete test. Otherwise, the test is incomplete and censoring exists where the usual statistical calculations for MTTF (mean time to failure) or μ (average of failure times) no longer make sense. Type I censoring occurs when a life test is terminated at a specified time, say T_0, before all N items fail. Type II censoring occurs when the life test is terminated at the time of a particular rth failure, $r < N$. A more complicated multiple censoring scheme contains some items removed during the test, in addition to failing.

In most reliability studies, the form of the distribution of the variable (time to failure or an operational characteristic such as distance traveled) is assumed or known. Occasionally, however, it is not possible to make an assumption concerning the form of the distribution, in which case the nonparametric distributions come into play. Both parametric and nonparametric methods will be studied. For both analyses, data may be grouped or ungrouped. *Ungrouped data* occur when individual component failures are recorded in a laboratory setting when the sample size is not large and sufficient instrumentation and personnel are available to measure the exact failure times. The opposite is true when many data exist with no adequate funds for personnel or equipment to record all failure times, and the only way to accommodate the failures is to stop at equal or unequal time increments to group them. Therefore, if the data consist of the number of failures within each time period with no information about the exact times of failure, they are classified as *grouped data*. Let's now look at these two types of empirical data [1,2,25,27,34].

Ungrouped Data Ungrouped data consist of a series of failure recordings, t_i, $i = 1, \ldots, N$, for the N units available. The order statistics are given in ascending order of magnitude: $t_{(1)} < t_{(2)} < t_{(3)} < \ldots < t_{(i)} < \ldots < t_{(N)}$. An estimate of the reliability function is denoted by $\hat{R}(t_{(i)}) = 1 - p_i$ with jumps at $t_{(i)}$, where $p_{(i)} = \hat{F}(t_{(i)})$. There are a number of formulas for $p_{(i)}$; for example, all but the first formula shown below are cases of the general formula $(i - a)/(N - 2a + 1)$ [4,26]

$$\frac{i-1}{N}, \quad \frac{i}{N+1}, \quad \frac{i-0.3}{N+0.4}(\text{small samples}),$$
$$\frac{i-3/8}{N+1/4}, \quad \frac{i-0.44}{N+0.12}, \quad \frac{i-0.5}{N}(\text{Hazen's for } N > 20) \tag{148}$$

are variations. For simplicity we will use the one with $\alpha = 0$, which gives us the second formula in the series. We will show ways to plot the reliability,

$$\hat{R}(t) = 1 - \hat{F}(t) = \frac{N+1-i}{N+1} = e^{-H(t)} \tag{149}$$

and the cumulative hazard,

$$\hat{H}(t) = -\log \hat{R}(t) = \ln(N+1) - \ln(N+1-i) \tag{150}$$

From a plot of $H(t)$, it is possible to judge if the hazard function is increasing, decreasing, or constant. A linear plot implies a constant hazard function, a convex plot an increasing hazard, and a concave plot a decreasing hazard. Direct estimation of the hazard function is given by

$$\hat{h}(t) = \frac{1}{(t_{(i+1)} - t_{(i)})(N - i + 1 - \alpha)} \tag{151}$$

where α takes the same value as that used to estimate $R(t)$. The density function is given by $\hat{f}(t) = \hat{h}(t)/\hat{R}(t)$ as in equation (2) in Section 1.1. The example below best explains use of the formulas above.

Ungrouped Example The failure times in months were recorded on certain equipment for $i = 1, \ldots, 9$ as 7.2, 9.7, 12.3, 13.5, 16, 18.2, 18.6, 19.8, and 21.3. The descending reliability and ascending cumulative hazard estimates at times t_i are, respectively, $\{R(0) = 1.00,\ H(0) = 0.0\}$, $\{R(7.2) = 0.9,\ H(7.2) = 0.105\}$, $\{R(9.7) = 0.8,\ H(9.7) = 0.223\}$, $\{R(12.3) = 0.7,\ H(12.3) = 0.357\}$, $\{R(13.5) = 0.6,\ H(13.5) = 0.511\}$, $\{R(16) = 0.5,\ H(16) = 0.693\}$, $\{R(18.2) = 0.4,\ H(18.2) = 0.916\}$, $\{R(18.6) = 0.3,\ H(18.6) = 1.204\}$, $\{R(19.8) = 0.2,$

TABLE 1.2 Ungrouped Complete Data Example

i	T_i	n_i	$R(t_i) = n_i/N$	$H(t_i) = -\ln R(t)$
0	0.0	9	1.00	0.0
1	7.2	8	0.90	0.105
2	9.7	7	0.80	0.223
3	12.3	6	0.70	0.357
4	13.5	5	0.60	0.511
5	16.0	4	0.50	0.693
6	18.2	3	0.40	0.916
7	18.6	2	0.30	1.204
8	19.8	1	0.20	1.609
9	21.3	0	0.10	2.303

$H(19.8) = 1.609\}$, $\{R(21.3) = 0.1,\ \ H(21.3) = 2.303\}$. Note that n_i represents the remaining units at the ith stage and $R(0) = 1$ as in Table 1.2.

Grouped Data As mentioned above, the data are monitored such that the only failure information available is the number of surviving items at times that are recorded in ascending order of magnitude, $t_{(1)} < t_{(2)} < \cdots < t_{(i)} < \cdots < t_{(N)}$, such that $\hat{R}(t) = n_i/N, i = 1, \ldots, K$. Therefore, combining into $\hat{H}(t_i) = \log\hat{R}(t_i) = -\ln N - \ln n_i$, we acquire plots for both the reliability and cumulative hazards. Additionally, we may want to estimate the mean and variance of the failure distribution for grouped data. Whereas for ungrouped data, the mean μ was simply the arithmetic average of the time intervals (differences) of the individually recorded failures and variance σ^2 denotes the sum of squares of the deviations of these time intervals from the mean all divided by $N - 1$; for the grouped data it is a different story. We approximate $f(t)$ by a histogram in which there are K intervals and the midvalue of each interval between i and $i - 1$ is M_i, with a frequency of f_i, where $\sum_1^K f_i = N$; then, G denoting the group

$$\mu_G = 1/N \sum_i^K f_i M_i \tag{152}$$

$$\sigma_G^2 = 1/N \sum_1^K f_i (M_i - \mu_G)^2 \tag{153}$$

To calculate the percentile, or the quantile of order P, we must locate the observation with rank $r = Pn + 0.5$. Then obtain the cumulative frequencies and determine the class that includes the percentile; it is the class whose cumulative frequency is the first to exceed r. Denote the lower and upper limits of this class by L and U, the frequency of this class by f, and the number of observations that

TABLE 1.3 Grouped Complete Data Example

i	T_i	n_i	$R(t_i) = n_i/N$	$H(t_i) = -\ln R(t)$
0	10	100	1.00	0
1	20	60	0.60	0.51082
2	30	40	0.40	0.91629
3	40	25	0.25	1.38629
4	50	10	0.10	2.30259
5	60	5	0.05	2.99573
6	70	0	0.00	—

are smaller than L by m; of course, $m < r$ [27]. Then the $(100P)$th percentile is given by

$$L + \frac{r - m}{f}(U - L) \tag{154}$$

Grouped Example In equal increments of 10 hours, the data in Table 1.3 were compiled in seven intervals with $N = 100$ units starting and n_i remaining units at the ith stage, where $R(0) = 1$.

1.4.2 Estimation Methods for Incomplete Data

Rather than wait for all N items to fail, it is often advantageous to halt the testing procedure earlier. This is done either by stopping to test when, for type I, the rth (r fixed) failure occurs, or for type II, at a given time T_0. Censoring types I and II are singly censored from the right, as we assume no censoring from the left in this section, like starting at a nonzero threshold value. We define the data to be multiply censored if units are removed at different times during the time of life testing. There may be a couple of reasons for multiply censored data. Either the units are removed, thus becoming unavailable (e.g., the death of a cancer patient, who is thus no longer available for clinical tests), or because a new or irrelevant mechanism that is not under analysis, and not known to us, caused the failure. In this section we assume negative exponential density governing the distribution of failure times. The point and interval estimates for the reliability are

$$R(t) = \exp\left(-\frac{t}{\hat{\mu}}\right) \qquad \text{where} \quad \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} t_i \tag{155}$$

$$\frac{2n\hat{\mu}}{\chi^2_{(\alpha/2),2n}} < \mu < \frac{2n\hat{\mu}}{\chi^2_{(1-\alpha/2),2n}} \tag{156}$$

which is a $100(1 - \alpha)\%$ confidence interval.

Singly (Right)-Censored Data for Replacement and Nonreplacement Data To summarize the formulas given below, note that all situations command that the

estimate of μ be given by the total good time divided by the number of failures, depending on the rule used for stopping the test. Rep. denotes replacement and NRep. denotes nonreplacement.

1. *Type I (NRep.).* The estimator of $\mu = \text{MTTF}$ when testing stops at a predetermined time T_0:

$$\hat{\mu} = \frac{1}{r}\left[\sum_{i=1}^{r} t_i + (n-r)T_0\right] \tag{157}$$

A $100(1-\alpha)\%$ confidence interval for μ is given by

$$\frac{-T_0}{\ln(1-p_U)} < \mu < \frac{-T_0}{\ln(1-p_L)} \tag{158}$$

where p_L and p_U are the confidence limits for a binomial parameter $p = 1 - R(T_0)$ with a nonparametric estimator $\hat{p} = 1 - r/n$ (good for $p \leq 0.5$), or if a negative exponential (constant failure rate) is assumed, use $\hat{p} = 1 - \exp(-t/\hat{\mu})$.

2. *Type I (Rep.)*

$$\hat{\mu} = \frac{1}{n}(NT_0) \tag{159}$$

is the type I (Rep.) estimator for the censored MTTF.

3. *Type II (NRep.).* The estimator of $\mu = \text{MTTF}$ when testing stops at the rth failure is

$$\hat{\mu} = \frac{1}{r}\left[\sum_{i=1}^{r} t_i + (n-r)t_r\right] \tag{160}$$

A $100(1-\alpha)\%$ confidence interval for μ is given by

$$\frac{2r\hat{\mu}}{\chi^2_{(\alpha/2),2n}} < \mu < \frac{2r\hat{\mu}}{\chi^2_{(1-\alpha/2),2n}} \tag{161}$$

4. *Type II (Rep.)*

$$\hat{\mu} = \frac{1}{n}Nt_r \tag{162}$$

is the type II (Rep.) estimator for the censored MTTF.

Equations are presented below for obtaining unbiased estimators of $R(t)$ for NRep. and Rep. tests. Even though $\hat{\mu}$ is an unbiased estimator of μ, their functions, such as $\hat{\lambda}$ (rate) $= \hat{\mu}^{-1}$ and reliability, $\hat{R}(t) = \exp(-t/\hat{\mu})$ are not unbiased estimators of λ and $R(t)$, respectively. An unbiased estimator of $R(t)$ for the

nonreplacement and replacement tests stopped at the rth failure is respectively given by

$$\hat{R}(t) = \begin{cases} \left(1 - \dfrac{t}{r\hat{\mu}}\right)^{r-1}, & t < r\hat{\mu} \\ 0, & \text{elsewhere} \end{cases} \tag{163}$$

$$\hat{R}(t) = \begin{cases} \left(1 - \dfrac{t}{r\hat{\mu}}\right)^{r}, & t < r\hat{\mu} \\ 0, & \text{elsewhere} \end{cases} \tag{164}$$

For MLE, by invariance property, the functions of $\hat{\mu}$ are still the MLE.

Multiply Censored Data We study the nonparametric analysis of multiply censored data where some units are removed from the test before failure for many different reasons, especially in the biomedical and medical (cancer) research community. The estimation technique used most to calculate the reliability function (or survival function, in human-related tests) is the product-limit or Kaplan–Meier estimate.

Ungrouped Data As in the individual failure times given below, the sequence consists of a series of times, $t_{(1)} < t_{(2)} < \cdots < t_{(i)} < \cdots < t_{(N)}$. Each of these times signifies either the failure or censored removal (indicated by an asterisk). To start estimating reliability, we need to derive a recursive relation for $R(t_i)$ in terms of $R(t_{i-1})$. With no censoring, as noted earlier,

$$\hat{R}(t_{i-1}) = \frac{N + 2 - i}{N + 1} \tag{165}$$

By taking the ratio

$$\frac{\hat{R}(t_i)}{R(t_{i-1})} = \frac{N + 1 - i}{N + 2 - i} \tag{166}$$

one obtains the recursive relationship

$$\hat{R}(t_i) = \frac{N + 1 - i}{N + 2 - i}\hat{R}(t_{i-1}) \tag{167}$$

an equation of conditional reliability. That is, the probability that a unit survives up to time t_i is the product of the probability that it survives up to time t_{i-1} multiplied by the conditional probability $(N + 1 - i)/(N + 2 - i)$ that it will not fail between t_{i-1} and t_i given that it is operating at time t_{i-1}. Now if a censoring action occurs at t_i, the reliability remains the same, such that $\hat{R}(t_i) = \hat{R}(t_{i-1})$. Therefore,

$$\hat{R}(t_i \mid t_{i-1}) = \begin{cases} \dfrac{N + 1 - i}{N + 2 - i}, & \text{failure at } t_i \\ 1, & \text{censoring at } t_i \end{cases} \tag{168}$$

This leads to the *product-limit* or *Kaplan–Meier estimator* as follows [1]:

$$\hat{R}(t_i) = \hat{R}(t_i \mid t_{i-1})\hat{R}(t_{i-1} \mid t_{i-2})\hat{R}(t_{i-2} \mid t_{i-3}) \cdots \hat{R}(t_1 \mid 0) \qquad \text{where} \quad R(0) = 1 \tag{169}$$

Ungrouped Example The following failure data are from a mechanical engineering experiment: {30, 53, 54*, 68, 83, 99*, 107, 116, 149*, 158}. The corresponding conditional reliabilities for the 10 sequential activities are calculated to be

$$\hat{R}(t_i \mid t_{i-1}) = \{0.909, 0.900, 1.0, 0.875, 0.857, 1.0, 0.8, 0.75, 1.0, 0.5\} \tag{170}$$

Finally, the Kaplan–Meier (reliability) product limits are

$$\hat{R}(t_i) = \{0.909, 0.818, 0.818, 0.716, 0.614, 0.614, 0.491, 0.368, 0.368, 0.184\} \tag{171}$$

Grouped Data Contrary to individual failure times, all the analyst has is the number of failures recorded at separate (disjoint) intervals. Suppose that the number of failed and removed (nonfailed) items is recorded. New expressions are derived for conditional reliability to apply to grouped data. Let d_i be the number of defectives (or failures) during the ith interval, and with c_i censored (unfailed but removed), there would be n_i (remaining) $= n_{i-1} - d_i - c_i$, where $n_0 = N$:

$$\hat{R}(t_i \mid t_{i-1}) = 1 - \frac{d_i}{n_{i-1} - q_i \mathrm{c_i}} \tag{172}$$

If removed items are recorded, a $q_i = 50\%$ fraction of the items removed are universally assumed to fail. However, the q_i proportions of failure (or death) may be entered differently at varying intervals at will, or a generic 50% for all intervals may be used. To estimate the reliability, the point reliability and conditional reliability values are updated at the termination of time intervals where failures have occurred. The removals will not change the reliability value.

TABLE 1.4 Grouped Multiply Censored Data Example

i	t_i (CPU seconds)	n_{i-1}	d_i	c_i	n_i	$\hat{R}(t_i \mid t_{i-1})$	$\hat{R}(t_i)$	$H(t_i)$
0	0	—	—	—	150	—	1.0	0.0
1	1000	150	5	0	145	0.966667	0.966667	0.033902
2	2000	145	19	0	126	0.868966	0.84	0.174353
3	3000	126	66	10	50	0.454545	0.381818	0.962811
4	4000	50	30	6	14	0.361702	0.138104	1.97975
5	5000	14	10	4	0	0.166667	0.023017	3.77152

Grouped Example The preceeding censored grouped data in Table 1.4 were collected by an independent quality-focused nonprofit organization on the failure of identical commercial software modules sold on the market for $0 < t < 5000$ CPU seconds at 1000-second intervals. The nonparametric reliability and cumulative hazard functions were asked ($q_i = 0.5$ for convenience).

1.5 REDUNDANCY IN SYSTEM RELIABILITY

In earlier sections of this chapter we dealt with life testing and reliability evaluations of the individual components that constitute a system. The same techniques could be expensive, inaccurate due to environment change, and mathematically intractable to perform and apply during a system analysis. Also, positive or negative correlations exist—even though hard to evaluate—between the individual components. This prohibits the convenient assumption of independence. Therefore, when more than one component is included to form a system and also to improve the reliability of a system, the concept of *redundancy* begins. But whether the type of redundancy preferred and designed is beneficial to the aims of the system analyst is a different matter.

Next we study a variety of systems in terms of their reliability. However, another way of looking at the quality of system performance is through its *availability*, defined as the proportion of time that a system is ready to be used. One can upgrade instantaneous reliability by adding reliability or by improving reparability and maintainability [28,29]. In the following subsections, the components are assumed to be statistically independent.

1.5.1 Series System Reliability

Series reliability simply assumes that the product of individual reliabilities in a system will result in lower reliability than that of the weakest component reliability in the system. Sometimes such systems are called nonredundant due to this negative effect. The formula depicts this reality:

$$R = R_1 R_2 R_3 R_n \cdots R_N, \qquad 0 < R_i < 1 \tag{173}$$

If $R_i = 0.9$, then $R = R_1 R_2 = 0.81$. If $R_i = 0.99$, then $\prod_{i=1}^{200} R_i = 0.134$, or down to 13.4% from 99% for a single component. However, series systems are sometimes good for purposes other than reliability, such as usability or convenience. Before the advent of calculators and computers, charts were the only means available to calculate the system reliability of N components in series. In the earlier stages of space missions, practitioners did not understand why payloads stacked on top of each other in a series format would not result in a more reliable performance in space [29]. This mystery lasted until someone figured out that the system reliability of series systems was worse than its weakest link. This indicates openly that reliability modeling should start in the design stage, not

after the fact, when it is both too late and too expensive to redeem. Eventually, reliability analysts turned to parallel redundancy to upgrade system reliability.

1.5.2 Active Parallel Redundancy

In these designs, two or more components will form a system without a switch, such that the failure of one will not lead to system breakdown as it did in series systems. The phenomenon of such duplication is defined as *redundancy*. There exist four basic active parallel topologies. Assume that independence prevails between components.

Simple Active Parallel System A simple parallel system is one in which if any individual component is a success, the system input–output or ingress–egress connection is a success. In a simple active (i.e., ready to operate without the need of a switch) parallel system, the individual components are positioned in parallel topology, where with n components, each of R_i, the system reliability is given as

$$R_a = 1 - \prod_{i=1}^{n}(1 - R_i) = 1 - (1 - R_i)^n \tag{174}$$

$$1 - R_a = (1 - R_i)^n \tag{175}$$

Reorganizing (174) as in (175) and taking the natural logarithm of both sides yields

$$n = \frac{\ln(1 - R_a)}{\ln(1 - R_i)} \tag{176}$$

If $R_i = 0.95$, then to raise the active parallel system to 99.5% reliability,

$$n = \frac{\ln(1 - 0.995)}{\ln(1 - 0.95)} = \frac{-5.3}{-3.0} \cong 2 \tag{177}$$

Therefore, if $Q = 1 - R$ for all, a measure of improvement called *relative efficiency* is given by

$$E = \frac{1 - Q^n}{1 - Q} \tag{178}$$

and the maximum improvement potential (or relative efficiency) as n goes to ∞ is then

$$E_{\max} = \frac{1}{1 - Q} = \frac{1}{R} = \frac{100\%}{R} \tag{179}$$

Series-in-Parallel System The series-in-parallel system is composed of k separate series subsystems arranged in active parallel in that if any series branch

with n components is a success, the overall system is a success. Such systems are also called *high-level redundant* (HL). The reliability is

$$R_{\mathrm{HL}} = 1 - (1 - R^n)^k \tag{180}$$

For example, given $R_1 = R_2 = R_3 = R_4 = 0.9$ and $k = 2, n = 2$, $R_{\mathrm{HL}} = 1 - (1 - 0.9^2)^2 = 0.9639$. For six identical components, for example, $R_{\mathrm{HL}} = 1 - (1 - R^n)^k = 2R^3 - R^6 = (2)(0.9)^3 - 0.9^6 = 0.9266$.

Parallel-in-Series System The parallel-in-series system is one for which maximum reliability can be obtained but in which reliability is hardest to maintain. In this system an individual component is replaced by k components in an active parallel system to increase reliability in a series of n branches. Such systems are also called *low-level redundant* (LL). This structure is to be used when very high reliability is necessary for a given length of time, as in a missile firing. The reliability is

$$R_{\mathrm{LL}} = [1 - (1 - R)^k]^n \tag{181}$$

For example, given $R_1 = R_2 = R_3 = R_4 = 0.9$ and $k = 2, n = 2$, $R_{\mathrm{LL}} = [1 - (1 - 0.9)^2]^2 = 0.9801$. For six identical components this time, $R_{\mathrm{LL}} = 1 - (1 - \mathrm{R}^n)^k = (2R - R^2)^3 = (2)(0.9)^3 - 0.9^6 = 0.9703$. We have $R_{\mathrm{LL}} - R_{\mathrm{HL}} = 6R^3(1 - R)^2 = 0.0437$ for our example of six identical components. Consequently, $R_{\mathrm{LL}} > R_{\mathrm{HL}}$.

Partial Parallel Topology This structure is designated by a group of n components in parallel, out of which any k are required to operate for the system to be successful. A popular example is when of three parallel engines in a jet plane, where $R + Q = 1$, at least two ($k = 2$) must work for the jet to operate. If one expands $(R + Q)^3 = R^3 + 3R^2Q + 3RQ^2 + Q^3$, the probability of success is given by $R^3 + 3R^2Q$. In the case of n unlike components, $\prod_{i=1}^{n}(R_i + Q_i) = 1$. We then get rid of those terms that represent failure scenarios for the defined "at least k" statement of system reliability.

1.5.3 Standby Redundancy

Standby redundancy is identical to the structure of an active parallel system except that it is termed "standby" redundant when a switch exists to direct the flow of current as desired [3]. Provided that the switch operates perfectly in a more efficient two-component standby system, the second unit is not activated until the first unit fails. The MTTF (reciprocal of λ, which is the rate of failure) of an active two-unit system is 1.5 times higher than that of a single component. On the other hand, the MTTF of a standby system is twice as high as that of a single unit. However, for $\lambda t \ll 1$, $R(t) = 1 - \exp(-\lambda t) \approx 1 - \lambda t$ for a single component, $R(t) \approx 1 - (\lambda t)^2$ for an active parallel system, and finally, for a standby system, $R(t) \approx 1 - 0.5(\lambda t)^2$. This can be interpreted as follows: For short time intervals

where $t \ll 1$, the standby system failure probability, $F = 1 - R$, is only one-half of the reliability of an active parallel system. That is, $F_a \approx (\lambda t)^2$ and $F_{\text{stby}} \approx 0.5(\lambda t)^2$. See equation (67) for a general statistical treatment of n components in standby [1–3].

$$R_{\text{stby}}(t) = e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}, \qquad t \geq 0, \quad \lambda > 0 \tag{182}$$

As an example, consider a system of two units with $\lambda = 0.01$. The reliability at $t = 10$ hours using equation (67) is $R_{\text{stby}} = [\exp(-0.01 \times 10) + 0.1][\exp(-0.01 \times 10)] = 0.90484 + (0.90484)(0.01) = 0.995324$. On the other hand, the reliability of two identical units in an active parallel system, when $R(10) = 0.90484$, is $R_a(10) = 1 - (1 - 0.9048)^2 = 0.99095$. In this two-component example, the advantage of a standby system over an active parallel structure is not significantly higher. This advantage is forfeited if the switching reliability is not perfect ($R_{\text{ss}} < 1$). For a two-unit standby,

$$R_{\text{stby}(t)} = (1 + R_{\text{ss}}\lambda t)\exp(-\lambda t) \tag{183}$$

Further, the reliability of an n-unit standby system with imperfect switching reliability [2,3] is

$$R_{\text{stby}}(t) = e^{-\lambda t}\left[1 + R_{\text{ss}} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}\right], \qquad t \geq 0, \quad \lambda > 0 \tag{184}$$

Additionally, a component in standby mode may fail before primary system failure. Such standby component failures can happen very rarely when the secondary, tertiary, or kth standby unit is necessitated, and may have deteriorated while waiting its turn of duty in the standby mode. For two components, the system reliability is then calculated as

$$R_{\text{stby}} = \left\{1 + R_{\text{ss}} \frac{\lambda_1}{\lambda_2}[1 - \exp(-\lambda_2 t)\right\} \exp(-\lambda_1 t) \tag{185}$$

where λ_2 is the failure rate of the second unit while standing by. What happens when the failure rate λ_3 of the second component, while standing by, differs from the first, λ_1? Then

$$R_{\text{stby}} = \exp(-\lambda_1 t) + R_{\text{ss}}[\exp(-\lambda_2 - \lambda_3)t - \exp(-\lambda_1 - \lambda_2 - \lambda_3)t] \tag{186}$$

Finally, if a system consists of a number of equal components in series supported by one or more spares, it cannot fail until the failure after the last spare is replaced. In this case, the system failure rate would be $N\lambda$, with N components in series with a constant failure rate λ. If n spares were on hand, however, the system

would not be operable until $n + 1$ failures happened. This results in a system MTTF of $(n + 1)/N\lambda$. Then

$$R_s(t) = \exp\left(-\frac{N\lambda t}{n+1}\right) \tag{187}$$

Example A jet plane has an airborne radar system with 30 identical integrated circuits (ICs), and the pilot has three spares for a 1-hour flight. If the IC hourly failure rate is $\lambda = 0.01$, the series system reliability with no spares at time $t =$ 1 hour is $R_s(1) = \exp[-N\lambda t/(n + 1)] = \exp(-0.3) = 0.74082 \approx 74\%$. With $n = 3$ spares, the reliability will be increased to

$$R_s(1) = \exp\left(-\frac{N\lambda t}{n+1}\right) = \exp\left(-\frac{0.3}{4}\right) = 0.928 \approx 93\% \tag{188}$$

When the standby component is too cold to be called upon, it may experience switching problems when activated. Therefore, in equation (188), the switching reliability R_{ss} and standby failure rate λ_2 will both be reduced. At the other extreme, when the standby is hot, the switching failure will be reduced, and switching reliability R_{ss} is higher. Also, when too hot, the standby reliability λ_2 will equal that of the primary component, λ_1. In that instance, when $\lambda_1 = \lambda_2 = \lambda$, the reliability equation for two components will change to

$$\begin{aligned} R_{stby} &= \{1 + R_{ss}[1 - \exp(-\lambda t)]\} \exp(-\lambda t) \\ &= (1 + R_{ss}) \exp(-\lambda t) - R_{ss} \exp(-2\lambda t) \end{aligned} \tag{189}$$

From this equation, if the switching reliability is almost perfect ($R_{ss} \cong 1$) due to hot standby, the equation converts to that of an active parallel equation, as if no standby switch existed:

$$R_{stby} = 2 \exp(-\lambda t) - \exp(-2\lambda t) = R_a \tag{190}$$

1.5.4 Other Redundancy Limitations: Common-Mode Failures and Load Sharing

The positive advantages of redundant systems are forfeited when dependencies between components create unexpected disadvantages [1].

Common-Mode Failures These occur when common connections or stresses influence the redundant components such that they fail simultaneously. This may be due to a bird fracturing a jet engine, in turn causing a commercial jet liner to crash. This would be like installing a component having a reliability R' in series with a parallel structure. This can be displayed as follows if $R = \exp(-\lambda t)$:

$$R'_a = (2R - R^2)R' \tag{191}$$

In the example of a twin-engine aircraft, if each engine had probability $p = 10^{-6}$, the common mode failure being $p' = 10^{-9}$, the system failure probability, $p'_a \approx 10^{-9}$, is dominated entirely by common-mode failure. If a subscript I denotes "independent" and C denotes "common mode", and if for $\lambda = \lambda_I + \lambda_C$, we define a factor $\beta = \lambda_C/\lambda$, then for an active parallel system,

$$R'_a = [2\exp(-\lambda_I t) - \exp(-2\lambda_I t)]\exp(-\lambda_C t) \tag{192}$$

and using $\lambda_C = \beta\lambda$ and $(1-\beta)\lambda = \lambda_I$, we can reformulate this as

$$R'_a = \{2 - \exp[-(1-\beta)\lambda t]\}\exp(-\lambda t) \tag{193}$$

The system reliability decreases with the increase in β, as in the rare event approximation

$$R'_a \approx 1 - \beta\lambda t - \left(1 - 2\beta + \frac{\beta^2}{2}\right)(-\lambda t)^2 + \cdots \tag{194}$$

as compared to $1 - (\lambda t)^2$ when no common-mode failure was present. Also,

$$\text{MTTF}_a = [2 - (2-\beta)^{-1}] \cdot \text{MTTF} \tag{195}$$

Load Sharing This limitation is another factor that degrades system reliability in active parallel systems. The failure rate of the second component, λ_L, will increase due to the stress of the first, which fails with λ, resulting in $\lambda_L > \lambda$. With no common-mode failures,

$$R_a = 2\exp(-\lambda_L t) + \exp(-2\lambda t) - 2\exp[-(\lambda + \lambda_L)t] \tag{196}$$

which defaults to the original equation for the active parallel structure if $\lambda_L = \lambda$. Now if $\lambda_L \to \infty$, equation (196) reduces to the reliability of two components placed in series:

$$R_a = \exp(-2\lambda t) = R_{\text{series}} \tag{197}$$

This means that if any component causes an instantaneous failure of the second component shared, the active parallel system failure rate will be great as that of a single unit.

1.6 REVIEW OF SOFTWARE RELIABILITY GROWTH MODELS

There are hundreds of papers and tens of models in a multitude of references on the subject of software reliability, a subject that has risen to unimaginable dimensions as the information age galloped full speed with a "fin de siecle" (end of the century) spirit [30–32]. It would be impractical to explain relevant

methods on an individual basis. However, for the general purpose of introducing first-time readers to these techniques, a classified breakdown of methods is in order. Due to space limitations, a sample handful of pioneering and representative techniques with mathematical–statistical modeling applications leading to applicable and practical software reliability engineering solutions are carefully chosen and included. Although the author would have liked to include them all, software reliability is only a piece of the puzzle of trustworthy computing concepts taken up in this book. In this chapter we describe reliability models based on the time domain only. The effort (time-independent) domain is studied later.

Software reliability models do not consider the same statistical techniques as those used in the hardware reliability models described earlier. Simplicity and practicality are two key factors in bridging the gap between the state of the art and that of applied software reliability modeling. The assumptions must be realistic and testable as well as applicable and accurate and valid from a predictive viewpoint. One should perform goodness-of-fit tests to assess how reasonably the model fits the data given. Examples are provided to help readers understand the comparisons.

A software reliability model is an essentially mathematical–statistical technique used to model an engineering phenomenon: specifically, obtain a quantitative measure of reliability such as the expected number of failures within a given or residual time interval, the failure intensity during operation, or the mean time between failures. These models are not simply a cookbook approach but require academic expertise in statistics and mathematics [33]. Some of these modeling concepts are outside the general discipline of computer science and therefore cannot easily be appreciated or interpreted by software developers. By *reliability in software* we mean the probability that the software will fulfill its intended function without failure(s) in a specified time interval. This definition is no different from that of its hardware counterpart when hardware is replaced by software. However, as the reader will observe, software models are considerably different from hardware models. Again, there have been many books and thousands of journal and conference proceeding papers on this broad topic since the inception of software reliability science and engineering in the early 1970s [34–40]. In a pioneering software reliability study in 1967, Hudson modeled software development as a Markovian birth (fault generation during the design or debugging stages) and death (failures resulting from the triggering of faults) process with transition probabilities from one to another [41]. He showed that the number of faults detected, which increased with time, displayed a binomial distribution whose mean value function had the form of a Weibull p.d.f. Other studies followed with the advances in software that took place in the late 1970s. Leading software reliability taxonomists have broken down the multiplicity of research papers into several general areas [31,34]:

1. *Time-between-failures models*. Some of the earliest examples of research on times between failures are those of Jelinski Moranda (1972, 1975), Shooman (1972), and Shick and Wolverton (1978) [42–45].

2. *Failure-counting models*. A representative group of failure-counting models is that of nonhomogeneous Poisson processes, where predictions can be made for future epochs. The eailier leading models in this category are the popularly used model of Goel and Okumoto of 1979 [46], and the Musa–Okumoto logarithmic Poisson in 1984 [47]. Discrete versions of this type of model have been studied by Ohba [48], Duane [49], and Littlewood [50], all in 1984; Yamada et al. in 1986 [51]; Knafl and Sacks in 1991 [52]; and Sahinoglu through his CPSRM (compound Poisson software reliability modeling) techniques [12–16] and Zhao and Xie [53], both in 1992. Musa's basic execution model of 1975 is also in this category [54].

3. *Bayesian models*. This type of model, a Bayesian estimation technique for models already studied, uses prior distribution to represent the view from past behavior, and thus a posterior distribution to integrate current data with past judgment. By way of posterior distributions, after deciding on the choice of loss functions and minimizing the loss expected, estimates of the unknown parameter are substituted in the reliability or hazard functions. For example, for a squared-error loss function, the best estimate is the mean of the posterior distribution; and for an absolute-value loss function, the median of the posterior is the best estimate. However, if the empirical Bayesian approach is used to derive more appropriate models, they can be classified as another modeling technique. The most popular model in this category is the Littlewood–Verral (1973) empirical Bayes model [55], which Mazzuchi and Soyer (1988) later modified using Bayesian principles [56]. There are many other papers on Bayesian treatment of the Jelinski–Moranda model: for example, those of Langberg and Singpurwalla [57] and Jewell [58], both in 1985. These models are difficult to apply without parameter estimation solutions.

4. *Static (nondynamic) models*. These models, which include complexity measures, failure injection, and fault seeding, do not deal with time. One of the first models was that of Nelson in 1978 [59]. An excellent must-read review paper for all interested beginners is that of Ramamoorthy and Bastani published in 1982 [60]. Bastani and Ramamoorthy later (1986) emphasized correctness estimation of software failures rather than time-dependent probability [61]. The latter publication describes a detailed study of correctness probability, which is estimated using a type of continuity assumption. Also discussed is a fuzzy set–based input domain model that is focused on developing more theoretical models. The earlier model of Nelson [59] was a special case of an input domain–based model, extended by Munson and Khosgoftaar in 1981 [62]; Hamlet [63] and Scott et al. [64], both in 1987; and Weiss and Weyuker in 1988 [65], in the area of software fault tolerance, a subject also studied by Littlewood and Miller in 1989 [66] and by Butler and Finelli in 1993 [67]. Software fault trees used as a conventional reliability engineering method were studied by Stalhane in 1989 [68], and Wohlin and Korner (1990) proposed a fault-spreading model [69]. The original seeding model discussed by Mills [70] has never been formally published other than as an IBM report, although Huang in 1984 [71], Duran and Wiorkowski in 1981 [72], and Schick and Wolverton in 1978 [45] have written on the topic.

5. *Others*. This group combines all the other topics, such as papers on the release time of software after testing by Xie in 1991 [30] and by Sahinoglu in 1995 and 2003, to name a few [13,15]. Model comparison papers have also been published, such as those by Keiller and Miller in 1991 [73], Khoshgoftaar and Woodcock in 1991 [74], and Lyu and Nikora in 1991 [75], in addition to Bendell and Mellor in 1986 [76] and Littlewood in 1987 [77]. A complete stochastic treatment that compared the predictive accuracy among competing reliability models using Bayesian principles was published by Sahinoglu in 2001 [16].

1.6.1 Software Reliability Models in the Time Domain

Next we study time-domain (not effort-based) models, in which time is either continuous (nonstop) or discrete (in distinct time units, such as days, weeks, or years). The basic goal is to model past failure data to predict behavior in the future (i.e., reliability projection) before software is released to the customer at the end of the development cycle. Reliability models are also useful to model failure patterns and provide input to maintain software before faults (defects) are triggered, causing failures.

The data consist of *failures per time period*, meaning the number of failures discovered in a time period, or *time between failures*, denoting the calendar or CPU time actually observed between software failures. We take up nontime- or effort-based models in Chapter 2, where efforts are made at equal intervals (e.g., days or weeks or months) or simply effort by effort, where the effort can be a test case or any input in a calendar time period. This approach can be likened to a time domain if efforts are made at equal intervals.

Any model used for prediction has to be tested for goodness of fit. In this book we do not distinguish between failures and faults, but recorded failures are actually triggered faults inherent in the software. There is another classification that may be used with respect to the type of statistical distribution that underlies the finite failure count within a given period. We consider the Poisson process over time for the countable finite quantity of failures, the binomial model, or other types. In the Poisson model we have a Poisson process over time where the total number of failures is not known in advance.

Poisson-type models assume that the number of failures detected within distinct time intervals are independent with separate means: (1) with the same rate of failure, the homogeneous Poisson process; (2) with a varying rate of failure, the nonhomogeneous Poisson process (NHPP), or (3) comprising a compound Poisson process beyond the HPP and NHPP if the failures occur in sizes or clusters rather than in terms of the conventional assumption of a single failure at a time.

Binomial-type models are based on similar assumptions: a binomial setting in which (1) a software defect will be removed whenever a failure occurs, (2) there is a known quantity of embedded defects or faults independent in the program in advance, and (3) the hazard rates are identical for all defects. Models that differ from these two types of count processes we call “other types.”

1.6.2 Classification of Reliability Growth Models

Again, for failure distribution over time, whether the distribution is negative exponential, Weibull, or other, the models differ from one another. Let's first study the negative exponential class of failure time models in Poisson, binomial, and other types. In software reliability we employ the mean value function, $\mu(t) = E[M(t)]$, to represent the expectation of failures with respect to time, where $M(t)$ is a random process to denote the number of failures achieved until time t.

On the other hand, the failure intensity function, $\lambda(t) = \mu'(t)$, is the first derivative of $\mu(t)$ with respect to (w.r.t.) time. $\lambda(t)$ denotes the instantaneous rate of change of the expectation of failures w.r.t. time t. Note that the hazard rate $h(t) = f(t)[R(t)]^{-1}$ is the conditional failure density given that there were no failures up to time t. Equations (20) and (25) to (29) showed these facts.

Negative Exponential Class of Failure Times In this class, the failure intensity $\lambda(t)$ is in the form of a negative exponential. The binomial types for this class have a constant hazard rate $h(t) = c$ and $\lambda(t) = Nc\exp(-ct)$. The Poisson types in this class also have a constant hazard rate of $h(t) = c$, but with a negative exponential time to failure $f(t) = c\ \exp(-ct)$. However, the number of failures that occur over a given period of time for either a HPP or an NHPP is Poisson. Next, let's look at models contained in this class.

Jelinski–Moranda (J-M) De-eutrophication Model (Binomial Type) A very early model proposed in 1972 by Jelinski and Moranda is the J-M time-between-failures (i.e., negative exponential) model [42]. The model assumes N faults (or potential failures) triggered randomly with equal probability. One also assumes that the failure fix ("as good as new") time is negligible, and this leads to the software's improvement by the same amount at each fix. Now the hazard function during the time $x_i = t_i - t_{i-1}$ between the $(i-1)$st and ith failures is given by

$$h(x_i) = \phi[N - i + 1] \tag{198}$$

where N is the total count of software faults at the very beginning, with ϕ a proportionality constant. The hazard function remains a constant between the failures but decreases in steps of ϕ after the removal of each fault, a fact that results in the improvement of the time between failures. Now, let's study the mathematical–statistical model in which $x_i = t_i - t_{i-1}$ are i.i.d. with a negative exponential p.d.f. with mean $\theta = [\phi(N - i + 1)]^{-1} f(x_i) = (1/\theta)\exp(-x_i/\theta)$ is the p.d.f. of interarrival times. $\mu(t) = N[1 - \exp(-\phi t)]$ is the finite mean value function, since $\lim_{t\to\infty} \mu(t) = N\lambda(t) = Nk\exp(-\phi t)$ is the failure intensity function.

For the model above, the estimates of the parameters and reliability prediction are given by

$$\sum_{i=1}^{n} \frac{1}{\hat{N} - i + 1} = \frac{n}{\hat{N} - \left(1/\sum_{i=1}^{n} x_i\right)\left[\sum_{i=1}^{n} (i-1)x_i\right]} \tag{199}$$

$$\hat{\phi} = \frac{n}{N\left(\sum_{i=1}^{n} x_i\right) - \left[\sum_{i=1}^{n} (i-1)x_i\right]} \tag{200}$$

First, N is estimated from the first nonlinear equation and then installed in the second nonlinear equation to estimate k. Then, after $n = i - 1$ faults have been observed, the estimate of the MTBF for the $(n + 1)$st fault is $\{z(t)\}^{-1} = \lfloor 1/\hat{\phi}(\hat{N} - n) \rfloor$. Shooman's safeguard reliability model (1972) is very similar to the J-M model [37]. These pioneering models have inspired others in sequence and have since been replaced by more modern methods.

Moranda's Geometric Model (Poisson Type) The geometric model proposed by Moranda is a variation of the original J-M model [43]. The interarrival time for failures is also a negative exponential, $f(x_i) = D\phi^{i-1} \exp(-D\phi^{i-1}x_i)$, whose mean decreases with respect to a *geometric* trend [i.e., $h(t) = D\phi^{i-1}$, $i = 1, 2, \ldots, n; 0 < \phi < 1$ at the $(i - 1)$st failure]. The expected time between failures is $E(X_i) = h^{-1}(t_{i-1})$. The hazard rate decreases in a geometric progression as each failure occurs. The functional form of the failure intensity (in terms of the expected number of failures) is geometric. The mean value and failure intensity functions, where $\beta = -\ln \phi$, $0 < \phi < 1$ in an infinite failure model, are

$$\mu(t) = \frac{1}{\beta} \ln\{[D\beta \exp(\beta)]t + 1\} \quad \text{and} \quad \lim_{t \to \infty} \mu(t) = \infty \tag{201}$$

$$\lambda(t) = \frac{D \exp(\beta)}{[D\beta \exp(\beta)]t + 1} \tag{202}$$

To estimate the parameters, we take the natural logarithm of the likelihood function $\prod_{i=1}^{n} f(X_i)$ and the partial derivatives with respect to ϕ and D. The maximum likelihood estimators (MLEs) are then solutions of the following pair:

$$\hat{D} = \frac{\hat{\phi} n}{\sum_{i=1}^{n} \hat{\phi}^i X_i} \quad \text{and} \quad \frac{\sum_{i=1}^{n} i\hat{\phi}^i X_i}{\sum_{i=1}^{n} \hat{\phi}^i X_i} = \frac{n + 1}{2} \tag{203}$$

Using these MLEs and their invariance property, the MLE of the failure intensity and mean value function can be estimated by inserting these MLEs in the equations for $\mu(t)$ and $\lambda(t)$.

Goel–Okumoto Nonhomogeneous Poisson Process (Poisson Type) This Poisson-type model was proposed by Goel and Okumoto in 1979 using the number of failures observed per unit time in groups [46] (see Goel's 1985 paper [78] for a well-done overview). They suggested that the cumulative count of failures $N(t)$ observed at time t can be modeled as an NHPP, a Poisson process with time-varying failure rate, which follows a negative exponential distribution:

$$P[N(t) = y] = \frac{[\mu(t)]^y}{y!} e^{-\mu(t)}, \qquad y = 0, 1, 2, \ldots \tag{204}$$

where

$$\mu(t) = N(1 - e^{-bt}) \tag{205}$$

is the mean value function for b, the per-fault detection rate. N, the expected number of faults, is not known (hence not of binomial type) and has to be estimated. The failure intensity function,

$$\lambda(t) = \mu'(t) = Nbe^{-bt} \tag{206}$$

is strictly decreasing for $t > 0$. It is not difficult to see that $\mu(t)$ and $\lambda(t)$ are the cumulative function, $F(t)$, and probability density function, $f(t)$, of the negative exponential, respectively. The MLEs of N and b can be estimated as solutions for the following pair of equations:

$$\hat{N} = \frac{\sum_{i=1}^{n} f_i}{1 - e^{-\hat{b}t_n}} \quad \text{and} \quad \frac{t_n e^{-\hat{b}t_n} \sum_{i=1}^{n} f_i}{1 - e^{-\hat{b}t_n}} = \frac{f_i(t_i e^{-\hat{b}t_i} - t_{i-1}e^{-\hat{b}t_{i-1}})}{e^{-\hat{b}t_{i-1}} - e^{-\hat{b}t_i}} \tag{207}$$

The second equation is solved for $\hat{b}$ by numerical (nonlinear) techniques. Then it is substituted into the first equation to calculate $\hat{N}$. One can then substitute these MLEs to find others, such as

$$\hat{\mu}(t) = \hat{N}(1 - e^{\hat{b}t}) \quad \text{and} \quad \hat{\lambda}(t) = \mu'(t) = \hat{N}\hat{b}e^{\hat{b}t} \tag{208}$$

and hence the estimated expected number of faults to be detected in the $(n + 1)$st observation period is given by

$$\hat{N}(e^{-\hat{b}t_n} - e^{-\hat{b}t_{n+1}}) \tag{209}$$

Okumoto and Goel also determined an optimal release time (ORT) of observation for a software product if the reliability desired is R for a specified operational period of T_o [79]:

$$\text{ORT} = \frac{1}{b}\left\{\ln[a(1 - e^{-bT_o}) - \left[\ln\left(\ln\frac{1}{R}\right)\right]\right\} \tag{210}$$

Earlier, Schneidewind (1975) adopted the same model by assuring that each time period T during which the software is observed is of the same length [80]. That is, $t_i = iT$, $i = 1, 2, \ldots, n$ for some constant $T > 0$, and $N = \alpha/\beta$, where α and β are Schneidewind's model parameters for his ith time period: $\mu(t) = \alpha/\beta(1 - e^{-\beta_i})$. Therefore, Schneidewind's model defaults to Goel–Okumoto's NHPP and therefore does not need a repetitious coverage of the same principles [81].

There is a trend toward diminishing defect rates or failures with the negative exponential assumption. However, in real life, there have been cases where the failure rate first increases (due to adding code, etc.) and then decreases (due to fixes) or sometimes cruises at a constant rate (adding code, and at the same

time, an equal effect of fixes). Goel's 1985 paper generalized the Goel–Okumoto NHPP model using a three-parameter Weibull model [78]:

$$\mu(t) = N(1 - e^{-bt^d}) \tag{211}$$

$$\lambda(t) = \mu'(t) = Nbde^{-bt^d}t^{d-1} \tag{212}$$

where the shape parameter $d = 1$ gives a negative exponential with a constant hazard rate, and $d = 2$ for the Rayleigh model. The shape parameter $d < 1$ denotes infancy, $d = 1$ denotes useful life, and $d > 1$ denotes the wear-out period in the traditional bathtub curve of the hazard function for most electronic components.

Example 1. For a generalized Goel–Okumoto NHPP model, given the data N (the number of failures expected at the end of mission time) $= 100$ and b (the fault detection rate per fault) $= 0.02$, calculate the mean value of failures at 80 hours. Take $d = 1$ and $d = 2$, respectively.

$$\mu(t) = \begin{cases} N(1 - e^{-bt^d}) = 100(1 - e^{-0.02(80)}) = 100(1 - e^{-1.6}) \\ \quad = 79.8 \text{ failures} \qquad d = 1 \end{cases} \tag{213}$$

$$\begin{cases} N(1 - e^{-bt^d}) = 100(1 - e^{-0.02(80)(80)}) \\ \quad \approx 100 \text{ failures} \qquad d = 2 \end{cases} \tag{214}$$

Musa's Basic Execution Time Model (Poisson Type) John D. Musa's model was one of the earliest to use the actual central processing unit (CPU) execution time rather than the clock or calendar time, which is actually irrelevant to the operating stress of the software environment [54]. The fundamental assumptions are:

1. The cumulative number of failures, $M(t)$, follows a nonhomogeneous Poisson process where the probability distribution functions of the random process, with mean value function $\mu(t) = \beta_0(1 - e^{-\beta_1^t})$, vary with time. It is a finite failure model: $\lim_{t\to\infty} \mu(t) = \beta_0$.
2. The interfailure times are piecewise negative exponentially distributed, implying that the hazard rate for a single fault is a constant:

$$\lambda(t) = \mu'(t) = \beta_0\beta_1(1 - e^{-\beta_1^t}) \tag{215}$$

The conditional reliability and hazard functions after $i - 1$ failures have occurred are

$$R(t \mid t_{i-1}) = \exp\{-[\beta_0 \exp(\beta_1 t_{i-1})][1 - \exp(-\beta_1 t)]\} \tag{216}$$

$$h(t \mid t_{i-1}) = \beta_0\beta_1 \exp(\beta_1 t_{i-1})[\exp(-\beta_1 t)] \tag{217}$$

Assume that n failures have occurred, that t_n is the last failure time, and that $t_n + x$ is the stopping time. The MLEs of β_0 and β_1, which possess the invariance

property needed to estimate other functions, such as reliability, hazard, and failure intensity, are given by

$$\hat{\beta}_0 = \frac{n}{1 - \exp[-\hat{\beta}_1(t_n + x)]} \tag{218}$$

$$\frac{n}{\hat{\beta}_1} - \frac{n(t_n + x)}{\exp[\hat{\beta}_1(t_n + x)] + 1} - \sum_{i=1}^{n} t_i = 0 \tag{219}$$

Example 2. Let us consider a software program with an initial failure density of 10 failures/hour and 100 total failures to be experienced in an infinite time. Determine the failure intensity, $\lambda(t)$, and number of failures predicted, $\mu(t)$, at $t = 10$ and 100 hours. Use the basic execution model.

SOLUTION At $t = 10$, $\lambda(t) = \lambda_0 \exp(-\lambda_0/\nu_0)t = 10\exp[-(10/100)(10)] = 10\exp(-1) = 3.68$ failures/CPU hour. Note that $\lambda_0 = \beta_0\beta_1 = 10$, and $\beta_1 = 0.1$, and the initial number of failures $\beta_0 = 100$. $\mu(t) = \beta_0(1 - e^{-\beta_1^t}) = 10(1 - e^{-0.1(10)}) = 100(1 - e^{-1}) = 100(1-0.368) = 63$ failures.

At $t = 100$, $\lambda(t) = \lambda_0 \exp[-(\lambda_0/\nu_0)t] = 10\exp(-(10/100)(100)] = 10\exp(-10) = 0.454 \times 10^{-6}$ failures/CPU hour. $\mu(t) = \beta_0(1 - e^{-}\beta_1^t) = 100(1 - e^{-0.1(100)}) = 100(1 - e^{-10}) \approx 100$ failures.

Musa–Okumoto Logarithmic Poisson Execution Time Model (Poisson Type)
This is similar to the G-O NHPP model, in which the number of failures experienced by a certain time t, $M(t)$, also follows a nonhomogeneous Poisson process with a negative exponentially decreasing intensity function, $\lambda(t) = \lambda_0 \exp[\theta\mu(t)]$, where $\mu(t) = (1/\theta)\ln(\lambda_0\theta t + 1)$ is the mean value function, $\theta > 0$ is the failure decay parameter (or rate of reduction in the normalized failure intensity per failure), and λ_0 is the initial failure rate [47]. Hence, when $\mu(t)$ is substituted, we obtain $\lambda(t) = \lambda_0/\lambda_0\theta t + 1$, since $\lim_{t\to\infty}\lambda(t) \to \infty$. This is an infinite failure model compared to the basic execution model's finite behavior. The rate of decrease explains the fact that earlier fixes of the failures detected reduced the failure rate of the latter part, thus causing fewer fixes by the end. The difference from the G-O NHPP is that its mean value function is different. It is defined to be logarithmic Poisson since the number of failures expected over time is a logarithmic function. The logarithmic Poisson process is thought to be superior for highly nonuniform distributions. The data needed are actual times, $t_i, i = 1, 2, \ldots,$ or interfailure times, $x_i = t_i - t_{i-1}$.

If we let $\beta_0 = \theta^{-1}$ and $\beta_1 = \lambda_0\theta$, which is the same as $\lambda_0 = \beta_0\beta_1$ in the basic execution model, the failure intensity and mean value functions become

$$\lambda(t) = \frac{\lambda_0}{\lambda_0\theta t + 1} = \frac{\beta_0\beta_1}{\beta_1 t + 1} \tag{220}$$

$$\mu(t) = \beta_0 \ln(\beta_1 \mathrm{t} + 1) \tag{221}$$

The conditional reliability and hazard rate functions at time t after the $(i-1)$st failure are

$$R(t \mid t_{i-1}) = \left[\frac{\beta_1 t_{i-1} + 1}{\beta_1 (t_{i-1} + t) + 1}\right]^{\beta_0} \tag{222}$$

$$h(t \mid t_{i-1}) = \frac{\beta_0 \beta_1}{\beta_1 (t_{i-1} + t) + 1} \tag{223}$$

Note that "|" denotes "given that" or "conditional upon." Use the reparametrized model to find the MLEs from the failure intensity and mean functions:

$$\hat{\beta}_0 = \frac{n}{\ln(1 + \hat{\beta}_1 t_n)} \tag{224}$$

$$\frac{1}{\hat{\beta}_1} \sum_{i=1}^{n} \frac{1}{1 + \hat{\beta}_1 t_i} = \frac{n(t_n)}{(1 + \hat{\beta}_1 t_n) \ln(1 + \hat{\beta}_1 t_n)} \tag{225}$$

As in the basic execution model, these MLEs calculated using their invariance property can be substituted in the failure intensity and mean value functions to estimate $\hat{\lambda}(t)$ and $\hat{\mu}(t)$.

Example 3. Let us consider a software program with an initial failure density of 10 failures/hour and 100 total failures to be experienced in infinite time. Find the failure intensity, $\hat{\lambda}(t)$, and number of failures predicted, $\hat{\mu}(t)$, at $t = 10$ and 100 execution hours. Also, $\theta = 0.02$ (two defects per 100 hours will decrease with time). Use the logarithmic Poisson model.

SOLUTION At $t = 10$ hours,

$$\lambda(t = 10) = \frac{\lambda_0}{\lambda_0 \theta t + 1} = \frac{10}{(10)(0.02)(10) + 1} = 3.33 \text{ failures/CPU hour}$$

Note that $\lambda_0 = \beta_0 \beta_1 = 10$ and $\beta_1 = 10/50 = 0.2$, since $\beta_0 = \theta^{-1} = 50$. Then

$$\lambda(t = 10) = \frac{\beta_0 \beta_1}{\beta_1 t + 1} = \frac{(50)(0.2)}{(0.2)(10) + 1} = \frac{10}{3} = 3.33 \text{ failures/CPU hour}$$

Also, $\mu(t = 10) = \theta^{-1} \ln(\lambda_0 \theta t + 1) = 50 \ln[(10)(0.02)(10) + 1] = 50 \ln(3) = 55$ failures, or $\mu(t = 10) = \beta_0 \ln(\beta_1 t + 1) = 50 \ln[(0.2)(10) + 1] = 50 \ln(3) = 55$ failures.

At $t = 100$ hours,

$$\lambda(t = 100) = \frac{\lambda_0}{\lambda_0 \theta t + 1} = \frac{10}{(10)(0.02)(100) + 1}$$

$$= \frac{10}{21} = 0.476 \text{ failure/CPU hour}$$

Also, $\lambda_0 = \beta_0\beta_1 = 10$ and $\beta_1 = 10/50 = 0.2$, since $\beta_0 = \theta^{-1} = 50$. Then

$$\lambda(t = 100) = \frac{\beta_0\beta_1}{\beta_1 t + 1} = \frac{(50)(0.2)}{(0.2)(100) + 1} = \frac{10}{21} = 3.33 \text{ failures/CPU hour}$$

Also, $\mu(t = 100) = \theta^{-1}\ln(\lambda_0\theta t + 1) = 50\ln[(10)(0.02)(100) + 1] = 50\ln(21) = 152$ failures, or $\mu(t = 100) = \beta_0\ln(\beta_1 t + 1) = 50\ln[0.2(100) + 1] = 50\ln(21) = 152$ failures.

Littlewood–Verral Bayesian Model This model is a result of a Bayesian approach by Littlewood and Verral (1973) in which they regarded software reliability measures as representing the strength of belief that a program is operating successfully [55]. This opposed the classical view taken by the majority of models in which the reliability is a measure of goodness or success in a given number of random trials. Whereas the hazard rate is a function of the number of defects remaining, the L-V model assumed that it was a random variable, a fact that has caused uncertainty in the effectiveness of the fault correction or failure prevention process. Therefore, even though failure time distributions are negative exponential (assumed in earlier classical models to behave with a certain failure rate), that rate is a random variable under the principles of Bayesian prior and posterior analysis. The distribution of this random failure rate powered by a gamma prior is also a gamma posterior distribution.

An identical Bayesian approach was adopted independently by Sahinoglu for the failure and repair rates of power generators [82] and was used in later research [11,21] to estimate their FOR (forced outage rate) in the estimation of the electric power system reliability index, LOLE (loss of load expected). Littlewood's differential fault model (1981), a variant of the original L-V model that uses the hazard rate as a random variable in a Bayesian framework [83], was a binomial model using a Pareto class of interfailure time distributions. However, the reliability growth is modeled in a process of two mechanisms, such as fault detection and fault correction, similar to some earlier models that adopted the same approach of differing stages. Later, Keiller et al. (1983) proposed a variation of the model very similar to their initial model, using the same randomness of hazard rate but employing a different parameter (the shape parameter, α, rather than the scale parameter, ξ) of that prior distribution to explain the effect of change on reliability [84]. Although their model used a negative exponential class of failure time distributions, it was neither of Poisson or binomial type, but "other." There are many other Bayesian approaches, such as Liu's Bayesian geometric model and Thompson and Chelson's Bayesian model, to name but two [85,86].

Formulation of the L-V model can be summarized as follows. The sequential failure times are assumed to be independent exponential random variables with parameter λ_i:

$$f(x_i) = \lambda\exp(-\lambda x_i), \qquad i = 1, 2, \ldots, a, \quad \lambda > 0, \quad x_i > 0 \tag{226}$$

Now let the software failure rate λ have a prior distribution from the gamma family:

$$\theta_1(\lambda) = \frac{\xi^c}{\Gamma(c)}\lambda^{c-1}\exp(-\lambda\xi), \qquad \lambda > 0 \tag{227}$$

The joint distribution of data and prior, assuming that all shape and scale parameters are identical, is given by

$$k(\mathbf{x}, \lambda) = f(x_1, x_2, \ldots, x_n; \lambda) = \frac{\xi^a}{\Gamma(a)}\lambda^{n+a-1}\exp[-\lambda(x_T + \xi)] \tag{228}$$

where n is the number of occurrences and $x_T = \sum_{i=1}^{n} x_i$ represents the total sampled failure times for n occurrences. Thus, the posterior distribution for λ can be derived as

$$h(\lambda \mid \mathbf{x}) = \frac{k(\mathbf{x}, \lambda)}{\int_\lambda f(\mathbf{x}, \lambda)\, d\lambda} = \frac{1}{\Gamma(n + a)}(x_T + \xi)\lambda^{n+a-1}\exp[-\lambda(x_T + \xi)] \tag{229}$$

which is Gamma$[n + a, (x_T + \xi)^{-1}]$. For $h(\lambda \mid x_i) \sim$ Gamma$[\alpha + 1, (x_i + \xi_i)^{-1}]$, $E(\lambda) = (\alpha + 1/x_i + 1)$ using a quadratic loss function. Recall that $\mathbf{x}$ denotes a vector of x_i. Then the marginal distribution of the random variable, $x_i > 0, i = 1, 2, \ldots, n$, given the gamma prior, can be derived as

$$f(x_i \mid \alpha, \xi_i) = \frac{\alpha(\xi_i)^a}{(x_i + \xi_i)^{\alpha+1}} \tag{230}$$

which is a Pareto distribution with joint density

$$f(x_1, x_2, \ldots, x_n) = \frac{\alpha^n \prod_{i=1}^{n} (\xi_i)^a}{\prod_{i=1}^{n}(x_i + \xi_i)^{\alpha+1}} \tag{231}$$

For model and reliability estimation, if one assumes that $\xi_i = \beta_0 + \beta_1 i$ (the linear form) or $\xi_i = \beta_0 + \beta_1 i^2$ (the quadratic form), then by using the foregoing marginal distribution for the x_i's, we calculate the MLEs for α, β_0, and β_1 as solutions to the following system of equations:

$$\frac{n}{\hat{\alpha}} + \sum_{i=1}^{n} \ln \hat{\xi}_i - \sum_{i=1}^{n} \ln(x_i + \hat{\xi}_i) = 0 \tag{232}$$

$$\hat{\alpha} \sum_{i=1}^{n} \frac{1}{\hat{\xi}_i} - (\hat{\alpha} + 1) \sum_{i=1}^{n} \frac{1}{x_i + \hat{\xi}_i} = 0 \tag{233}$$

$$\hat{\alpha} \sum_{i=1}^{n} \frac{i'}{\hat{\xi}_i} - (\hat{\alpha} + 1) \sum_{i=1}^{n} \frac{i'}{x_i + \hat{\xi}_i} = 0 \tag{234}$$

where $\xi_i = \beta_0 + \beta_1 i'$ and $i' = i$ or $i' = i^2$. Using a uniform prior $U(a, b)$ for the shape parameter α, Littlewood and Verrall derived the marginal distribution of the x_i's as a function of β_0 and β_1 only. Once the three unknowns α, β_0, and β_1 are estimated, the linear intensity function is, for example,

$$\lambda_{\text{linear}}(t) = \frac{\alpha - 1}{\sqrt{\beta_0^2 + 2\beta_1 t(\alpha - 1)}} \tag{235}$$

A final procedure is to estimate the least-squares estimates using the fact that for a Pareto p.d.f., $E(X_i) = \xi_i/(\alpha - 1)$. Once the parameters are estimated, reliability measures such as reliability and failure intensity functions can be estimated. Additionally, the mean time to failure for the ith failure can be estimated as $E(X_i) = \text{MTTF} = \xi_{i'}/(\alpha - 1)$, where i' is the linear or quadratic assumed term for i. Again, recall that $\xi_i = \beta_0 + \beta_1 i'$ and $i' = i$ or $i' = i^2$.

A later paper by Mazzuchi and Soyer (1988) suggested that α, β_0, and β_1 are all random variables with selected priors to estimate these unknown parameters [56]. Musa and Okumoto in 1984 proposed that ξ_i be a function related inversely to the number of failures remaining, inspired by an efficient debugging process: $\xi_i = N(\alpha + 1)/\lambda_0(N - i)$, where N is the number of defects expected as the time lengthens, λ_0 is the initial failure intensity function, i is the failure index, and α is the shape parameter of the gamma prior for the rate λ [87]. It shows that the scale parameter increases as the number of remaining failures decreases with diminishing i.

Sahinoglu's Poisson^Geometric and Poisson^Logarithmic Series Models A generalized compound Poisson process model is proposed for estimation of the residual count of software failures in references 12 to 16. It is observed that conventional nonhomogenous Poisson process models do not allow for the possibility of multiple counts, and the compound Poisson model is superior when clumping of failures exists at any given epoch [31]. Specifically, a model called *Poisson^geometric* (or *stuttering Poisson*) is studied in which the underlying failure process is assumed to be Poisson while a geometrically distributed number of failures may be detected at each failure epoch. The model proposed is validated using a few of Musa's data sets. Further, the *Poisson^logarithmic* series (equivalent to negative binomial given certain assumptions) is studied similarly, where the compounding p.d.f. is logarithmic whereas the counting process is the the same as before, NHPP [12,15]. The CD-ROM comprises both programs used to calculate the reliability and failure functions. The results from these programs can easily be used to obtain compound Poisson plots [88].

GENERALIZATIONS OF THE POISSON MODEL The Poisson theorem asserts that a counting process is a Poisson process if the jumps in all intervals of the same length are identically distributed and independent of past jumps (an assumption of stationary and independent increments), and the events occur singly at each epoch (an assumption of orderliness) [89]. Failure interarrival times may be

negative-exponentially distributed, but this is not sufficient to prove that the counting process is Poisson [90, p. 434]. Let us observe two generalizations (sometimes called *degenerations*) of the Poisson process [91,92]: The first is the well-known NHPP, obtained by dropping the "stationary increments" property in the Poisson theorem and replacing it with the "time-dependent increments" property, where the Poisson failure arrival rate β varies with time t (e.g., in software testing or unexpected ambulance calls on an ordinary day). The second is the less popularly known compound Poisson process (CPP), which is the process obtained if the orderliness property is dropped from the conventional Poisson theorem and replaced with that of stationary jumps: Let Z_n be the size of the nth jump, where $\{Z_n, n = 1, 2, \ldots\}$ are i.i.d. random variables. Let $J(t)$ be the size of jumps that occur during $(0,t]$; then $N(t)$ is a compound Poisson process with $N(T) = Z_1 + Z_2 + \cdots + Z_{J(t)}, t \geq 0$ [89].

The discrete compound Poisson p.d.f. in this section is one of two types. It may be of geometric density type with its forgetfulness property to govern the failure size ($x > 1$) distribution, whereas the conventional Poisson is a special case when q (= variance/mean) = 1 [93–96]. The symbol ^ designates that the parent Poisson distribution to the left of ^ is compounded by the compounding distribution to the right of ^ [12]. A similar publication by Sahinoglu on the Poisson^geometric p.d.f. reports on a study of the limiting sum of Markov Bernoulli variables [17]. Or, if the forgetfulness property does not exist, there is a positive or negative correlation between the failures in a clump upon arrival. The author uses a logarithmic-series distribution (LSD) for jump sizes with a true-contagion property (positive correlation). The sum of LSD random variables governed by a Poisson counting process results in a Poisson^logarithmic series, which simply defaults to a negative binomial distribution (NBD) given that a certain mathematical assumption holds [97,102].

TRUNCATED POISSON^GEOMETRIC (STUTTERING POISSON) A compound Poisson with a specific compounding distribution has negative exponentially distributed failure interarrival times with rate β. This implies that the p.d.f. of negative exponential interarrival times is independent of or not influenced by the earlier arrival epochs—hence the forgetfulness property of the Poisson process. Suppose that each Poisson arrival dictates a positive discrete amount x of failures that are i.i.d. as $\{f_x\}$. Then the total number of demands follow a CP distribution within an affixed time interval given by [88]

$$P(X) = \sum_{Y=0}^{\infty} \frac{(\beta t)^Y e^{-\beta t}}{Y!} f^{Y^*}(X) X = 0, 1, 2, \ldots, \quad \beta > 0 \tag{236}$$

where $f^{Y^*}(x)$ is the Y-fold convolution of $\{f_x\}$ when $f_x = 1$ for $x = 1$ and $f_x = 0$ for $x = 0$ for a conventional Poisson process. Therefore, this equation reduces to a Poisson distribution in the case of a single failure per arrival. On the other hand, the geometric distribution is given as

$$f_x(x) = (1 - r)r^{x-1}, \qquad x = 1, 2, 3, \ldots \tag{237}$$

Thus, a special case of CP distribution is the Poisson^geometric. The rate β of the Poisson process is the average number of arrivals per unit time, and r is the probability of finding the next independent failure in the batch or clump within each arrival. Then $p = 1 - r$ is the probability of starting the Poisson process for the next arrival. In summary, the total count of failures $X = \sum x_i$ within time interval t is a Poisson^geometric distribution [12], where $P(X = 0|Y = 0) = e^{-\beta t}$, or $e^{-\beta}$ if $t = 1$:

$$P(X|Y) = \sum_{Y=1}^{X} \frac{(\beta t)^Y e^{-\beta t}}{Y!} \binom{X-1}{Y-1} r^{X-Y}(1-r)^Y,$$
$$X = 1, 2, 3 \ldots, \quad 0 < r < 1, \quad \beta < 0 \tag{238}$$

From equation (238), the joint distribution of $P[X(t), Y(t)]$ for unit time $t = 1$ is given in Table 1.5. The expected value of the compound Poisson random variable for the marginal distribution of X is obtained by multiplying the first row by 0, the second row by 1, the third row by 2, and so on, and adding the columns. Summing over X rows of 1 to Y columns, we get

$$E(X) = \frac{\beta}{1-r} = \frac{\beta}{p} \tag{239}$$

TABLE 1.5 Joint Distribution of Poisson^Geometric X and Poisson Y

	Y					Expected,
X	0	1	2	3	...	$E(X) = \sum XP(X)$
0	$e^{-\beta t}$	—	—	—		0
1	—	$\beta e^{-\beta}(1-r)$	—	—		$\beta e^{-\beta}(1-r)$
2	—	$\beta e^{-\beta} r(1-r)$	$\frac{\beta^2 e^{-\beta}}{2!}\binom{1}{1}(1-r)^2$	—		$2\beta e^{-\beta} r(1-r) + \beta^2 e^{-\beta}(1-r)^2$
3	—	$\beta e^{-\beta} r^2(1-r)$	$\frac{\beta^2 e^{-\beta}}{2!}\binom{2}{1} r(1-r)^2$	$\frac{\beta^3 e^{-\beta}}{3!}\binom{2}{2}(1-r)^3$		$3\beta e^{-\beta} r^2(1-r) + 3\beta^2 e^{-\beta}(1-r)^2 + \frac{\beta^3 e^{-\beta}}{2}(1-r)^3$
4	—	$\beta e^{-\beta} r^3(1-r)$	$\frac{\beta^2 e^{-\beta}}{2!}\binom{3}{1} r^2(1-r)^2$	$\frac{\beta^3 e^{-\beta}}{3!}\binom{3}{2} r(1-r)^3$		...
5	—	$\beta e^{-\beta} r^4(1-r)$	$\frac{\beta^2 e^{-\beta}}{2!}\binom{4}{1} r^3(1-r)^2$	$\frac{\beta^3 e^{-\beta}}{3!}\binom{4}{2} r^2(1-r)^3$		...
⋮	—	⋮	⋮	⋮		⋮
Sum[a]	0	$\beta e^{-\beta}[(1-r)](1 + 2r + 3r^2 + 4r^3 + 5r^4 + \cdots)$	$\beta^2 e^{-\beta}(1-r)^2(1 + 3r + 6r^2 + 10r^3 + \cdots)$	...		$E(X) = \frac{\beta}{1-r}$ by math series expansion

Source: [12].

[a]Each term in the summation is the product of Poisson p.d.f. and geometric p.m.f. values that add up to $E(X) = \beta/(1-r)$ of the Poisson^geometric.

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{\beta(1+r)}{(1-r)^2} = \frac{\beta(1+r)}{(1-r)(1+r)} = \frac{\beta}{1-r} \tag{240}$$

which is identical to $E(X)$. Therefore, $E(X^2) = \beta/p + \beta^2/p^2$. Similarly, $E(Y) = \text{Var}(Y) = \beta$ in the Poisson process. Note that geometric p.m.f. is a discrete analog of the continuous negative exponential p.d.f. and has a similar nonaging Markov forgetfulness property. Consequently, for the $X \sim$ Poisson$\hat{}$geometric, and $x \sim$ geometric(r) processes,

$$\begin{aligned} Q &= \frac{\text{variance}}{\text{mean}} = \frac{E(x^2)}{E(x)} = \frac{(q+q^2)p^{-2}}{qp^{-1}} \\ &= \frac{q+q^2}{qp} = \frac{1+q}{p} = \frac{1+1-p}{p} = (2-p)p^{-1} \end{aligned} \tag{241}$$

The probabilities and moments of the zero-truncated P$\hat{}$G(μ, Q) are $(1 - e^{-}\beta t)^{-1}$ times the nontruncated:

$$\mu = E(X) = \beta t[(1-r)(1-e^{-\beta t})]^{-1} \tag{242}$$

TRUNCATED POISSON$\hat{}$LOGARITHMIC SERIES Of a number of chance mechanisms generating the negative binomial distribution (NBD), it can also be defined to be a Poisson sum of logarithmic-series-distributed (LSD) random variables [101,102]. Let $X = x_1 + x_2 + x_3 + \cdots + x_n$, where the x_i are LSD random variables with a p.m.f.

$$f_x = \frac{a\theta^x}{x}, \qquad x = 1, 2, 3, \ldots, \quad a = \frac{1}{\ln(1-\theta)}, \quad 0 < \theta < 1 \tag{243}$$

Then a randomly stopped sum of $x_i \sim \text{LSD}(\theta)$ has an NBD(k, q) with parameters

$$k = \frac{\beta}{\ln q} \quad (\text{or} \quad \beta = k \ln q) \tag{244}$$

$$q = \frac{1}{1-\theta} \tag{245}$$

where $E(x) = a\theta/(1-\theta)$. The stopping rule $N(t)$ is a compound Poisson. Let $x \sim$ LSD be reparametrized. Let $\theta = p/q$, and $q = p + 1 = (1-\theta)^{-1}$, $p = \theta(1-\theta)^{-1}$, and $a = -1/\ln(1-q) = 1/\ln q$; therefore, inserting for $p = \theta(1-\theta)^{-1} = q - 1$ and a in equation (245), $E(x) = (q-1)/\ln q$. Therefore,

$$\begin{aligned} &f_x = (x \ln q)^{-1} \left(\frac{p}{q}\right)^x, \quad x = 1, 2, 3, \ldots, \quad a = \frac{1}{\ln q}, \quad 0 < \theta < 1, \\ &q = p + 1 > 1, \quad q = (1-\theta)^{-1} \end{aligned} \tag{246}$$

Then $E(X) = kp$ for $X \sim$ NBD, where $k = \beta(\ln q)^{-1}$. Therefore, the probability of X failures for the decapitated (truncated) NBD is

$$P(X) = \frac{(k + X - 1)!}{(k - 1)!X!} \frac{p^X}{q^{k+X} - q^x} \tag{247}$$

See Tables 1.6 and 1.7.

Gamma, Weibull, and Other Classes of Failure Times In the classes in which the failure intensity $\lambda(t)$ is not in the form of a negative exponential but gamma,

TABLE 1.6 Cross Section of Historical Failure Times of Data Sets T1(X, Y) to T5(X, Y) in CPU[a]

T1: 21,700 object instructions delivered in 92 calendar days and 24.6 exec. hours of real time ($X = 136$, $Y = 133$): 130 singletons, 3 doubles
3, 33, 146, 227, 342, 351, 353, 444, 556, 571, 709, . . . , *5089, 5089*, . . . , 11811, *12559,12559*, . . . , 42188, *42996, 42996*, . . . , 81542, 82702, 85566, 88682 (last)

T2: 27,700 object instructions delivered in 72 calendar days and 30.2 exec. hours of real time ($X = 54$, $Y = 52$): 51 singletons, 1 triplet
191, 413, 693, 983, 1273, 1658, 2228, 2838, 3203, 3593, 3858, 4228, 5028, 6238, 6645, . . . , *62361, 62361, 62361*, 62661, 71682, 74201, 81091, 84339, . . . , 108708 (last)

T3: 23,400 object instructions delivered in 55 calendar days and 18.7 exec. hours of real time ($X = 38$, $Y = 37$): 36 singletons, 1 double
115, 115, . . . , 198, 376, 570, 706, . . . , 36818, 37381, 40151, . . . , 58065, 64789, 67335(last)

T4: 33,500 object instructions delivered in 71 calendar days and 14.6 exec. hours of real time ($X = 53$, $Y = 50$): 47 singletons, 3 doubles
5, 78, 219, 710, 715, 720, 748, 886, 1364, . . . , 3148, *4572, 4572*, . . . , 4664, . . . , 12850, *13021, 13021*, 13664, 14551, . . . , 15885, *16489, 16489*, 17263, . . . , 50896, 52422 (last)

T5: 2,445,000 object instructions delivered in 173 calendar days and 1785 exec. hours of real time ($X = 831$, $Y = 810$): 794 singletons, 13 doubles, 2 triplets, 1 quintiplet
37320, . . . , 2712322, *2715360, 2715360, 2715360*, 2861760, . . . , *3014160, 3014160*, 3104338, . . . , 3277760, *3291260, 3291260, 3291260, 3291260, 3291260*, 3294140, 3296120, 3299960, *3337820, 3337820*, 3369440, . . . , 7108021, *7249520, 7249520*, 7251320, . . . , 7407920, *7488620, 7488620*, 7493936, . . . , *10712788, 10712788*, 10729436, . . . , 15491208, *15566088, 15566088*, . . . , 15573288, . . . , *15837888, 15837888*, 16073188, *16139088, 16139088*, 16175088, . . . , *16277688,16277688*, . . . ,16279488, *16502688, 16502688*, . . . , 16750473, *16754343, 16754343, 16754343*, . . . , *16847943, 16847943*, . . . , 18855143, 18696039, 18747761, 18772252, 19608286, . . . , 27080864, *27080924, 27080924*, *20901716, 20901716*, . . . , 21120288 (last)

[a]The underlined entries are the failure repetitions upon one arrival, where $X =$ total number of failures, $Y =$ total number of arrivals [12,103,104].

TABLE 1.7 Statistical Analysis of Data Sets T1 to T5 for Poisson^Geometric and Poisson^Logarithmic

Data Set	No. of failures: X	No. of stops Y	First Moment: $E(x) = X/Y$	Second Moment: $E(x^2)$	Geometric Variable: $p = 1 - r$: $p = Y/X$	Empirical: $q =$ Var/Mean, $E(x^2)/E(x)$	q(Poisson^ Geometric): $q = (2 - p)/p$	q (NBD) from Nonlinear $E(x) = \frac{q-1}{\ln q}$
T1	136	133	1.0226	1.0677	0.9779	1.0441	1.0451	1.0219
T2	54	52	1.0385	1.1539	0.9630	1.1111	1.0769	1.0546
T3	38	37	1.0270	1.0841	0.9737	1.0555	1.0541	1.0275
T4	53	50	1.06	1.18	0.9434	1.1132	1.12	1.0556
T5	831	810	1.0259	1.0975	0.9747	1.0698	1.0523	1.0345

Weibull, or other distributions, the number of failures to occur over a given period of time is either homogeneous or nonhomogeneous Poisson, binomial, or other. Next, we look at the models that fall in these classes.

Yamada's Delayed and Ohba's Inflection S and Hyperexponential Models (Poisson Type) The interfailure distribution is gamma p.d.f., but the failure count per unit time is a Poisson type of model, not binomial. Ohba, Yamada, and joint authors proposed a software reliability growth model based on the assumption that there exist two types of defects, one of which is easier to detect [48,51]. Yamada et al. (1986) claimed that a testing process is a combination of a defect detection process and a defect isolation process. They indicated that their model is more reasonable and useful when defects can be classified into two such categories. Since the model employs more parameters than those of the simple Goel–Okumoto model, it becomes more complicated to apply this technique in practice.

With this method there can be a significant delay between the time of the first failure observation and the time of reporting it. The authors proposed a delayed S-shaped reliability model where the experienced growth curve of the cumulative count of defects that are detected becomes S-shaped. Their model is based on an NHPP to govern the failure count process, but their mean value function (similar to the Goel–Okumoto), whose limit is finite [i.e., $\lim_{t\to\infty} \mu(t) = \kappa$ (total count of defects) $< \infty$], reflects the delay in reporting the defects, as the team members become familiar with the software, usually followed by growth and then decay as the residual defects get more difficult to discover. This results in an S-shaped curve, unlike the exponential growth of the Goel–Okumoto model:

$$\mu(t) = \kappa[1 - (1 + \lambda t)e^{-}\lambda^{t}] \tag{248}$$

and the inter-failure-time distribution is gamma:

$$f(t) = \lambda^2 t e^{-\lambda t} \tag{249}$$

where t is the time of occurence, λ is the error detection rate, and κ is the total count of defects. Ohba later (1984) proposed another S-shaped reliability model, also an NHPP, which he called the *inflection S model* [48]. The new model explains a software failure detection process, where the defects detected are mutually dependent. This implies that the more failures that one detects, the more undetected failures become easily recognizable. This proposal introduces a more practical approach than that of earlier models, which invariably adopt the independence assumption of defects in a software program by default. The mean value function for this approach, on the other hand, is

$$\mu(t) = \kappa \frac{1 - e^{-\lambda t}}{1 + ie^{-\lambda t}} \tag{250}$$

where t is the time, λ is the error detection rate, i is the infection factor, and κ is the total count of defects, or total cumulative defect rate (count per unit time). On the other hand, the failure intensity function can be found to be $\lambda(t) = \mu'(t)$ for the first derivative of $\mu(t)$ with respect to time.

Yamada's delayed S and Ohba's inflection S model are both considered to account for the learning or training period (reflected as delayed or inflection patterns of the respective methods) that analysts experience at the infancy level of the software testing process. The mean value function (c.d.f.) and failure intensity function (p.d.f.) curves of both models are different from the negative exponential model. The exponential model assumes that the peak defect arrival is time zero (beginning) of the test phase and decays afterward. The delayed S model assumes a slightly delayed peak, and the inflection S model assumes a sharper peak later in a symmetrical shape.

The hyperexponential model is an extended exponential model. That is, the various sections of the software experience varying rates of failure, as when differing groups of people do too many different things in different languages under different circumstances. In mathematical–statistical analysis, the sum of these varying exponential curves can best be formulated by a hyperexponential. The cumulative count of failures by time t, $M(t)$, follows a Poisson process with a mean value function $\mu(t) = N \sum_{i=1}^{k} p_i[1 - \exp(-\beta_i t)]$, where $0 < \beta_i < 1$, $\sum_{i=1}^{k} p_i = 1$, and $0 < p_i < 1$; N is finite in a finite failure model. The defect counts in each testing interval, the f_i's, are given as input data, as well as the completion time of each period of software observation, t_i's. If $k = 1$, it is an NHPP. Since $\lambda(t) = \mu'(t)$ and Np_i is the expected number of faults for the ith class,

$$\lambda(t) = N \sum_{i=1}^{k} p_i \beta_i [1 - \exp(-\beta_i t)] \tag{251}$$

is strictly decreasing for $t > 0$. For model estimation and reliability prediction, estimate N_i as the number of defects in each class from the MLE equations presented in the NHPP model. The MLE of N is then the sum of the MLEs over the k classes. If the practice suggests that there exist only two classes,

such as new versus old or easy versus difficult, the new model is called the *modified exponential software reliability growth model* [51]. Also, Laprie and Kanoun designed a hyperexponential model for $k = 2$ with an equivalent failure rate function [105], identical to $h(t)$ formula on page 46 in 1.3.3.

$$\lambda(t) = \frac{p_1\beta_1 e^{-\beta_1 t} + p_2\beta_2 e^{-\beta_2 t}}{p_1 e^{-\beta_1 t} + p_2 e^{-\beta_2 t}} \tag{252}$$

and then they derived a system availability model to integrate the two classes: $k = 1$ for hardware and $k = 2$ for software.

Schick–Wolverton Model (Binomial Type) Schick and Wolverton modified the J-M model by assuming that the failure rate is not only proportional to the number of operating errors, but also increases linearly with operating time t, where the hazard function is [45]

$$h(t) = k(N - i + 1)t \tag{253}$$

The interfailure times are assumed to be with respect to the *Weibull* p.d.f., with the negative exponential being a special case. Later, they proposed a more general model [106] in which the per-fault hazard rate is parabolic instead of a linear function as in their original model:

$$h(t) = k(N - i + 1)(-b_1 t^2 + b_2 t + b_3) \tag{254}$$

Critics have pointed out that the model is no longer valid since $h(t)$ should decrease over time. The main reason is that the latter errors are hidden and difficult to encounter in operation in modern software operations.

Duane and AMSAA Model (Poisson Type) This model was originally proposed as a hardware design by Duane [49], who discovered that the cumulative failure rate or cumulative hazard function $H(t)$, versus the accumulated testing time, resulted in a straight line on log-log plotting paper [49]. It is also an NHPP, where the fact that the failure intensity function has the same rate for a *Weibull* p.d.f. has been used for some software systems. It is a reliability growth model later adopted by AMSAA (Army Material Systems Analysis Activity), which uses a relationship between a cumulative test time and cumulative failures. It is an infinite failure model, since $\lim_{t\to\infty} \mu(t) = \infty$. The reason it is also referred to as a *power model* (Section 1.3.3) is because the mean value function for the cumulative number of failures by time t is taken as a power of t:

$$\mu(t) = at^b \qquad \text{for } a > 0, b > 0$$
$$(b = 1 \text{ implies a homogeneous Poisson process}) \tag{255}$$

The actual times to failure need to be given or the elapsed time between failures, where $t_0 = 0$. The cumulative count of failures, $M(t)$, follows a Poisson process

with a mean value function $\mu(t) = at^b$. If we divide the right- and left-hand sides by total testing time T and take the natural log of both sides, we obtain

$$Y = \ln \frac{\mu(t)}{T} = \ln \frac{at^b}{T} = \ln a + (b-1) \ln T \tag{256}$$

One plots this equation versus T on ln-ln plotting paper to get a straight line. On the other hand, $\lambda(t) = abt^{b-1}$ is the failure intensity function strictly increasing for $b > 1$ (no reliability growth recorded), strictly decreasing for $0 < b < 1$ (reliability growth recorded), and constant for $b = 1$ (homogeneous Poisson process with a constant rate). Ref. 107 derived the MLEs for a and b to be, where $t_n = T$,

$$\hat{a} = \frac{n}{T^{\hat{b}}} \quad \text{and} \quad \hat{b} = \frac{n-1}{\sum_{i=1}^{n} \ln(T/t_i)} \tag{257}$$

which when inserted in $\hat{\mu}(t) = \hat{a}t^{\hat{b}}$ and $\hat{\lambda}(t) = \hat{a}\hat{b}t^{\hat{b}-1}$ give the MLEs for their respective functions. In 1974 in the AMSAA model, Ref. 107 also derived the MLE for MTTF $= \hat{\mu} = t_{(n)}/n\hat{b}$ for the time to the $(n+1)$st failure, and constructed confidence intervals for the MTTF reliability measure for unrepairable systems.

APPENDIX 1A: 500 COMPUTER-GENERATED RANDOM NUMBERS

0.6953	0.5247	0.1368	0.9850	0.7467	0.3813	0.5827	0.7893	0.7169	0.8166
0.0082	0.9925	0.6874	0.2122	0.6885	0.2159	0.4299	0.3467	0.2186	0.1033
0.6799	0.1241	0.3056	0.5590	0.0423	0.6515	0.2750	0.8156	0.2871	0.4680
0.8898	0.1514	0.1826	0.0004	0.5259	0.2425	0.8421	0.9248	0.9155	0.9518
0.6515	0.5027	0.9290	0.5177	0.3134	0.9177	0.2605	0.6668	0.1167	0.7870
0.3976	0.7790	0.0035	0.0064	0.0441	0.3437	0.1248	0.5442	0.9800	0.1857
0.0642	0.4086	0.6078	0.2044	0.0484	0.4691	0.7058	0.8552	0.5029	0.3288
0.0377	0.5250	0.7774	0.2390	0.9121	0.5345	0.8178	0.8443	0.4154	0.2526
0.5739	0.5181	0.0234	0.7305	0.0376	0.5169	0.5679	0.5495	0.7872	0.5321
0.5827	0.0341	0.7482	0.6351	0.9146	0.4700	0.7869	0.1337	0.0702	0.4219
0.0508	0.7905	0.2932	0.4971	0.0225	0.4466	0.5118	0.1200	0.0200	0.5445
0.4757	0.1399	0.5668	0.9569	0.7255	0.4650	0.4084	0.3701	0.9446	0.8064
0.6805	0.9931	0.4166	0.1091	0.7730	0.0691	0.9411	0.3468	0.0014	0.7379
0.2603	0.7507	0.6414	0.9907	0.2699	0.4571	0.9254	0.2371	0.8664	0.9553
0.8143	0.7625	0.1708	0.1900	0.2781	0.2830	0.6877	0.0488	0.8635	0.3155
0.5681	0.7854	0.5016	0.9403	0.1078	0.5255	0.8727	0.3815	0.5541	0.9833
0.1501	0.9363	0.3858	0.3545	0.5448	0.0643	0.3167	0.6732	0.6283	0.2631
0.8806	0.7989	0.7484	0.8083	0.2701	0.5039	0.9439	0.1027	0.9677	0.4597
0.4582	0.7590	0.4393	0.4704	0.6903	0.3732	0.6587	0.8675	0.2905	0.3058
0.0785	0.1467	0.3880	0.5274	0.8723	0.7517	0.9905	0.8904	0.8177	0.6660
0.1158	0.6635	0.4992	0.9070	0.2975	0.5686	0.8495	0.1652	0.2039	0.2553
0.2762	0.7018	0.6782	0.4013	0.2224	0.4672	0.5753	0.6219	0.6871	0.9255
0.9382	0.6411	0.7984	0.0608	0.5945	0.3977	0.4570	0.9924	0.8398	0.8361
0.5102	0.7021	0.4353	0.3398	0.8038	0.2260	0.1250	0.1884	0.3432	0.1192

(*Continued*)

APPENDIX 1A (*Continued*)

0.2354	0.7410	0.7089	0.2579	0.1358	0.8446	0.1648	0.3889	0.5620	0.6555
0.9082	0.7906	0.7589	0.8870	0.1189	0.7125	0.6324	0.1096	0.5155	0.3449
0.6936	0.0702	0.9716	0.0374	0.0683	0.2397	0.7753	0.2029	0.1464	0.8000
0.4042	0.8158	0.3623	0.6614	0.7954	0.7516	0.6518	0.3638	0.3107	0.2718
0.9410	0.2201	0.6348	0.0367	0.0311	0.0688	0.2346	0.3927	0.7327	0.9994
0.0917	0.2504	0.2878	0.1735	0.3872	0.6816	0.2731	0.3846	0.6621	0.8983
0.8532	0.4869	0.2685	0.6349	0.9364	0.3451	0.4998	0.2842	0.0643	0.6656
0.8980	0.0455	0.8314	0.8189	0.6783	0.8086	0.1386	0.4442	0.9941	0.6812
0.8412	0.8792	0.2025	0.9320	0.7656	0.3815	0.5302	0.8744	0.4584	0.3585
0.5688	0.8633	0.5818	0.0692	0.2543	0.5453	0.9955	0.1237	0.7535	0.5993
0.5006	0.1215	0.8102	0.1026	0.9251	0.6851	0.1559	0.1214	0.2628	0.9374
0.5748	0.4164	0.3427	0.2809	0.8064	0.5855	0.2229	0.2805	0.9139	0.9013
0.1100	0.0873	0.9407	0.8747	0.0496	0.4380	0.5847	0.4183	0.5929	0.4863
0.5802	0.7747	0.1285	0.0074	0.6252	0.7747	0.0112	0.3958	0.3285	0.5389
0.1019	0.6628	0.8998	0.1334	0.2798	0.7351	0.7330	0.6723	0.6924	0.3963
0.9909	0.8991	0.2298	0.2603	0.6921	0.5573	0.8191	0.0384	0.2954	0.0636
0.6292	0.4923	(1.0276	0.6734	0.6562	0.4231	0.1980	0.6551	0.3716	0.0507
0.9430	0.2579	0.7933	0.0945	0.3192	0.3195	0.7772	0.4672	0.7070	0.5925
0.9938	0.7098	0.7964	0.7952	0.8947	0.1214	0.8454	0.8294	0.5394	0.9413
0.4690	0.1395	0.0930	0.3189	0.6972	0.7291	0.8513	0.9256	0.7478	0.8124
0.2028	0.3774	0.0485	0.7718	0.9656	0.2444	0.0304	0.1395	0.1577	0.8625
0.6141	0.4131	0.2006	0.2329	0.6182	0.5151	0.6300	0.9311	0.3837	0.7828
0.2757	0.8479	0.7880	0.8492	0.6859	0.8947	0.6246	0.1574	0.4936	0.8077
0.0561	0.0126	0.6531	0.0378	0.4975	0.1133	0.3572	0.0071	0.4555	0.7563
0.1419	0.4308	0.8073	0.4681	0.0481	0.2918	0.2975	0.0685	0.6384	0.0812
0.3125	0.0053	0.9209	0.9768	0.3584	0.0390	0.2161	0.6333	0.4391	0.6991

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EXERCISES

1.1 At the end of one year of service, the reliability of a certain software product during its useful life period after the debugging process (assuming a constant failure rate) is 0.8.

(a) What is the failure rate of this software product in hours?

(b) If four of these products are put in series and active parallel independently, what are the annual reliability figures in series and active parallel systems, respectively?

(c) For active parallel, if 30% of the component failure rate may be attributed to common-mode failures, what will the annual reliability become for the two components in parallel?

(d) Suppose that the failure rate for a software component is given as 0.08 per hour. How many components must be placed in active parallel form if a distributed system of modules will have to run for 100 hours with a system reliability of no less than 95%?

(e) Assuming now that the annual reliability of the software module is improved to 0.95, a series system of four components is formed. A second set of four components is bought and a redundant system is built. What is the reliability of the new redundant system with (1) high-level redundancy by drawing the representation in numbered blocks, and (2) low-level redundancy by drawing the representation in numbered blocks?

1.2 **(a)** A wear test is run on 10 PC hard drives and the following times in months found:

$$27, 39+, 40, 54, 68+, 85, 93, 102, 135+, 144$$

Using the product limit (Kaplan–Meier) technique to account for censoring, make a nonparametric plot of the reliability and hazard functions.

(b) A nonreplacement test is run for 60 hours on 40 microprocessors. Five failures occur at 12, 19, 28, 39, 47 hours. Estimate the value of the constant failure rate.

Also find approximate upper and lower bounds for the MTTF.

1.3 The reliability of an operating system in time is given by

$$R(t) = \begin{cases} (1 - 0.2t)^2, & 0 \le t \le 5 \\ 0, & t > 5 \end{cases}$$

where the original p.d.f. of failure time was $f(t) = (0.4)(1 - 0.2t)$ for $0 < t \le 5$.

(a) Verify $R(t)$ using $f(t)$. Calculate the MTTF.

(b) Find the failure or hazard rate $h(t)$ Is the failure rate increasing or decreasing? Justify your result.

(c) How often should it be updated if failures are to be held to no more than 5%?

1.4 A constant-failure-rate device (a desktop PC) has a MTTF of 2000 hours. The vendor offers a one year warranty. What fraction of the PCs will fail during the warm-up period?

1.5 A software module being marketed is tested for two months and found to have a reliability of 0.99; the module is known to have a constant failure rate.

(a) What is the failure rate?

(b) What is the MTTF?

(c) What is the reliability of this product four years into its operation if it is in continuous use?

(d) What should the warranty time be to achieve an operational reliability of 95%?

1.6 The reliability growth models are outlined in Section 1.6.2. Among the most popular are the generalized Goel–Okumoto NHPP (when $c = 1$, the Weibull becomes exponential, and when $c = 2$, it becomes the Raleigh model) and the Musa–Okumoto logarithmic Poisson execution time model, which have different mean value functions to predict the number of failures at the end of a mission time t. Given the following input data, calculate and compare the mean values of the number of failures expected to be predicted by the end of a time t for each model. These models are all NHPP. Use the following data as necessary: t (the time at which to predict) $= 80$ CPU hours; a (the number of failures expected at the end of the mission time) $= \upsilon_0 = \kappa = 100$; $b = \theta$ (the fault detection rate per fault) $= 0.02$; and λ_0 (the initial failure intensity) $= 10$ per hour.

1.7 Table E1.7 lists uncensored grouped data on the failure of identical commercial software modules collected by an independent quality focused nonprofit organization. Draw a nonparametric plot of the reliability and cumulative hazard functions versus time.

TABLE E1.7

Interval (CPU seconds)	Number of Failures
$0 < t \leq 6000$	5
$6000 < t \leq 12{,}000$	19
$12{,}000 < t \leq 18{,}000$	61
$18{,}000 < t \leq 24{,}000$	27
$24{,}000 < t \leq 30{,}000$	20
$30{,}000 < t \leq 36{,}000$	17

1.8 Calculate the source–target (s-t) reliabilities of the following systems:

(a) Take $s = 1$, $t = 13$ in Figure E1.8(a).

(b) Take $s = 5$, $t = 9$ in Figure E1.8(b).

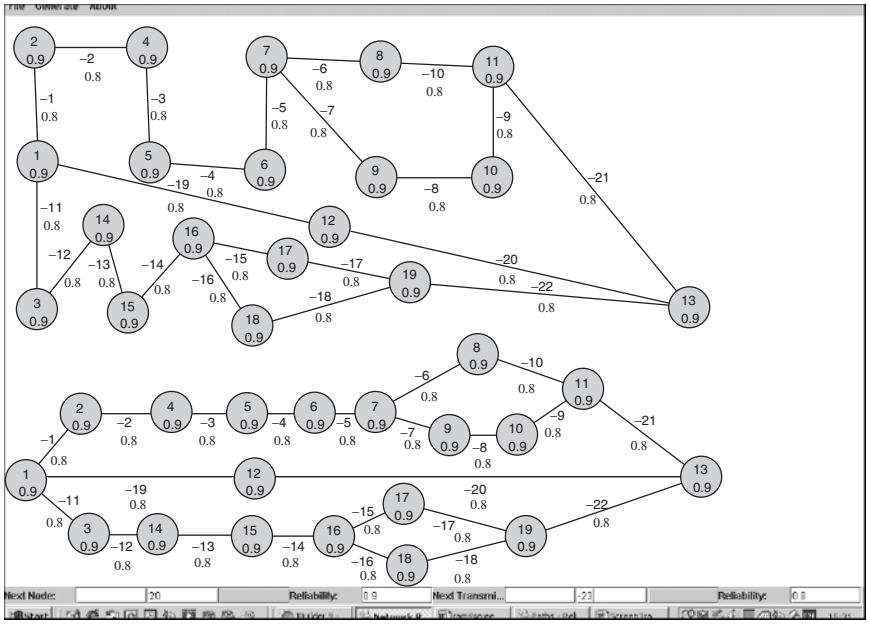


FIGURE E1.8(a)

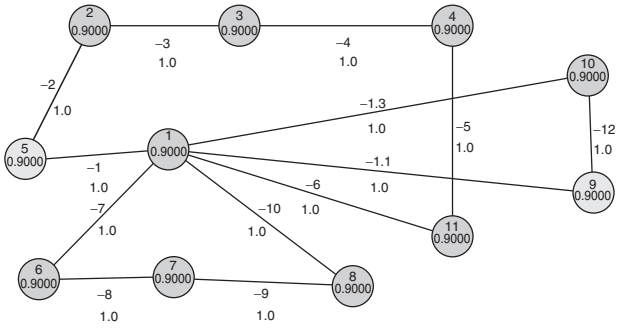


FIGURE E1.8(b)

1.9 A disk drive has a constant failure rate λ with an MTTF of 5000 hours. What is the probability of failure for one year of operation? What is the probability of failure for one year of operation if two of the drives are placed in active parallel mode with failures assumed to be independent? What if in series?

1.10 What is the probability of failure for one year of active parallel operation if the common-mode errors are characterized by $\beta = 0.2$?

1.11 Suppose that a system consists of two components placed in series, each with a failure rate $\lambda = 1$ per year. A redundant system is built consisting of four identical components. Derive and calculate expressions for the system's failure rates after $t = 1$ year of operation:

(a) for high-level redundancy

(b) for low-level redundancy

1.12 Suppose that without preventive maintenance, the failure (hazard) rate of a computer hard drive is given by $\lambda(t) = (1.1)(10^{-6})t + (1.0)(10^{-9})t^2$, where t is given in CPU hours and $\lambda(t)$ is per year.

(a) Calculate the design-life reliability of the hard drive for a design life of 10,000 CPU hours assuming that no preventive maintenance is performed.

(b) Suppose that by overhaul the hard drive is returned to as-good-as-new condition. How frequently should the PC company perform overhauls to accomplish a design reliability of at least 0.90?

1.13 Suppose that a computer keyboard is life-tested and has an MTTF of 68 hours and an MTTR (mean time to repair) of 1.5 hours. What is the availability with corrective maintenance? If the MTTR is reduced to 1 hour with extra measures, what MTTF can be tolerated without altering the machine availability?

1.14 An annual life test revealed that 92% of computer hard disks were found to be operable. According to these data, what is the long-term availability of these disks? If one is not happy with this availability figure and desires to reach an availability value of 0.98, how often must one perform testing and replacement procedures?

1.15 At the end of one year of service, the unreliability of certain General Electric light bulbs, assuming a constant failure rate, is 10%.

(a) What is the failure rate in hours?

(b) If two bulbs are put independently in series and active parallel, what are the annual reliability figures, respectively?

(c) For active parallel, if 20% of the component failure rate may be attributed to common-mode failures, what will the annual reliability become for the two components in active parallel?

(d) Suppose that the design failure rate for the component is given to be 0.008 per hour. How many bulbs must be placed in active parallel if a system of lamps will have to run for 100 hours with a system reliability of no less than 95%.

(e) Assuming now that the annual reliability of the bulb is improved to 0.96, a series system of three bulbs is formed. A second set of three

components is bought and a redundant system is built. What is the reliability of the new redundant system with (1) high-level redundancy; (2) low-level redundancy?

1.16 Let us consider a software program with an initial failure intensity of 10 failures/hour and 100 total failures to be experienced in infinite time. What is the failure intensity $\lambda(\tau)$ going to be at $\tau = 10$ and 100 execution hours (**a**) for Musa's basic execution model and (**b**) for the Musa–Okumoto logarithmic Poisson model? Having calculated those, estimate the predicted number of failures, $\mu(\tau)$ at $\tau = 10$ and $\tau = 100$ execution hours for parts (a) and (b).

1.17 For the grouped (clustered) failure data formulated in Table E1.17(a) and presented in Table E1.17(b) for system T38 [34], use Sahinoglu's compound Poisson software reliability model to estimate the number of failures to be expected by the end of the mission time (at the end of eleventh interval) at step 7 by applying (**a**) the Poisson^geometric distribution model; (**b**) the Poisson^logarithmic series distribution model; (**c**) the Goel–Okumoto model; (**d**) the Musa–Okomoto model.

TABLE E1.17(a) Grouped Failure Data Format

Interval Number	Total Test Time After the Interval	Duration of the Interval	Number of Failures in the Interval	Cumulative Number of Failures
1	x_1	x_1	y_1	z_1
2	x_2	$x_2 - x_1$	y_2	z_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p	x_p	$x_p - x_{p-1}$	y_p	z_p

1.18 Consider the following multiply censored data without replacement for the computer hard disks in months:

30 39 40+ 54 68+ 85 93+ 102 135 144+

(a) Prepare a spreadsheet for these ungrouped data showing time, reliability function, and hazard function columns. Make a nonparametric plot of the reliability and hazard functions. Estimate the MTTF for the pumps.

(b) Suppose that these were the initial $n = 10$ failures recorded at the end of the experiment (type II: stop at nth failure) for a total of $N = 20$ disks. Estimate the MTTF and find the upper and lower 90% confidence interval estimates. Calculate for (1) nonreplacament and (2) replacement.

1.19 Given the failure times 5.2, 6.8, 11.2, 16.8, 17.8, 19.6, 23.4, 25.4, 32.0, and 44.8 hours.

TABLE E1.17(b) Grouped Failure Data for System T38

Interval Number	Total Test Time at the End of the Interval (CPU hours)	Duration of the Test Interval (CPU hours)	Number of Failures in the Interval	Cumulative Number of Failures
1	5	5	1	1
2	15	10	0	1
3	25	10	16	17
4	35	10	1	18
5	45	10	1	19
6	50	5	0	19
7	65	15	1	20
8	75	10	3	23
9	95	20	2	25
10	120	25	7	32
11	125	5	0	32

(a) Using a Weibull probability p.d.f., determine the scale and shape parameters.

(b) Using a negative exponential p.d.f., estimate the failure rate, θ.

(c) Which is a better estimate? Why?

1.20 A constant-failure-rate device (PC monitor) has MTTF = 2000 hours. The vendor offers a one-year warranty. What fraction of the PC monitors will fail during the warranty period?

1.21 Given the time-domain data sets T1, T2, T3, T4, and T5 in the data bank of this book's CD-ROM use five software reliability models of your choice to predict the expected value and hazard functions at the end of the mission. Explain which is best, and why.

1.22 Given the grouped data for WD1, WD2, WD3, WD4, and WD5 in the data bank of the CD-ROM, use five software reliability models of your choice to predict expected value and hazard functions at the end of the mission. Which is best, and why?

1.23 Given $q = 3$ and mean $(M) = 10$, plot two curves for the p.d.f and survival function using the CD-ROM for both Poisson^geometric and Poisson^log series.

1.24 Using the uniform random table in Appendix 1A from right to left, generate at least a pair of random variables from each of (**a**) the p.m.f.'s in Section 1.3 and (**b**) the pdf's in Section 1.3 from Table 1.1 only.

1.25 Generate a pair of random deviates from all the rest of p.d.f.'s and p.m.f.'s not covered in Exercise 1.24 using Appendix 1A.